

# Gravitational Waves: Introduction

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# Who am I?

- PhD (2002) at Institute of Astronomy, University of Cambridge with Donald Lynden-Bell.
- Postdoc at Caltech then junior research fellow and Royal Society Research Fellow back in Cambridge.
- Now: Professor in the School of Mathematics at the University of Edinburgh.
- Research interests cover all aspects of gravitational wave source modelling, detection and exploitation in all frequency bands - LIGO, Pulsar Timing and LISA.

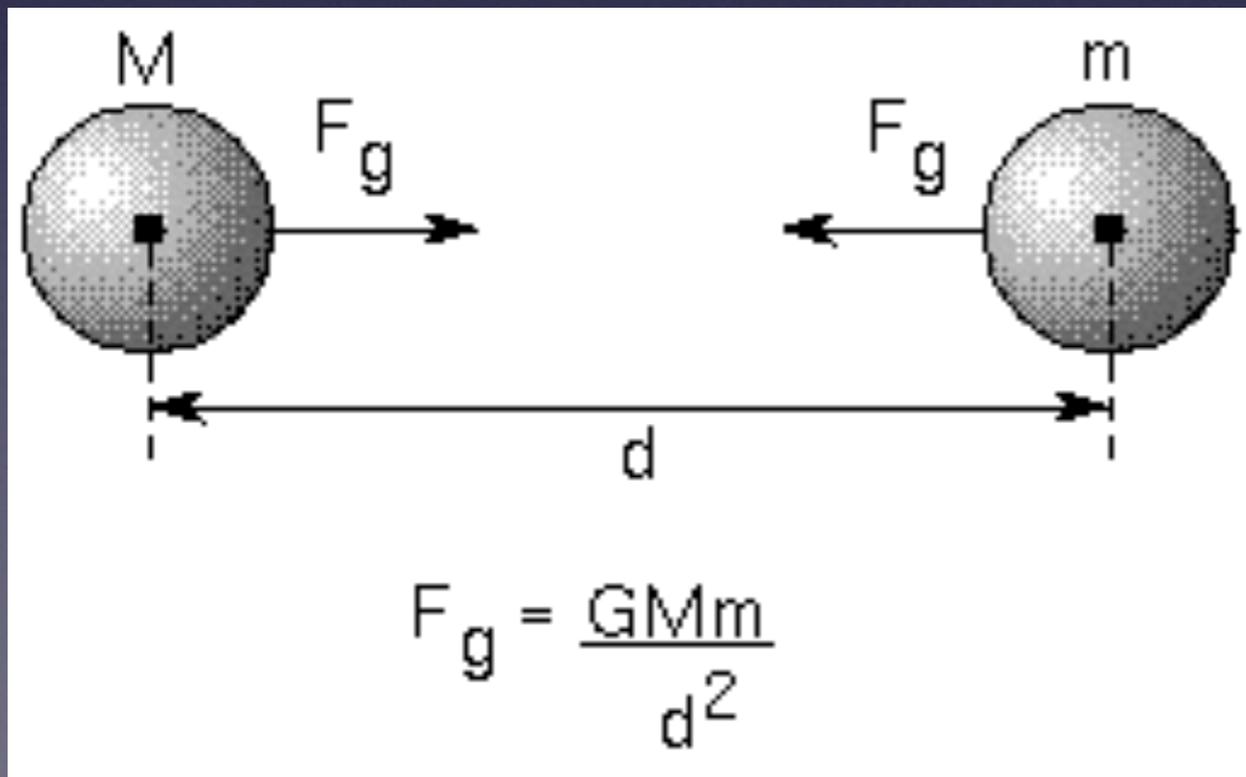
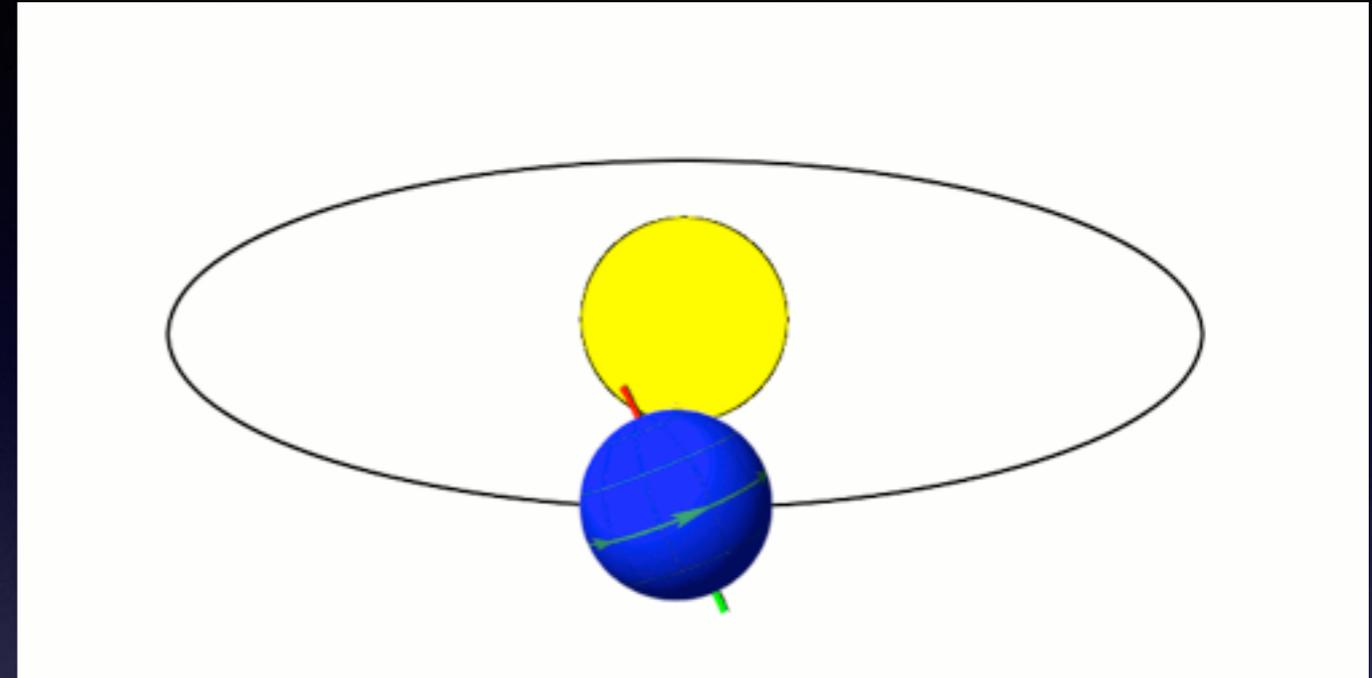


# Lecture Outline

- What are gravitational waves?
- Principles of gravitational wave detection.
- Current and planned gravitational wave detectors.
- Principles of signal analysis.
- Understanding gravitational wave detector noise/sensitivity curves.
- Estimating gravitational wave amplitudes/detectability for astrophysical sources.

# The nature of gravity

- Newton
  - Gravity is a universal force
  - “Action at a distance”
  - Newton’s law describes effect of gravity but does not explain its origin.



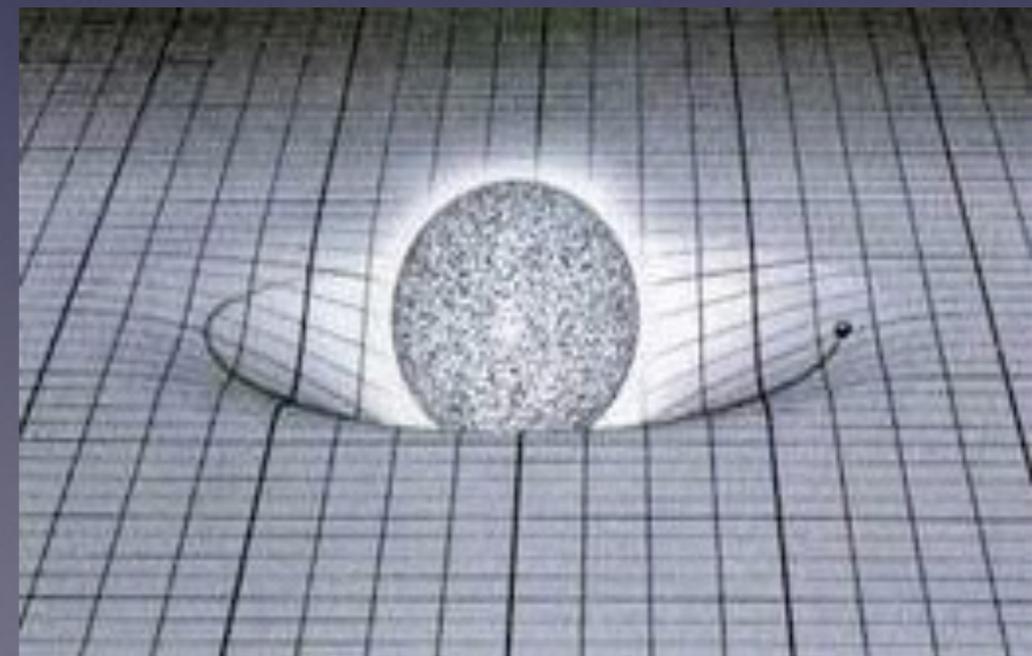
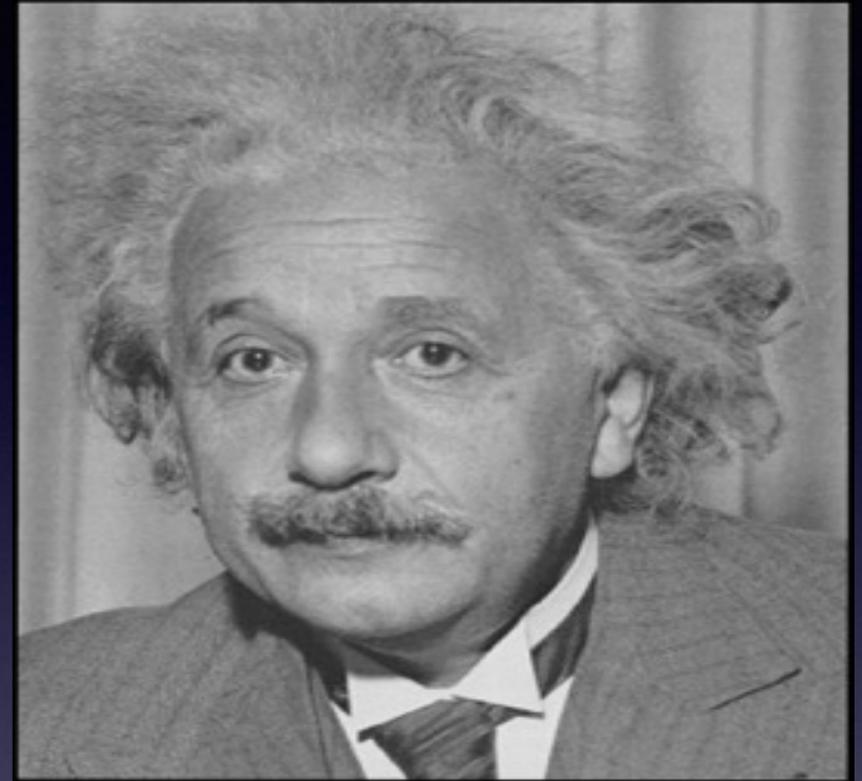
# The nature of gravity

- Einstein

- Gravity is spacetime curvature
- Any mass/energy bends spacetime in its vicinity
- Freely falling objects follow the background curvature.
- “Matter tells space how to curve, space tells matter how to move.”
- Einstein’s equations are simple to write

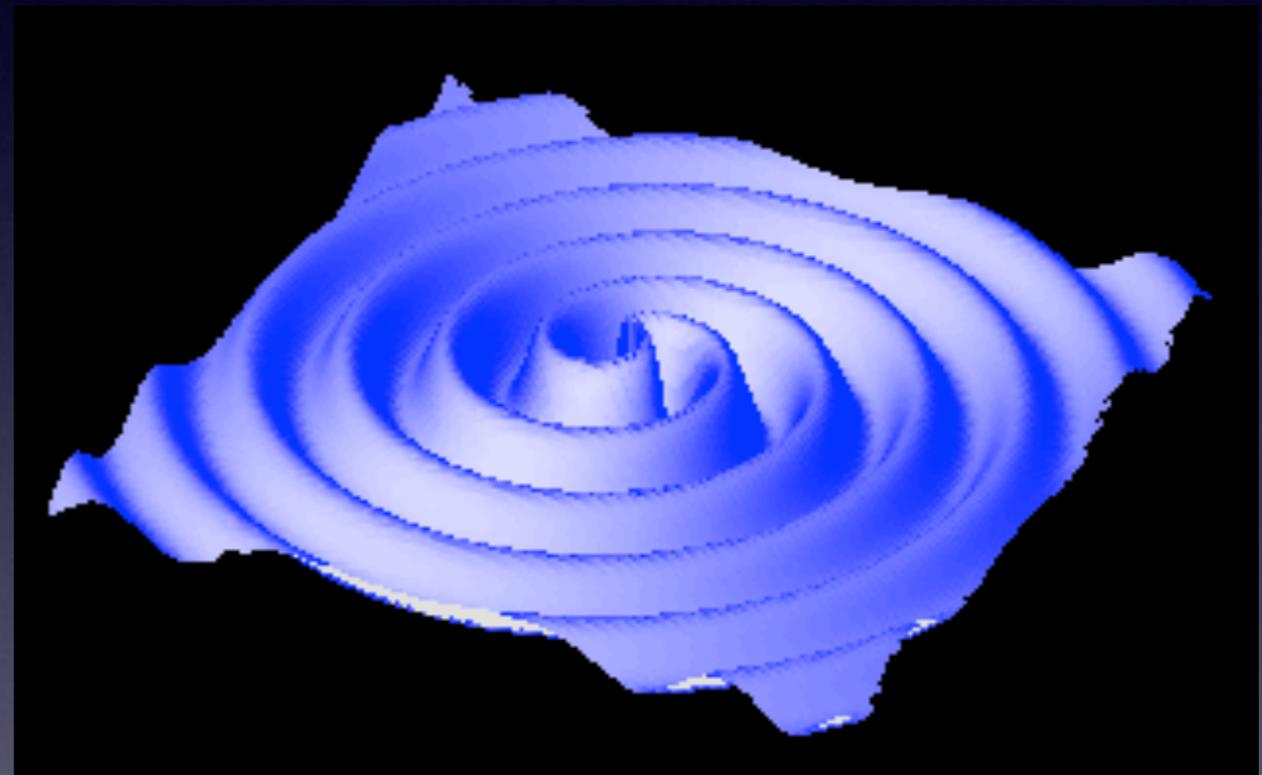
$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- but hard to solve!
- Incorporates finite propagation speed for gravitational information



# Gravitational Waves

- “Ripples in Spacetime” - rapidly moving mass creates fluctuations in spacetime curvature.
- These fluctuations propagate away at the speed of light - these are gravitational waves.
- GWs are very weakly interacting. Coupling constant  $G/c^4 \sim 10^{-44}$ .
- No absorption, but hard to detect!



# Gravitational Waves: key properties

- Conservation of mass and momentum mean that it is the *quadrupole moment* that radiates

$$I_{ij} = \int \rho x_i x_j dV$$

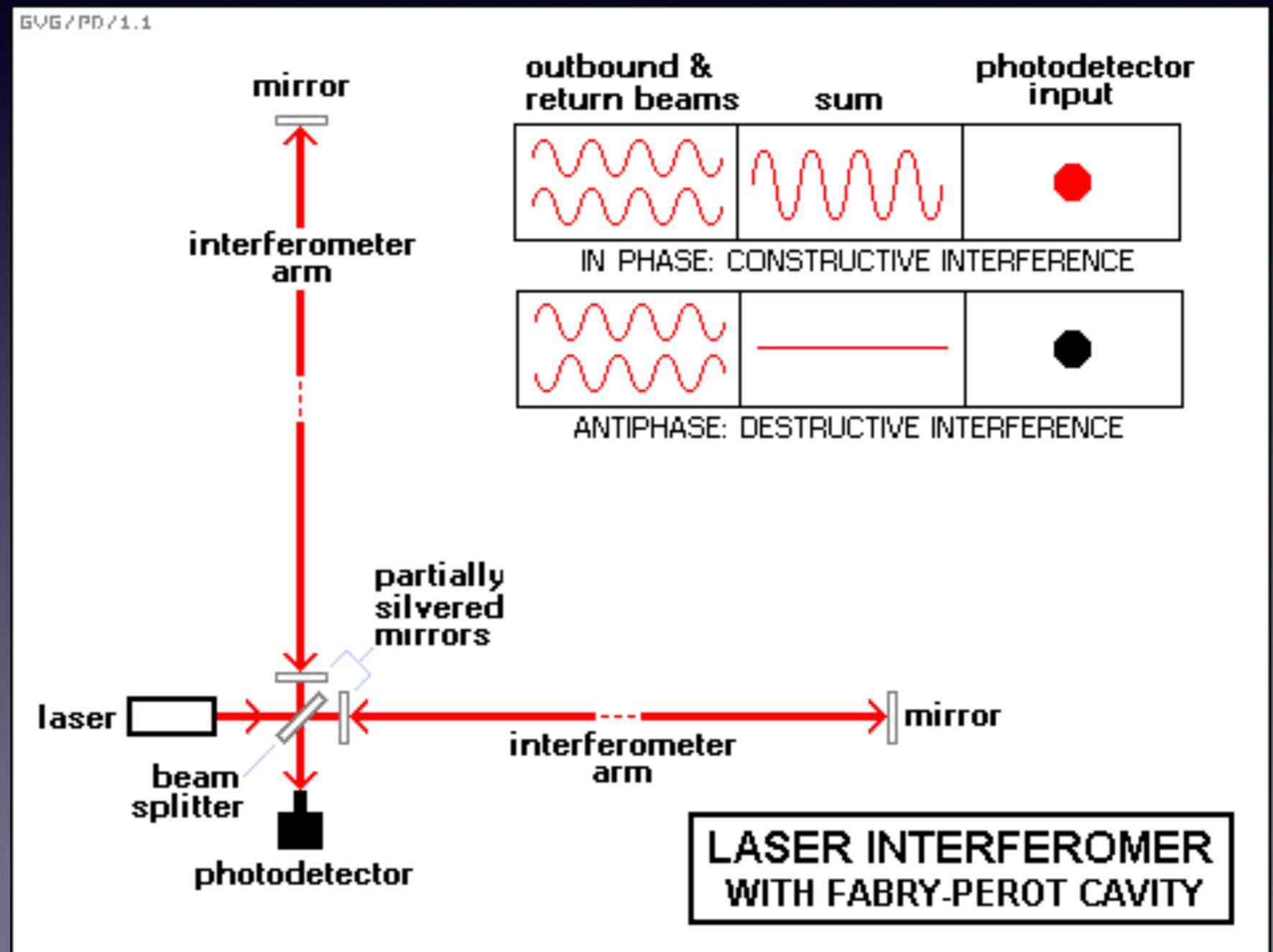
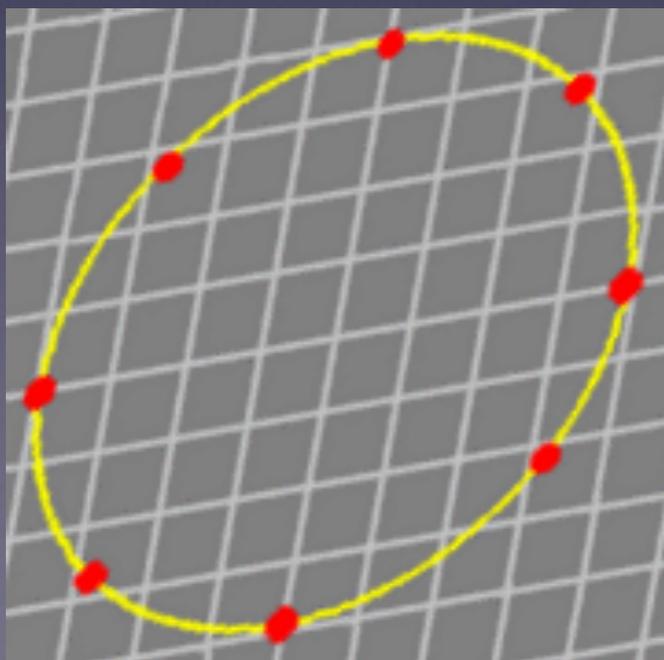
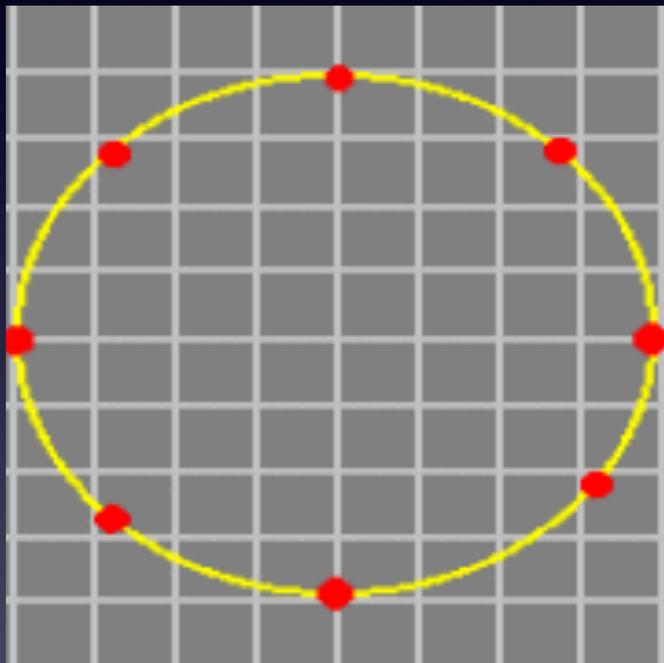
- A gravitational wave is a fluctuation in the spacetime metric - changes *distances* between objects.
- Characterised by a *strain*: fractional change in distance.
- The gravitational wave strain and rate of energy loss are

$$h \sim \frac{\ddot{I}_{ij}}{D} \qquad \dot{E} \sim \ddot{I}_{ij}^2$$

- Quadrupole waves - distances at right angles oscillate half a cycle out of phase.

# Gravitational Wave Detection - Interferometry

- Interferometers use lasers to measure distances.
- They exploit quadrupole nature of gravitational waves.



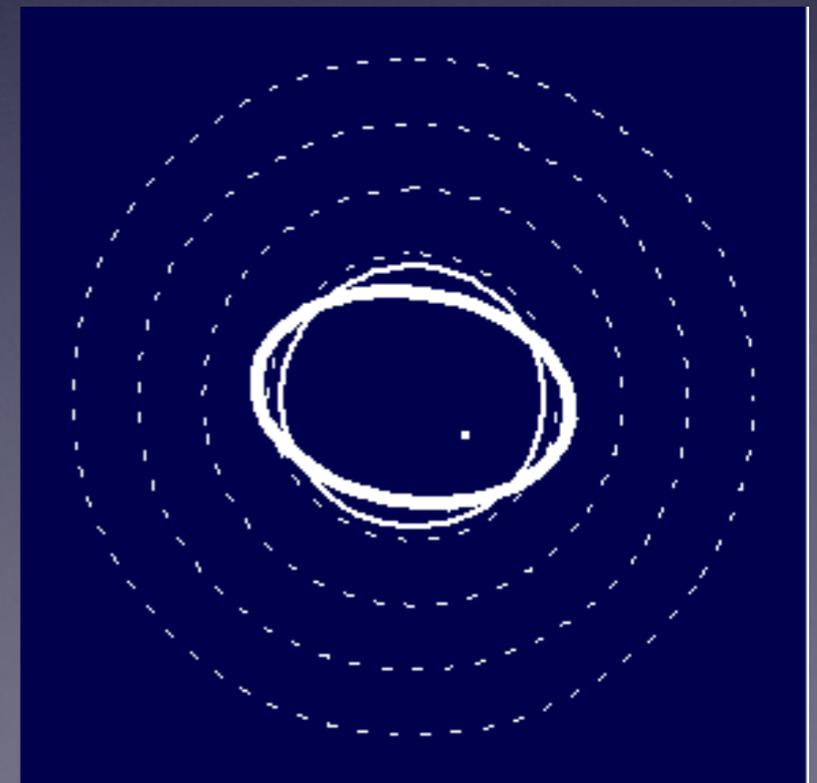
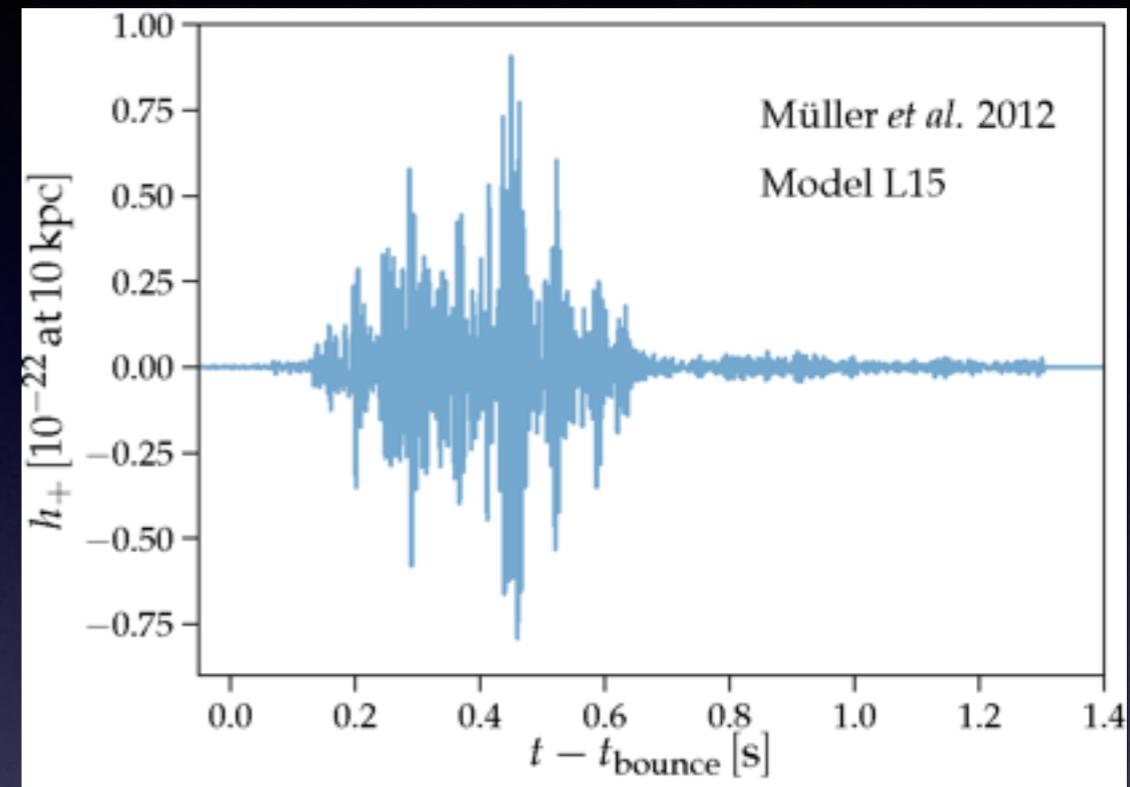
# Ground Based Interferometers

- A network of ground-based interferometers has been constructed over the last 15 years
  - LIGO: 2 4km detectors in the US. Hanford, Washington; Livingston, Louisiana.
  - Virgo: 3km detector near Pisa, Italy.
  - GEO: 600m detector near Hanover, Germany.
  - KAGRA: 4km underground detector in Japan.
- Plans for third generation detectors in 15+ years.



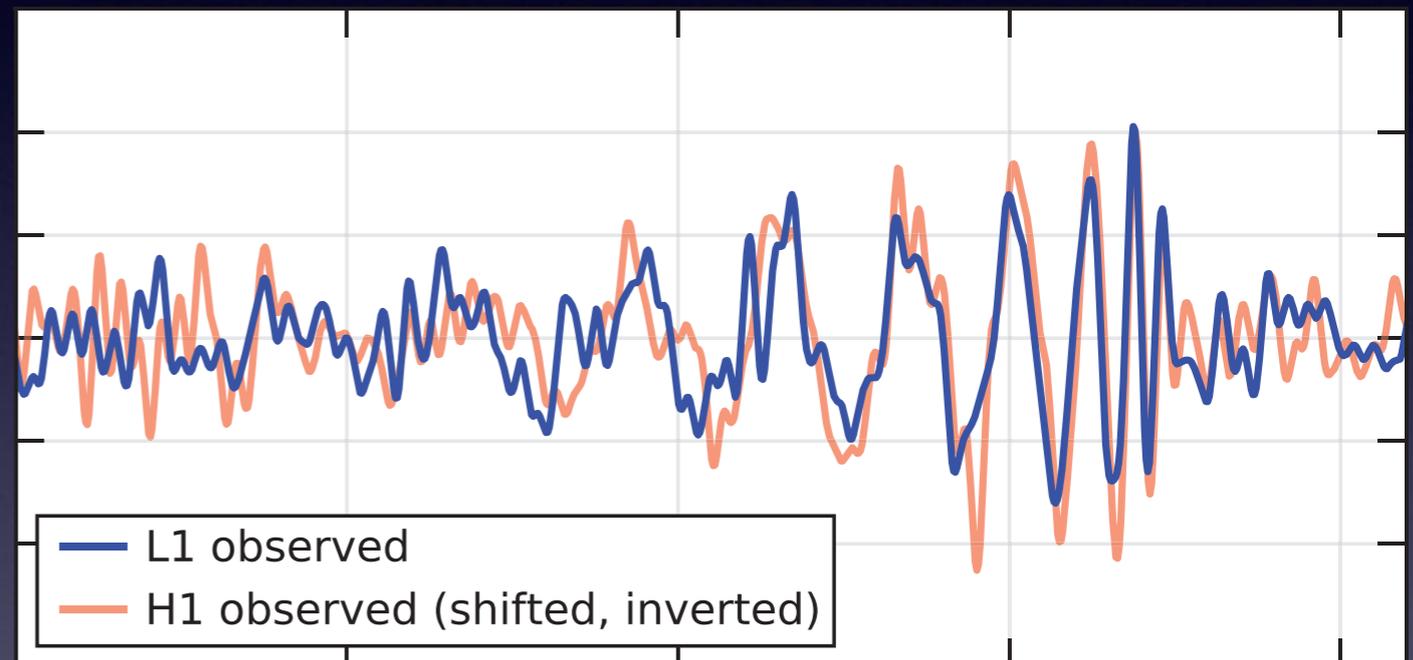
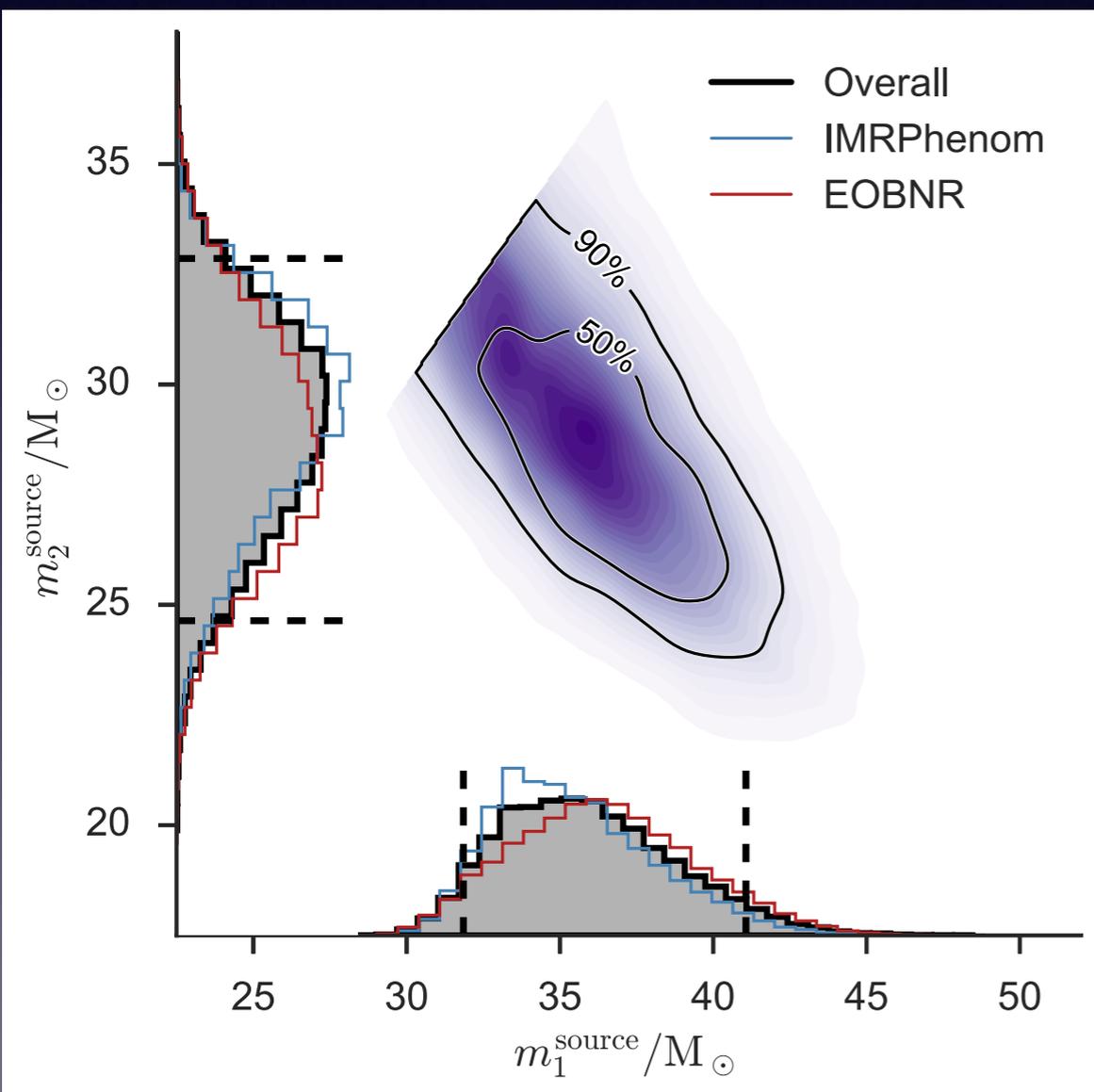
# Ground Based Interferometers: Sources

- *Compact binary coalescence*
  - BH-BH, NS-BH, NS-NS. Components formed from stars, from one to a few 10s of solar masses.
- *Bursts*
  - Supernovae; cosmic string cusps and kinks.
- *Continuous waves*
  - Monochromatic emission from deformed rotating neutron stars; possible causes include crustal deformation, fluid modes, free-precession.
- *Stochastic backgrounds*
  - Generated in early Universe or produced by astrophysical populations.



# Ground Based Interferometers: Results

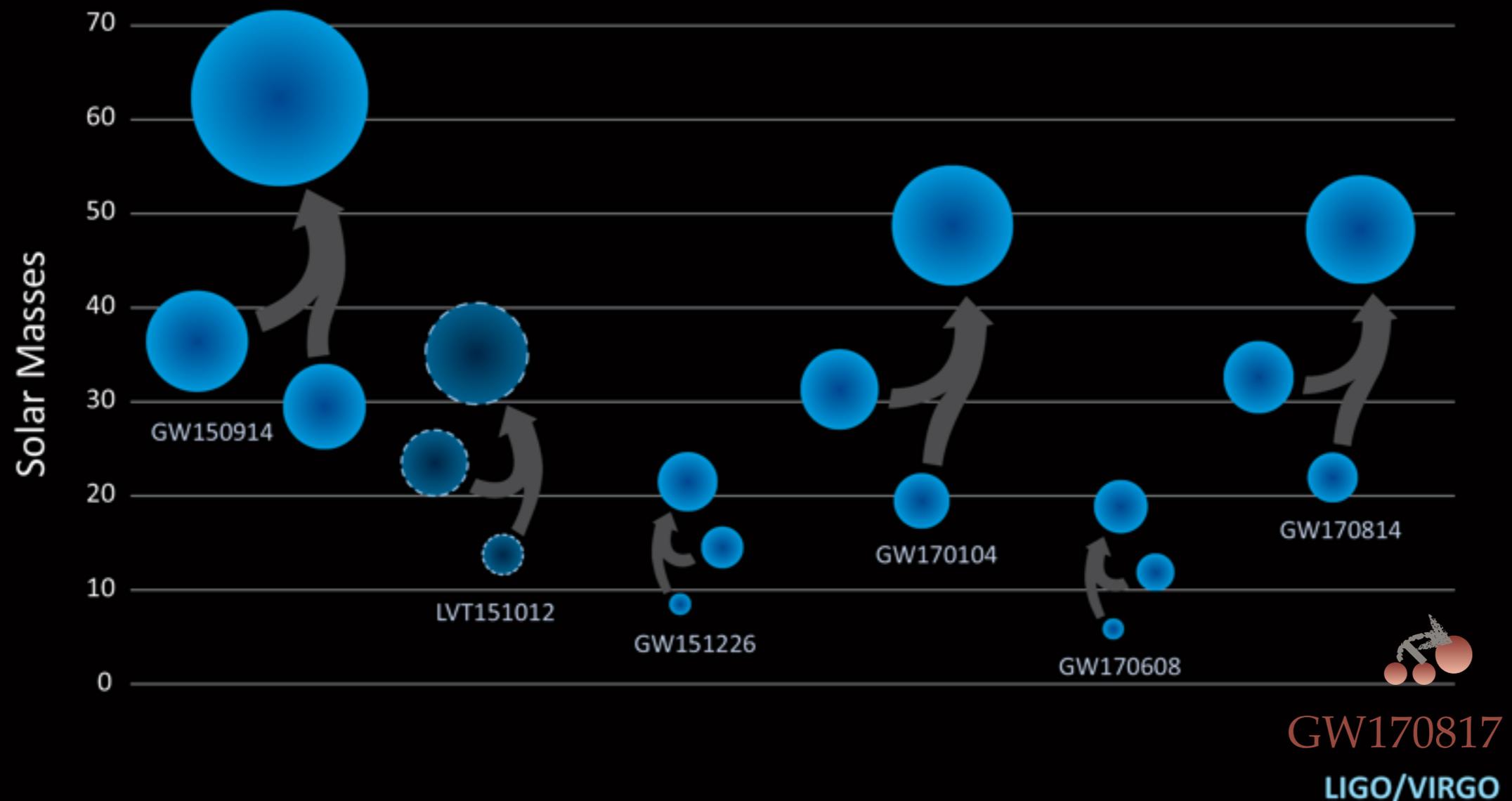
- First gravitational wave detection was made on September 14th 2015 at 09:50:45 UTC - GW150914. Source was a merger of a stellar origin black hole binary.



Primary black hole mass	$36_{-4}^{+5} M_{\odot}$
Secondary black hole mass	$29_{-4}^{+4} M_{\odot}$
Final black hole mass	$62_{-4}^{+4} M_{\odot}$
Final black hole spin	$0.67_{-0.07}^{+0.05}$
Luminosity distance	$410_{-180}^{+160} \text{ Mpc}$
Source redshift $z$	$0.09_{-0.04}^{+0.03}$

# Ground Based Interferometers: Results

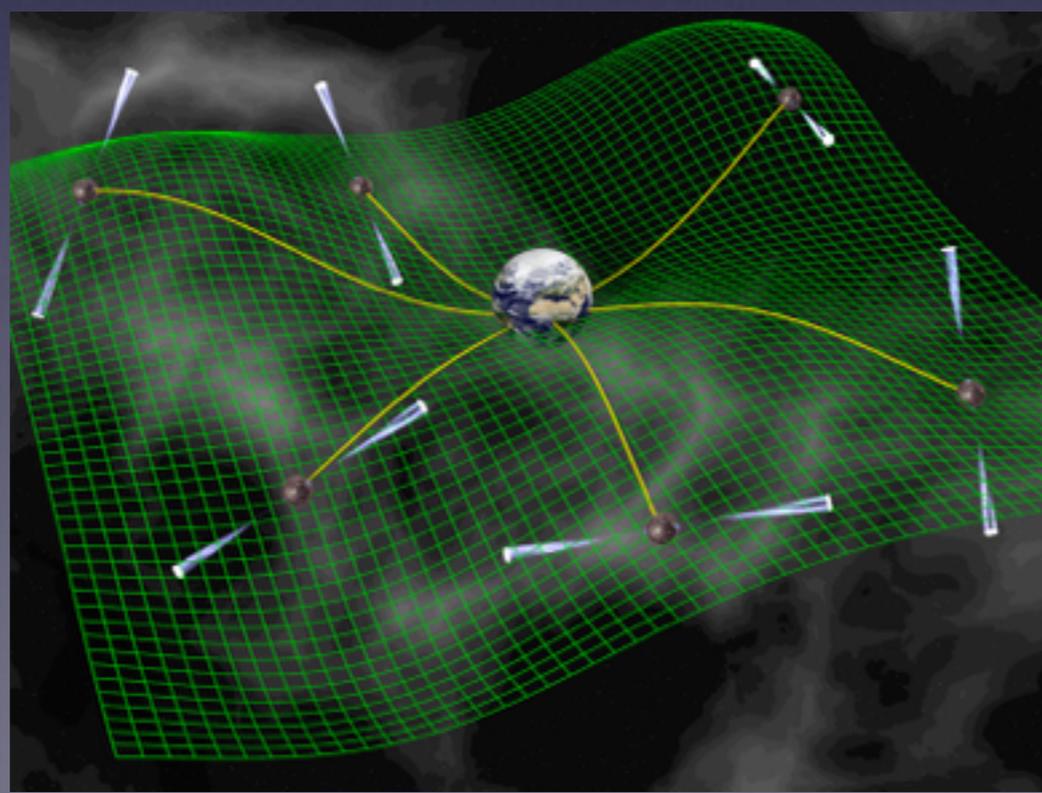
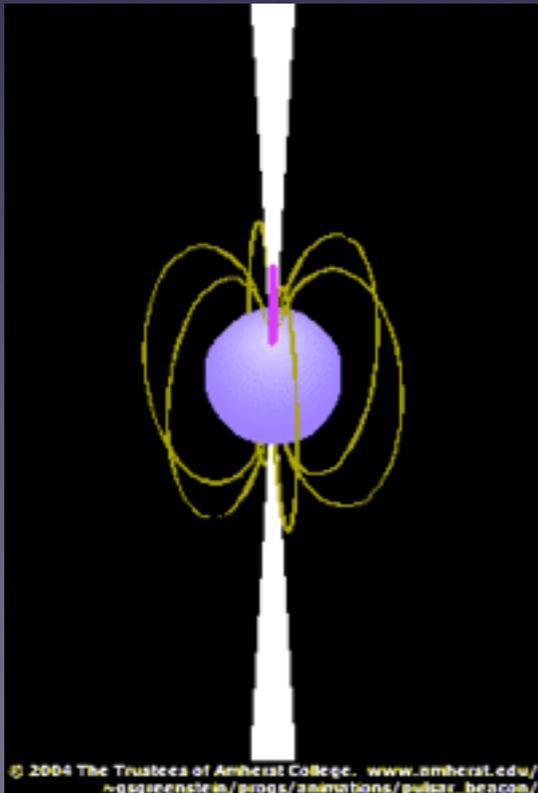
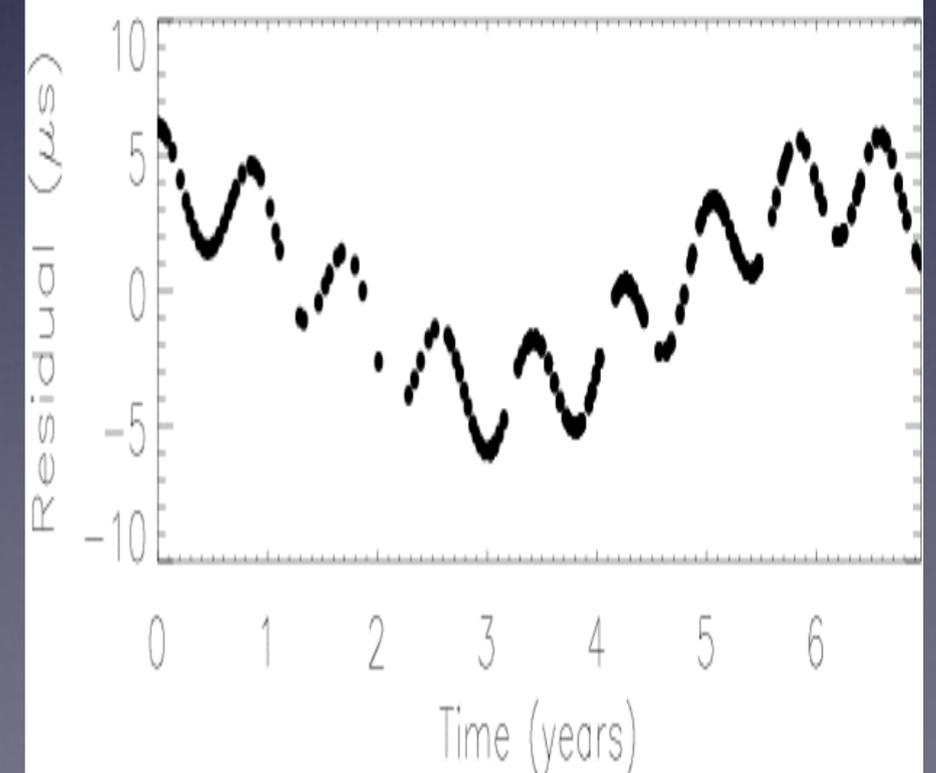
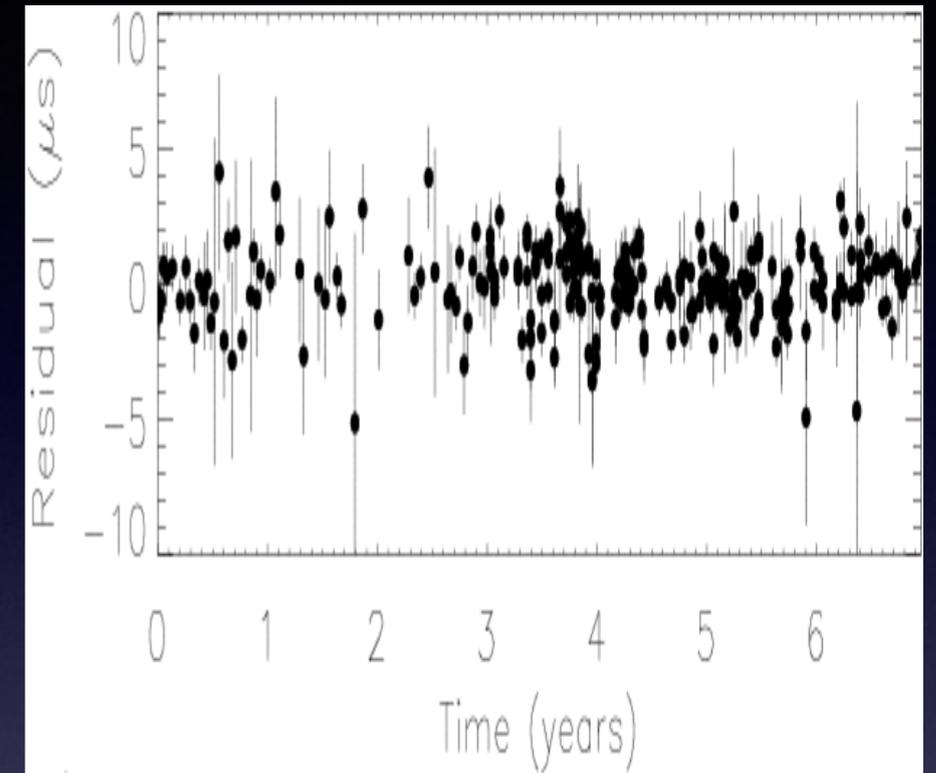
## Black Holes of Known Mass



- More details in Samaya Nissanke's lecture.

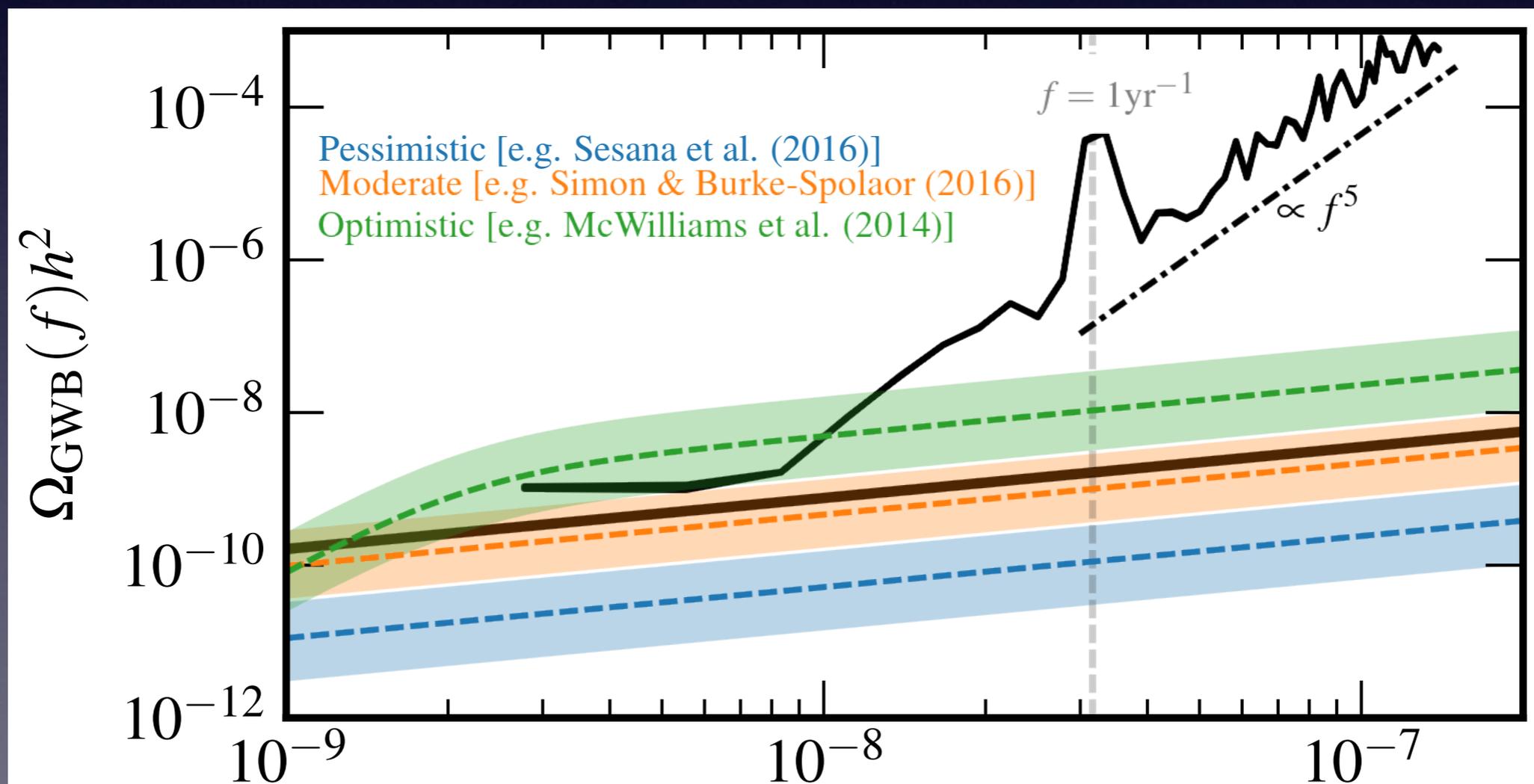
# Pulsar Timing Arrays

- Pulsars are very accurate clocks.
  - GW passing between source and observer changes pulse time of arrival.
  - Use a network (array) of pulsars to reduce noise.
  - Ongoing international effort using various radio telescopes.



# Pulsar Timing Arrays: Results

- The primary source for pulsar timing arrays is a stochastic background generated by supermassive black hole binaries.
- No detection as yet, but upper limits are approaching levels where detection is likely.

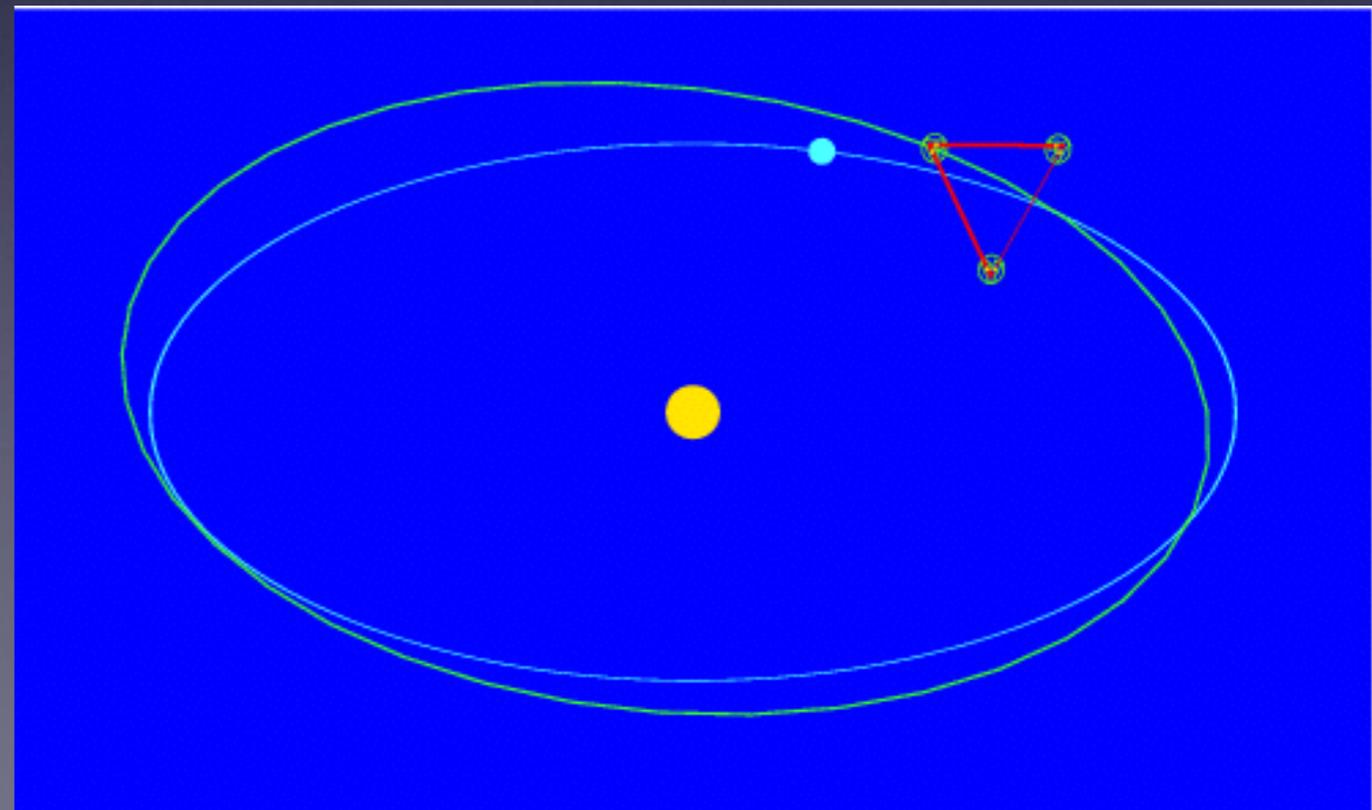
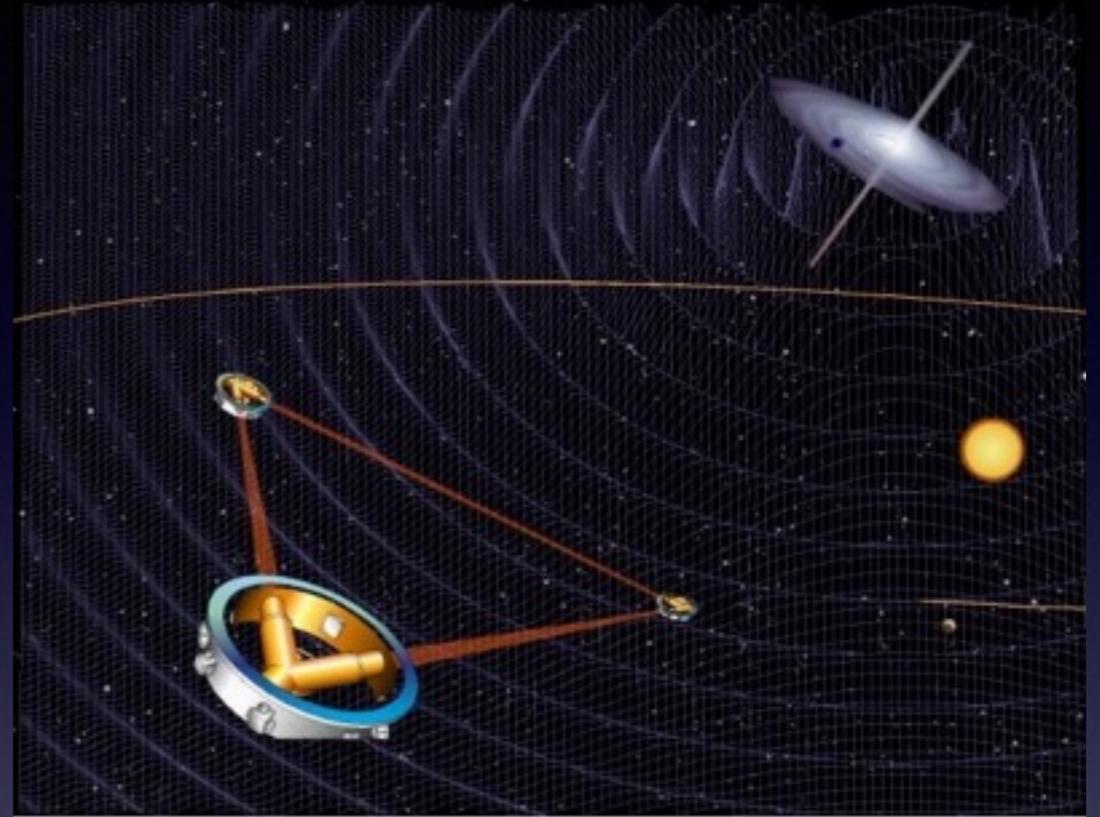


Arzoumanian et al. (2018)

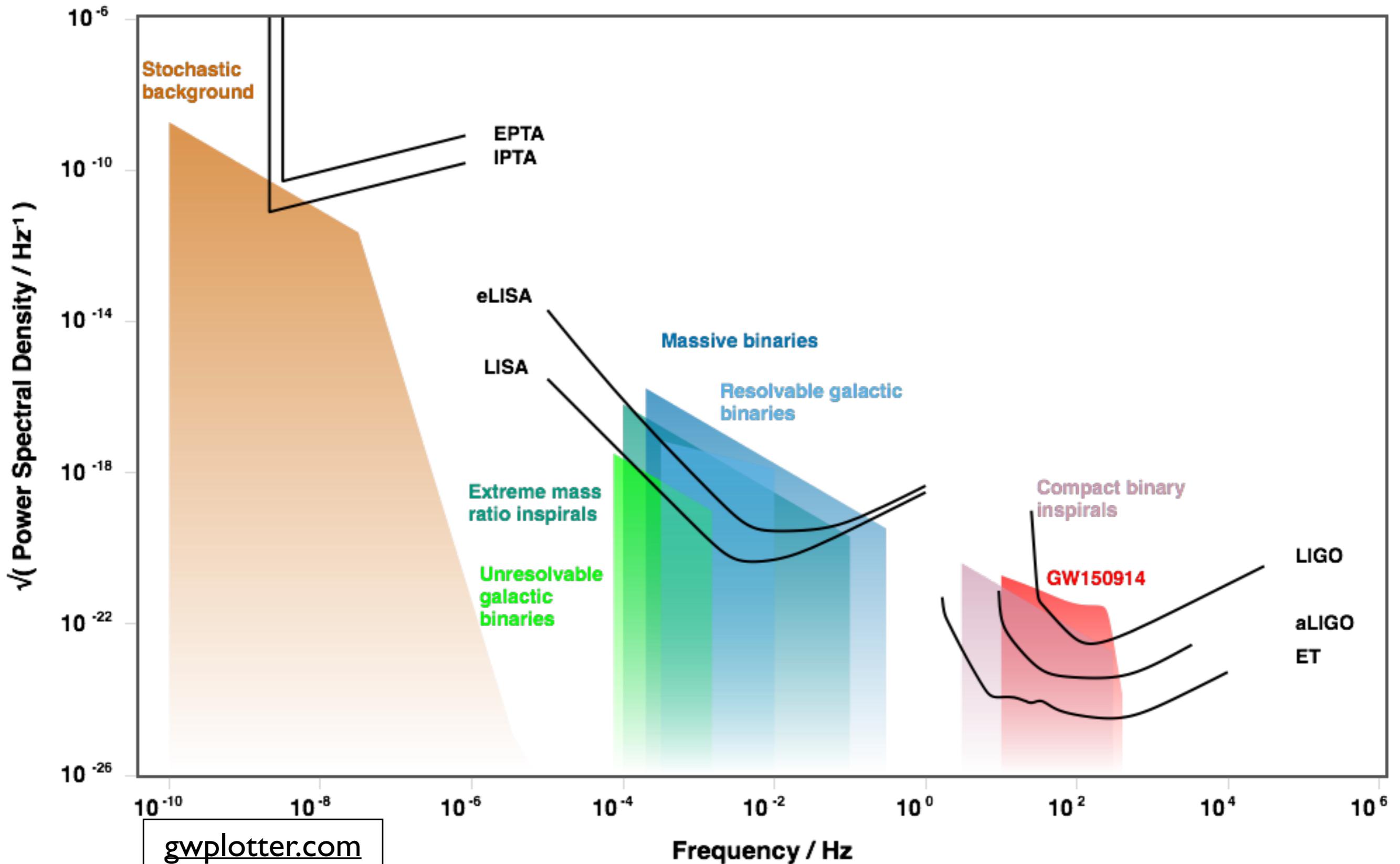
Observed GW Frequency,  $f$  [Hz]

# Space Based Interferometers

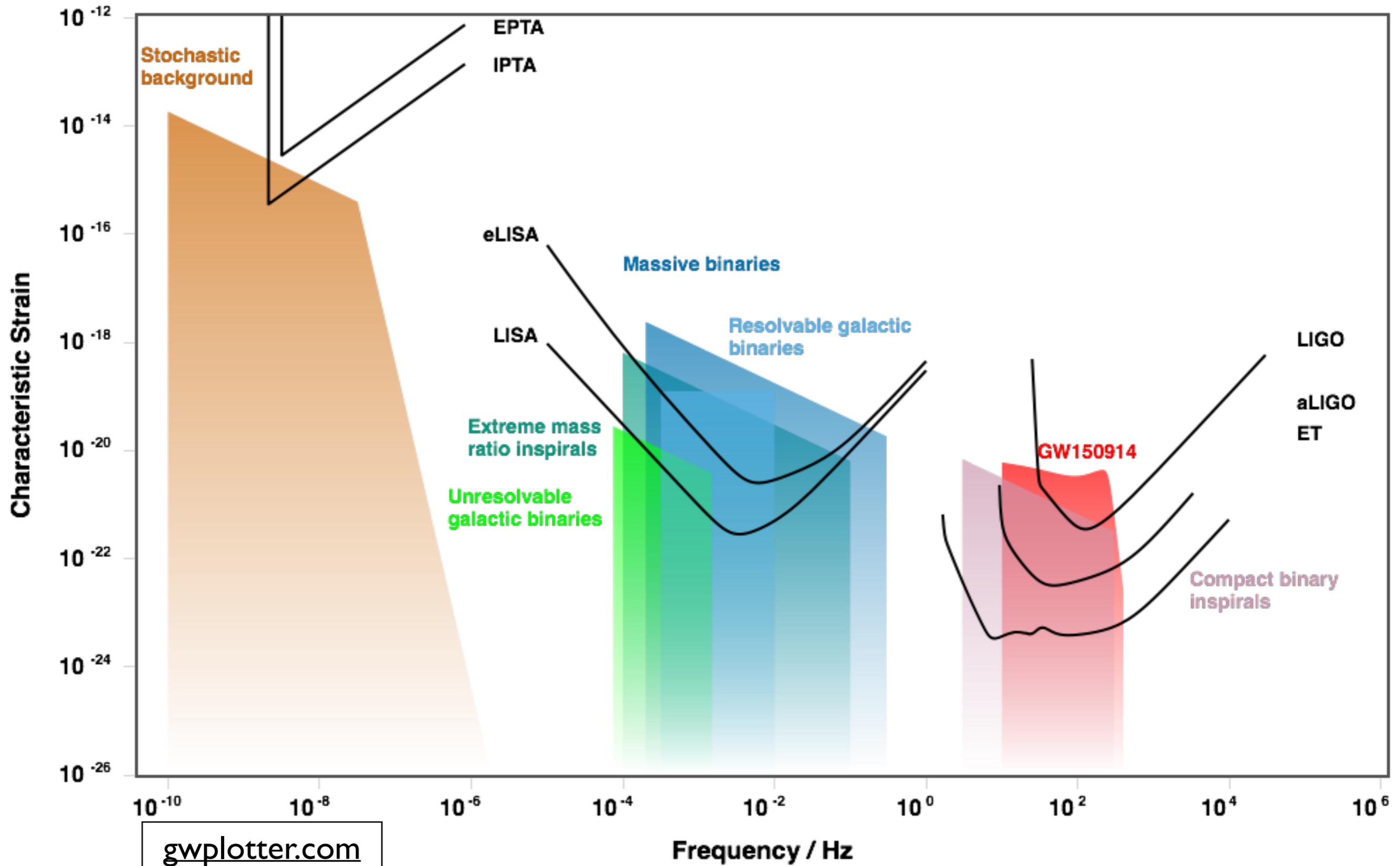
- There are plans for a space-based interferometer, LISA. Due to launch in 2034
  - Operating in millihertz band.
  - Three satellites, 2.5 million km apart, in heliocentric, Earth-trailing orbit. 6 laser links.
  - ESA-led project with NASA involvement. Just entered Phase A (first industrial study).
  - New sources: supermassive black holes, extreme-mass-ratio inspirals, galactic binaries.



# The Gravitational Wave Spectrum



# The Gravitational Wave Spectrum



# Principles of Signal Analysis

# Principles of Signal Analysis

- Gravitational wave detectors are intrinsically noisy. The output  $s(t)$  will consist of a (possible) signal  $h(t)$  plus noise fluctuations  $n(t)$ .

$$s(t) = h(t) + n(t)$$

- The noise is a random process.
  - ➔ Future values are not uniquely determined by initial data, but evolves according to some probabilistic model.
  - ➔ We suppose the random process is drawn from an *ensemble of random processes characterised by probability distributions*

$$p_N(n_N, t_N; \dots; n_2, t_2; n_1; t_1) dn_N \dots dn_2 dn_1$$

# Principles of Signal Analysis

- We typically make various useful assumptions about the properties of a random process

→ *Stationarity*: A stationary process is one for which the probability distributions depend only on time *differences*, not absolute time.

$$p_N(n_N, t_N + \tau; \dots; n_2, t_2 + \tau; n_1; t_1 + \tau) = p_N(n_N, t_N; \dots; n_2, t_2; n_1; t_1) \quad \forall \tau$$

→ *Gaussianity*: A process is Gaussian if and only if *all* of its (absolute) probability distributions are Gaussian.

$$p_N(n_N, t_N; \dots; n_2, t_2; n_1; t_1) = A \exp \left[ -\frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n \alpha_{jk} (y_j - \bar{y})(y_k - \bar{y}) \right]$$

→ *Ergodicity*: An ensemble of stationary random processes is ergodic if for any process  $n(t)$  drawn from the ensemble, the new ensemble  $\{n(t+KT): K \text{ an integer}\}$  has the same probability distributions.

# Principles of Signal Analysis

- We are interested in how large the random fluctuations are about the mean value. We'll assume this is zero here, which can be arranged by subtracting a constant.
- The fluctuations can be characterised by the power in a certain time interval  $-T/2 < t < T/2$

$$\int_{-T/2}^{T/2} |n(t)|^2 dt$$

- For stationary random processes this increases linearly with time. So, we instead use the mean power (or mean square fluctuations)

$$P_n = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |n(t)|^2 dt$$

# Principles of Signal Analysis

- Defining  $n_T(t) = n(t)\mathbb{I}[|t| < T/2]$  and using Parseval's theorem we have

$$\int_{-T/2}^{T/2} [n(t)]^2 dt = \int_{-\infty}^{\infty} [n_T(t)]^2 dt = \int_{-\infty}^{\infty} |\tilde{n}_T(f)|^2 df = 2 \int_0^{\infty} |\tilde{n}_T(f)|^2 df$$

$$P_n = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} [n(t)]^2 dt = \lim_{T \rightarrow \infty} \frac{2}{T} \int_0^{\infty} |\tilde{n}_T(f)|^2 df$$

- This motivates defining the spectral density,  $S_n(f)$ , via

$$S_n(f) = \lim_{T \rightarrow \infty} \frac{2}{T} \left| \int_{-T/2}^{T/2} n(t) \exp(2\pi i f t) dt \right|^2$$

- This is the *one-sided* spectral density which assumes the time series is real and we only consider positive frequencies. The *two-sided* spectral density is half this.

# Principles of Signal Analysis

- The spectral density represents the average power in the process at a particular frequency

$$P_n = \int_0^{\infty} S_n(f) df$$

- The auto-correlation function of a (zero mean) time series is defined by

$$C(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} n(t)n(t + \tau) dt$$

- For an ergodic (and hence stationary) random process this is equivalent to the expectation value over the ensemble

$$C(\tau) = \langle n(t)n(t + \tau) \rangle$$

- The auto-correlation function is the Fourier transform of the spectral density (the Wiener-Khintchine theorem).

# Principles of Signal Analysis

- For stationary processes a consequence of the Wiener-Khintchine theorem is that

$$\langle \tilde{n}^*(f) \tilde{n}(f') \rangle = S_n(f) \delta(f - f')$$

- where  $\sim$  denotes the Fourier transform, and  $*$  denotes complex conjugation.
- Examples of spectral densities include -

<i>white noise spectrum</i>	$S_n(f) = \text{const.}$
<i>flicker noise spectrum</i>	$S_n(f) \propto 1/f$
<i>random walk spectrum</i>	$S_n(f) \propto 1/f^2$

# Understanding Gravitational Wave Detector Sensitivity Curves

# Principles of Signal Analysis

- For a Gaussian, stationary random process the spectral density conveys all the information about the statistical properties of the process.
- For gravitational wave detectors, it is natural therefore to plot the spectral density to characterise the detector sensitivity. But - how then do we represent sources on the same diagram?
- There is no unique way to do this. Different types of source are best represented in different ways.

# Signal Sensitivity: Bursts

- A transient *burst* of gravitational waves can be characterised by its frequency,  $f$ , its duration,  $\Delta t$ , its *bandwidth*,  $\Delta f$  and its mean square amplitude, a proxy for signal power

$$\bar{P}_h = \frac{1}{\Delta t} \int_0^{\Delta t} |h(t)|^2 dt = h_c^2$$

- The square root of this defines the *characteristic amplitude* of the burst,  $h_c$ .
- The power in the noise in the same bandwidth is

$$\Delta f S_n(f)$$

# Signal Sensitivity: Bursts

- The square root of the ratio of the signal power to the noise power is the *signal-to-noise ratio*.

$$\left(\frac{S}{N}\right)^2 = \frac{\bar{P}_h}{\Delta f S_n(f)} = \frac{h_c^2}{\Delta f S_n(f)}$$

- This is a measure of detectability. If we window and bandpass the time series, this is the ratio of the root-mean-square signal contribution to the root-mean-square noise contribution.
- For a broad-band burst with  $\Delta f \sim f$ , the signal-to-noise ratio is approximately

$$\left(\frac{S}{N}\right)^2 = \frac{h_c^2}{f S_n(f)}$$

- This motivates plotting  $f S_n(f)$  instead of PSD. Height above this curve is a measure of burst detectability.

# Signal Sensitivity: Continuous Waves

- Consider now a monochromatic GW source

$$h(t) = h_0 \exp(2\pi i f_0 t)$$

- The signal power is constant over time and given by

$$P_h = \lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} |h(t)|^2 dt = \frac{1}{2} h_0^2$$

- However, this power is concentrated at  $f_0$ . With finite time series of length  $T$  we can resolve frequency to a precision

$$\Delta f \sim 1/T$$

- Noise power in this bandwidth is  $S_n(f)/T$ .

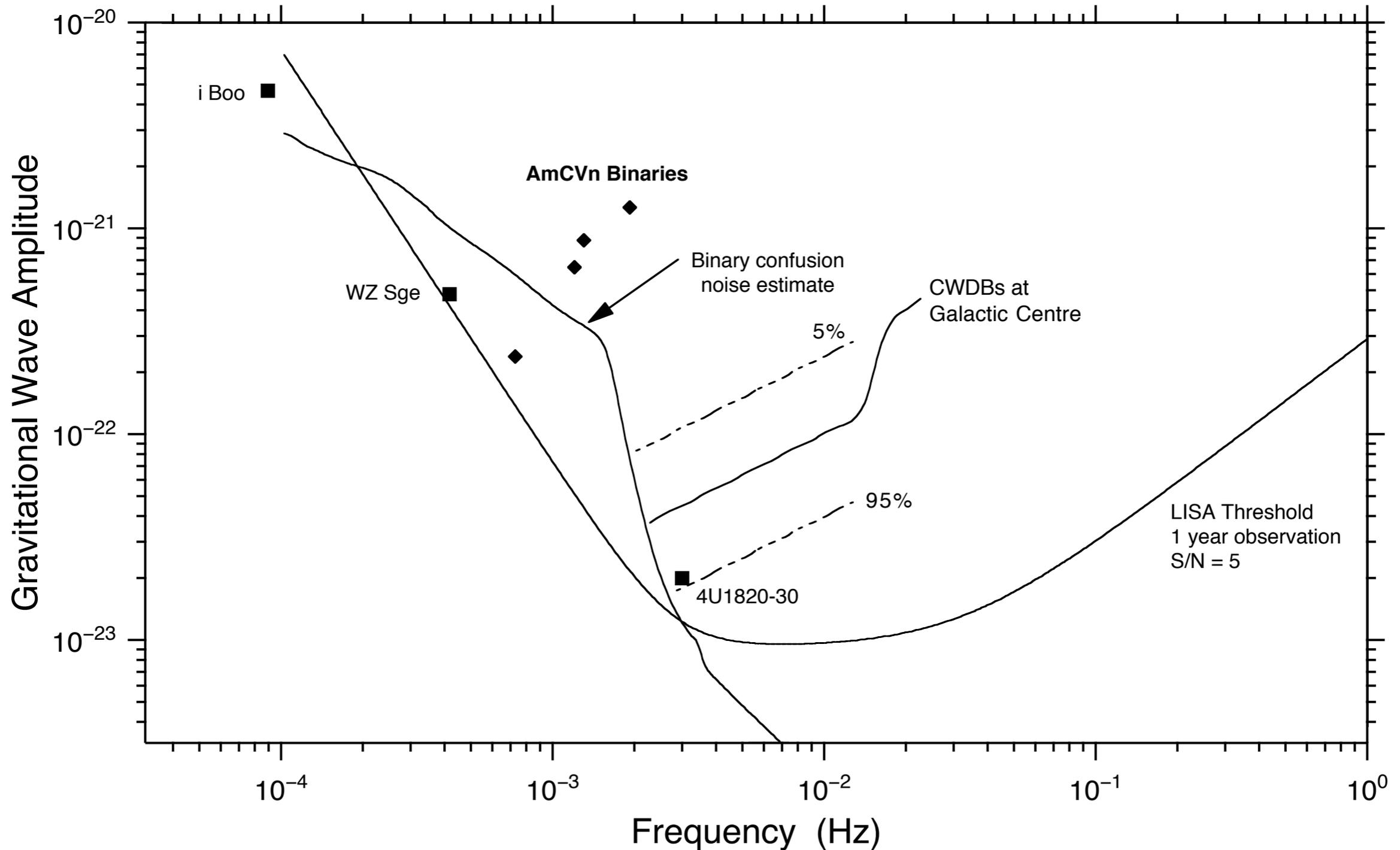
# Signal Sensitivity: Continuous Waves

- This motivates representing sensitivity by plotting

$$\sqrt{S_n(f)/T} \quad \text{or} \quad \rho_{\text{thresh}} \sqrt{S_n(f)/T}$$

- where  $\rho_{\text{thresh}}$  is the estimated threshold S/N needed for detection. This is the *strain spectral density*.
- Advantage: for a monochromatic source, height above curve gives expected S/N or, with specified threshold, an easy assessment of whether source is detectable or not.
- Disadvantage: must specify length of observation. Not appropriate for ongoing experiments, e.g., LIGO. But - can produce this after each observing run.

# Signal Sensitivity: Continuous Waves



# Signal Sensitivity: Inspiring Sources

- For an inspiraling source, the total energy emitted in each frequency band is finite and so is the Fourier transform.

- Hence

$$\frac{1}{\sqrt{T}} \tilde{h}(f) \Rightarrow 0 \quad \text{as } T \rightarrow \infty$$

- and so the spectral density is zero (over all time).
- Band passing and windowing can recover some of the power, but can we do better than this?
- Yes - using *filtering*.

# Filtering

- A filtered time series is defined using a *kernel*  $K(t - t')$ .

$$w(t) = \int_{-\infty}^{\infty} K(t - t')s(t')dt'$$

- We now apply a slightly modified definition of S/N. We compare the *amplitude* output of the filter due to the signal to the rms output of the filter due to the noise.

$$\left(\frac{S}{N}\right)(t) = \frac{\int_{-\infty}^{\infty} K(t - t')h(t')dt'}{\sqrt{\left\langle \left| \int_{-\infty}^{\infty} K(t - t')n(t')dt' \right|^2 \right\rangle}}$$

- The rms output of the filter  $S + N$  is the signal amplitude to within an rms fractional error  $N/S$ , which is the reciprocal of the signal to noise ratio.

# Optimal Filter

- We can ask what choice of filter maximises the value of S/N at zero-lag, i.e.,  $t=0$ .
- From the convolution theorem for Fourier transforms we have

$$\tilde{w}(f) = \tilde{K}(f)\tilde{h}(f)$$

- The expression for S/N can thus be written

$$\frac{S}{N} = \frac{\int \tilde{K}(f)\tilde{h}(f)df}{\sqrt{\int |\tilde{K}(f')|^2 S_n(f')df'}}$$

# Optimal Filter

- This motivates a natural inner product,  $(\mathbf{h}_1 | \mathbf{h}_2)$ , on the space of signals of the form

$$(\mathbf{h}_1 | \mathbf{h}_2) = 2 \int_0^\infty \frac{\tilde{\mathbf{h}}_1^*(f) \tilde{\mathbf{h}}_2(f) + \tilde{\mathbf{h}}_1(f) \tilde{\mathbf{h}}_2^*(f)}{S_n(f)} df$$

- in terms of which we have 
$$\frac{S}{N} = \frac{(S_n K | h)}{\sqrt{(S_n K | S_n K)}}$$
- which is maximised by the choice

$$\tilde{K}(f) \propto \frac{\tilde{h}(f)}{S_n(f)}$$

- This is the *Weiner optimal filter*. In the frequency domain the optimal kernel is equal to the signal weighted by the spectral density of the noise.

# Optimal Filter

- A search using the optimal filter then amounts to taking the inner product  $(s|h)$  of the data stream,  $s$ , with a *template* of the signal  $h$ . This is *matched filtering*.

- The signal to noise ratio of a matched filtering search is

$$\frac{S}{N}[\mathbf{h}] = \frac{(\mathbf{h}|\mathbf{h})}{\sqrt{\langle(\mathbf{h}|\mathbf{n})(\mathbf{h}|\mathbf{n})\rangle}} = (\mathbf{h}|\mathbf{h})^{1/2}$$

- which follows from the fact that  $\langle(\mathbf{h}_1|\mathbf{n})(\mathbf{h}_2|\mathbf{n})\rangle = (\mathbf{h}_1|\mathbf{h}_2)$
- For a monochromatic source, the matched filter is just a Fourier transform, so this agrees with the previous result. In that case, the signal to noise ratio increases like the square root of the observation time.

# Optimal Filter

- The matched filtering S/N squared is

$$\left(\frac{S}{N}\right)^2 = 4 \int_0^\infty \frac{|\tilde{h}(f)|^2}{S_n(f)} df$$

- which can also be written as

$$\left(\frac{S}{N}\right)^2 = 4 \int_0^\infty \frac{f|\tilde{h}(f)|^2}{S_n(f)} d \ln f = 4 \int_0^\infty \frac{f^2|\tilde{h}(f)|^2}{f S_n(f)} d \ln f$$

- These expressions aid “integration by eye” in a logarithmic plot.
- For a source which has amplitude  $h_0$  at frequency  $f$  and corresponding frequency derivative  $\dot{f}$ , we have

$$\tilde{h}(f) \sim \frac{h_0}{\sqrt{\dot{f}}}$$

# Characteristic Strain

- The analogy with a broad-band burst therefore motivates the definition of a *characteristic strain*,  $h_c$ , for inspiraling sources (e.g., Finn and Thorne 2000).

$$h_c = h_o \sqrt{\frac{2f^2}{df/dt}}$$

- The characteristic strain is a measure of the SNR accumulated while the frequency sweeps through a bandwidth equal to frequency. If we also plot the rms noise in a bandwidth equal to frequency,  $h_n(f)$

$$h_n(f) = \sqrt{f \langle S_n(f) \rangle_{SA}} \quad \left( \frac{S}{N} \right)_{f \rightarrow 2f}^2 = \left[ \frac{h_c(f)}{h_n(f)} \right]^2$$

- Plots of  $h_c(f)$  and  $h_n(f)$  allow us to see directly how the SNR of an evolving source builds up over the evolution.

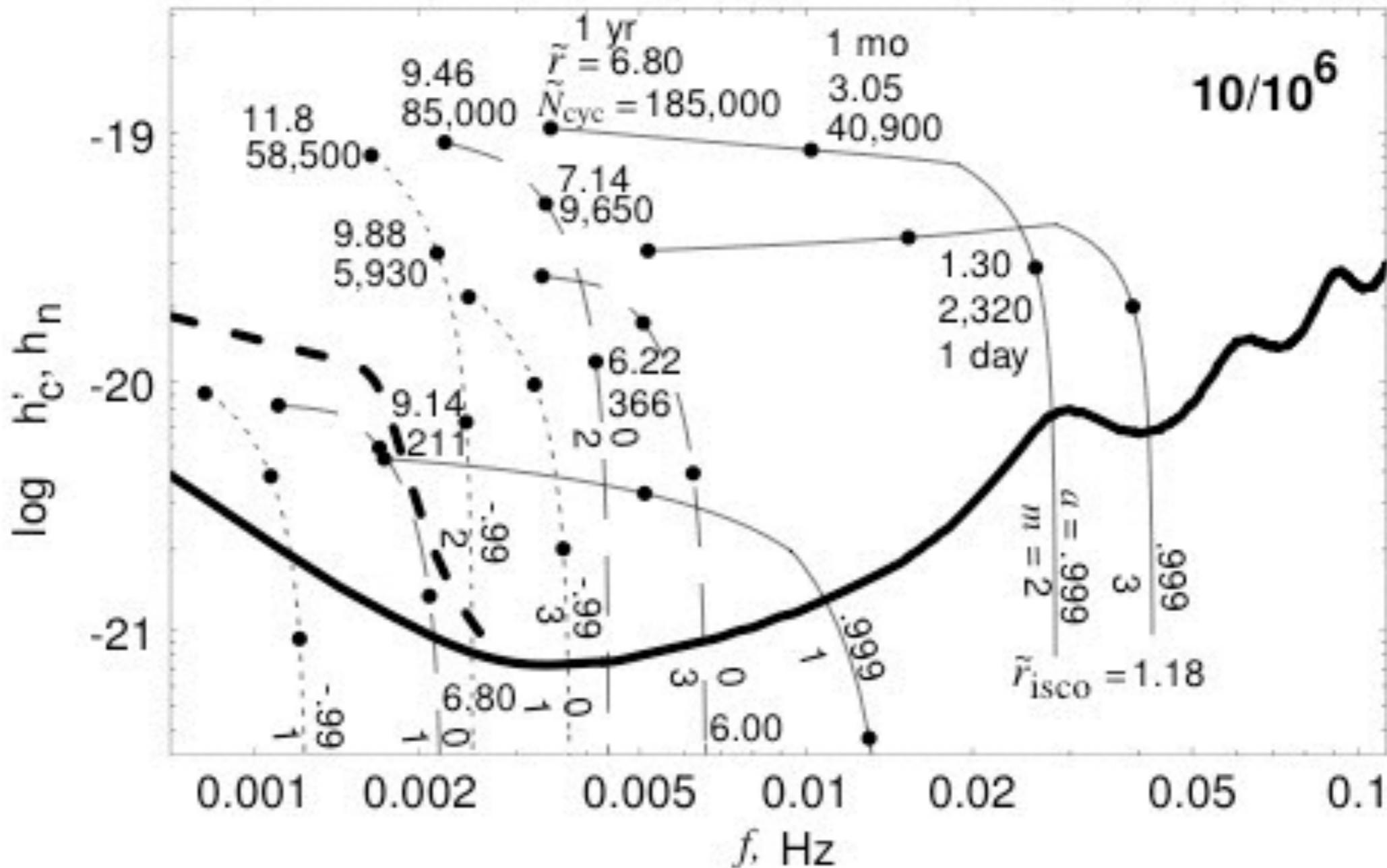
# Characteristic Strain

- In the definition of *characteristic strain*

$$h_c = h_o \sqrt{\frac{2f^2}{df/dt}}$$

- the term inside the square root is equal to the number of cycles the inspiral spends in the vicinity of the frequency  $f$ .
- You will read papers in which people talk about S/N being enhanced by the number of cycles spent in the vicinity of a certain frequency. This is what they are referring to.
- **Note:** plotting characteristic strain only makes sense if you are *also* plotting  $f S_n(f)$ . If you are plotting  $S_n(f)$  directly your strain should be a factor of  $\sqrt{f}$  lower.

# Characteristic Strain



Build up of SNR for EMRIs observed by LISA  
(Finn & Thorne 2000)

# Signal Sensitivity: Sky Averaging

- SNRs also depend on the *sky position* and *orientation* of a source. This can be folded into the spectral density by using a sky and orientation averaged sensitivity, and using the strain of an optimally positioned and oriented source.

$$\langle S_n(f) \rangle_{SA}^{\text{LIGO}} \approx 5S_n(f)$$

$$\langle S_n(f) \rangle_{SA}^{\text{LISA}} \approx 20/3S_n(f)$$

# Representing Stochastic Backgrounds

- *Stochastic backgrounds* are characterised by a spectral density, so it is natural to compute the power spectral density and plot it on the same axes as the detector PSD.
- There are two caveats.
- Firstly, the “power” we have been talking about so far has not been a power in a physical sense since we have not specified any units for the time series (and indeed for GW strain this is dimensionless). Better to use something that represents a physical energy density if possible.
- Plotting two PSDs does not convey any information about their distinguishability. Can we represent backgrounds in a way that allows the reader to assess detectability at a glance?

# Representing Stochastic Backgrounds

- The energy density carried by a gravitational wave is

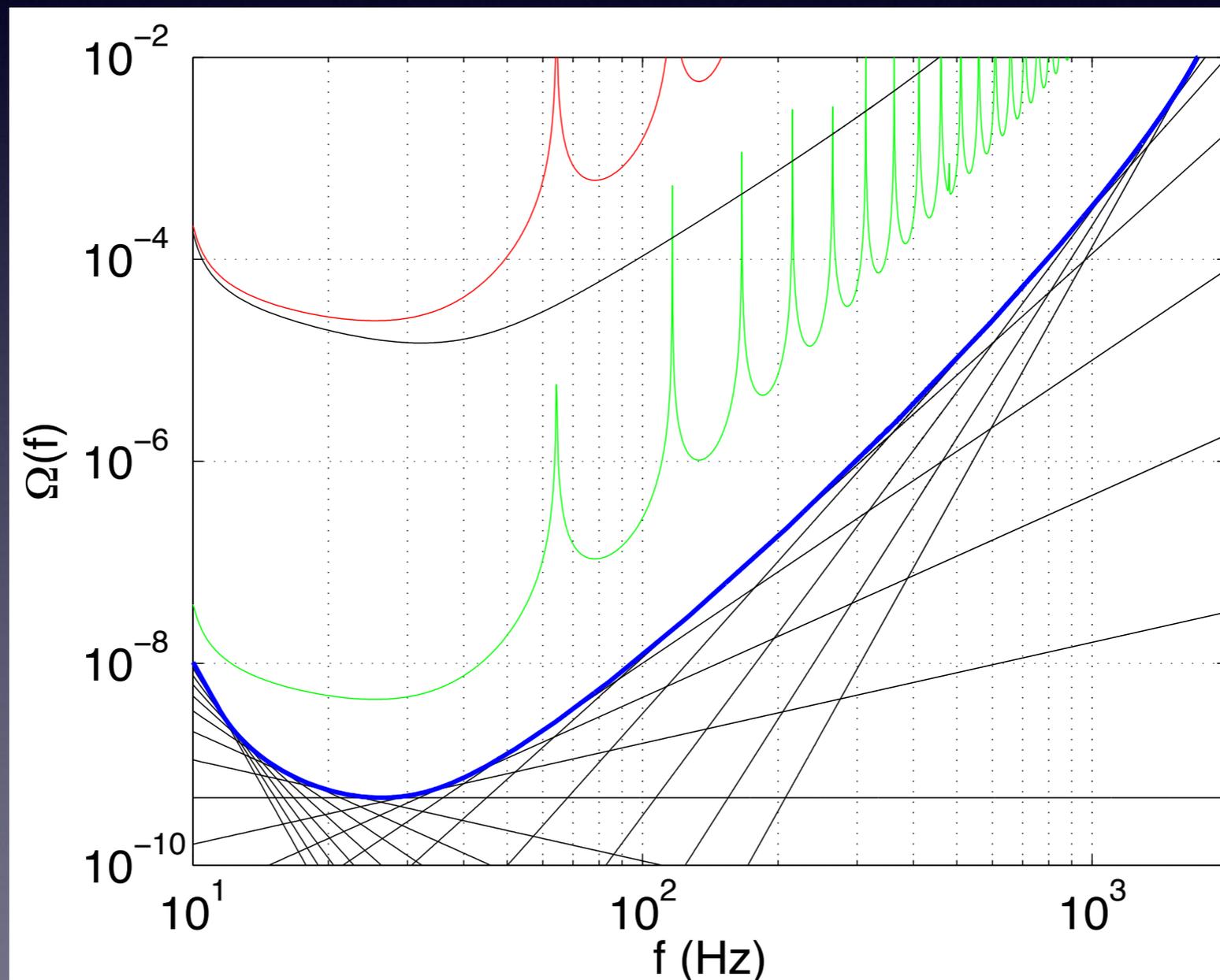
$$\frac{dE}{dt dA} \propto \dot{h}_+^2 + \dot{h}_\times^2$$

- Therefore, we should consider the time derivative of the strain series to get a physical energy.
- The corresponding spectral density is  $f^2 S_n(f)$  and fluctuations in a bandwidth equal to frequency are  $f^3 S_n(f)$ .
- Energy densities in astrophysical and cosmological backgrounds are often expressed as a fraction of the *closure density* of the Universe

$$\Omega_{\text{GW}} = \frac{8\pi G}{3H_0^2} \frac{dE_{\text{GW}}}{d \ln f} \propto f^2 h_c^2(f)$$

# Representing Stochastic Backgrounds

- Quick assessment of background detectability can be derived from *power-law sensitivity curves* (Thrane & Romano 2013). Requires assumptions about data analysis procedures.



# Sensitivity Curves Summary

- To summarise, there are four different types of sensitivity curve you might see in figures.
- *Power Spectral Density* - summarises statistical properties of noise

$$S_n(f)$$

- *Strain spectral density*

$$S_n(f)/T \quad - \quad \text{for monochromatic sources}$$

$$f S_n(f) \quad - \quad \text{for inspirals and bursts}$$

- *Energy spectral density* - for backgrounds

$$f^3 S_n(f)$$

# Estimating Detectability of Sources

# Typical strains of astrophysical sources

- Recall that the gravitational wave strain is given by

$$h \sim \frac{\ddot{I}_{ij}}{D} \quad I_{ij} = \int \rho x_i x_j dV$$

- and the rate of energy loss is given by

$$\dot{E} \sim \ddot{I}_{ij}^2$$

- Order of magnitude of quadrupole moment can be estimated from virial theorem

$$\ddot{I}_{ij} \sim 2 \int \rho v_i v_j dV \Rightarrow |\ddot{I}_{ij} n_i n_j| \lesssim 2M \phi_{\text{int}} \Rightarrow |h| \lesssim \phi_N \phi_{\text{int}} / c^4$$

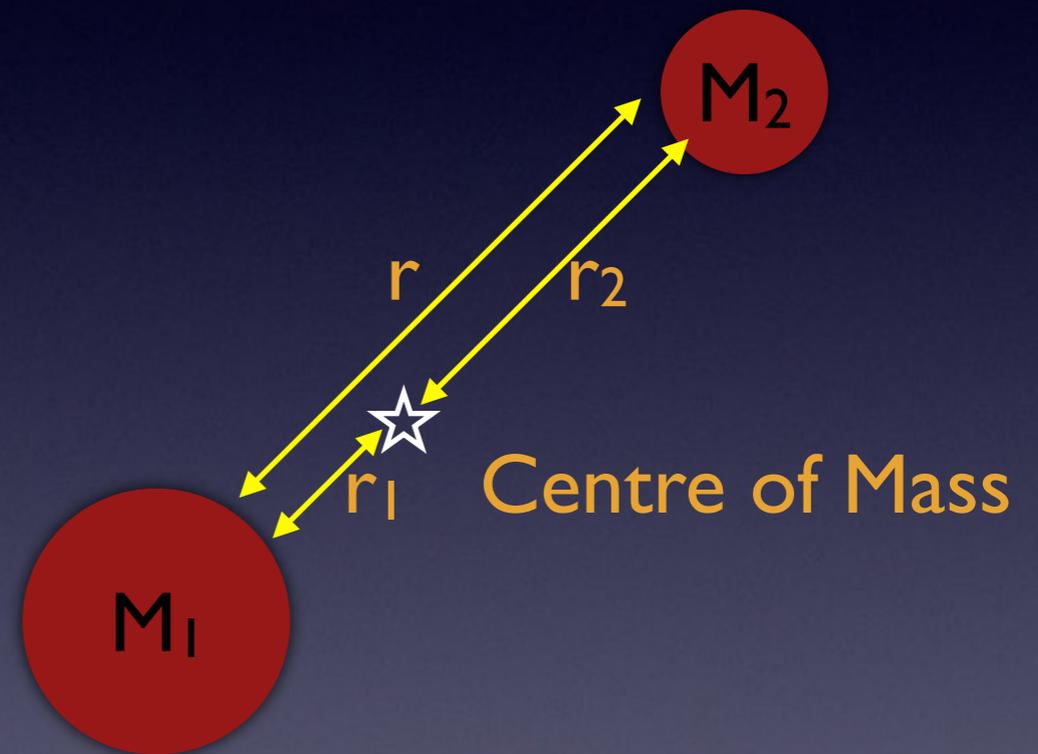
# Typical strains of astrophysical sources

- Newtonian circular binary system with masses  $M_1, M_2$  and separation  $r$  is equivalent to object of mass  $\mu$  orbiting in a fixed potential of mass  $M$  at distance  $r$ .

$$\mu = \frac{M_1 M_2}{M_1 + M_2}$$

$$M = M_1 + M_2$$

$$r_1 M_1 = r_2 M_2 = \mu r$$



- Frequency and energy are given by

$$\omega^2 = \left( \frac{2\pi}{T} \right)^2 = (2\pi f)^2 = \frac{M}{r^3} \quad E = -\frac{M\mu}{2r}$$

# Typical strains of astrophysical sources

- We deduce the quadrupole moment scales as

$$I \sim \mu r^2 \cos 2\omega t \sim \frac{M_1 M_2}{(M_1 + M_2)^{\frac{1}{3}}} \omega^{-\frac{4}{3}}$$

- So emissions is at  $f_{\text{GW}} = \omega/\pi$ , with strain amplitude

$$h \sim \frac{1}{D} \frac{M_1 M_2}{(M_1 + M_2)^{\frac{1}{3}}} \omega^{\frac{2}{3}}$$

- From these we deduce

$$\dot{E} \sim \mu^2 M^{\frac{4}{3}} \omega^{\frac{10}{3}} \quad \dot{\omega} \sim \frac{M_1 M_2}{(M_1 + M_2)^{\frac{1}{3}}} \omega^{\frac{11}{3}} = M_c^{\frac{5}{3}} \omega^{\frac{11}{3}}$$

- where we have introduced the *chirp mass*

$$M_c = \frac{M_1^{\frac{3}{5}} M_2^{\frac{3}{5}}}{(M_1 + M_2)^{\frac{1}{5}}}$$

# Typical strains of astrophysical sources

- We may also deduce expressions for the Fourier domain amplitude of the signal and the characteristic strain

$$\tilde{h}(f) \sim \frac{1}{D} M_c^{\frac{5}{6}} f^{-\frac{7}{6}} \quad h_c(f) \sim \frac{1}{D} M_c^{\frac{5}{6}} f^{-\frac{1}{6}}$$

- Reintroducing dimensional and numerical factors the expressions for strain and luminosity are

$$h \sim \frac{G^{\frac{5}{3}}}{2c^4} \frac{1}{|\mathbf{x}|} \frac{M_1 M_2}{(M_1 + M_2)^{\frac{1}{3}}} \omega^{\frac{2}{3}} \quad L \sim \frac{G^{\frac{7}{3}}}{5c^5} \frac{M_1^2 M_2^2}{(M_1 + M_2)^{\frac{2}{3}}} \omega^{\frac{10}{3}}$$

# Typical strains of astrophysical sources

- Putting in physical units we have

$$h \sim 3 \times 10^{-16} \left( \frac{|\mathbf{x}|}{1\text{pc}} \right)^{-1} \left( \frac{M_1}{M_\odot} \right) \left( \frac{M_2}{M_\odot} \right) \left( \frac{M_\odot}{M_1 + M_2} \right)^{\frac{1}{3}} \left( \frac{f}{100\text{Hz}} \right)^{\frac{2}{3}}$$

$$L \sim 5 \times 10^{43} \text{Js}^{-1} \left( \frac{M_1}{M_\odot} \right)^2 \left( \frac{M_2}{M_\odot} \right)^2 \left( \frac{M_\odot}{M_1 + M_2} \right)^{\frac{2}{3}} \left( \frac{f}{100\text{Hz}} \right)^{\frac{10}{3}}$$

- For the Earth/Sun system, viewed at 1 parsec

$$f \sim 3 \times 10^{-8} \text{Hz} \quad h \sim 5 \times 10^{-28} \quad L \sim 10 \text{Js}^{-1}$$

- For a  $1M_\odot + 1M_\odot$  binary at merger

$$f = \frac{1}{2\pi} \sqrt{\frac{GM}{(6GM/c^2)^3}} = \frac{c^3}{12\sqrt{6}\pi GM} \sim 1000 \text{Hz} \quad L \sim 10^{47} \text{Js}^{-1}$$

# Typical strains of astrophysical sources

- At a distance of 1 Mpc, we find  $h \sim 10^{-21}$  at merger. This scales like  $M^{2/3}$ , so a  $10^6 M_{\odot} + 10^6 M_{\odot}$  SMBH binary at 10 Gpc generates the same strain in the Solar System, with merger frequency of  $\sim 1$  mHz.
- A  $1M_{\odot} + 1M_{\odot}$  millihertz compact binary at 1 kpc has
$$f \sim 3 \times 10^{-4} \text{ Hz} \quad h \sim 5 \times 10^{-23} \quad L \sim 10^{25} \text{ J s}^{-1}$$
- These are all sources we expect to be able to detect with current and future detectors. LIGO operates in the 10Hz to 10kHz range, while LISA will operate in the 0.1 mHz to 0.1Hz range.

# Eccentric binaries

- Eccentric orbits have emission at all frequencies, with relative contribution

$$\dot{E}_n = \frac{32}{5} \mu^2 M^{\frac{4}{3}} (2\pi f)^{\frac{10}{3}} g(n, e)$$

$$g(n, e) = \frac{n^4}{32} \left\{ \left[ J_{n-2}(ne) - 2eJ_{n-1}(ne) + \frac{2}{n}J_n(ne) + 2eJ_{n+1}(ne) - J_{n+2}(ne) \right]^2 + (1 - e^2) [J_{n-2}(ne) - 2J_n(ne) + J_{n+2}(ne)]^2 + \frac{4}{3n^2} [J_n(ne)]^2 \right\}$$

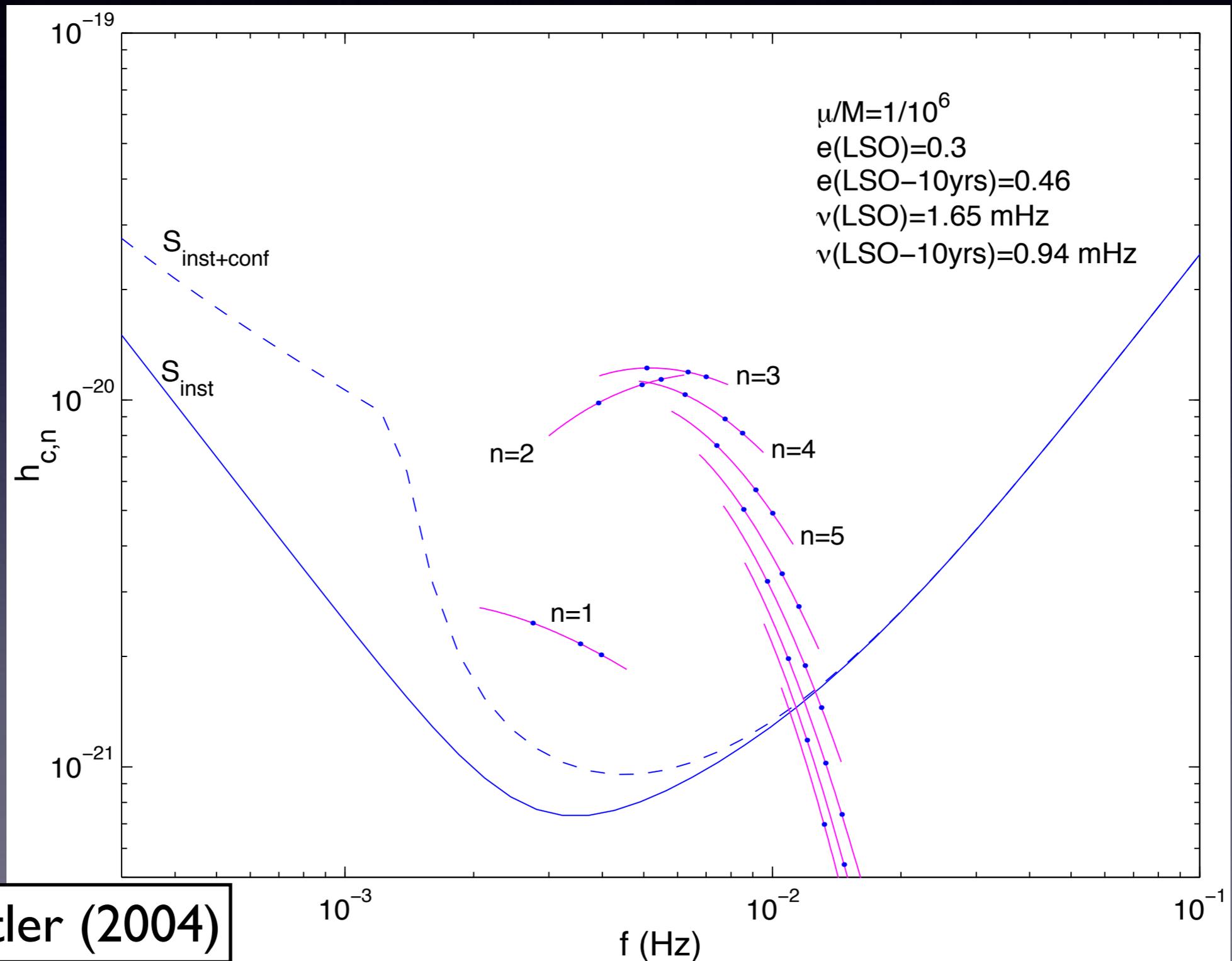
- Can compute a characteristic strain for a particular mode

$$h_{c,n}(f) = \frac{1}{\pi D} \sqrt{\frac{2\dot{E}_n(f/n)}{nf(f/n)}} \sim M_c^{\frac{5}{6}} f^{-\frac{7}{6}} n^{\frac{2}{3}} \sqrt{g(n, e)}$$

- This is the characteristic strain of the n'th harmonic at frequency  $f$ . The argument  $(f/n)$  indicates the functions are to be evaluated when the orbital frequency is equal to  $f/n$ .

# Eccentric binaries

- Represent these on a single “waterfall plot”.



# Stochastic Backgrounds

- Recall definition

$$\Omega_{\text{GW}} = \frac{8\pi G}{3H_0^2} \frac{dE_{\text{GW}}}{d \ln f} \propto f^2 h_c^2(f)$$

- Suppose background is generated by a astrophysical population of sources with comoving volume density  $N(z)$ . Then, total energy density in background today is

$$\mathcal{E}_{\text{GW}} = \int_0^\infty \rho_c c^2 \Omega_{\text{GW}} d \ln f = \int_0^\infty \int_0^\infty N(z) \frac{1}{(1+z)} \frac{dE}{df} f \frac{df}{f} dz$$

- We deduce (Phinney 2001, astro-ph/0108028)

$$\rho_c c^2 \Omega_{\text{GW}} = \frac{\pi}{4} \frac{c^2}{G} f^2 h_c^2(f) = \int_0^\infty \frac{N(z)}{1+z} \left( f_r \frac{dE}{df_r} \right)_{|f_r=f(1+z)} dz$$

# Stochastic Backgrounds

- For a population of inspiraling binaries we have

$$f \frac{dE}{df} \sim M_c^{\frac{5}{3}} f^{\frac{2}{3}}$$

- From which we deduce the scaling of the energy density

$$\Omega_{\text{GW}}(f) \sim M_c^{\frac{5}{3}} f^{\frac{2}{3}} \int_0^\infty \frac{N(z)}{(1+z)^{\frac{1}{3}}} dz$$

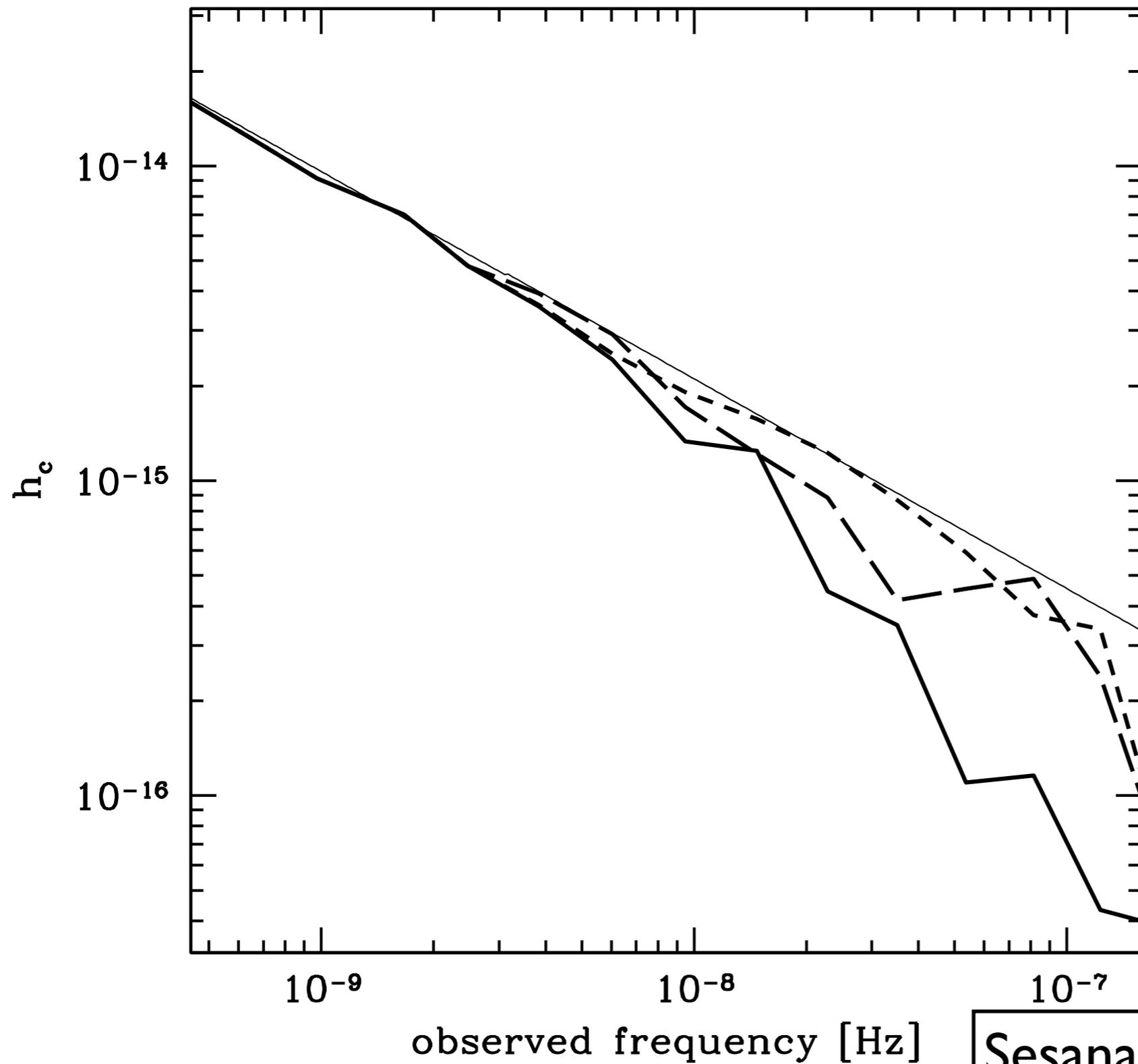
- and of the characteristic strain and spectral density

$$h_c(f) \sim \sqrt{\Omega_{\text{GW}}(f)} / f \sim M_c^{\frac{5}{6}} f^{-\frac{2}{3}}$$

$$S_h(f) \sim \Omega_{\text{GW}}(f) / f^3 \sim M_c^{\frac{5}{3}} f^{-\frac{7}{3}}$$

- Note the difference to the inspiraling source case. The latter is enhanced by matched filtering.

# Stochastic Backgrounds



Sesana, Vecchio & Colacino (2008)

# Typical strains of astrophysical sources

- Supernovae

- ➔ Collapse of the core of a massive star. If collapse is asymmetric, expect gravitational waves to be produced.
- ➔ Internal potential of system is potential of pre-collapse core, which can be modelled as a Neutron star, for which  $\phi_{\text{int}}/c^2 \sim 0.2$ .
- ➔ At a distance of 10 Mpc, the Newtonian potential is
$$|\phi_{\text{N}}/c^2| = \frac{GM_{\odot}/c^2}{d_{10} \times 10 \text{Mpc}} \sim 5 \times 10^{-21} d_{10}^{-1}$$
- ➔ And we estimate a strain of  $|h_{\text{SN}}| \lesssim 10^{-21} d_{10}^{-1}$ .
- ➔ 10 Mpc is the distance to the Virgo Cluster, in which volume the supernovae rate is  $\sim$ one event per month.

# Typical strains of astrophysical sources

- Rotating Neutron Stars

- ➔ Pulsars are rapidly rotating Neutron Stars.

- ➔ If the Neutron Star is deformed, i.e., it has a “mountain” on the surface, then this rotation leads to a time varying quadrupole moment and hence gravitational radiation.

- ➔ The quadrupole moment of a “mountain” of mass  $\delta M$  on the surface of the star can be approximated as

$$I_{ij} \sim \delta M R_{\text{NS}}^2 = \epsilon M_{\text{NS}} R_{\text{NS}}^2$$

- ➔ Hence, for a  $1M_{\odot}$  neutron star with  $R_{\text{NS}} = 10\text{km}$ , rotating at a rate of  $1\text{kHz}$ , the luminosity and strain are

$$h \sim 10^{-20} \epsilon (d/1\text{kpc})^{-1} \quad L \sim 10^{41} \epsilon^2 \text{J s}^{-1}$$