Copenhagen 1: Neutrino Mass and Oscillations Problems

1. For vacuum oscillations we determined an oscillation length of

$$L_0 = \frac{4\pi\hbar c E_\nu}{\delta m_{21}^2 c^4} \quad \Rightarrow \quad (\mathrm{L/km}) = \mathrm{X}\ \frac{(E_\nu/\mathrm{GeV})}{(\delta m_{21}^2 c^4/\mathrm{eV}^2)}$$

What is X? If you are interested in probing neutrino oscillations for $\delta m_{21}^2 c^4 \sim 7.5 \times 10^{-5}$ eV^2 using a reactor beam of $\bar{\nu}_e$s, what is the nearest distance at which you would see the maximum effect of oscillations? (Reactor antineutrino spectra are broad, but peak around 3 MeV.) Answer the same question if $\delta m_{21}^2 c^4$ is replaced by $\delta m_{32}^2 c^4 \sim 2.3 \times 10^{-3}$ eV^2 and a 5 GeV FermiLab ν_μ beam is used. (The FermiLab beam will also be broad, but use this one energy for your calculation.) Finally, for solar neutrinos of energy ~ 1 MeV, determine the $\delta m^2 c^4$ that corresponds to a vacuum oscillation maximum coincident with the earth-Sun distance (thus determining the minimum $\delta m^2 c^4$ that might be tested with solar neutrinos).

2. Consider matter-enhanced neutrino oscillations where right-hand side of the time-dependent Schroedinger equation involved the mass^2 matrix (c=1 here)

$$\begin{pmatrix} -\delta m_{21}^2 \cos 2\theta_{12} + 2\sqrt{2} E G_F \rho(t) & \delta m_{21}^2 \sin 2\theta_{12} \\ \delta m_{21}^2 \sin 2\theta_{12} & \delta m_{21}^2 \cos 2\theta_{12} - 2\sqrt{2} E G_F \rho(t) \end{pmatrix}$$

This matrix is written in the favor basis, e.g., the upper left entry is the electron-electron component, the lower right the muon-muon component.

a) Show that the local mass eigenvalues – the instantaneous eigenvalues of the matrix above – are

$$\pm \delta m_{21}^2 \sqrt{X^2(\rho) + \sin^2 2\theta_{12}} \equiv \pm \delta m_{21}^2(\rho) \quad \text{where} \quad X(\rho) = 2\sqrt{2} G_F \rho(t) E/\delta m_{21}^2 - \cos 2\theta_{12}$$

b) Show that the corresponding local mass eigenstates are

$$|\nu_1(\rho)\rangle = \cos\theta_{12}(\rho)|\nu_e\rangle - \sin\theta_{12}(\rho)|\nu_\mu\rangle \qquad |\nu_2(\rho)\rangle = \sin\theta_{12}(\rho)|\nu_e\rangle + \cos\theta_{12}(\rho)|\nu_\mu\rangle \quad \text{where}$$

$$\sin 2\theta_{12}(\rho) = \frac{\sin 2\theta_{12}}{\sqrt{X^2(\rho) + \sin^2 2\theta_{12}}} \qquad \cos 2\theta_{12}(\rho) = -\frac{X(\rho)}{\sqrt{X^2(\rho) + \sin^2 2\theta_{12}}}$$

c) Show that by applying the unitary transformation

$$U = \begin{pmatrix} \cos\theta_{12}(\rho) & -\sin\theta_{12}(\rho) \\ \sin\theta_{12}(\rho) & \cos\theta_{12}(\rho) \end{pmatrix} \qquad U \begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} |\nu_1(\rho)\rangle \\ |\nu_2(\rho)\rangle \end{pmatrix}$$

one transforms to the mass eigenstate basis.

d) If you have Mathematica or any similar tool, please make some graphs. Let $\sin^2 2\theta_{12} = .0001$. Let $X(\rho)$ evolve linearly from 5 to $-\cos 2\theta_{12}$. Graph the corresponding $\theta_{12}(\rho)$ as a function of $X(\rho)$. Graph the two eigenvalues as a function of $X(\rho)$. Plot the electron probability and muon probability for the eigenstate that is heaviest at large $X(\rho)$ as a function of $X(\rho)$. Do the same for the second eigenstate.

3. As discussed in the lecture, there are two conditions for matter-enhanced neutrino oscillations: a level crossing must occur and that the level crossing must be adiabatic. Consider a world in which

$$\sin 2\theta_{12} = 0.1 \qquad \delta m_{21}^2 c^4 = 2 \cdot 10^{-6} \text{ eV}^2$$

We consider the effects of matter on solar neutrino oscillations. The electron number density of the Sun is approximately given by

$$\rho(r) = \rho(0) e^{-r/r_0} \qquad r_0 = 0.1 R_\odot \qquad R_\odot = 7 \cdot 10^{10}\text{cm} \qquad \rho(0) = 6 \cdot 10^{25}/\text{cm}^3$$

Another quantity you will need to know is $G_F/(\hbar c)^3 = 1.166 \times 10^{-5}/\text{GeV}^2$ (GeV = 10^9 eV). Consider neutrino oscillations for three neutrino energies representative of the solar neutrino spectrum: E = 0.5 MeV, 3 MeV, and 15 MeV. Start the neutrinos at the center of the Sun, and allow them to propagate to a point near the solar surface, where the density is effectively zero. Calculate three quantities at each energy:

a) The critical electron density ρ_c where the level crossing occurs, as a fraction of $R_\odot$. Is there a level crossing in the Sun for all three energies?
b) The adiabatic parameter γ_c at the level crossing. How would you classify the crossings in terms of being more or less adiabatic? Please calculate P_{hop} in making this classification.
c) The quantity $\sin 2\theta_{12}(\rho_0)$, the local mixing angle at the center of the Sun. Compare these angles with the corresponding vacuum quantity.

Consequently, calculate from the Landau-Zener formula the electron neutrino survival probability at the surface of the Sun. Qualitatively, how would the solar neutrino spectrum be distorted under this so-called "small mixing angle" MSW solution? In fact the survival probability for low energy neutrinos is higher than that of higher energy neutrinos, as determined by experiment. What do you conclude?