Copenhagen 1: Neutrino Mass and Oscillation Solutions

1. The oscillation length is

$$L_0 = \frac{4\pi\hbar c E_\nu}{\delta m_{21}^2 c^4} = \frac{4\pi(0.197\text{ GeV f})(E_\nu/\text{GeV})\text{GeV}}{(m_{21}^2 c^4/\text{eV}^2)\text{eV}^2}\left(\frac{10^{18}\text{eV}^2}{\text{GeV}^2}\right)\left(\frac{\text{km}}{10^{18}\text{f}}\right) \Rightarrow$$

$$L_0/\text{km} = 2.48\frac{(E_\nu/\text{GeV})}{(\delta m_{21}^2 c^4/\text{eV}^2)}$$

For a reactor spectrum, the peak is near 4 MeV. So

$$E_\nu = 3\text{ MeV} = 0.004\text{ GeV} \quad \delta m_{21}^2 c^4 = 7.5\times 10^{-5}\text{ eV}^2 \Rightarrow L_0 = 99.2\text{ km}$$

And for the FermiLab beam

$$E_\nu = 5\text{ GeV} \quad \delta m_{21}^2 c^4 = 2.3\times 10^{-3}\text{ eV}^2 \Rightarrow L_0 = 5391\text{ km}$$

The nearest distance where the oscillation is at a maximum is $L_0/2$, so these distances are

$$L_0^{\text{reactor}}/2 = 49.6\text{ km} \qquad L_0^{\text{FermiLab}}/2 = 2695\text{ km}$$

Now the distance from the Sun to the earth is 1.496×10^8 km. So this would correspond to the first maximum of oscillations for $L_0 = 2.992\times 10^8$ km. So

$$2.992\times 10^8 = \frac{2.48(.001)}{(\delta m_{21}^2 c^4/\text{eV}^2)} \Rightarrow \delta m_{21}^2 c^4 = 8.29\times 10^{-12}\text{ eV}^2$$

So solar neutrino experiments are sensitive to neutrino masses $\delta m_{21}^2 c^4 \gtrsim 8.29\times 10^{-12}$ eV^2.

2. The matrix has the generic form

$$\begin{pmatrix} -a & b \\ b & a \end{pmatrix} \quad \text{where } a = -\delta m_{21}^2\left[2\sqrt{2}EG_F\rho(t)/\delta m_{21}^2 - \cos 2\theta_{12}\right] \equiv -\delta m_{21}^2 X(\rho) \quad b = \delta m_{12}^2 \sin 2\theta_{12}$$

a) Diagonalizing such a matrix yields

$$(-a-\lambda)(a-\lambda) - b^2 = 0 \Rightarrow \lambda = \pm\sqrt{a^2+b^2}$$

$$\lambda = \pm\delta m_{12}^2(\rho) \quad \text{where} \quad \delta m_{21}^2(\rho) \equiv \delta m_{21}^2\sqrt{X^2(\rho)+\sin^2 2\theta_{12}}$$

b) Solving for the normalized eigenvectors

$$\lambda = \delta m_{12}^2(\rho) \quad \Rightarrow \quad \frac{1}{\sqrt{2}\sqrt{a^2+b^2-a\sqrt{a^2+b^2}}}\begin{pmatrix} \sqrt{a^2+b^2}-a \\ b \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}\,[a^2+b^2]^{1/4}}\frac{1}{\sqrt{\sqrt{a^2+b^2}-a}}\begin{pmatrix} \sqrt{a^2+b^2}-a \\ \sqrt{(\sqrt{a^2+b^2}-a)(\sqrt{a^2+b^2}+a)} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}\,[a^2+b^2]^{1/4}} \left(\begin{array}{c} \sqrt{\sqrt{a^2+b^2}-a} \\ \\ \sqrt{\sqrt{a^2+b^2}+a} \end{array} \right) \equiv \left(\begin{array}{c} \sin\theta_{12}(\rho) \\ \\ \cos\theta_{12}(\rho) \end{array} \right)$$

$$\lambda = -\delta m_{12}^2(\rho) \quad \Rightarrow \quad \frac{1}{\sqrt{2}\sqrt{a^2+b^2+a\sqrt{a^2+b^2}}} \left(\begin{array}{c} -(\sqrt{a^2+b^2}+a) \\ b \end{array} \right)$$

$$= \frac{1}{\sqrt{2}\,[a^2+b^2]^{1/4}} \frac{1}{\sqrt{\sqrt{a^2+b^2}+a}} \left(\begin{array}{c} -(\sqrt{a^2+b^2}+a) \\ \\ \sqrt{(\sqrt{a^2+b^2}-a)(\sqrt{a^2+b^2}+a)} \end{array} \right)$$

$$= \frac{1}{\sqrt{2}\,[a^2+b^2]^{1/4}} \left(\begin{array}{c} -\sqrt{\sqrt{a^2+b^2}+a} \\ \\ \sqrt{\sqrt{a^2+b^2}-a} \end{array} \right) \equiv \left(\begin{array}{c} -\cos\theta_{12}(\rho) \\ \\ \sin\theta_{12}(\rho) \end{array} \right)$$

Finally

$$\sin 2\theta_{12}(\rho) \equiv 2\sin\theta_{12}(\rho)\cos\theta_{12}(\rho) = \frac{2b}{2\sqrt{a^2+b^2}} = \frac{\sin 2\theta_{12}}{\sqrt{X^2(\rho)+\sin^2 2\theta_{12}}}$$

$$\cos 2\theta_{12}(\rho) \equiv 1-2\sin^2\theta_{12}(\rho) = \frac{1}{2\sqrt{a^2+b^2}} \left[2\sqrt{a^2+b^2} - 2\left(\sqrt{a^2+b^2}-a\right)\right]$$

$$= \frac{a}{\sqrt{a^2+b^2}} = -\frac{X(\rho)}{\sqrt{X^2(\rho)+\sin^2 2\theta_{12}}}$$

c) Our matrix can be written

$$M^2 = \delta m_{12}^2 \left(\begin{array}{cc} X(\rho) & \sin 2\theta_{12} \\ \sin 2\theta_{12} & -X(\rho) \end{array} \right)$$

Thus

$$UM^2U^{-1} = \delta m_{12}^2 \left(\begin{array}{cc} \cos\theta_{12}(\rho) & -\sin\theta_{12}(\rho) \\ \sin\theta_{12}(\rho) & \cos\theta_{12}(\rho) \end{array} \right) \left(\begin{array}{cc} X(\rho) & \sin 2\theta_{12} \\ \sin 2\theta_{12} & -X(\rho) \end{array} \right) \left(\begin{array}{cc} \cos\theta_{12}(\rho) & \sin\theta_{12}(\rho) \\ -\sin\theta_{12}(\rho) & \cos\theta_{12}(\rho) \end{array} \right)$$

$$= \delta m_{12}^2 \left(\begin{array}{cc} -\sin 2\theta_{12}\sin 2\theta_{12}(\rho) + X(\rho)\cos 2\theta_{12}(\rho) & \sin 2\theta_{12}\cos 2\theta_{12}(\rho) + \sin\theta_{12}(\rho)X(\rho) \\ \sin 2\theta_{12}\cos 2\theta_{12}(\rho) + \sin 2\theta_{12}(\rho)X(\rho) & \sin 2\theta_{12}\sin 2\theta_{12}(\rho) - X(\rho)\cos 2\theta(\rho) \end{array} \right)$$

Now from our previous results

$$\sin 2\theta_{12}\cos 2\theta_{12}(\rho) + \sin\theta_{12}(\rho)X(\rho) = -\frac{\sin 2\theta_{12}X(\rho)}{\sqrt{X^2(\rho)+\sin^2 2\theta_{12}}} + \frac{\sin 2\theta_{12}X(\rho)}{\sqrt{X^2(\rho)+\sin^2 2\theta_{12}}} = 0$$

$$-\sin 2\theta_{12}\sin 2\theta_{12}(\rho) + X(\rho)\cos 2\theta_{12}(\rho) = -\frac{\sin^2 2\theta_{12}}{\sqrt{X^2(\rho)+\sin^2 2\theta_{12}}} - \frac{X^2(\rho)}{\sqrt{X^2(\rho)+\sin^2 2\theta_{12}}}$$

$$= -\sqrt{X^2(\rho)+\sin^2 2\theta_{12}}$$

So we have

$$UM^2U^{-1} = \left(\begin{array}{cc} -\delta m_{12}^2(\rho) & 0 \\ 0 & \delta m_{12}^2(\rho) \end{array} \right)$$

showing that U diagonalizes M^2, yielding the local mass eigenstates.

3. a) As $\sin 2\theta_{12} = 0.1 \Rightarrow \cos 2\theta_{12} = 0.995$. The critical density is

$$\rho_c = \frac{\delta m_{21}^2 c^4}{E_\nu} \frac{\cos 2\theta_{12}}{2\sqrt{2} G_F/(\hbar c)^3} (\hbar c)^3 = \frac{2 \times 10^{-6} \mathrm{eV}^2}{E_\nu} \frac{.995}{2\sqrt{2}(1.166 \times 10^{-5})/\mathrm{GeV}^2} \frac{1}{(\hbar c)^3}$$
$$= \frac{7.86 \times 10^{24}/\mathrm{cm}^3}{E_\nu/\mathrm{MeV}}$$

$$\Rightarrow \frac{\rho_c}{\rho(0)} = \frac{0.1309}{E_\nu/\mathrm{MeV}}$$

Thus the critical densities are

$$\frac{\rho_c(0.5 \text{ MeV})}{\rho(0)} = 0.262 \qquad \frac{\rho_c(3.0 \text{ MeV})}{\rho(0)} = 0.0436 \qquad \frac{\rho_c(15.0 \text{ MeV})}{\rho(0)} = 0.00873$$

As

$$\log \frac{\rho(r)}{\rho(0)} = -\frac{r}{r_0} = -\frac{r}{0.1 R_\odot} \quad \Rightarrow \quad r = -0.1 R_\odot \log \frac{\rho(r)}{\rho(0)}$$

So

$$r_c(0.5 \text{ MeV}) = 0.134 R_\odot \qquad r_c(3.0 \text{ MeV}) = 0.313 R_\odot \qquad r_c(15.0 \text{ MeV}) = 0.474 R_\odot$$

Thus there is a level crossing for all three energies, given these neutrino parameters, so one of the two criteria for an adiabatic level crossing is satisfied.

b) The adiabatic parameter at the crossing point is

$$\gamma_c = \frac{\sin^2 2\theta}{\cos 2\theta} \left(\frac{\delta m_{21}^2 c^4}{2E\hbar c} \right) \frac{1}{\left| \frac{1}{\rho_c} \frac{d\rho(r)}{dr} \Big|_{r=r_c} \right|} = \frac{\sin^2 2\theta}{\cos 2\theta} \left(\frac{\delta m_{21}^2 c^4}{2E\hbar c} \right) 0.1 R_\odot = \frac{3.566}{(E/\mathrm{MeV})}$$

So

$$\gamma_c(0.5 \text{ MeV}) = 7.132 \qquad \gamma_c(3.0 \text{ MeV}) = 1.189 \qquad \gamma_c(15.0 \text{ MeV}) = 0.2377$$

As

$$P_{hop} = e^{-\pi \gamma_c/2}$$

$$P_{hop}(0.5 \text{ MeV}) = 1.36 \times 10^{-5} \qquad P_{hop}(3.0 \text{ MeV}) = 0.1545 \qquad P_{hop}(15.0 \text{ MeV}) = 0.6884$$

So the low energy neutrino have a highly adiabatic crossing, remain on the local mass eigenstate trajectory, and are strongly converted to muon neutrinos; the medium energy solar neutrinos have a significant correction due to hopping to the low-mass trajectory, and thus are less strongly converted to muon neutrinos; while the high energy neutrinos mostly jump to the low-mass trajectory, and thus still remain primarily electron neutrinos after emerging from the Sun.

c) But we can do this calculation more precisely. We note

$$X(\rho(0)) = 2\sqrt{2}\ G_F\ \rho(0)\ \frac{E}{\delta m_{21}^2 c^4} - \cos 2\theta_{12} = 7.599 \frac{E}{\text{MeV}} - .9950 \Rightarrow$$

$$X(E = 0.5\text{MeV}, \rho(0)) = 2.804 \quad X(E = 3.0\text{MeV}, \rho(0)) = 21.80 \quad X(E = 15\text{MeV}, \rho(0)) = 113.0$$

Using the formulas derived in problem 2 we find

$$\sin 2\theta_{12}(\rho(0)) \sim (.0356, .00459, .000885) \ \text{ for } \ E_\nu = (0.5, 3.0, 15.0)\ \text{MeV}$$

$$\cos 2\theta_{12}(\rho(0)) \sim (-.999365, -.999989, -1.) \ \text{ for } \ E_\nu = (0.5, 3.0, 15.0)\ \text{MeV}$$

Thus in each case, $\theta_{12}(\rho(0)) \sim \pi/2$. Using our Landau-Zener formula for the probability of measuring an electron neutrino after emerging from the Sun

$$P_{\nu_e} = \frac{1}{2} + \frac{1}{2} \cos\left[2\theta_{12}(\rho(0))\right]\ \cos 2\theta_{12}(1 - 2P_{hop})$$

we finally have

$$P_{\nu_e}(0.5\ \text{MeV}) = 0.00284 \qquad P_{\nu_e}(3.0\ \text{MeV}) = 0.156 \qquad P_{\nu_e}(15.0\ \text{MeV}) = 0.687$$

This so-called "small mixing angle" solution doesn't fit the data, which require a larger survival probability for the lowest energy pp neutrinos. This solution produces an energy-dependent survival probability, but does so because the level crossing is adiabatic for low-energy neutrino (so these oscillate) but nonadiabatic for higher energy ones (so more of the high energy neutrinos survive). The MSW solution that fits the data produces the energy dependence by exploiting the second level-crossing condition, that there be a level crossing. In that solution the lowest energy neutrinos survive because the density at the Sun's center is not sufficient to cause a level crossing for these neutrinos. But the high energy neutrinos do have a level crossing, and thus oscillate.