Synchrotron Radiation

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1 Electrodynamics

1.1 Basic Electrodynamics

1.1.1 The Lienard–Wiechert potentials

Consider the case of a point charge (i.e., a charged particle) with charge q at position $\vec{x}_0(t)$ and with velocity $\vec{v}(t)$ and some acceleration $\vec{a}$, seen by an observer at location $\vec{x}$. This situation is shown in Fig. 1.1. We define $\vec{r} \equiv \vec{x} - \vec{x}_0$ and $\hat{\vec{k}} \equiv \vec{r}/r$.

For an unaccelerated charge at rest (i.e., $v = 0$, $a = 0$), the observer measures the potentials

$$\phi = \frac{q}{|\vec{x} - \vec{x}_0|} = \frac{q}{r} \qquad \text{and} \qquad \vec{A} = 0 \tag{1.1}$$

However, if we set the charge in motion, we must now *Lorentz transform* the four-potential into the observer's frame.

Information about the fields travels at the speed of light, so at any given point in time, the local potential is given by the position of the charge *one "light travel time" ago*.

The resulting Lorentz-transformed four-potential is then given by:

$$\phi = \left[\frac{q}{\kappa r}\right]_{\text{ret}} \qquad \text{and} \qquad \vec{A} = \left[\frac{q\vec{v}}{c\kappa r}\right]_{\text{ret}} \tag{1.2}$$

where the notation $[...]_{\text{ret}}$ indicates that a quantity inside the brackets is to be evaluated at the **retarded time** defined such that $t - t_{\text{ret}} = r(t_{\text{ret}})/c$. We have used the usual definition of $\vec{\beta} \equiv \vec{v}/c$. We also define

$$\kappa_{\text{ret}} \equiv 1 - \frac{\hat{\vec{r}}_{\text{ret}} \cdot \vec{v}_{\text{ret}}}{c} = 1 - \frac{\hat{\vec{k}} \cdot \vec{v}_{\text{ret}}}{c} = 1 - \frac{v_{\text{ret}}}{c} \cos\vartheta_{\text{ret}} \tag{1.3}$$

where ϑ is the angle between the direction of motion of the particle and the line of sight.

N.B.: $\vec{A}$ and ϕ are the **Lienard-Wiechert** potentials of a moving charge.

As expected, the meaning of the retardation is that the potential at a particular point is set by where the particle was a light crossing time r_{ret}/c ago[1]. That is: the fields transport information about the particle's position at exactly the speed of light.

Two things stand out:

[1] Because t_{ret} depends on r_{ret}, which itself depends on t_{ret}, both r_{ret} and t_{ret} have to be solved for self-consistently.

A) The term $\hat{\vec{k}}\vec{v}/c$ in κ is negligible for $v \ll c$. It is only important for relativistic charges and, together with the time retardation factor γ, accounts for relativistic Doppler boosting. It implies that field lines from an approaching charge get bunched together, making a stronger field.

B) The potentials are evaluated at the retarded time $t_{\rm ret} = t - r/c$. This implies that the potential at the current position is given by the point where the particle crosses the observer's "light cone". Information about the potential travels exactly with the speed of light. The retarded time for a sample particle trajectory is shown in the sketch in Fig. 1.1.

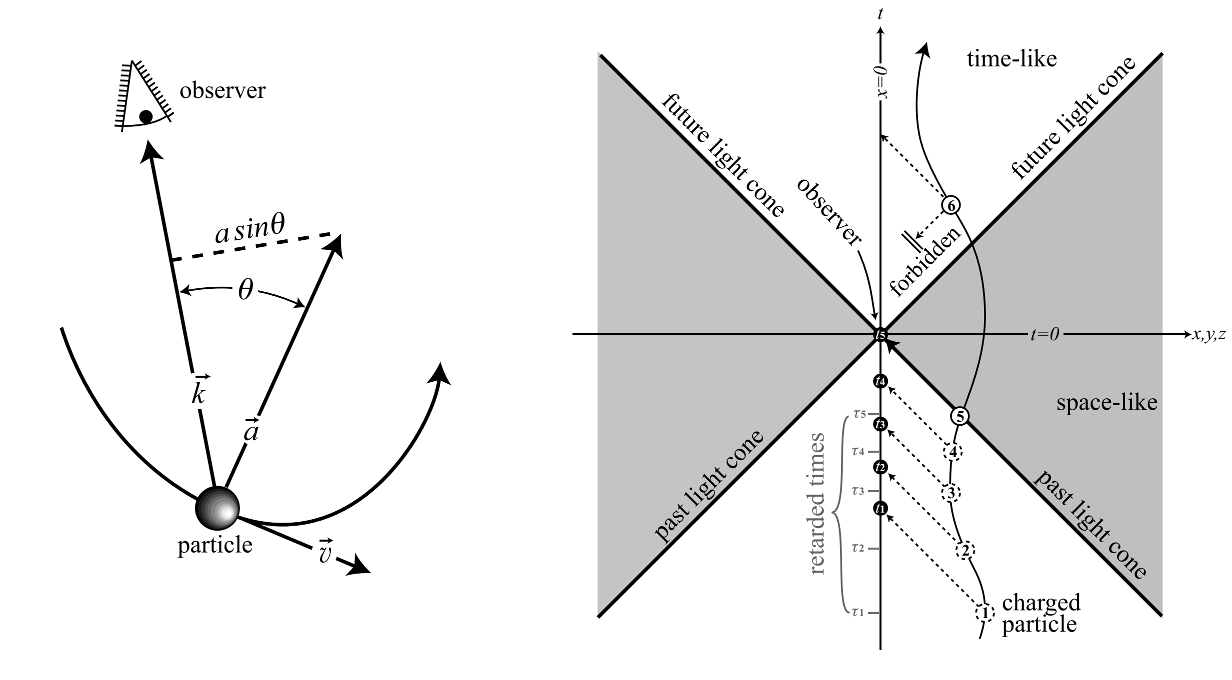

Figure 1.1: Light cone and retarded time τ for a charged particle, as seen by the observer (stationary at $x = 0$, observed at time t). In the figure, as is customary in relativity, $c = 1$.

1.1.2 The electromagnetic field of accelerated charges

Recall from graduate E&M (e.g., Rybicki & Lightman) that the radiation part ($\propto r^{-1}$) of the electric field for an accelerated charge is

$$\vec{E} = q\left[\frac{\left(\hat{\vec{k}} - \vec{\beta}\right)(1-\beta^2)}{\kappa^3 r^2}\right] + \frac{q}{\kappa^3 c^2 r}\hat{\vec{k}} \times \left[\left(\hat{\vec{k}} - \frac{\vec{v}}{c}\right) \times \vec{a}\right] \tag{1.4}$$

and

$$\vec{B} = \hat{\vec{k}} \times \vec{E} \tag{1.5}$$

Note that all quantities are evaluated at the retarded time. Since the first term on the right hand side of eq. (1.4) goes as r^{-2}, while the second goes as r^{-1}, so at sufficiently large distances, it completely dominates. This is the radiation part of the field.

N.B.: The instantaneous B-field is perpendicular to $\vec{a}$ and $\hat{\vec{k}}$: the radiation is 100% polarized. The electric field vector is perpendicular to $\hat{\vec{k}}$ and $\vec{B}$, but not necessarily perpendicular to $\vec{a}$ (it is perpendicular to $\vec{a}$ only for the direction $\vec{k} \perp \vec{a}$).

The spectrum is obtained by Fourier-transforming the electric field. For our discussion of synchrotron radiation, this will be important.

1.1.3 The Larmor Formula

Consider a non-relativistic particle with $\beta \ll 1$. We can then neglect the relativistic terms κ and β and the retardation **radiated power:** The Poynting flux (power per unit area) is

$$\vec{S} = \frac{c}{4\pi}\vec{E}_{\rm rad} \times \vec{B}_{\rm rad} = \frac{c}{4\pi}B^2\hat{\vec{k}} = \frac{q^2a^2\sin^2(\theta)}{4\pi c^3 r^2}\hat{\vec{k}} \tag{1.6}$$

Using $dA/d\Omega = r^2$, the power per unit solid angle is

$$\frac{dW}{dt\,d\Omega} = \frac{dW}{dt\,dA}\frac{dA}{d\Omega} = \frac{q^2a^2\sin^2(\theta)}{4\pi c^3} \tag{1.7}$$

where W is the radiated energy, following Rybicki & Lightman's notation.

The integrated power is

$$\begin{aligned} P &= \int d\Omega \frac{dW}{d\Omega \partial t} = \int d\phi \sin(\theta) d\theta \frac{q^2a^2\sin^2(\theta)}{4\pi c^3} & (1.8)\\ &= \frac{2\pi q^2 a^2}{4\pi c^3}\underbrace{\left[\int_{-1}^{1} d\cos(\theta)\sin^2(\theta)\right]}_{=4/3} & (1.9)\\ &= \frac{2q^2a^2}{3c^3} & (1.10)\end{aligned}$$

N.B.: $P = (2q^2a^2)\,/\,(3c^3)$ is called the **Larmor formula**, which is easily memorized and used frequently throughout these notes. Remember: the 2 in the numerator goes with the squares of q and a, and the 3 in the denominator goes with the cube of c.

Note also that no radiation is emitted in the direction of acceleration and the emitted power is forward-backward symmetric (only valid in the non-relativistic case).

2 Doppler Boosting

Suppose we denote the frame of emission of radiation as the primed frame moving at velocity β along the x-axis, and the observer's frame as the un-primed frame

The photon four-momentum is a null-vector: $P_\mu P^\mu = 0$. We write

$$P^\mu = \hbar \begin{pmatrix} \omega \\ k_x \\ k_y \\ k_z \end{pmatrix} \tag{2.11}$$

WLOG, we can boost along the x-axis with β to find

$$P'^\mu = \gamma\hbar \begin{pmatrix} \omega - \beta k_x \\ k_x - \beta\omega \\ k_y \\ k_z \end{pmatrix} \tag{2.12}$$

so the energy of the photon transforms as

$$h\nu' = \gamma h\nu \left(1 - \vec{\beta}\cdot\hat{\vec{k}}\right) = \gamma h\nu \left(1 - \beta\cos\left(\theta\right)\right) \tag{2.13}$$

and by symmetry

$$h\nu = \gamma h\nu' \left(1 + \beta\cos\theta'\right) = \frac{h\nu'}{\gamma(1-\beta\cos\theta)} \equiv \frac{h\nu'}{\delta} \tag{2.14}$$

where we defined the Doppler parameter

$$\delta \equiv \gamma\left(1 - \beta\cos\theta\right) \tag{2.15}$$

We can then derive the angle of the transformed photon:

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'} \qquad \cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta} \tag{2.16}$$

and

$$\sin\theta = \frac{\sin\theta'}{\gamma\left(1 + \beta\cos\theta'\right)} \qquad \sin\theta' = \frac{\sin\theta}{\gamma\left(1 - \beta\cos\theta\right)} \tag{2.17}$$

Photons emitted at $\theta' = 90°$ move at angle

$$\sin\theta = \frac{1}{\gamma} \tag{2.18}$$

and for $\gamma \gg 1$, this means that half of all photons emitted in the rest frame are beamed into a narrow cone with opening angle $1/\gamma$.

Now let's derive the transformation of the intensity:

Consider invariant scalar $dx_\mu dP^\mu$. In CM frame of particles, $dP'^0 = 0$ (all particles have same energy, since energy is second order in dP_i), so $dx'_\mu dP'^\mu = \sum_{i=1}^{3} dx'_i dP'_i$. In frame we want to evaluate phase space volume, we must choose $dt = 0$, so

$$dV_6 = dx_\mu dP^\mu = \sum_{i=1}^{3} dx_i dP_i = dx'_\mu dP'^\mu = \sum_{i=1}^{3} dx'_i dP'_i = dV'_6 \tag{2.19}$$

that is, phase space volume is invariant.

Since particle number is invariant, phase space density is also invariant:

$$f = \frac{dN}{dV_6} = f' \tag{2.20}$$

In particular, for photons, where $p = h\nu/c$, the energy differential

$$dW = h\nu f d\Omega p^2 dp = d\Omega h^4 \nu^3 d\nu f \tag{2.21}$$

and the specific energy density per unit solid angle

$$\frac{du_\nu}{d\Omega d\nu} = h^4 \nu^2 f \tag{2.22}$$

Finally, the intensity is

$$I_\nu = \frac{du_\nu}{d\Omega d\nu}\frac{1}{c} = \frac{h^4 \nu^3 f}{c} \tag{2.23}$$

and since f is invariant, we know that

$$\frac{I_\nu}{\nu^3} = \frac{I'_{\nu'}}{\nu'^3} \tag{2.24}$$

is invariant, or

$$I(\nu) = \frac{\nu^3}{\nu'^3} I'_{\nu'} = \frac{1}{\delta^3} I'_{\nu'} \tag{2.25}$$

It is clear that δ becomes very small when $\Gamma \gg 1$ and $\theta \ll 1$. Then, we expand both terms to first order:

$$\frac{1}{\delta} \sim \frac{1}{\Gamma\left[1 - \left(1 - \frac{1}{2\Gamma^2}\right)\left(1 - \frac{\theta^2}{2}\right)\right]} = \frac{1}{\Gamma\left(\frac{1}{2\Gamma^2} + \frac{\theta^2}{2}\right)} \tag{2.26}$$

and we can see that for any $\theta \leq \frac{1}{\Gamma}$,

$$\delta_{\theta \leq 1/\Gamma} \geq \Gamma \tag{2.27}$$

and the intensity is strongly boosted.

The three factors of δ come from (a) the Doppler boosting of photons into a forward cone of angle $1/\Gamma$, which increases the intensity by the ratio of solid angles for observers within that angle by a factor of roughly Γ^2, and (b) the Doppler boosting of photon frequencies.

And because optical depth just counts the number of absorptions along a path length, it must be invariant:

$$\tau_\nu = \tau'_{\nu'} \tag{2.28}$$

and from the equation of radiative transfer

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu = \frac{dI'_{\nu'}}{d\tau'_{\nu'}} \tag{2.29}$$

we can see that the source function must transform the same way the intensity does:

$$S_\nu = \frac{\nu^3}{\nu'^3} S'_{\nu'} \tag{2.30}$$

Finally, we need to understand how emissivities transform. For that, we have to understand how photon path-lengths trasnform Since the propagation angle of a photon changes under Lorentz transform, and since longitudinal lengths are Lorentz-contracted, we need to work out the change in path length from the combination of both.

Since perpendicular lengths are unchanged under Lorentz transform, we can calculate the total path length as

$$ds = \frac{\sin\theta'}{\sin\theta} ds' = \frac{ds'}{\gamma\left(1 - \beta\cos\theta\right)} \tag{2.31}$$

Since optical depth is invariant, we can write

$$\tau_\nu = \alpha_\nu ds = \tau'_{\nu'} = \alpha'_{\nu'} ds' \tag{2.32}$$

and

$$\alpha_\nu = \alpha'_{\nu'} \frac{ds'}{ds} = \frac{\alpha'_{\nu'}}{\gamma\left(1-\beta\cos\theta\right)} = \frac{\alpha'_{\nu'}}{\delta} \tag{2.33}$$

which, given the definition of the source function, brings us to

$$j_\nu = S_\nu \alpha_\nu = \delta^2 j'_{\nu'} \tag{2.34}$$

3 Synchrotron radiation

Consider a charged particle with mass m and charge q gyrating in a magnetic field of strength B with pitch angle α between $\vec{v}$ and $\vec{B}$.

Assume $v \approx c$, such that $\gamma = 1/\sqrt{1 - v^2/c^2} \gg 1$. Taylor expansion gives:

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2} + ... \tag{3.35}$$

Gyro orbit: Balance Lorentz force and centrifugal force

$$-\frac{q}{c}\vec{v} \times \vec{B} + \gamma m \vec{r}\omega_{\mathrm{B}}^2 = -\frac{q}{c}\omega_{\mathrm{B}} B\vec{r} + \gamma m \omega_{\mathrm{B}}^2 \vec{r} = 0 \tag{3.36}$$

which gives helical orbits with gyro frequency and Larmor radius:

$$\omega_{\mathrm{B}} = \frac{1}{\gamma}\frac{qB}{mc} = \frac{\omega_{\mathrm{L}}}{\gamma} \qquad \text{and} \qquad R_{\mathrm{L}} = \frac{v_\perp}{\omega_{\mathrm{B}}} = \frac{\gamma mc}{qB} v \sin\alpha \tag{3.37}$$

The gyro acceleration is perpendicular to the direction of motion,

$$a = R_{\mathrm{L}}\omega_{\mathrm{B}}^2 = \frac{qB}{\gamma m}\frac{v}{c}\sin\alpha \tag{3.38}$$

Note: In the frame of the electron, the Lorentz-transformed B-field induces an electric field of strength

$$\vec{E}' = \gamma\vec{\beta} \times \vec{B} \tag{3.39}$$

and leads to an acceleration perpendicular to both $\vec{\beta}$ and $\vec{B}$:

$$a'_\perp = \frac{qE_\perp}{m_{\mathrm{e}}} = \frac{q\gamma\beta B\sin\alpha}{m_{\mathrm{e}}} \tag{3.40}$$

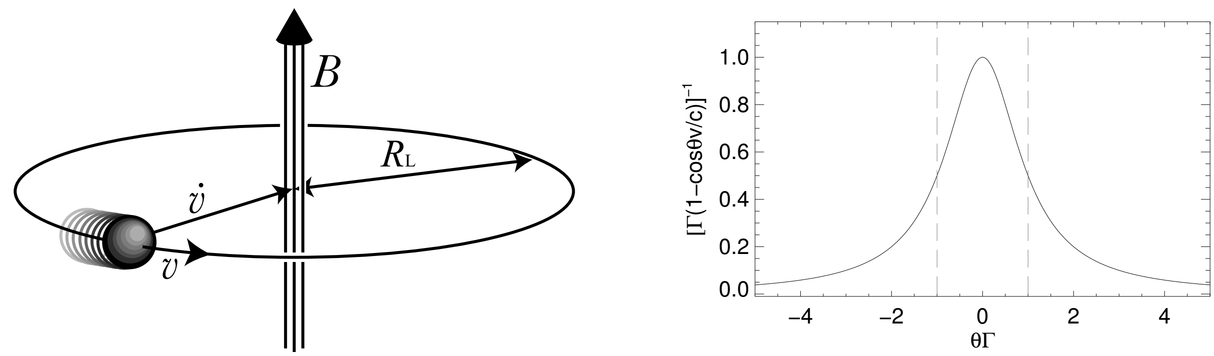

Figure 3.2: *Left:* sketch of cyclotron/synchrotron-gyro orbit. *Right:* Plot of the beaming function $[1 - \cos(\theta)v/c]^{-1}$ for $\gamma \gg 1$, normalized to a peak value of 1.

3.1 Lorentz transformed accelerations

Consider a relativistic particle undergoing acceleration. We can decompose the acceleration into two components: Along the direction of motion, $a_{\|}$, and perpendicular to it, $a_{\perp}$.

From undergraduate physics, recall Lorentz contraction and time dilation imposed by the Lorentz transform. Lengths along the direction of motion are Lorentz contracted in the observer's frame by a factor of γ, while the time measured in the observer's frame is longer by a factor of γ.

First consider the perpendicular acceleration, $a_{\perp}$. The transverse dimensions are not Lorentz contracted, however, the observer sees a time dilation, $dt_{\rm obs} = \gamma dt_{\rm part}$.

Suppose the observer measures an acceleration $a_{\perp}$, then in the particle frame the transverse acceleration would be

$$a_{\perp,\rm part} = \frac{d}{dt_{\rm part}}\frac{dl_{\perp,\rm part}}{dt_{\rm part}} = \gamma^2 \frac{d}{dt_{\rm obs}}\frac{dl_{\perp,\rm obs}}{dt_{\rm obs}} = \gamma^2 a_{\perp,\rm obs} \tag{3.41}$$

Now consider the longitudinal acceleration, $a_{\|}$. We have the same time dilation factor of γ^2, but additionally, we have to take account of the Lorentz contraction, which makes $dl_{\|,\rm part} = \gamma dl_{\|,\rm obs}$. Thus,

$$a_{\|,\rm part} = \frac{d}{dt_{\rm part}}\frac{dl_{\|,\rm part}}{dt_{\rm part}} = \gamma^3 \frac{d}{dt_{\rm obs}}\frac{dl_{\|,\rm obs}}{dt_{\rm obs}} = \gamma^3 a_{\|,\rm obs} \tag{3.42}$$

We now have the acceleration in the particle frame in terms of the acceleration measured in the observer's frame (which is what we have calculated above for Larmor gyration).

3.2 Lorentz invariance of the Larmor formula

From E&M, recall that the Larmor formula provides the energy emitted per unit time interval by a single particle,

$$\frac{dW}{dt} = \frac{2q^2a^2}{3c^3} \tag{3.43}$$

Since particle numbers are Lorentz invariant, the only two things that we have to consider under Lorentz transform in this formula are energy and time.

We already know that Lorentz transforms induce a time dilation,

$$dt_{\text{part}} = \frac{dt_{\text{obs}}}{\gamma} \tag{3.44}$$

In the comoving frame, the emitted radiation is forward-backward symmetric, as the power depends only on $\sin^2\theta$. We now have to consider the total energy of the light emitted. Consider the emission forward-backward symmetric in the particle frame (i.e., no net momentum is transported in any direction by the radiation in the particle's frame). The total energy emitted is dW_{part}. The net four-momentum of the emitted photons is $P^\mu_{\text{part}} = (dE_{\text{part}}, 0, 0, 0)$. In the observer's frame, the energy of the photos is

$$P^0_{\text{obs}} = \gamma P^0_{\text{part}} + \gamma\beta P^1_{\text{part}} = \gamma dE_{\text{part}} \tag{3.45}$$

and so

$$\frac{dE}{dt}_{\text{obs}} = \frac{dE}{dt}_{\text{part}} = \frac{2q^2a^2_{\text{part}}}{3c^3} \tag{3.46}$$

i.e., the Larmor formula is invariant. Note, however, that the acceleration has to be measured in the rest frame of the particle.

Given that $a^2_{\text{part}} = \gamma^4 a^2_{\perp,\text{obs}} + \gamma^6 a_{\|,\text{obs}}$, we finally have the Larmor formula for an accelerated particle, expressed in terms of quantities entirely measured in the observer's frame:

$$\frac{dW_{\text{obs}}}{dt_{\text{obs}}} = \frac{2q^2}{3c^3}\left(\gamma^4 a^2_{\perp,\text{obs}} + \gamma^6 a_{\|,\text{obs}}\right) \tag{3.47}$$

Inserting the expression for Larmor gyration from above and noting that the acceleration is perpendicular to $\vec{v}$, we have the synchrotron power emitted per particle:

$$\frac{dW}{dtdN} = \frac{2q^2}{3c^3}\left(\frac{qB\beta\sin\alpha}{\gamma\, m}\right)^2\gamma^4 = \frac{2q^4B^2\beta^2\sin^2\alpha\,\gamma^2}{3c^3m^2} \tag{3.48}$$

3.3 Doppler boosting

Equation (3.47) gives the total power emitted by the particle measured in the observers frame, integrated over frequency and solid angle. However, the emission is not uniform in either ν or Ω.

This is apparent from the relativistic expression for the Poynting flux:

$$\vec{S}_{\rm rad} = \left[\frac{q^2}{c^2\kappa^6}\hat{\vec{k}}\left(\left(\vec{\beta}-\hat{\vec{k}}\right)\times\dot{\vec{\beta}}\right)^2\right]_{\rm ret} \propto \left[\frac{\left(\left(\vec{\beta}-\hat{\vec{k}}\right)\times\dot{\vec{\beta}}\right)^2}{(1-\cos\theta\beta)^6}\right]_{\rm ret} \tag{3.49}$$

$$\propto \left[\frac{(\beta-\cos\theta)^2+\sin^2\theta\,\sin^2\phi}{(1-\cos\theta\,\beta)^6}\right]_{\rm ret} \tag{3.50}$$

where θ is the angle between the line of sight $\hat{\vec{k}}$ and the velocity $\vec{\beta}$ and ϕ is the angle between $\hat{\vec{k}}$ and $\vec{\beta}\times\dot{\vec{\beta}}$. All of this is evaluated at the retarded time.

The term $1-\cos(\theta)v/c$ in the denominator is **exceedingly small** when $\cos(\theta)v/c \sim 1$. This can only be the case when $\beta \approx 1$ (i.e., $\gamma \gg 1$) and $\cos\theta \approx 1$ (i.e., $\theta \ll 1$). The part of the gyro-orbit where this is the case will dominate the total emission.

To see how the emission is peaked around $\theta = 0$, we expand the denominator in $1/\gamma \ll 1$:

$$1-\cos\theta\,\frac{v}{c} \approx 1-\left(1-\frac{\theta^2}{2}\right)\left(1-\frac{1}{2\gamma^2}\right) \approx \frac{\theta^2}{2}+\frac{1}{2\gamma^2} \tag{3.51}$$

which is **very** small (see Fig. 3.2) when both of the following conditions are satisfied:

$$\gamma \gg 1 \qquad \text{and} \qquad \theta \lesssim \frac{1}{\gamma} \tag{3.52}$$

N.B.: Emission from relativistic particles is concentrated into a "**beaming cone**" around their direction of motion with a half-opening angle of $\sim 1/\gamma$.

Because only particles whose beaming cone sweeps across the observer contribute significantly, only particles at a pitch angle

$$\alpha = \theta_{B,\rm LOS} \pm 1/\gamma \tag{3.53}$$

contribute, where $\theta_{B,\rm LOS}$ is the angle between the line of sight $\hat{\vec{k}}$ and the direction of the magnetic field. Thus, α can be regarded as the angle between the line of sight and the B-field in the following.

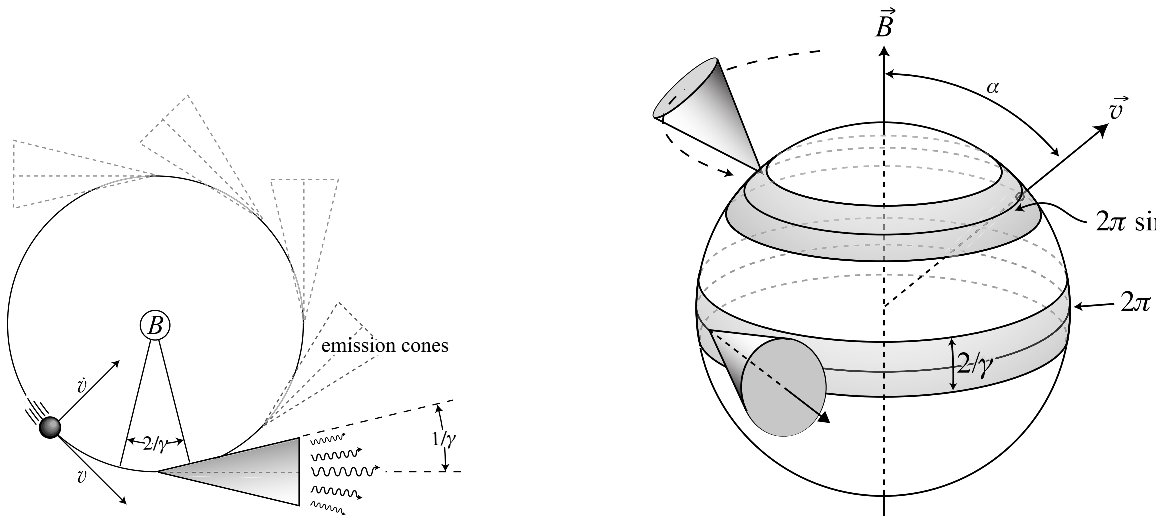

Figure 3.3: *Left:* Beaming cone rotating around gyro orbit: Only observers within $1/\gamma$ from direction of motion see pulse of radiation. *Right:* Solid angle illuminated by particle: $2/\gamma \cdot 2\pi \sin\alpha$.

3.4 The characteristic synchrotron emission frequency

The "beaming cone" is rotating with the particle orbit (see Fig. 3.3) and an observer will not stay inside the beaming cone for very long (since the cone itself has a very narrow opening angle).

The angular velocity with which the particle beaming cone sweeps across the observer is

$$\frac{d\theta}{dt} = \frac{2\pi \sin\alpha}{T_{\rm orb}} = \sin\alpha\, \omega_{\rm B} \tag{3.54}$$

Since the full width of the opening angle of the beaming cone is $\theta \sim 2/\gamma$, the time the observer is inside the beaming cone for a particle whose beaming cone does sweep over the observer is of order

$$\Delta\tau \approx \frac{\theta}{d\theta/dt} = \frac{2}{\gamma \sin\alpha\, \omega_{\rm B}} \tag{3.55}$$

The distance the particle travels in that time is $\Delta\tau v$.

The distance the photon travels in that time is $\Delta\tau c$.

The distance the photon emitted at beginning of pulse is ahead of the particle at the end of the pulse is $\Delta\tau(c - v)$.

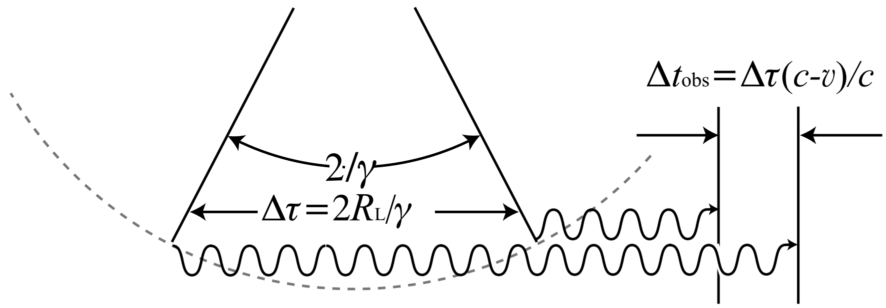

Figure 3.4: Duration of pulse seen by observer: Combination of beaming angle and light-travel time.

The arrival time difference between beginning and end of the pulse (see Fig. 3.4) is the characteristic time scale during which the observer receives the bursts of radiation (one per particle orbit):

$$\Delta\, t_{\rm pulse} \;=\; \Delta t_{\rm obs} = \Delta\tau \frac{c-v}{c} = \frac{2}{\gamma \sin\alpha\, \omega_{\rm B}} \left(1 - \frac{v}{c}\right) \tag{3.56}$$

$$\approx\; \frac{2}{\gamma \sin\alpha\, \omega_{\rm B}} \frac{1}{2\gamma^2} = \frac{1}{\sin\alpha\, \gamma^3 \omega_{\rm B}} \tag{3.57}$$

The fundamental frequency of the pulse is:

$$\nu_{\rm c} \approx \frac{1}{2\Delta\tau_{\rm pulse}} = \frac{\sin\alpha\gamma^3\omega_{\rm B}}{2} = \frac{\gamma^2 q \sin\alpha B}{2mc} \qquad \text{R\&L}: \quad \frac{3\gamma^2 qB\sin\alpha}{2\pi mc} \tag{3.58}$$

The spectrum emitted by each particle is broad, but we will simplify by assuming each particle only emits at exactly $\nu_{\rm c}$.

This relates the γ of a particle to the energy it emits:

$$\gamma = \left(\frac{2\pi\, m\, c\, \nu}{3\, q\, B\, \sin\alpha}\right)^{1/2} \qquad \text{and} \qquad \frac{d\gamma}{d\nu} = \frac{\pi\, m\, c}{3\, q\, B\, \sin\alpha} \frac{1}{\gamma} \tag{3.59}$$

3.5 Synchrotron spectra:

Suppose we define the distribution of particles (and in particular the case of a powerlaw distribution)

$$\frac{dN(\alpha)}{d\gamma} = f(\gamma) = \frac{N_0(\alpha)}{4\pi} \gamma^{-s} \tag{3.60}$$

such that the number density of particles at pitch angle α is given by

$$n(\alpha) = \int_{\gamma_{\min}}^{\gamma_{\max}} d\gamma \frac{N_0(\alpha)}{4\pi}\gamma^{-s} \tag{3.61}$$

and the total number density of particles is

$$n = \int d\Omega n(\alpha) = \int d\alpha\, 2\pi \sin(\alpha) n(\alpha) \tag{3.62}$$

Then the power emitted per frequency per unit solid angle is

$$\begin{aligned}
\frac{dW}{dt\,d\nu\,d\Omega} &= \frac{dW}{dt\,dN}\frac{dN}{d\gamma}\frac{d\gamma}{d\nu} \approx \frac{2q^4B^2\beta^2\sin^2\alpha\,\gamma^2}{3c^3m^2}\frac{N_0}{4\pi}\gamma^{-s}\frac{\pi\,m\,c}{3\,q\,B\,\sin\alpha}\gamma^{-1} && (3.63)\\
&\approx \frac{2\pi\,B\sin\alpha}{9c^2\,m}\frac{N_0}{4\pi}\gamma^{1-s} \approx \frac{2\pi\,B\sin\alpha}{9c^2\,m}N_0\left(\frac{2\pi\,m\,c\,\nu}{3\,q\,B\,\sin\alpha}\right)^{-\frac{s-1}{2}} && (3.64)\\
&\propto (\sin\alpha\,B)^{1+\alpha}\frac{N_0}{4\pi}\nu^{-\alpha} && (3.65)
\end{aligned}$$

which is the classic synchrotron powerlaw with spectral index $\alpha = (s-1)/2$. For typical powerlaw particle spectra, $s \sim 2$, so $\alpha \sim 0.5$.

3.6 Polarization

It is intuitive that synchrotron radiation must be polarized, given that there is a stronly preferred direction to the problem. To calculate the proper synchrotron emission coefficients, one must Fourier transform the electric field in both polarization directions. This is straight forward but mathematically slightly tedious.

We quote the specific power emitted per particle is

$$\frac{dP_\perp}{d\nu} = 2\pi\frac{dP_\perp}{d\omega} = \frac{\sqrt{3}q^3B\sin\alpha}{2mc^2}\left[F(x)+G(x)\right] \tag{3.66}$$

and

$$\frac{dP_\parallel}{d\nu} = 2\pi\frac{dP_\perp}{d\omega} = \frac{\sqrt{3}q^3B\sin\alpha}{2mc^2}\left[F(x)-G(x)\right] \tag{3.67}$$

with

$$F(x) = x\int_x^\infty d\zeta\, K_{\frac{5}{3}}(\zeta) \tag{3.68}$$

and

$$G(x) = xK_{\frac{2}{3}}(x) \tag{3.69}$$

where

$$x \equiv \frac{\nu}{\nu_{\mathrm{c}}} \tag{3.70}$$

To calculate the spectrum, these functions must be integrated over the electron distribution function

$$\epsilon_{\nu,\perp,\|} = \int_{\gamma_{\min}}^{\gamma_{\max}} d\gamma\, f(\gamma)\, P_{\perp,\|}(\nu/\nu_{\mathrm{c}}(\gamma)) \tag{3.71}$$

For an isotropic powerlaw distribution, this yields the emission coefficient

$$j_\nu(\alpha) = \frac{dW}{dt\, d\nu\, d\Omega}(\alpha) \tag{3.72}$$

$$= \frac{\sqrt{3}q^3 \frac{N_0}{4\pi} B \sin\alpha}{mc^2\,(s+1)} \Gamma\left(\frac{s}{4} + \frac{19}{12}\right) \Gamma\left(\frac{s}{4} - \frac{1}{12}\right) \left(\frac{2\pi mc\nu}{3qB\sin\alpha}\right)^{-\frac{s-1}{2}} \tag{3.73}$$

N.B.: α is now the angle between $\hat{\vec{k}}$ and $\vec{B}$ and only particles with that pitch angle contribute to the emission. Thus: No synchrotron emission along the field direction.

N.B.: The nomalization N_0 in eq. 3.73 may be pitch-angle dependent. Generally, the assumption is that the pitch angle distribution is isotropic. In this case, $N_0 = const$ and properly normalized such that $n = \int d\gamma N_0 \gamma^{-s}$.

The polarization fraction is

$$\Pi(x) = \frac{P_\perp - P_\|}{P_\perp + P_\|} = \frac{G(x)}{F(x)} \tag{3.74}$$

For powerlaw distributions, this can be evaluated to give

$$\frac{s+1}{s+7/3} =_{s=2} \frac{9}{13} \approx 69\% \tag{3.75}$$

3.6.1 Synchrotron-self absorption

We discussed non-thermal synchrotron radiation (which is by far the most common).

$\rightarrow$ cannot use Kirchhoff's law to calculate α_ν

The Einstein relations still hold, however, so we can determine B_{12} from j_ν. This is a great demonstration of detailed balance, so it is worth sketching out.

The effective absorption coefficient can be written in terms of the Einstein coefficients as

$$\alpha_\nu = \frac{h\nu}{4\pi} \int d^3p_1 \int d^3p_2 \left[f(p_1 B_{12} g_1 - f(p_2) B_{21} g_2 \right] \delta(E_2 - E_1 - h\nu) \tag{3.76}$$

where the delta function ensures that only electrons at the appropriate energy for transitions at the frequency in question are counted.

The power emitted per particle can be written as

$$P_\nu = h\nu \int dE_1 A_{21} \phi_\nu \tag{3.77}$$

Now recall the Einstein relations from detailed balance:

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21} \tag{3.78}$$

and

$$B_{21} = \frac{g_1}{g_2} B_{12} \tag{3.79}$$

Then we can rewrite the power emitted per particle as

$$P_\nu = \frac{2\,(h\nu^3)\,h\nu}{c^2} B_{21} \phi_\nu \tag{3.80}$$

We can now substitute for the Einstein B coefficients, which gives the two components of α_ν. For stimulated emission we have

$$\begin{aligned} \alpha_{\nu,\mathrm{stim}} &= -\frac{h\nu}{4\pi} \int d^3p_2 f(p_2) B_{12} \delta(p_2 - p_1 - h\nu/c) && (3.81) \\ &= -\frac{c^2}{8\pi h\nu^3} \int dp_2 f(p_2) P_\nu(E_2) && (3.82) \end{aligned}$$

and for the absorption part:

$$\alpha_{\nu,\mathrm{abs}} = \frac{c^2}{8\pi h\nu^3} \int dp_2 f(p_1) P_\nu(E_2) \tag{3.83}$$

We use

$$f(p) d^3 p = n(E) dE \tag{3.84}$$

and $E_2 = E_1 + h\nu$ to arrive at

$$\begin{aligned} \alpha_{\nu,\mathrm{eff}} &= \frac{c^2}{8\pi h\nu^3} \int d^3 p_2 \left[f(p_1) - f(p_2)\right] P_\nu(E_2) & (3.85) \\ &= \frac{c^2}{8\pi h\nu^3} \int dE_2 E_2^2 \left[\frac{n(E_2 - h\nu)}{(E_2 - h\nu)^2} - \frac{n(E_2)}{E_2^2}\right] P_\nu(E_2) & (3.86) \end{aligned}$$

For photon energies below the electron energy $h\nu \ll E_2$, which is generally a good assumption, we can Taylor expand:

$$\alpha_\nu = -\frac{c^2}{8\pi h\nu^3} \int dE E^2 \frac{\partial}{\partial E} \left[\frac{n(E)}{E^2}\right] h\nu P_\nu(E) \tag{3.87}$$

For a powerlaw distribution of electrons, the emission coefficient becomes

$$\alpha_\nu = \frac{\sqrt{3} q^3 c^2}{8\pi} \left(\frac{3q}{2\pi mc}\right)^{s/2} N_0 \left(B \sin\alpha\right)^{(s+2)/2} \Gamma\left(\frac{3s+2}{12}\right) \Gamma\left(\frac{3s+22}{12}\right) \nu^{-(s+4)/2} \tag{3.88}$$

or, highlighting the proportionalities:

$$\alpha_\nu = C_{\mathrm{abs}} N_0 \left(B \sin\alpha\right)^{(s+2)/2} \nu^{-(s+4)/2} \tag{3.89}$$

N.B.: The normalization C used in Rybicki & Lightman is very misleading. It differs by a factor mc^2 from the normalization C used in their eq. (6.36). The above normalization is consistent with our previous discussion.

The most important thing here is the different powerlaw index of $-(s+4)/2$, compared to the emission coefficient. We can see that most of this comes from the $1/\nu^2$ from the Einstein relations. The additional factor of energy comes from the Taylor expansion of $n(E)$ and the fact that the characteristic emission frequency $\nu_{\mathrm{c}} \propto E^2$

The source function for synchrotron emission by a powerlaw of electrons is:

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{\epsilon_\nu}{4\pi\alpha_\nu} \propto \nu^{5/2} B^{-1/2} \tag{3.90}$$

Two things to note:

A) The powerlaw index of 2.5 for optically thick synchrotron emission is independent of s and different from the powerlaw index of low frequency thermal emission (the thermal case is 2, not 2.5).

B) The density cancels out, so we can constrain the magnetic field strength if we can in principle measure the intensity of an optically thick source.

The spectrum from a synchrotron source without cooling is therefore a powerlaw with index 2.5 at low energies that breaks to an index of $-(s-1)/2 \sim 0.5$ at optically thin frequencies.

For a syncrotron-emitting object of a given transverse size L, the frequency at which the spectrum becomes optically thin can be derived from the condition

$$\tau_\nu = 1 \tag{3.91}$$

or

$$\alpha_\nu L = 1 \tag{3.92}$$

which gives

$$\nu_{\tau=1} = \left[L C_{\rm abs} N_0 \left(B \sin\alpha\right)^{(s+2)/2}\right]^{2/(s+4)} \tag{3.93}$$

Since N_0 and B are proportional to particle and magnetic pressure, the transition frequency moves up for higher pressure. It moves up for larger sources as well.

4 Scale Invariance in Jets

To calculate the emission from a jet, we have to integrate the radiative transfer equation over the entire jet. The jet changes $\vec{B}$, N_0, and its cross section R constantly, and this is going to be complicated... How can we understand that simple relations should hold across different objects?

In the following, we will sketch out the discussion in Heinz & Sunyaev (2003). Scale invariance implies that the spatial variation of important jet quantities (such as the shape of the jet, i.e., its lateral cross section, the orientation of magnetic field lines, the field strength, etc.) depends only on the dimensionless variable $\mathbf{r}/r_{\rm g}$. Thus, a given variable f should be proportional to a function

$\psi(\mathbf{r}/r_{\rm g})$ which depends on $\mathbf{r}$ only through $\mathbf{r}/r_{\rm g}$. In other words, we can scale a jet model for mass M_1 by a length factor M_2/M_1 and some spatially independent normalization factor and arrive at a jet model for mass M_2.

In mathematical terms, this can be expressed as the condition that we can write any dynamically relevant quantity f, such as the magnetic field $B(r)$, as the product of two decoupled functions:

$$\begin{aligned} f(M,\dot{m},a,r) &= \phi_f(M,\dot{m},a)\,\psi_f\left(\frac{r}{r_{\rm g}},a\right) \\ &= \phi_f(M,\dot{m},a)\,\psi_f(\chi,a) \end{aligned} \tag{4.94}$$

where r is the distance to the central engine measured along the jet, ϕ_f describes the dependence of f on the central engine mass M, and ψ_f describes the spatial dependence of f on the similarity variable $\chi \equiv r/r_{\rm g}$ for a given set of $\dot{m}$ and a. Note that this is a *requirement* we place on the jet model, inspired by the observational similarity between jets from different kinds of objects. Not all possible jet model must necessarily satisfy this relation. However, those models that do satisfy it span an important sub-class of jet models and all of them will obey the relations derived below. One important example of such a model is the Blandford-Koenigl model.

The normalization functions ϕ_f reflect the dependence of the conditions at the base of the jet on the central mass M. Since jets are launched above accretion disks, it is natural to assume that these functions ϕ_f can be adopted from accretion disk models.

The synchrotron self-absorption coefficient is

$$\alpha_\nu = C_{\rm abs}\,N_0\,B^{(s+2)/2}\nu^{-(s+4)/2} \tag{4.95}$$

where $C_{\rm abs}$ is a proportionality constant weakly dependent on p.

For ease of expression, we will present the following analysis in the case of a jet viewed from side on, however, extension to the case of arbitrary viewing angles is straight forward, and the result we derive is fully general. In the perpendicular case, the expression for τ_ν takes on a particularly simple form:

$$\begin{aligned} \tau_\nu &= R_{\rm jet}\,\alpha_\nu = R_{\rm jet}\,C_{\rm abs}\,N_0\,B^{(s+2)/2}\nu^{-(s+4)/2} \\ &= C_{\rm abs}\,M\,\phi_{N_0}(M,\dot{m},a)\,\left[\phi_B(M,\dot{m},a)\right]^{\frac{s+2}{2}}\,\nu^{-\frac{s+4}{2}} \\ &\qquad \psi_R(\vec{\chi},a)\,\psi_{N_0}(\vec{\chi},a)\,\left[\psi_B(\vec{\chi},a)\right]^{\frac{s+2}{2}} \\ &= \Phi(M,\dot{m},a,\nu)\;\Psi(\vec{\chi},a) \end{aligned} \tag{4.96}$$

where we define

$$\Phi\left(M,\dot{m},a,\nu\right) \equiv M\phi_{N_0}(M,\dot{m},a)\left[\phi_B(M,\dot{m},a)\right]^{\frac{s+2}{2}}\,\nu^{-\frac{s+4}{2}} \tag{4.97}$$

$$\Psi\left(\vec{\chi},a\right) \equiv C_{\rm abs}\psi_R(\vec{\chi},a)\psi_{N_0}(\vec{\chi},a)\left[\psi_B(\vec{\chi},a)\right]^{\frac{s+2}{2}} \tag{4.98}$$

The optically thin synchrotron emissivity for a powerlaw distribution of electrons (well away from the lower and upper cutoff in the energy distribution) follows

$$\begin{aligned} j_\nu &= J_p\, N_0\, B^{\frac{s+1}{2}}\, \nu^{-\frac{s-1}{2}} \\ &= J_p \phi_{N_0}(M, \dot{m}, a) \left[\phi_B(M, \dot{m}, a)\right]^{\frac{s+1}{2}} \nu^{-\frac{s-1}{2}} \\ &\qquad \psi_{N_0}(\vec{\chi}, a) \left[\psi_B(\vec{\chi}, a)\right]^{\frac{s+1}{2}} \end{aligned} \tag{4.99}$$

where J_p is a constant weakly dependent on p.

For simplicity, we will neglect the dependence on the viewing angle ϑ due to Doppler beaming and optical depth effects. Because the viewing angle ϑ and the Lorentz factor Γ are independent of M, it follows that any function of viewing angle must also be independent of M, which justifies this approach in what follows.

Recall the solution of the radiative transfer equation:

$$I_\nu(\tau_\nu) = e^{-\tau_\nu} I_\nu(0) + \int_0^\tau d\tau'_\nu e^{\tau'_\nu - \tau_\nu} S_\nu \tag{4.100}$$

The jet flux F_ν is then simply the solid-angle integral over I_ν, with $d\Omega = R dr/D^2$:

$$\begin{aligned} F_\nu = \frac{1}{D^2} \int_{r_{\rm g}}^\infty dr dz e^{-\tau_\nu} \int d\tau e^{\tau_\nu} S_\nu(r) &= \frac{1}{D^2} \int_{r_{\rm g}}^\infty dz \int_{-R}^R dr e^{\tau_\nu} \int_{-R}^R ds j_\nu(r) e^{-\tau_\nu(s)} \\ = \frac{r_{\rm g}^3}{D^2} \int_1^\infty d\chi_z \int_{-\psi_r}^{\psi_r} d\chi_z e^{-\tau_\nu} \int_{-\psi_r}^{\psi_r} d\chi_s j_\nu(\vec{\chi}) e^{\tau(\vec{\chi})} \\ \propto M^3 \phi_{N_0}\, \phi_B^{\frac{s+1}{2}}\, \nu^{-\frac{s-1}{2}} \int d\chi_z \int d\chi_r e^{-\Phi\Psi} \int d\chi_s \psi_{N_0} \psi_B^{\frac{s+1}{2}}\, \mathrm{e}^{\Phi\Psi(\vec{\chi})} \\ \propto M^3\, \phi_{N_0}\, \phi_B^{\frac{s+1}{2}}\, \nu^{-\frac{s-1}{2}}\, \Theta\left[\Phi(M, \dot{m}, a, \nu), a, \Psi(\vec{\chi})\right] \end{aligned} \tag{4.101}$$

The integral term Θ depends on M, and ν only through the combination Φ from eq. (4.97).

4.1 The relation between F_ν and M

From eq. (4.101), we can now work out the non-linear dependence of F_ν on the central engine mass M. The spectral index $\alpha \equiv -\partial \ln(F_\nu)/\partial \ln(\nu)$ of the jet emission is given by

$$\frac{\partial \ln(F_\nu)}{\partial \ln(\nu)} = -\frac{s-1}{2} + \frac{\partial \ln(\Theta)}{\partial \ln(\Phi)} \frac{\partial \ln(\Phi)}{\partial \ln(\nu)} \tag{4.102}$$

$$= -\frac{s-1}{2} - \frac{\partial \ln(\Theta)}{\partial \ln(\Phi)} \left(\frac{s+4}{2}\right) \equiv -\alpha \tag{4.103}$$

Now taking the partial derivative of eq. (4.101) with respect to M and substituting $\partial \ln(\Theta)/\partial \ln(\Phi)$ from eq. (4.103), we can write

$$\begin{aligned}
\frac{\partial \ln(F_\nu)}{\partial \ln(M)} &= 3 + \frac{\partial \ln \phi_{N_0}}{\partial \ln(M)} + \frac{\partial \ln \phi_B^{\frac{s+1}{2}}}{\partial \ln(M)} + \frac{\partial \ln(\Theta)}{\partial \ln(\Phi)} \frac{\partial \ln(\Phi)}{\partial \ln(M)} \\
&= \frac{2s + 13 + 2\alpha}{s+4} + \frac{\partial \ln(\phi_B)}{\partial \ln(M)} \left(\frac{2s + 3 + \alpha s + 2\alpha}{s+4} \right) \\
&\quad + \frac{\partial \ln(\phi_{N_0})}{\partial \ln(M)} \left(\frac{5 + 2\alpha}{s+4} \right) \equiv \xi_M
\end{aligned} \tag{4.104}$$

Quite generally, the functions ϕ_{N_0} and ϕ_B will be simple powers of M — for our fiducial assumptions, $\phi_{N_0} = M^{-1}$ and $\phi_B = M^{-1/2}$, and thus the index ξ_M will simply be a constant:

$$\begin{aligned}
\xi_M &= \frac{2s + 13 + 2\alpha}{s+4} - \frac{1}{2} \left[\frac{2s + 3 + (s+2)\alpha}{s+4} \right] - \frac{5 + 2\alpha}{s+4} \\
&\sim \frac{17}{12} - \frac{\alpha}{3} \approx 1.42 - 0.33\alpha
\end{aligned} \tag{4.105}$$

where the approximate expressions assume $p = 2$. Thus, for any given set of $\dot{m}$, a, and ϑ, F_ν will follow a simple powerlaw relation in M with powerlaw index ξ_M

$$F_\nu \propto M^{\xi_M} \sim M^{1.42 - 0.33\alpha}. \tag{4.106}$$

Variations in the other source parameters $\dot{m}$, a, the viscosity parameter α_{visc}, and ϑ will only cause a *mass independent* scatter around this relation.

Remarkably, this result is entirely independent of the functions ψ_f. Given a set of functions ϕ_f, which describe the dependence of the input conditions in the inner disk on M, and given an observed jet spectrum with spectral index α, eq. (4.106) *predicts* the scaling of jet flux F_ν with M for any jet model that reproduces this spectral slope. The only assumptions that went into the derivation of this result are that *a)* the relevant parameters can be decomposed following eq. (4.94), *b)* that the high (low) energy cutoffs in the spectrum are far above (below) the observed spectral band, and *c*) that the function Θ is analytic. This is what was meant when we required the functions ψ_f to be mathematically well behaved.

Typically, the radio emission from core dominated jets follows a flat spectrum over many decades in frequency, i.e., $\alpha \sim 0$. In this case, it follows for our fiducial parameters that the radio flux F_{r} depends non-linearly on the mass to the $\xi_M = 17/12 \sim 1.42$ power, once again, *independent* of the jet model, which manifests itself only through ψ_f.

The typical assumption is that

$$B \propto M^{-1/2} \tag{4.107}$$

and

$$N_0 \propto B^2 \tag{4.108}$$

which gives $\xi_M = 17/12$ for a flat-spectrum source.

As mentioned before, the fiducial $B^2 \propto N_0 \propto M^{-1}$ scaling arises in a number of standard scenarios for the inner accretion disk: both in high efficiency, radiation pressure dominated inner disks and in low efficiency ADAFs. The value of $\xi = 17/12 - \alpha/3$ is therefore a very general result, which depends only weakly on the spectral index α.

It is worth noting that this analysis holds even for the case of jets composed of discrete ejections or internal shocks, if we define F_ν as the time averaged flux or the peak flux. In fact, because the derivation of eqs. (4.101-4.104) did not assume any specific jet-like geometry, they hold for any synchrotron emitting plasma with powerlaw spectrum if the source parameters can be described by eq. (4.94).

4.2 mdot-dependence

Now consider the dependence on accretion rate:

$$\begin{aligned}\frac{\partial \ln(F_\nu)}{\partial \ln(\dot{m})} = & \frac{\partial \ln(\phi_B)}{\partial \ln(\dot{m})}\left(\frac{2s+3+\alpha(s+2)}{s+4}\right) \\ & + \frac{\partial \ln(\phi_C)}{\partial \ln(\dot{m})}\left(\frac{5+2\alpha}{s+4}\right) \equiv \xi_{\dot{m}}\end{aligned} \tag{4.109}$$

following the same derivation as in eq. (4.104).

For our fiducial assumption $\phi_C \propto \phi_B^2 \propto \dot{m}$ (from ADAF type accretion, or the Ansatz $W_{\rm jet} \propto L_{\rm disk}$) we get

$$\xi_{\dot{m}} = \frac{2s+(s+6)\alpha+13}{2(s+4)} \sim \frac{17}{12} + \frac{2\alpha}{3} \approx 1.42 + 0.67\alpha \tag{4.110}$$

4.3 Power dependence

We can ask how the radio luminosity depends on jet power, which itself depends on $P_{\rm jet} \propto \dot{M} = M\dot{m}$:

$$L_{\rm r} \propto M^{\xi_M}\dot{m}^{\xi_{\dot{m}}} \propto \dot{M}^{\frac{17+4\alpha}{12}}\dot{m}^{\frac{17+8\alpha}{12}} \propto \dot{M}^{\frac{17+8\alpha}{12}}M^{-\alpha} \propto P_{\rm jet}^{\frac{17+8\alpha}{12}}M^{-\alpha} \tag{4.111}$$

This is a clear prediction and indications are that it holds in the class of AGN jets for which jet power can be measured from X-ray cavities in clusters (see left panel of Fig. 4.5).

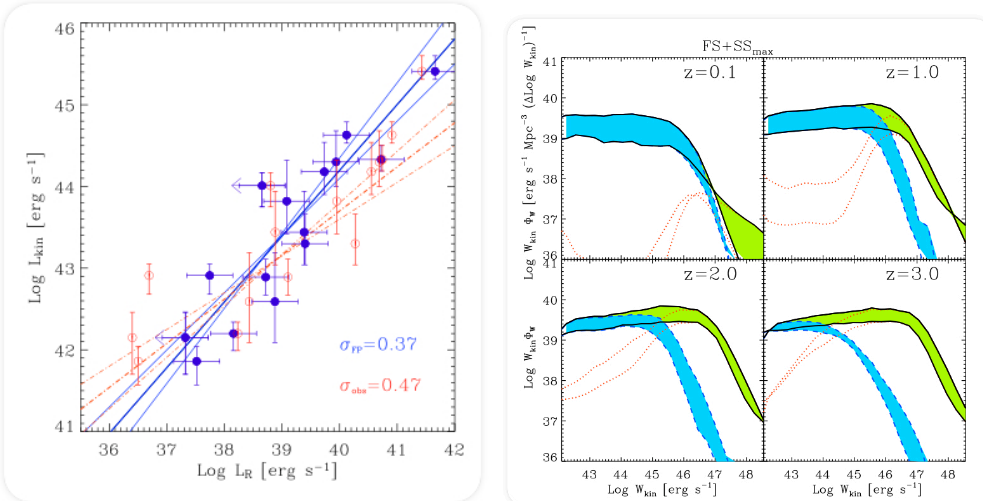

Figure 4.5: *Left:* Correlation between jet core power and large scale kinetic power determined from X-ray cavities (Merloni & Heinz, 2007); *Right:* From the jet-power–jet-core luminosity correlation, we can derive the kinetic luminosity function of jets (Merloni & Heinz, 2008), which shows that there is roughly equal jet power per decade of power.

With this correlation, we can convert the radio luminosity function of flat spectrum radio sources into a kinetic luminosity function, as shown in the right panel of Fig. 4.5.