Niels Bohr Institute Dark Cosmology Centre
Multi-Messenger Astrophysics Summer School 2018

Gravitational Waves Question 1: Computing Background Characteristics

If you have questions, please contact Jonathan Gair, j.gair@ed.ac.uk

---

This question is focussed on calculating properties of gravitational wave source populations. For two different classes of source you will estimate properties such as the characteristic strains, signal-to-noise ratios and background spectrum generated by a population of such sources. There are two sub-parts to this question. At the end of each there is a starred part which is more open ended and involves some numerical investigation. I suggest you leave that to explore in your own time.

1. This question is concerned with properties of a background generated by a population of massive black holes. For the first few parts of this question you can assume that these binaries are identical, i.e., they all have the same values for the two masses, m_1 and m_2, and hence the derived quantities of total mass, $M = m_1 + m_2$, reduced mass, $\mu = m_1 m_2 / M$, and chirp mass,

$$\mathcal{M}_c = \frac{m_1^{\frac{3}{5}} m_2^{\frac{3}{5}}}{M^{\frac{1}{5}}}.$$

In the final part of this question you'll be asked to think about and play around with different assumptions about the mass distribution in the binaries.

We will use geometric units throughout, i.e., we set $c = G = 1$ so we don't need to worry about keeping track of these factors.

(a) Assuming that the binary is Newtonian and circular, derive the following characteristic properties of the emitted gravitational waves [1].

i. The GW amplitude scales like

$$h \sim \frac{1}{D} \mathcal{M}_c^{\frac{5}{3}} f^{\frac{2}{3}}. \quad (1)$$

ii. The GW energy loss scales like

$$\dot{E}_{\rm GW} \sim \mathcal{M}_c^{\frac{10}{3}} f^{\frac{10}{3}}. \quad (2)$$

iii. The rate of change of frequency scales like

$$\dot{f} \sim \mathcal{M}_c^{\frac{5}{3}} f^{\frac{11}{3}}. \quad (3)$$

iv. The Fourier transform of $h(t)$ scales like

$$\tilde{h} \sim \frac{1}{D} \mathcal{M}_c^{\frac{5}{6}} f^{-\frac{7}{6}}. \quad (4)$$

v. The characteristic strain scales like

$$h_c \sim \frac{1}{D} \mathcal{M}_c^{\frac{5}{6}} f^{-\frac{1}{6}}. \quad (5)$$

---

[1] You may find it useful to recall that the energy of a Newtonian binary of semi-major axis a is $E = -M\mu/(2a)$ and the frequency is related to the semi-major axis via $2\pi f = \sqrt{M/a^3}$.

vi. The energy density of a GW background generated by a population of these sources scales like

$$\Omega_{\rm GW}(f) \sim \mathcal{M}_c^{\frac{5}{3}} f^{\frac{2}{3}} \tag{6}$$

In the above f denotes the orbital frequency of the binary and D the distance to the source.

If you made good notes during the lecture and are confident in those arguments, feel free to ski[this and move straight on to the next part.

(b) We now suppose that there is an additional process driving the evolution of the binaries, stellar hardening. This leads to an evolution of the semi-major axis, a, of the binary of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a}\right) = k\frac{\rho_*}{\sigma^3}\frac{m_2}{a} \tag{7}$$

where k is a numerical constant, ρ_* is the stellar density and σ is the velocity dispersion of the stars. Show that this equation implies the energy loss from stellar hardening scales like

$$\dot{E}_{\rm hard} \propto \frac{\rho m_2 \mu}{\sigma^3} M^{\frac{2}{3}} f^{\frac{2}{3}}. \tag{8}$$

(c) Derive an expression for the corresponding GW background energy density, $\Omega_{\rm GW}(f)$.

(d) Assuming that the sources are at a common redshift (or the population is dominated by sources at a particular redshift), show that the spectrum is a broken power-law and find the asymptotic slopes at low and high frequency.

(e) If a broken power law background were detected, what would it tell you about the relevant physical processes influencing the population? How is that information encoded in the background?

(f) * Try varying the assumptions about the population, e.g., the masses of the binary components or the properties of the stellar system or the redshift distribution of the sources. Compute the background from such populations numerically and explore how these assumptions change the results qualitatively and quantitatively.

2. We now consider a population of burst sources. We will represent these as sine-Gaussians with a four parameter waveform family

$$h(t) = \frac{A}{D}\cos(2\pi f_0 t)\mathrm{e}^{-\frac{1}{2}Qt^2}.$$

(a) Find the average waveform power

$$\langle h^2 \rangle = \frac{1}{2T}\int_{-T}^{T} h^2(t)\mathrm{d}t$$

and show that, for a reasonable choice of T, e.g., $T = 2/\sqrt{Q}$, it scales like A^2/D^2 as expected. Why is this a 'reasonable choice' for T?

(b) Compute the Fourier transform of $h(t)$. Hence deduce the bandwidth of the source is $\Delta f \sim \sqrt{Q}/\pi$.

(c) Estimate the SNR that might be possible in a burst search using

$$\left(\frac{S}{N}\right)^2 \approx \frac{\langle h^2 \rangle}{\Delta f S_n(f)}.$$

You may assume white noise, $S_n(f) = \sigma^2$, for simplicity.

(d) Compute the SNR that would be obtained if the source was found by matched filtering and compare it to the burst search SNR. Comment on your answer.

(e) Compute the energy distribution for one of these bursts, $\mathrm{d}E/\mathrm{d}f$.

(f) Find an expression for the GW background energy density produced by a population of these bursts, assuming that A and Q are constant and the distribution of f_0 is a power law $\mathrm{d}n/\mathrm{d}f_0 \propto f_0^\alpha$ with $\alpha > -1$.

(g) Assuming the sources are at a common redshift, show that the asymptotic slope of the background is f^3 at low frequency and $f^{3+\alpha}$ at high frequency.

(h) * Use simulations on a computer to explore how things change under modifications of these various assumptions.