Niels Bohr Institute Dark Cosmology Centre
Multi-Messenger Astrophysics Summer School 2018

Solution to Gravitational Waves Question 1: Computing Background Characteristics

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1. As in the question description we denote the two masses by m_1 and m_2, the total mass by$M = m_1 + m_2$, the reduced mass by $\mu = m_1 m_2/M$, and the chirp mass by

$$\mathcal{M}_c = \frac{m_1^{\frac{3}{5}} m_2^{\frac{3}{5}}}{M^{\frac{1}{5}}}.$$

We will use geometric units throughout, i.e., we set $c = G = 1$ so we don't need to worry about keeping track of these factors.

(a) For a Newtonian binary, the motion is equivalent to that of a body of mass μ orbiting in a fixed Newtonian potential with mass M. Denoting the orbital radius by a (it is also the semi-major axis for a circular binary), the orbital frequency is given by

$$2\pi f = \sqrt{\frac{M}{a^3}}$$

and the total energy of the binary is

$$E = -\frac{M\mu}{2a}.$$

i. The GW amplitude is determined by the quadrupole moment of the space-time

$$h \sim \frac{\ddot{I}_{jk}}{D}, \quad I_{jk} = \int \rho x_i x_j \mathrm{d}V.$$

For a binary, the density is only non-zero at the location of the objects. Using the effective-one-body analogy we deduce

$$I \sim \mu a^2 \exp(2\pi i f t)$$

where the frequency is now twice the orbital frequency because we are taking squares of positions, which vary at that frequency. It follows that

$$h \sim \frac{1}{D} f^2 \mu a^2 \sim \frac{1}{D} f^2 \mu \left(\frac{M}{f^2}\right)^{\frac{2}{3}} = \frac{1}{D} f^{\frac{2}{3}} \frac{m_1 m_2}{M^{\frac{1}{3}}} = \frac{1}{D} \mathcal{M}_c^{\frac{5}{3}} f^{\frac{2}{3}}.$$

ii. The GW energy loss is determined by

$$\dot{E}_{\rm GW} \sim D^2 \dot{h}^2 = \dddot{I}^2 \sim \mu^2 a^4 f^6 \sim \mu^2 f^6 \left(\frac{M}{f^2}\right)^{\frac{4}{3}} = \mu^2 M^{\frac{4}{3}} f^{\frac{10}{3}} = \mathcal{M}_c^{\frac{10}{3}} f^{\frac{10}{3}}.$$

iii. The rate of change of frequency is given by

$$\dot{f} \sim \sqrt{\frac{M}{a}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a}\right) \sim \frac{1}{M\mu}\sqrt{\frac{M}{a}}\dot{E} \sim \mu M^{\frac{1}{3}}\left(Mf\right)^{\frac{1}{3}} f^{\frac{10}{3}} = \mu M^{\frac{2}{3}} f^{\frac{11}{3}} = \mathcal{M}_c^{\frac{5}{3}} f^{\frac{11}{3}}.$$

iv. The Fourier transform of $h(t)$ is given approximately by

$$\tilde{h} \sim \frac{h}{\sqrt{\dot{f}}} \sim \frac{1}{D}\frac{\mathcal{M}_c^{\frac{5}{3}} f^{\frac{2}{3}}}{\mathcal{M}_c^{\frac{5}{6}} f^{\frac{11}{6}}} = \mathcal{M}_c^{\frac{5}{6}} f^{-\frac{7}{6}}.$$

v. The characteristic strain is given by

$$h_c \sim h\sqrt{\frac{f^2}{\dot{f}}} \sim \frac{1}{D}\mathcal{M}_c^{\frac{5}{3}} f^{\frac{2}{3}} \frac{f}{\mathcal{M}_c^{\frac{5}{6}} f^{\frac{11}{6}}} = \frac{1}{D}\mathcal{M}_c^{\frac{5}{6}} f^{-\frac{1}{6}}.$$

vi. The energy density of a GW background generated by a population of these sources is given by

$$\rho_c \Omega_{\mathrm{GW}}(f) = \int_0^\infty \frac{N(z)}{1+z}\left(f_r \frac{\mathrm{d}E}{\mathrm{d}f_r}\right)_{f_r = f(1+z)} \mathrm{d}z. \sim \mathcal{M}_c^{\frac{5}{3}} f^{\frac{2}{3}}$$

For the inspiraling binaries the previous results give

$$f\frac{\mathrm{d}E}{\mathrm{d}f} \sim f\frac{\dot{E}}{\dot{f}} \sim \mathcal{M}_c^{\frac{5}{3}} f^{\frac{2}{3}}$$

and so we find

$$\Omega_{\mathrm{GW}}(f) \sim \mathcal{M}_c^{\frac{5}{3}} f^{\frac{2}{3}} \int_0^\infty \frac{N(z)}{(1+z)^{\frac{1}{3}}}\mathrm{d}z.$$

(b) The energy of the binary is proportional to $1/a$, hence we have

$$\dot{E}_{\mathrm{hard}} \propto \mu M \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a}\right) = k\mu M \frac{\rho_*}{\sigma^3}\frac{m_2}{a} \propto k\frac{\rho_* m_2 \mu}{\sigma^3}(Mf)^{\frac{2}{3}} = k\frac{\rho m_2 \mu}{\sigma^3} M^{\frac{2}{3}} f^{\frac{2}{3}}.$$

(c) The previous derivation of the background energy density assumed that all of the energy loss driving the frequency evolution was due to GW emission. If there are other processes driving energy loss and hence frequency evolution, the background is suppressed because not all of the orbital energy lost is emitted as gravitational waves. In general we have $f = f(E)$ and hence $\dot{f} = (\mathrm{d}f/\mathrm{d}E)\dot{E}$ and therefore

$$\frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}f} = \frac{\dot{E}_{\mathrm{GW}}}{(\mathrm{d}f/\mathrm{d}E)[\dot{E}_{\mathrm{GW}} + \dot{E}_{\mathrm{other}}]} = \frac{\dot{E}_{\mathrm{GW}}}{\dot{E}_{\mathrm{GW}} + \dot{E}_{\mathrm{other}}}\left(\frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}f}\right)_{\text{pure GW}}.$$

The final bracketed expression denotes the background energy density in the pure GW-driven evolution case. In the case of stellar hardening we therefore find a modified expression for the GW background energy density

$$\rho_c \Omega_{\mathrm{GW}}(f) = \mathcal{M}_c^{\frac{5}{3}} f^{\frac{2}{3}} \int_0^\infty \frac{N(z)}{(1+z)^{\frac{1}{3}}} \frac{\mathcal{M}_c^{\frac{10}{3}}}{\mathcal{M}_c^{\frac{10}{3}} + k(\rho m_2 \mu/\sigma^3) M^{\frac{2}{3}} f^{-\frac{8}{3}} (1+z)^{-\frac{8}{3}}}\mathrm{d}z.$$

This can be simplified a bit more — for example, we notice that the factor $\mu M^{\frac{2}{3}}$ in the hardening term is just $\mathcal{M}_c^{\frac{5}{3}}$ — but the above result is all we need to answer the next few questions.

(d) If the sources are at a common redshift, z_0, we can replace $N(z)$ by a delta function, $\delta(z - z_0)$, and do the integral explicitly. It is then clear that we have

$$\Omega_{\rm GW}(f) \sim \frac{f^{\frac{2}{3}}}{1 + \lambda f^{-\frac{8}{3}}}$$

where

$$\lambda = k(\rho m_2/\sigma^3)\mathcal{M}_c^{-\frac{5}{3}}(1+z_0)^{-\frac{8}{3}}.$$

This is a broken power-law, as required. For $f \ll 1$ the term $f^{-\frac{8}{3}}$ dominates in the denominator and we have $\Omega_{\rm GW} \sim f^{\frac{10}{3}}$. This is the stellar hardening dominated regime. For $f \gg 1$ the constant term dominates in the denominator and we find $\Omega_{\rm GW} \sim f^{\frac{2}{3}}$. This is the GW dominated regime and this is the standard result for GW backgrounds.

(e) If a broken power law background were detected, it tells us about the processes that drive the inspiral of the binary. In this example the power at low frequencies (where hardening dominates) is suppressed relative to that of a pure GW background (see Figure 1). The low frequency slope is characteristic of whatever process drove the early evolution of the binaries — a measurement of this tells you which physical process was important at that time. The high frequency slope tells us about the late evolution of the binary, and in this case the value $f^{\frac{2}{3}}$ is consistent with GW-driven inspiral. The turn over point tells us about the relative efficiencies of the two processes. In this example it occurs where $f \approx \lambda^{\frac{3}{8}}$ and so a measurement of that value tells us about the parameters that go into λ, such as σ, ρ and the typical source redshift, z_0.

(f) * No results here. If there is a distribution over masses, then the background energy density involves an integral over the mass distribution as well as the redshift. Try playing around with different choices. Try also including some dependence of ρ and σ on the binary properties. The GW background in the PTA regime may well be suppressed by stellar processes of the type described here. If we see that suppression we will want to be able to interpret it in the context of models of the binary population.

2. (a) The average waveform power is

$$\langle h^2 \rangle = \frac{1}{2T}\int_{-T}^{T} h^2(t)\mathrm{d}t = \frac{1}{2\sqrt{Q}T}\frac{A^2}{D^2}\int_{-\sqrt{Q}T}^{\sqrt{Q}T} \cos^2\left(\frac{2\pi f_0}{\sqrt{Q}}u\right)\mathrm{e}^{-u^2}\mathrm{d}u.$$

We see that beyond $\sqrt{Q}T \sim$ few, the waveform is exponentially suppressed. Hence, the duration of the signal is order $\sim 1/\sqrt{Q}$. We take $|\sqrt{Q}T| \lesssim 2$ as a reasonable approximation.

For this choice, we find

$$\langle h^2 \rangle = \frac{A^2}{D^2}\frac{\sqrt{\pi}}{8}\left(\mathrm{erf}(2) + \mathrm{e}^{-\left(\frac{2\pi f_0}{\sqrt{Q}}\right)^2}\mathrm{Re}\left[\mathrm{erf}\left(2 + i\frac{2\pi f_0}{\sqrt{Q}}\right)\right]\right) \sim \frac{A^2}{D^2}$$

with a pre-factor that is order 0.few.

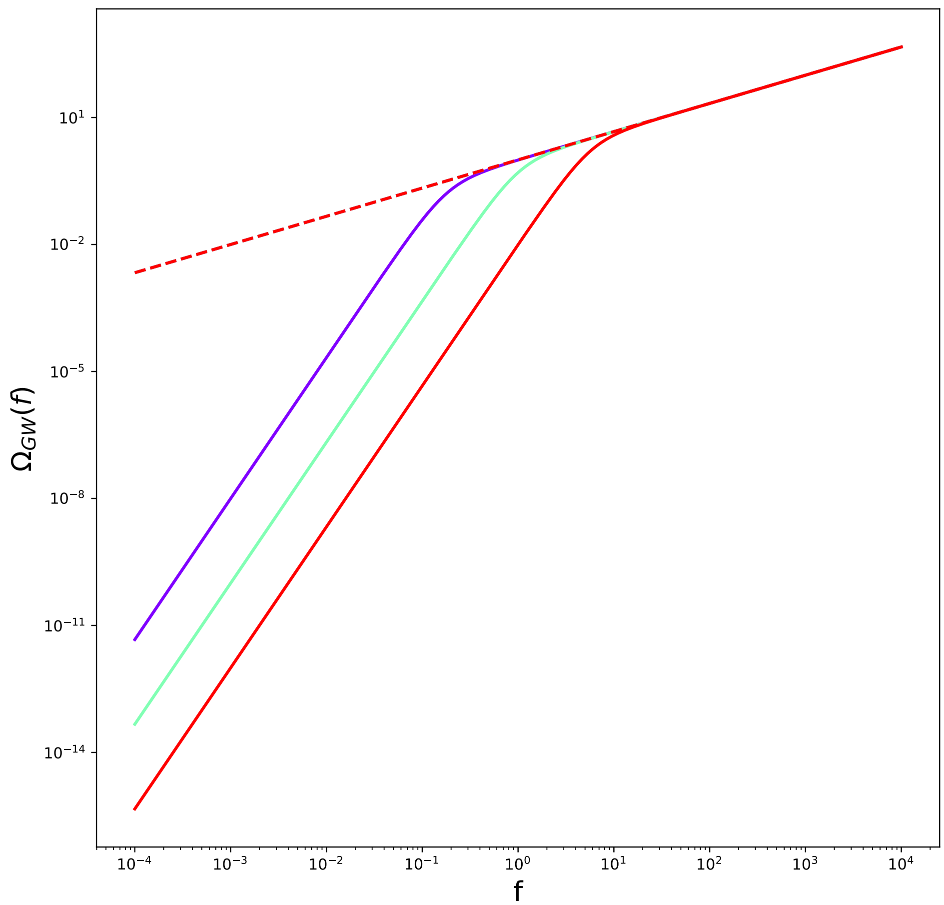

Figure 1: Example backgrounds. We show $\Omega_{\mathrm{GW}}(f)$ as a function of frequency for $\lambda = 0.01$ (purple), $\lambda = 1$ (green) and $\lambda = 100$ (red). Also shown, as a dashed red line, is the background in the absence of stellar hardening.

(b) Using standard results for Fourier transforms, $\mathcal{F}[g] = \tilde{g}(f)$, including $\mathcal{F}[\exp(-t^2)] = \sqrt{\pi}\exp(-\pi^2 f^2)$, $\mathcal{F}[g(\alpha t)] = \tilde{g}(f/\alpha)/|\alpha|$ and $\mathcal{F}[\exp(2\pi i f_0 t) g(t)] = \tilde{g}(f - f_0)$, we find

$$\tilde{h}(f) = \frac{A}{2D}\sqrt{\frac{\pi}{Q}}\left(\mathrm{e}^{-\frac{\pi^2}{Q}(f-f_0)^2} + \mathrm{e}^{-\frac{\pi^2}{Q}(f+f_0)^2}\right).$$

We can use the fact that the time series is real to wrap onto only positive frequencies and then we have

$$\tilde{h}(f) = \frac{A}{D}\sqrt{\frac{\pi}{Q}}\mathrm{e}^{-\frac{\pi^2}{Q}(f-f_0)^2}.$$

We see that the Fourier transform is also proportional to a Gaussian which goes to zero exponentially when $\pi^2(f - f_0)^2/Q \sim$few. Hence the bandwidth is $\Delta f \sim \sqrt{Q}/\pi$.

(c) Using the power ratio formula

$$\left(\frac{S}{N}\right)^2 \approx \frac{\langle h^2 \rangle}{\Delta f S_n(f)}$$

and assuming white noise, $S_n(f) = \sigma^2$, we have

$$\left(\frac{S}{N}\right)^2 \approx k\frac{A^2}{D^2\sqrt{Q}\sigma^2}$$

where k is a constant of order unity. This SNR could be achieved by windowing the data (to the time range $|\sqrt{Q}T| \lesssim$a few) and bandpassing it (to the frequency range $\pi|f - f_0|/\sqrt{Q} \lesssim$a few) and then comparing the signal power to the average off-source noise power.

(d) Using the Fourier transform obtained above, the matched filtering SNR is

$$\left(\frac{S}{N}\right)^2 = 4\int_0^\infty \frac{|\tilde{h}(f)|^2}{S_n(f)}\mathrm{d}f = \frac{4}{\sigma^2}\frac{A^2\pi}{4D^2Q}\mathrm{e}^{-\frac{2\pi^2}{Q}(f-f_0)^2}\mathrm{d}f \approx \frac{A^2}{2D^2\sigma^2\sqrt{Q}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x$$

which is also equal to $A^2/(D^2\sigma^2\sqrt{Q})$ times a constant of order unity.

We have found that the matched filtering SNR is essentially the same as the burst search SNR, so we are not gaining anything by doing matched filtering. We argued in lectures that matched filtering gained over a burst search by a factor of the square root of the number of cycles spent near a particular frequency. These sine-Gaussian sources are peculiar in that as Q decreases so that the source spends more time near frequency f_0, the bandwidth also decreases so the burst power is increasingly concentrated — we effectively have only '1 cycle' in the vicinity of each relevant frequency.

This result does not necessarily mean matched filtering is no better than a burst search — the SNR does not directly translate to a false alarm probability. There may be many instrumental artefacts that could give broadband power in the frequency domain which looks burst like, but those artefacts would look nothing like the specific sine-Gaussian form of the matched filter. Nonetheless, this problem illustrates why excess power searches are quite effective for sources that are burst-like, even if models are available.

(e) The energy distribution can be found from

$$\int \frac{\mathrm{d}E}{\mathrm{d}f}\mathrm{d}f = \int_{-\infty}^{\infty} D^2 \dot{h}^2(t)\mathrm{d}t = \int_{-\infty}^{\infty} D^2 f^2 \tilde{h}^2(f)\mathrm{d}f.$$

We find

$$\frac{\mathrm{d}E}{\mathrm{d}f} = A^2 \frac{f^2\pi}{2Q} \exp\left(-\frac{\pi^2}{Q}(f-f_0)^2\right).$$

(f) Assuming the number of objects per unit comoving volume with redshift between z and $z+\mathrm{d}z$ and with f_0 between f_0 and $f_0+\mathrm{d}f_0$ is $N(z)\mathrm{d}f_0\mathrm{d}z$, the background energy density is

$$\rho_c\Omega_{\mathrm{GW}}(f) = \int_0^{\infty}\int_0^{\infty} N(z)(1+z)^2 A^2 \frac{f^3\pi}{2Q} \exp\left(-\frac{\pi^2}{Q}(f(1+z)-f_0)^2\right) f_0^{\alpha}\mathrm{d}f_0\mathrm{d}z.$$

(g) The common redshift assumption allows us to replace the integral over z by evaluation of the integrand at z_0 as before. We then have

$$\rho_c\Omega_{\mathrm{GW}}(f) = N_0(1+z_0)^2 A^2 \frac{\pi}{2Q} f^3 \int_0^{\infty} \exp\left(-\frac{\pi^2}{Q}(f(1+z_0)-f_0)^2\right) f_0^{\alpha}\mathrm{d}f_0\mathrm{d}z.$$

The integral over f_0 takes the form

$$\int_0^{\infty} x^{\alpha}\exp\left[-(x-\lambda f)^2\right]\mathrm{d}x$$

where $\lambda = \pi(1+z_0)/\sqrt{Q}$. This integral can be written down as a combination of hypergeometric functions

$$\begin{aligned}\int_0^{\infty} x^{\alpha}\exp\left[-(x-\lambda f)^2\right]\mathrm{d}x &= \frac{1}{2}\mathrm{e}^{-\lambda^2 f^2}\left[\alpha\lambda f\Gamma\left(\frac{\alpha}{2}\right){}_1F_1\left(\frac{\alpha}{2}+1;\frac{3}{2};\lambda^2 f^2\right)\right.\\ &\quad \left. +\Gamma\left(\frac{\alpha+1}{2}\right){}_1F_1\left(\frac{\alpha}{2}+1;\frac{1}{2};\lambda^2 f^2\right)\right].\end{aligned}$$

The exact background computed from this expression is shown in Figure 2, but we can also find analytic approximations for the low and high frequency behaviour. If $f \ll 1$, then the integral is approximately

$$\int_0^{\infty} x^{\alpha}\exp\left[-x^2\right]\mathrm{d}x = \frac{1}{2}\Gamma\left(\frac{\alpha+1}{2}\right)$$

with corrections of order λf. Hence, the dominant behaviour is a constant and $\Omega_{\mathrm{GW}}(f) \sim f^3$ due to the factor out the front of the expression.

For $f \gg 1$ we can make a change of variable in the integral

$$\begin{aligned}\int_0^{\infty} x^{\alpha}\exp\left[-(x-\lambda f)^2\right]\mathrm{d}x &= \int_{-\lambda f}^{\infty}(u+\lambda f)^{\alpha}\exp\left[-u^2\right]\mathrm{d}u\\ &\approx \lambda^{\alpha} f^{\alpha}\int_{-\infty}^{\infty}\left(1+\frac{u}{\lambda f}\right)^{\alpha}\exp\left[-u^2\right]\mathrm{d}u\\ &= \sqrt{\pi}\lambda^{\alpha} f^{\alpha}\left(1+O\left(\frac{1}{f}\right)\right).\end{aligned}$$

So we deduce $\Omega_{\mathrm{GW}} \sim f^{3+\alpha}$.

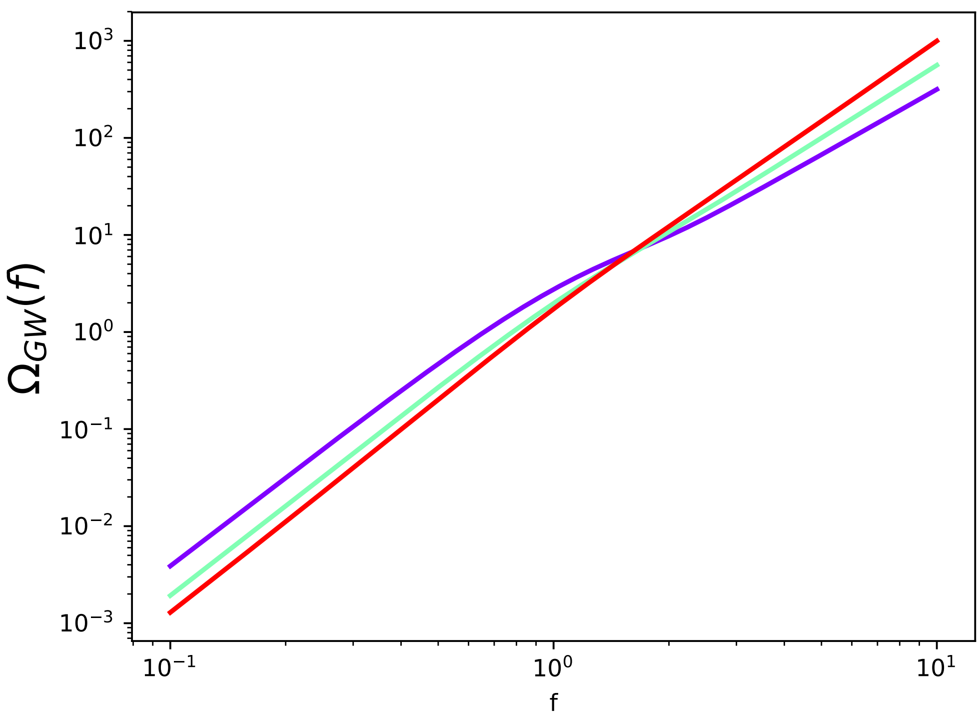

Figure 2: Example backgrounds for the burst population model. We show $\Omega_{\rm GW}(f)$ as a function of frequency for $\lambda = 1$ and three choices of α: $\alpha = -0.75$ (purple), $\alpha = -0.5$ (green) and $\alpha = -025$ (red).

(h) * No results here again, but things to explore would be how the introduction of a redshift distribution modifies things, what happens if the distribution of f_0 is changed, e.g., by introducing a cut-off in the frequency range, what happens if we add a distribution for Q etc.