Exercises on Gravitational Waves (GWs): Nissanke Lecture 1

I. Estimates for gravitational Waves (GWs) generated on Earth

In this exercise you will make rough, order-of-magnitude estimates to get a feel for the numbers involved in GWs. Recall the quadrupole formula for the GW power is

$$\dot{E}_{\rm GW} \sim \frac{G}{5c^5} \dddot{Q}_{ij} \dddot{Q}_{ij} \tag{1}$$

where Q_{ij} is the quadrupole moment of the source. Note, however, that for the purpose of the estimates below you can ignore the tensorial nature of Q_{ij} and approximate $Q_{ij} \sim ML^2$, where M and L are the mass and size of the source. Useful numbers are:

$$\frac{G}{c^5} \sim 3 \times 10^{-60} {\rm cm}^3 {\rm s}^{-2} {\rm g}^{-1}, \qquad \frac{G}{c^2} \sim 7.4 \times 10^{-29} {\rm cm\, g}^{-1}, \qquad \hbar \sim 10^{-27} {\rm erg\, s}^{-1} \tag{2}$$

1. An angry politician shakes his fist during a heated debate. He goes on in this way for 10 seconds, shaking his fist twice per second. The length of his fist and forearm is 50cm and their mass is 2kg.
   (a) Roughly, what is the kinetic energy expended per shake, and for the total number of shakes?
   (b) What is the GW power he generates?
   (c) What is the total energy that goes into GWs during this process, and what percentage of the expended energy is it?
   (d) In some cases it can be useful to think of GWs as a stream of gravitons, quanta somewhat similar to photons. For a source whose non-spherical variations occur at a frequency ω, the emitted gravitons have energy $E_{\rm graviton} = \hbar\omega$.
   How many gravitons are emitted by the shaking?
2. Next, assume that the politician picks up a laser pointer that he moves up an down on a timescale of $T \sim$ 1s. The laser pointer has a metal barrel of length $\ell \sim$ 10cm, diameter $d \sim$ 0.5cm, and thickness 0.25cm. Due to the motion, some charges will be centrifuged to the far end of the barrel and depleted in the other, so the pointer becomes an electric dipole.
   (a) By balancing the electrostatic force electrostatic force $e\nabla\Phi$ (where e is the charge of an electron and Φ the potential) and the acceleration due to the motion for an electron $\sim m_e L/T^2$ obtain an approximation for the induced charge density by using that $\rho \sim \nabla^2\Phi$.
   (b) Determine approximately the electric dipole moment $D_{\rm dipole} \sim qL$ where $q \sim \rho A$ is the charge and A the cross-sectional area of the metal part of the barrel.
   (c) Compute the energy flux from electric dipole radiation $\dot{E} \sim \ddot{D}^2_{\rm dipole}/(3c^3)$ and compare to the GW flux from the first problem.

II. Event rate estimates

Make an estimate of the rate per volume of BH-BH mergers (expressesd in number per ${\rm Gpc}^3{\rm yr}^{-1}$) based on the 3 events reported during the O1 run of LIGO. The run had 49 total days in which both detectors were taking data. Three events can be considered: GW150914 a distance of ∼420 Mpc and with a signal to noise ratio of 23.7; GW151226 at a distance of 440 Mpc and with a signal to noise ratio of 13.0, and LVT151012 (which was a marginal detection but still contributes to these estimates) was at a distance of 1 Gpc and had a signal to noise ratio of 9.7. Suppose that the threshold for announcing a detection is a signal to noise ratio of 12.0.

1. Using the fact that for a given event the distance scales as the reciprocal of the signal to noise ratio, compute out to what distance each of the events could have been seen and what the corresponding visible volume of the universe was (neglecting cosmology).
2. Use these results together with the length of LIGO's observing time to estimate the BH-BH merger rate based on each of the events individually. Also compute the combined rate obtained by adding the results from the individual events.

III. Detecting Gravitational waves from Black Hole Binaries using Pulsar Timing Array

A binary supermassive black hole in another galaxy has an orbital period of one year. Out to what distance D could the gravitational radiation from the binary be detected with a timing array of 100 millisecond pulsars? [Consider just a 1-sigma detection threshold, and assume the average pulse arrival times can be measured to a precision of 1 microsecond.]