Half-BPS Wilson line and defect 1d CFT

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Based mainly on SG, Roiban, Tseytlin arXiv: 1706.00756
SG, Komatsu arXiv: 1802.05201

Wilson loops in *N*=4 SYM

- In N=4 SYM, it is natural to study Wilson loop operators that include couplings to the six adjoint scalars Φ^I (“Maldacena-Wilson” loop)

$$W = \mathrm{tr} P e^{\oint dt \left(i \dot{x}^\mu A_\mu + |\dot{x}| \theta^I \Phi^I \right)}$$

where $x^\mu(t)$ is a loop in spacetime and $\theta^I(t)$ a unit 6-vector.

- Special choices of (x^μ, θ^I) lead to families of Wilson loop operators preserving various fractions of the superconformal symmetry

Zarembo ’02

Drukker, SG, Ricci, Trancanelli ‘07

Half-BPS Wilson loop

- The most supersymmetric case is the 1/2-BPS Wilson loop:
  - $x^\mu(t)$: an infinite straight line, or circle (related by conformal transformation)
  - θ^I : a constant unit 6-vector
- E.g. take the line $x^0=t$, and $\theta^I = \delta^{I6}$

$$W = \mathrm{tr} P e^{\int dt \left(iA_t + \Phi^6 \right)}$$

- This preserves 8 Q's and 8 S's (superconformal charges): 1/2-BPS. (Similarly for a circle, but it preserves 16 lin. combinations of Q and S)

Correlators on the Wilson loop

- We will be interested in the following observables: given some local operators $O_i(t)$ in the adjoint of the gauge group, consider

$$
\begin{aligned}
\langle\langle O_1(t_1)O_2(t_2)\cdots O_n(t_n)\rangle\rangle &\equiv \frac{\langle \mathrm{tr}P\left[O_1(t_1)\ e^{\int dt(iA_t+\Phi^6)}\ O_2(t_2)\ e^{\int dt(iA_t+\Phi^6)}\ldots\ O_n(t_n)\ e^{\int dt(iA_t+\Phi^6)}\right]\rangle}{\langle W\rangle} \\
&\equiv \frac{\langle \mathrm{tr}P\left[O_1(t_1)O_2(t_2)\cdots O_n(t_n)e^{\int dt(iA_t+\Phi^6)}\right]\rangle}{\langle W\rangle}
\end{aligned}
$$

- Gauge invariant, due to path-ordering and Wilson-line factors
- For the 1/2-BPS straight line <W>=1 and normalization is trivial. But when we work on the circle, then <W> is non-trivial and normalization is important (the conformal symmetry maps correlators on the line and circle normalized by <W>)

Correlators on the Wilson loop

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$$
\begin{aligned}
\langle\langle O_1(t_1)O_2(t_2)\cdots O_n(t_n)\rangle\rangle \;&\equiv\; \frac{\langle \mathrm{tr} P\left[O_1(t_1)\; e^{\int dt(iA_t+\Phi^6)}\; O_2(t_2)\; e^{\int dt(iA_t+\Phi^6)} \cdots\; O_n(t_n)\; e^{\int dt(iA_t+\Phi^6)}\right]\rangle}{\langle W\rangle} \\
&\equiv\; \frac{\langle \mathrm{tr} P\left[O_1(t_1)O_2(t_2)\cdots O_n(t_n) e^{\int dt(iA_t+\Phi^6)}\right]\rangle}{\langle W\rangle}
\end{aligned}
$$

- Correlators of this kind arise naturally when we consider small deformations of the Wilson loop (*Drukker, Kawamoto '06*)
- Knowledge of these correlation functions therefore encodes information on the expectation value of Wilson loops of more general shape

Half-BPS Wilson line as conformal defect

- To understand the structure of these correlators, it is useful to recall the symmetries preserved by the 1/2-BPS Wilson line. The bosonic symmetries are
  - SO(3): rotations in the directions orthogonal to the line ($i=1,2,3$)
  - SO(5): R-symmetry rotations of the five scalars Φ^a, $a=1,\ldots,5$ that do not couple to the Wilson line
  - SL(2,R): translations, dilatations and special conformal transformation on the line. **1d conformal symmetry**
- Together with the 16 supercharges, these combine into the 1d superconformal group $OSp(4^*|4) \supset SL(2,R)\times SO(3)\times SO(5)$
- Since it preserves a 1d (super)conformal subgroup of the 4d conformal symmetry, the 1/2-BPS Wilson loop can be regarded as a conformal defect of the 4d theory

Correlators on the defect

- The operators $O_i(t)$ inserted on the WL are the defect operators living on the line. As usual, they can be organized in defect primaries and descendants
- Defect primaries are labelled by their scaling dimension Δ and SO(3)xSO(5) representation, e.g. $(\Delta, j\ ; m_1,m_2)$
- Correlation functions of defect primaries are constrained by the SL(2,R) conformal symmetry as usual in CFT_d

$$\langle O(t_1)O(t_2)\rangle = \frac{1}{t_{12}^{2\Delta}}$$

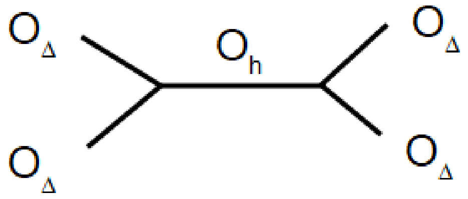
$$\langle O_1(t_1)O_2(t_2)O_3(t_3)\rangle = \frac{c_{123}}{t_{12}^{\Delta_1+\Delta_2-\Delta_3}t_{23}^{\Delta_2+\Delta_3-\Delta_1}t_{31}^{\Delta_3+\Delta_1-\Delta_2}}$$

Correlators on the defect

- For 4-point functions $\langle O_\Delta(t_1)O_\Delta(t_2)O_\Delta(t_3)O_\Delta(t_4)\rangle = \frac{1}{(t_{12}t_{34})^{2\Delta}}\mathcal{G}(\chi)$

  With χ the 1d conformal cross ratio $\chi = \frac{t_{12}t_{34}}{t_{13}t_{24}}$
- It has an OPE expansion

$$\mathcal{G}(\chi) = \sum_h c_{\Delta,\Delta;h}\, \chi^h\ {}_2F_1(h,h,2h,\chi)$$



where ${}_2F_1(h,h,2h,\chi)$ is the exact d=1 conformal block,
and $c_{\Delta,\Delta;h} = C^2_{O_\Delta O_\Delta O_h}/(C^2_{O_\Delta O_\Delta} C_{O_h O_h})$ the OPE coefficients

- As usual, finding all scaling dimensions and structure constants would amount to solving the system of correlators on the Wilson line
- Perhaps simple enough 1d system to be studied and solved by bootstrap techniques

(Liendo, Meneghelli '16; Liendo, Meneghelli, Mitev '18)

The super-displacement multiplet

- Among the possible defect primaries, a special role is played by a set of 8_B+8_F "elementary insertions" forming a short multiplet of Osp(4*|4).
- The 8 bosonic insertions are
  - The 5 scalars not coupled to the loop $\Phi^a, \quad a = 1, \ldots, 5 \qquad \Delta = 1$
  - The "displacement operator" $\mathbb{F}_{ti} \equiv iF_{ti} + D_i\Phi^6, \quad i = 1, 2, 3 \qquad \Delta = 2$
- These operators have protected scaling dimensions, due to being in short multiplet

(The displacement operator, which is related to deformations of the defect in the transverse directions, has in fact protected scaling dimension Δ=2 for any line defect more generally, independently from supersymmetry)

Two-point functions

- Because they have protected scaling dimensions, their exact 2-point functions take the form

$$\langle\langle \Phi^a(t_1)\Phi^b(t_2)\rangle\rangle = \delta^{ab}\frac{C_\Phi(\lambda)}{t_{12}^2}\,, \qquad \langle\langle \mathbb{F}_{ti}(t_1)\mathbb{F}_{tj}(t_2)\rangle\rangle = \delta_{ij}\frac{C_\mathbb{F}(\lambda)}{t_{12}^4}$$

- The normalization factors are related to the "Brehmsstrahlung function", and can be determined exactly using localization *(Correa, Maldacena, Sever '12)*. In the planar limit:

$$C_\Phi(\lambda) = 2B(\lambda)\,, \qquad C_\mathbb{F}(\lambda) = 12B(\lambda)$$

$$B(\lambda) = \frac{\sqrt{\lambda}\, I_2(\sqrt{\lambda})}{4\pi^2\, I_1(\sqrt{\lambda})}$$

Four-point functions

- 3-point functions of these elementary protected insertions vanish by symmetry
- 4-point functions, on the other hand, have a non-trivial dependence on the coupling constant and conformal cross ratio. They encode in particular scaling dimensions and structure constants of unprotected operators appearing in the OPE
- At weak-coupling, these 4-point functions are known up to 1-loop order *(Cooke, Dekel, Drukker, '17; Kyriu, Komatsu, to appear)*
- At strong coupling, they can be computed from string theory using the AdS_2 worldsheet dual to the Wilson loop *(SG, Roiban, Tseytlin '17)*

Wilson loop from string theory

- In AdS/CFT dictionary, the Wilson loop operator is dual to a minimal string surface ending on the contour defining the operator at the boundary
- The bosonic part of the $AdS_5 x S^5$ string action reads (taking Poincare coordinates and using Nambu-Goto form)

$$S_B = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \sqrt{\det\left[\frac{1}{z^2}\left(\partial_\mu x^r \partial_\nu x^r + \partial_\mu z \partial_\nu z\right) + \frac{\partial_\mu y^a \partial_\nu y^a}{(1+\frac{1}{4}y^2)^2}\right]}$$

where $\sigma^\mu=(t,s)$ are worldsheet coordinates, r=(0,i), i=1,2,3 label the coordinates of the (Euclidean) boundary, and a=1,…,5 are S^5 directions

AdS_2 minimal surface

- The minimal surface dual to the 1/2-BPS Wilson line is given by

$$z = s \;, \qquad x^0 = t \;, \qquad\qquad x^i = 0 \;, \qquad y^a = 0$$

- The induced metric is just that of AdS_2 in Poincare coordinates

$$ds_2^2 = \frac{1}{s^2}\left(dt^2 + ds^2\right)$$

- Similarly, one can describe the minimal surface for the circular Wilson loop, which is given by AdS_2 with the hyperbolic disk coordinates

$$ds_2^2 = d\rho^2 + \sinh^2\rho\, d\tau^2$$

AdS_2 minimal surface

- So the minimal surface dual to 1/2-BPS Wilson loop is an AdS_2 embedded in AdS_5, and sitting at a point on S^5
- It preserves the same superconformal symmetry $OSp(4^*|4)$
- In particular, the $SL(2,R)$ is realized as the isometry of AdS_2
- The $SO(3)\times SO(5)$ correspond to rotations of the transverse coordinates $x^i(t,s)$ $(i=1,2,3)$ and $y^a(t,s)$ $(a=1,\ldots,5)$
- By expanding the string sigma model around this minimal surface, we can study the dynamics of small fluctuations of the worldsheet

Worldsheet fluctuations as fields in AdS_2

- It is convenient to adopt a static gauge where x^0 and z (which coincide with the AdS_2 coordinates) do not fluctuate
- Then we get a Lagrangian for the transverse fluctuations $x^i(t,s)$ and $y^a(t,s)$, which can be viewed as fields propagating in AdS_2

$$S_B = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \sqrt{g} L_B$$

$$L_B = 1 + L_2 + L_4 + \dots$$

$$L_2 = \tfrac{1}{2} g^{\mu\nu} \partial_\mu x^i \partial_\nu x^i + x^i x^i + \tfrac{1}{2} g^{\mu\nu} \partial_\mu y^a \partial_\nu y^a$$

$$L_{4y} = -\tfrac{1}{4}(y^b y^b)(g^{\mu\nu} \partial_\mu y^a \partial_\nu y^a) + \tfrac{1}{8}(g^{\mu\nu} \partial_\mu y^a \partial_\nu y^a)^2 - \tfrac{1}{4}(g^{\mu\nu} \partial_\mu y^a \partial_\nu y^b)\,(g^{\rho\kappa} \partial_\rho y^a \partial_\kappa y^b)$$

etc.

Worldsheet fluctuations as fields in AdS_2

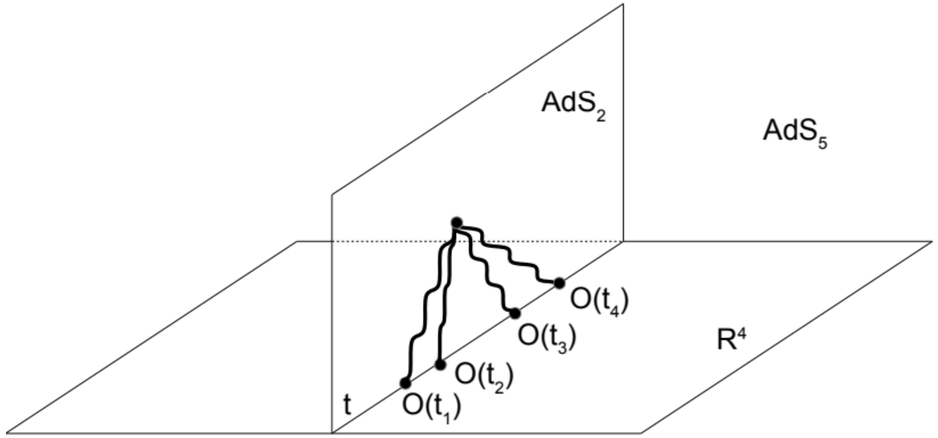
- From the quadratic Lagrangian $L_2 = \frac{1}{2} g^{\mu\nu}\partial_\mu x^i \partial_\nu x^i + x^i x^i + \frac{1}{2} g^{\mu\nu}\partial_\mu y^a \partial_\nu y^a$ we find
  - 5 massless scalars y^a
  - 3 scalars x^i with $m^2=2$
- Since these may be viewed as scalar fields in AdS_2, they should be dual to operators inserted at the d=1 boundary, with dimension given by $\Delta(\Delta-1)=m^2$
- Hence, we recover the eight bosonic operators in the super-displacement multiplet

$$y^a \quad \leftrightarrow \quad \Phi^a \quad \Delta = 1$$

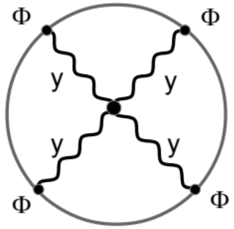
$$x^i \quad \leftrightarrow \quad \mathbb{F}_{ti} \quad \Delta = 2$$

Four-point functions

- The four-point functions of the dual operators at strong coupling can then be obtained from familiar AdS/CFT techniques by computing Witten diagrams in AdS_2



- E.g., the leading tree level connected term just involves contact 4-point interactions, with Witten diagram



Some comments

- These calculations are technically very similar to Witten diagram calculations in SUGRA in $AdS_5 x S^5$, but the interpretation is quite different
- In the SUGRA case, one computes correlation functions of single trace local operators like, $\mathrm{tr}Z^J$, dual to closed string states. The expansion parameter is $G_N \sim 1/N^2$
- In our case, we compute correlators of insertions inside the Wilson loop trace (it is an expectation value of a single trace operator), dual to open string fluctuations. The expansion parameter is the worldsheet coupling $1/\sqrt{\lambda}$

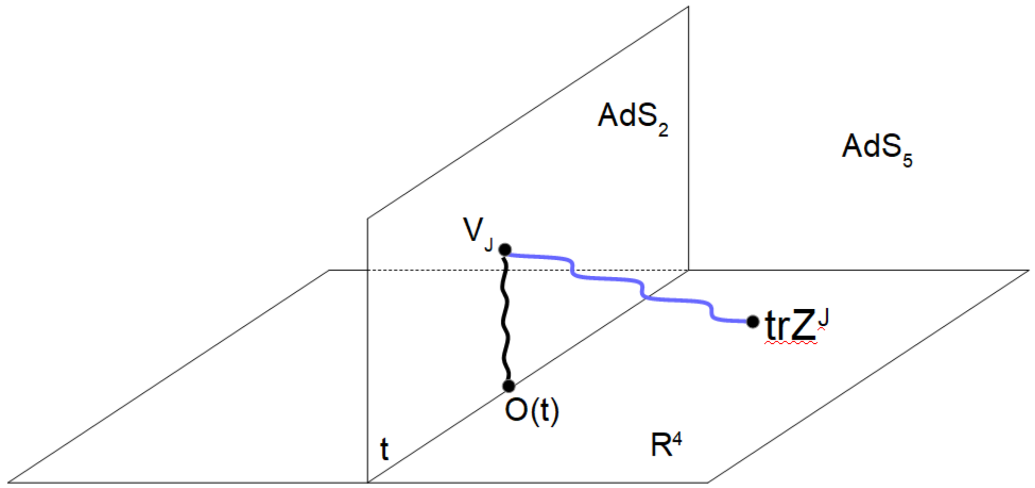
Some comments

- Because the string theory is UV finite, the Witten diagram calculations on the worldsheet should be in principle well-defined to all orders in $1/\sqrt{\lambda}$ (of course, one would need to include fermions)
- In particular, loop calculations should make sense and this could be an interesting toy model where to apply recent techniques to compute loops in AdS
- For instance, from loop corrections to the two-point functions ("boundary-to-boundary" propagator) of the transverse coordinates one should recover the strong coupling expansion of the Bremsstrahlung function

Some comments

- Recall that the AdS_2 worldsheet is embedded in a higher dimensional theory. Here I am focusing mainly on correlators of defect operators, which should be captured by the AdS_2 worldsheet theory
- But more generally one can consider also "bulk-defect" correlators: correlation functions of the Wilson loop (with or without extra insertions in it) and single-trace operators inserted away from the loop, e.g. $\langle W \, \mathrm{tr}Z^J \rangle$, $\langle W[O(t)] \, \mathrm{tr}Z^J \rangle$ etc. ("bulk" here means away from the defect, but still in the 4d gauge theory)
- This correspond to an "open-closed" string amplitude of the schematic form (to leading order):

These are 1/N suppressed compared purely defect correlators



Summary of 4-point function result

- Let us consider just the 4-point function of the S^5 fluctuations y^a , dual to the Δ=1 operator insertions Φ^a .
- It is convenient to multiply y^a (or Φ^a) by an auxiliary null polarization 5-vector Y^a . The result then takes the form

$$\langle Y_1\cdot y(\tau_1)Y_2\cdot y(\tau_2)Y_3\cdot y(\tau_3)Y_4\cdot y(\tau_4)\rangle^{\text{conn.}}_{\text{AdS}_2}=$$
$$=\frac{\left(\frac{\sqrt{\lambda}}{2\pi^2}\right)^2 Y_1\cdot Y_2\, Y_3\cdot Y_4}{(t_{12}t_{34})^2}\frac{1}{\sqrt{\lambda}}\left[G_S^{(1)}(\chi)-\frac{2}{5}G_T^{(1)}(\chi)+\xi(G_T^{(1)}(\chi)+G_A^{(1)}(\chi))+\zeta(G_T^{(1)}(\chi)-G_A^{(1)}(\chi))\right]$$

- The 3 functions of χ correspond to the singlet (S), symmetric traceless (T) and antisymmetric (A) channels, and the cross ratios are

$$\chi=\frac{t_{12}t_{34}}{t_{13}t_{24}}\qquad \xi=\frac{Y_1\cdot Y_3\, Y_2\cdot Y_4}{Y_1\cdot Y_2\, Y_3\cdot Y_4},\qquad \zeta=\frac{Y_1\cdot Y_4\, Y_2\cdot Y_3}{Y_1\cdot Y_2\, Y_3\cdot Y_4}$$

Summary of 4-point function result

- Explicitly, they are found to be *(SG, Roiban, Tseytlin)*

$$
\begin{aligned}
G_S^{(1)}(\chi) = & -\frac{2\left(\chi^4-4\chi^3+9\chi^2-10\chi+5\right)}{5(\chi-1)^2}+\frac{\chi^2\left(2\chi^4-11\chi^3+21\chi^2-20\chi+10\right)}{5(\chi-1)^3}\log|\chi| \\
& -\frac{2\chi^4-5\chi^3-5\chi+10}{5\chi}\log|1-\chi| \,, \\
G_T^{(1)}(\chi) = & -\frac{\chi^2\left(2\chi^2-3\chi+3\right)}{2(\chi-1)^2}+\frac{\chi^4\left(\chi^2-3\chi+3\right)}{(\chi-1)^3}\log|\chi|-\chi^3\log|1-\chi| \,, \\
G_A^{(1)}(\chi) = & \frac{\chi\left(-2\chi^3+5\chi^2-3\chi+2\right)}{2(\chi-1)^2}+\frac{\chi^3\left(\chi^3-4\chi^2+6\chi-4\right)}{(\chi-1)^3}\log|\chi|-(\chi^3-\chi^2-1)\log|1-\chi|
\end{aligned}
$$

- One can check that this result for the 4-point satisfies the relevant superconformal Ward identity *(Liendo, Meneghelli, Mitev '18)*
- From the small χ expansion we can read off the anomalous dimensions and OPE coefficients of "two-particle" operators appearing in the OPE

Scaling dimensions

- In the symmetric-traceless channel, we see a protected operator with Δ=2. This corresponds to the operator $\Phi^{(a}\Phi^{b)}$ in the symmetric traceless of SO(5): it sits in a short multiplet and has protected dimension. [More generally, there are defect primaries $(Y\cdot\Phi)^J$ with Δ=J, analog to chiral primaries (though J=1 here is allowed)]
- In the singlet sector, we find the anomalous dimension of the unprotected 2-particle operator $y^a\, y^a$ (this heads a long supermultiplet)

$$\Delta_{y^a y^a} = 2 - \frac{5}{\sqrt{\lambda}} + \dots .$$

The dimension of Φ^6

- At weak coupling, the lowest dimension unprotected operator in the defect primary spectrum is Φ^6: the insertion of the scalar that appears in the Wilson loop exponent
- Its dimension is known to 1-loop order *(Alday, Maldacena '07; Polchinski,Sully '11)*

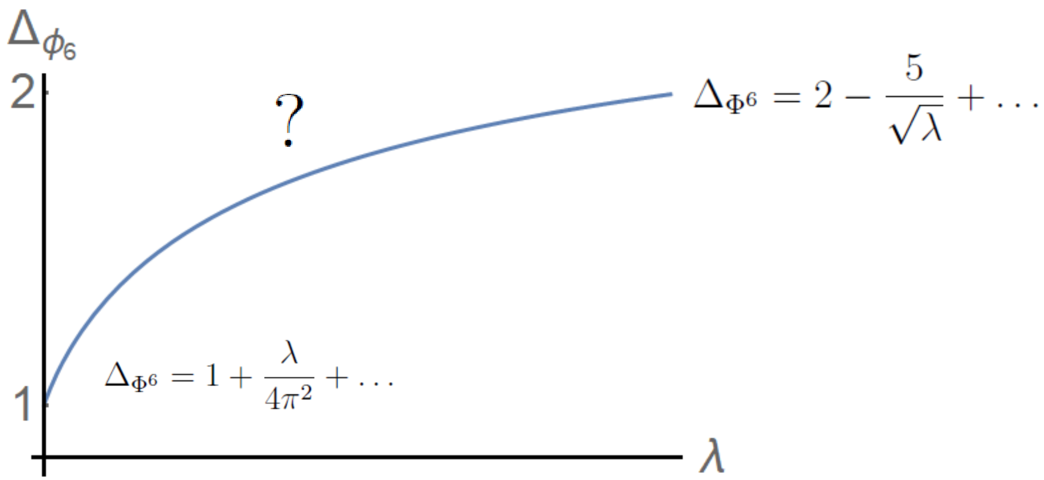
$$\Delta_{\Phi^6} = 1 + \frac{\lambda}{4\pi^2} + \ldots$$

- Assuming no level crossing, it is natural to expect that this operator goes to the lowest unprotected singlet at strong coupling: this is the "2-particle" operator $y^a\, y^a$. So we expect at strong coupling

$$\Delta_{\Phi^6} = 2 - \frac{5}{\sqrt{\lambda}} + \ldots$$

The dimension of Φ^6

- These results are consistent with a smooth interpolation from weak to strong coupling



- Can the exact function of λ be fixed by integrability?
- More generally, can we use integrability to fully solve for the spectrum of insertions on the Wilson loop and their correlation functions?
- This is somewhat analogous to the problem of finding the spectrum of excitations and S-matrix on top of the GKP string *(Basso '10)* . But here instead of S-matrix we have Witten diagrams in AdS_2/correlators in d=1 CFT. Is there an AdS_2 analog of e.g. S-matrix factorization of integrable theories? Mellin representation?

Exact results from localization

- It turns out to be possible to derive a number of exact results for the correlators of a special type of protected insertions on the Wilson loop
- To use localization, we consider the 1/2-BPS circular loop rather than straight line. Correlators on the circle are related to those on the line by a conformal transformation, e.g.

$$\langle\langle O_\Delta(t_1) O_\Delta(t_2)\rangle\rangle_{\rm line} = \frac{C_O}{t_{12}^{2\Delta}} \quad \rightarrow \quad \langle\langle O_\Delta(\tau_1) O_\Delta(\tau_2)\rangle\rangle_{\rm circle} = \frac{C_O}{\left(2\sin\frac{\tau_{12}}{2}\right)^{2\Delta}}$$

and similarly for 4-point functions, with cross ratio now given by

$$\chi = \frac{\sin\frac{\tau_{12}}{2}\sin\frac{\tau_{34}}{2}}{\sin\frac{\tau_{13}}{2}\sin\frac{\tau_{24}}{2}}$$

Exact results from localization

- Note that the expectation value of the circular loop is non-trivial, and given at large *N* by the well-known expression

$$\langle W_{\text{circle}} \rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

and in mapping the correlators from line to circle as above, we have to normalize by this

- To use localization in our system, we need to embed the circular loop into a family of 1/8-BPS Wilson loops constructed in *Drukker, SG, Ricci, Trancanelli '07*
- These Wilson loops are defined on generic contours on an S^2 subspace of R^4 (or S^4), and couple to three of the six scalar fields, say Φ^1, Φ^2, Φ^3

The 1/8-BPS Wilson loops

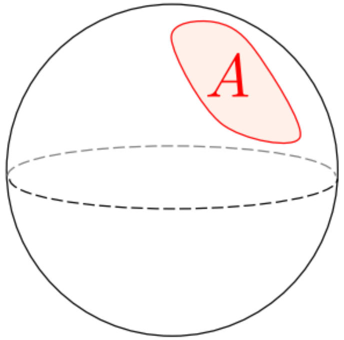
- Explicitly, take an S^2 given by $x_1^2 + x_2^2 + x_3^2 = 1$ in Cartesian coordinates, and define the Wilson loop operator

$$\mathcal{W} \equiv \frac{1}{N} \mathrm{tr}\, \mathrm{P} \left[e^{\oint_C \left(iA_j + \epsilon_{kjl} x^k \Phi^l \right) dx^j} \right]$$

- This preserves 1/8 of the superconformal symmetries for generic contour
- The 1/2-BPS circle is a special case: it corresponds to the contour being a great circle of S^2 . E.g. taking the equator at $x^3=0$, we get the 1/2-BPS loop which couples to Φ^3 (what I was calling Φ^6 before)
- It was conjectured in *Drukker et al '07*, and essentially proved in *Pestun '09* by localization, that their expectation value (as well as correlators of any number of Wilson loops on the S^2) is captured by 2d YM theory

The 1/8-BPS Wilson loops

- This in particular implies that the expectation value only depend on the area singled out by the loop on S^2



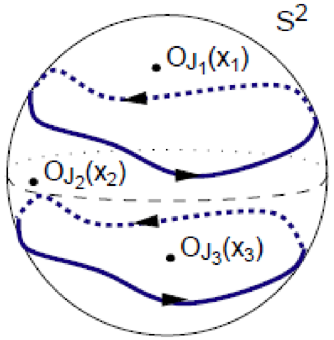
- The expectation value is given by the same function as for the 1/2-BPS circular loop, but with a rescaled coupling constant. E.g. in the planar limit:

$$\langle W(A)\rangle = \frac{2}{\sqrt{\lambda'}} I_1(\sqrt{\lambda'})\,, \qquad \lambda' \equiv \frac{A(4\pi - A)}{4\pi^2}\lambda$$

with $A=2\pi$ being the 1/2-BPS case

1/8-BPS Wilson loops and local operators

- More generally, localization applies to general correlation functions of Wilson loops and local operators *(SG, Pestun '09-'12)*



$$\langle W_{R_1}(\mathcal{C}_1) W_{R_2}(\mathcal{C}_2) \cdots O_{J_1}(x_1) O_{J_2}(x_2) \cdots \rangle_{4d}$$
$$= \langle W_{R_1}^{2d}(\mathcal{C}_1) W_{R_2}^{2d}(\mathcal{C}_2) \cdots \mathrm{tr}\, F_{2d}^{J_1}(x_1)\, \mathrm{tr}\, F_{2d}^{J_2}(x_2) \cdots \rangle_{\text{2d YM}}$$

- The relevant local operators may be inserted outside or inside the loop, and they involve the position-dependent combination of scalars

$$(x_1\Phi_1 + x_2\Phi_2 + x_3\Phi_3 + i\Phi_4)^J \equiv \tilde{\Phi}^J, \qquad x_1^2 + x_2^2 + x_3^2 = 1$$

1/8-BPS Wilson loops and local operators

- These are just chiral primaries of the form $(Y \cdot \Phi)^J$, with Y a null vector which is taken to be position dependent. They were first studied in *Drukker, Plefka '09*
- A crucial property is that their correlation functions are *position independent* (with or without Wilson loops)
- In the localization setup, they are mapped to insertions of powers of the Hodge dual of 2d YM

$$\tilde{\Phi} \quad \Leftrightarrow \quad i * F_{2d}$$

Correlators on the Wilson loop

- Now focusing on our problem of correlators on the circular loop, it means that localization allows us to study correlators of insertions of

$$\tilde{\Phi}^J = (Y_i(\tau_i) \cdot \Phi(\tau_i))^J \qquad\qquad Y_i = (\cos\tau_i, \sin\tau_i, 0, i, 0, 0)$$

- These operators form a topological subsector of the defect CFT, as their *n*-point correlation functions

$$\langle \tilde{\Phi}^{L_1}(\tau_1)\tilde{\Phi}^{L_2}(\tau_2)\cdots\tilde{\Phi}^{L_n}(\tau_n)\rangle_{\text{circle}}$$

are completely position independent

Defect CFT data from topological correlators

- Since 2-point and 3-point functions of the general defect chiral primaries are completely fixed by symmetries up to overall functions of the coupling

$$\langle\!\langle (Y_1\cdot\vec{\Phi})^{L_1}(\tau_1)\;\;(Y_2\cdot\vec{\Phi})^{L_2}(\tau_2)\rangle\!\rangle_{\text{circle}} = n_{L_1}(\lambda,N)\times\frac{\delta_{L_1,L_2}(Y_1\cdot Y_2)^{L_1}}{(2\sin\frac{\tau_{12}}{2})^{2L_1}}$$

$$\langle\!\langle (Y_1\cdot\vec{\Phi})^{L_1}(\tau_1)\;\;(Y_2\cdot\vec{\Phi})^{L_2}(\tau_2)\;\;(Y_3\cdot\vec{\Phi})^{L_3}(\tau_3)\rangle\!\rangle_{\text{circle}} = c_{L_1,L_2,L_3}(\lambda,N)\times\frac{(Y_1\cdot Y_2)^{L_{12|3}}(Y_2\cdot Y_3)^{L_{23|1}}(Y_3\cdot Y_1)^{L_{31|2}}}{\left(2\sin\frac{\tau_{12}}{2}\right)^{2L_{12|3}}\left(2\sin\frac{\tau_{23}}{2}\right)^{2L_{23|1}}\left(2\sin\frac{\tau_{31}}{2}\right)^{2L_{31|2}}}$$

we can use localization for the topological operators to find the exact 2-point normalization and structure constants in this protected sector. Note that these are non-trivial functions of the coupling.

- Of course, for higher-point functions, one cannot fully reconstruct the general correlators from the topological ones.

Correlators from localization

- Let us start from the correlators of "length 1" insertions. Using the correspondence

$$\tilde{\Phi} \quad \Leftrightarrow \quad i * F_{2d}$$

and area-preserving invariance in 2d YM, one can simply obtain the field strength insertions by taking area-derivatives of the Wilson loop VEV

$$\langle \underbrace{\tilde{\Phi} \cdots \tilde{\Phi}}_{L} \rangle \big|_{\text{circle}} = \left. \frac{\partial^L \langle \mathcal{W} \rangle}{(\partial A)^L} \right|_{A=2\pi}$$

- E.g. for the 2-point function, we get for the normalized correlator

$$\langle\langle \tilde{\Phi}(\tau_1) \tilde{\Phi}(\tau_2) \rangle\rangle = \frac{\partial^2}{\partial A^2} \log \langle W(A) \rangle \Big|_{A=2\pi} = -\frac{\sqrt{\lambda} I_2(\sqrt{\lambda})}{4\pi^2 I_1(\sqrt{\lambda})}$$

which gives the Bremsstrahlung function

Four-point function from localization

- For the 4-point function

$$\langle\langle\tilde{\Phi}\tilde{\Phi}\tilde{\Phi}\tilde{\Phi}\rangle\rangle \;=\; \frac{\frac{\partial^4}{\partial A^4}\langle\mathcal{W}\rangle}{\langle\mathcal{W}\rangle}\Bigg|_{A=2\pi} = \frac{3\lambda}{16\pi^4} + \frac{3}{2\pi^4} - \frac{3\sqrt{\lambda}I_0(\sqrt{\lambda})}{4\pi^4 I_1(\sqrt{\lambda})}$$

- This can be compared to our Witten diagram string calculation above. We had

$$\langle Y_1\cdot y(\tau_1)Y_2\cdot y(\tau_2)Y_3\cdot y(\tau_3)Y_4\cdot y(\tau_4)\rangle^{\text{conn.}}_{\text{AdS}_2} =$$

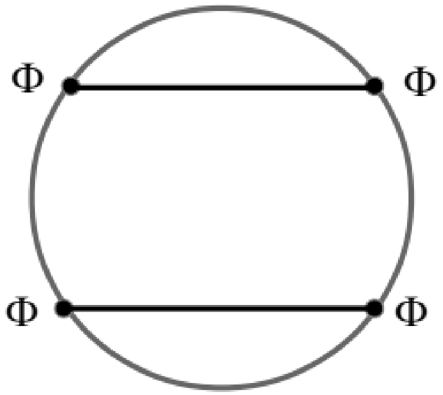
$$= \frac{\left(\frac{\sqrt{\lambda}}{2\pi^2}\right)^2 Y_1\cdot Y_2\, Y_3\cdot Y_4}{(4\sin\frac{\tau_{12}}{2}\sin\frac{\tau_{34}}{2})^2}\frac{1}{\sqrt{\lambda}}\left[G_S(\chi) - \frac{2}{5}G_T(\chi) + \xi(G_T(\chi)+G_A(\chi)) + \zeta(G_T(\chi)-G_A(\chi))\right]$$

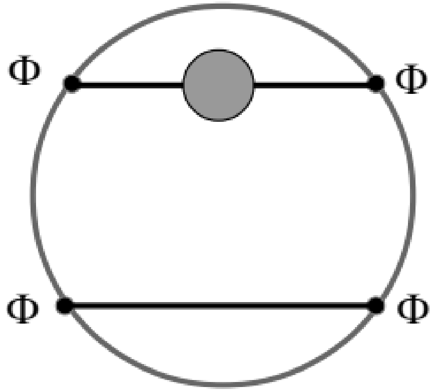
Four-point function

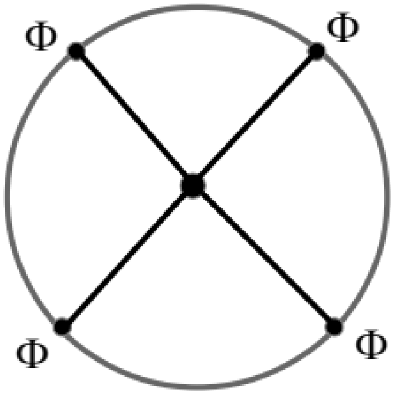
- Specializing to the topological configuration by setting $Y_i = (\cos\tau_i, \sin\tau_i, 0, i, 0, 0)$ this reduces as expected to the position independent result

$$\langle \tilde{y}(\tau_1)\tilde{y}(\tau_2)\tilde{y}(\tau_3)\tilde{y}(\tau_4)\rangle^{\text{conn.}}_{\text{AdS}_2} = -\frac{3\sqrt{\lambda}}{16\pi^4}$$

- Including the contribution of disconnected diagrams to the same order, the AdS_2 result can be seen to match the strong coupling expansion of the localization result (alternatively we can directly select the connected part of the localization result by taking derivatives of the log of <W>)







The general operators

- For general length operators $\tilde{\Phi}^J = (Y_i(\tau_i) \cdot \Phi(\tau_i))^J$, one needs to understand how to define the properly normal-ordered operators (that in particular have zero one-point functions and diagonal 2-point functions).
- Recall the correlator of *L* single-letter insertions is given by area derivatives

$$\langle \underbrace{\tilde{\Phi} \cdots \tilde{\Phi}}_{L} \rangle|_{\text{circle}} = \left. \frac{\partial^L \langle \mathcal{W} \rangle}{(\partial A)^L} \right|_{A=2\pi}$$

- Using the topological natural of the correlators, we can bring some of the operators together without affecting the correlator, and use the OPE to rewrite the product in terms of a basis of normal ordered operators
- E.g. at length 2:

$$\overbrace{\tilde{\Phi} \quad \tilde{\Phi}}^{\text{OPE}} = \tilde{\Phi}^2 + c_1 \, \tilde{\Phi} + c_0 \, \mathbf{1}$$

The general operators

- In other words, we can get the normal ordered operator by taking two area derivatives and subtracting the extra terms

$$:\tilde{\Phi}^2: = \tilde{\Phi}\tilde{\Phi} - c_1\tilde{\Phi} - c_0\mathbf{1}$$

where the constants are fixed by

$$c_1 = \frac{\langle\tilde{\Phi}\tilde{\Phi}\tilde{\Phi}\rangle|_{\text{circle}}}{\langle\tilde{\Phi}\tilde{\Phi}\rangle|_{\text{circle}}} = \left.\frac{\partial_A^3\langle\mathcal{W}\rangle}{\partial_A^2\langle\mathcal{W}\rangle}\right|_{A=2\pi} = 0$$

$$c_0 = \frac{\langle\tilde{\Phi}\tilde{\Phi}\mathbf{1}\rangle|_{\text{circle}}}{\langle\mathbf{1}\mathbf{1}\rangle|_{\text{circle}}} = \left.\frac{\partial_A^2\langle\mathcal{W}\rangle}{\langle\mathcal{W}\rangle}\right|_{A=2\pi}$$

General operators from Gram-Schmidt

- In practice, what we are doing is constructing a set of orthogonal operators with the requirements

    - $\langle :\tilde{\Phi}^L : :\tilde{\Phi}^M : \rangle|_{\text{circle}} \propto \delta_{LM}$.
    - $:\tilde{\Phi}^L:$ is a linear combination of $\underbrace{\tilde{\Phi}\cdots\tilde{\Phi}}_{M}$ with $M \leq L$.
    - The coefficient of the $M = L$ term is 1. Namely $:\tilde{\Phi}^L: = \underbrace{\tilde{\Phi}\cdots\tilde{\Phi}}_{L} + \cdots$

- The operator basis with these properties can be conveniently constructed using the Gram-Schmidt orthogonalization procedure, starting from the set of single-letter products $\{\mathbf{1}\,,\,\tilde{\Phi}\,,\,\tilde{\Phi}\tilde{\Phi},\ldots\}$

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- Note: this construction works in the planar limit, where we can focus on single-trace operators given by operator insertions inside the Wilson loop trace only. At non-planar level, it appears to be necessary to enlarge the set of operators to include double (and multi) trace objects like $\mathcal{W}[\tilde{\Phi}\cdots\tilde{\Phi}]\prod_k \mathrm{tr}\tilde{\Phi}^{J_k}$ *(SG, Komatsu, in progress)*

General operators from Gram-Schmidt

- Applying the Gram-Schmidt algorithm, one gets the explicit form of the length-*L* operators as a determinant

$$:\tilde{\Phi}^L: = \frac{1}{D_L}\begin{vmatrix} \langle\mathcal{W}\rangle & \langle\mathcal{W}\rangle^{(1)} & \cdots & \langle\mathcal{W}\rangle^{(L)} \\ \langle\mathcal{W}\rangle^{(1)} & \langle\mathcal{W}\rangle^{(2)} & \cdots & \langle\mathcal{W}\rangle^{(L+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \langle\mathcal{W}\rangle^{(L-1)} & \langle\mathcal{W}\rangle^{(L)} & \cdots & \langle\mathcal{W}\rangle^{(2L-1)} \\ \mathbf{1} & \tilde{\Phi} & \cdots & \underbrace{\tilde{\Phi}\cdots\tilde{\Phi}}_{L} \end{vmatrix} \qquad D_L = \begin{vmatrix} \langle\mathcal{W}\rangle & \langle\mathcal{W}\rangle^{(1)} & \cdots & \langle\mathcal{W}\rangle^{(L-1)} \\ \langle\mathcal{W}\rangle^{(1)} & \langle\mathcal{W}\rangle^{(2)} & \cdots & \langle\mathcal{W}\rangle^{(L)} \\ \vdots & \vdots & \ddots & \vdots \\ \langle\mathcal{W}\rangle^{(L-1)} & \langle\mathcal{W}\rangle^{(L)} & \cdots & \langle\mathcal{W}\rangle^{(2L-2)} \end{vmatrix}$$

with $\langle\mathcal{W}\rangle^{(k)} \equiv (\partial_A)^k \langle\mathcal{W}\rangle$

- In particular this gives the 2-point functions as ratio of determinants

$$\langle :\tilde{\Phi}^L::\tilde{\Phi}^M: \rangle = \frac{D_{L+1}}{D_L}\delta_{LM}$$

Some explicit 2-point function results

- For example, one finds

$$\langle\langle :\tilde{\Phi}::\tilde{\Phi}: \rangle\rangle = -\frac{\sqrt{\lambda} I_2\left(\sqrt{\lambda}\right)}{4\pi^2 I_1\left(\sqrt{\lambda}\right)}$$

$$\langle\langle :\tilde{\Phi}^2::\tilde{\Phi}^2: \rangle\rangle = \frac{3\lambda}{16\pi^4} - \frac{\lambda I_0\left(\sqrt{\lambda}\right)^2}{16\pi^4 I_1\left(\sqrt{\lambda}\right)^2} - \frac{\sqrt{\lambda} I_0\left(\sqrt{\lambda}\right)}{2\pi^4 I_1\left(\sqrt{\lambda}\right)} + \frac{5}{4\pi^4}$$

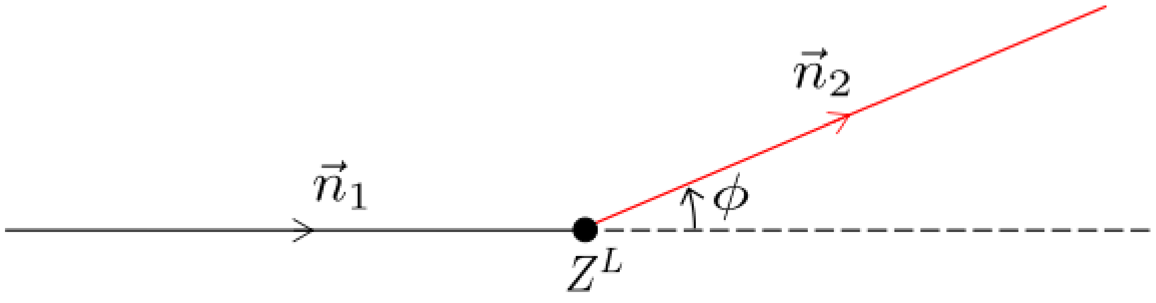
$$\langle\langle :\tilde{\Phi}^3::\tilde{\Phi}^3: \rangle\rangle =$$

$$= -\frac{3\sqrt{\lambda}(5\lambda+72) I_0\left(\sqrt{\lambda}\right)^2}{64\pi^6 I_1\left(\sqrt{\lambda}\right) I_2\left(\sqrt{\lambda}\right)} + \frac{3(13\lambda+144) I_0\left(\sqrt{\lambda}\right)}{32\pi^6\left(I_0\left(\sqrt{\lambda}\right) - \frac{2I_1\left(\sqrt{\lambda}\right)}{\sqrt{\lambda}}\right)} - \frac{3(\lambda(32-3\lambda)+288) I_1\left(\sqrt{\lambda}\right)}{64\pi^6\sqrt{\lambda}\left(I_0\left(\sqrt{\lambda}\right) - \frac{2I_1\left(\sqrt{\lambda}\right)}{\sqrt{\lambda}}\right)}$$

Generalized Bremsstrahlung

- As an explicit application, we can reproduce from localization the results for the "Generalized Bremsstrahlung function" previously obtained from integrability *(Gromov, Sever '12; Gromov, Levkovich-Maslyuk, Sever '13)*

$$n_1 = (1,0,0,0,0)\,, \qquad n_2 = (\cos\theta, \sin\theta, 0,0,0,0)$$



$$\langle \mathcal{W}_L(\theta,\phi)\rangle \sim \left(\frac{\epsilon_{\rm UV}}{r_{\rm IR}}\right)^{\Gamma_L(\theta,\phi)}$$

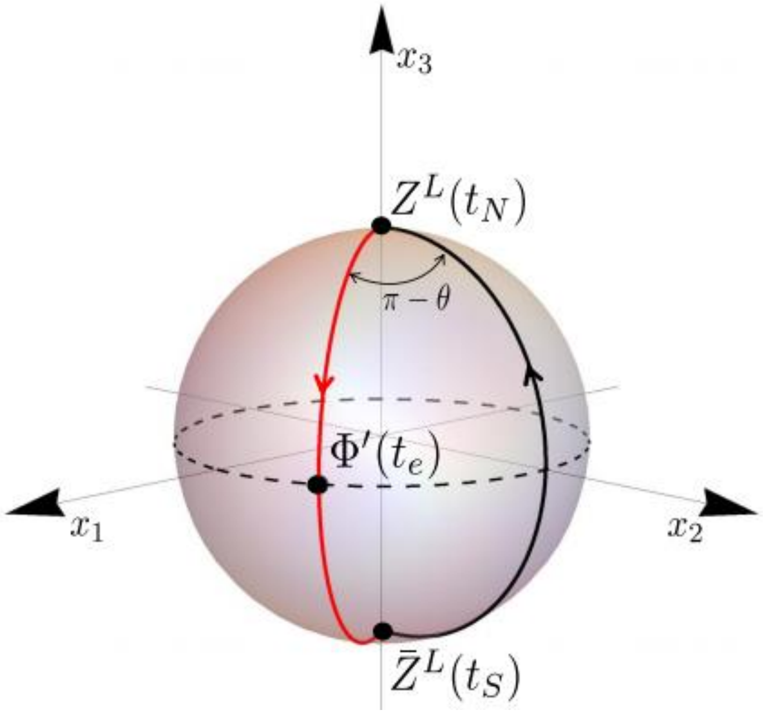
Near BPS:

$$\Gamma_L(\theta,\phi) = (\theta-\phi)H_L(\theta) + O((\theta-\phi)^2)$$

$$H_L(\theta) = \frac{2\theta}{1-\frac{\theta^2}{\pi^2}} B_L(\theta)$$

Generalized Bremsstrahlung

- By mapping to a cusped Wilson loop on S^2 , we can relate the near BPS expansion of the cusp anomalous dimension to our localization correlators



$$\Gamma_L(\theta,\phi)=(\theta-\phi)H_L(\theta)+O((\theta-\phi)^2)$$

$$H_L(\theta)=\frac{\langle:\tilde{\Phi}^L:\ :\tilde{\Phi}^L:\ \tilde{\Phi}\rangle}{\langle:\tilde{\Phi}^L:\ :\tilde{\Phi}^L:\rangle}=\partial_A\log\langle:\tilde{\Phi}^L:\ :\tilde{\Phi}^L:\rangle=-\frac{1}{2}\partial_\theta\log\frac{D_{L+1}}{D_L}$$

$$A=2\pi-2\theta$$

- The localization result can be seen to precisely match the integrability one

General correlators in the planar limit

- In the planar limit, the Wilson loop VEV and its derivatives may be represented by the contour integral

$$\langle \mathcal{W} \rangle = \oint d\mu\,, \qquad d\mu = \frac{dx}{2\pi i x^2} \frac{\sinh(2\pi g(x+1/x)) e^{ga(x-1/x)}}{2\pi g}$$

$$\langle \mathcal{W} \rangle^{(n)} \left(\equiv (\partial_A)^n \langle \mathcal{W} \rangle \right) = \oint d\mu \;\left(g(x - x^{-1}) \right)^n \qquad g \equiv \frac{\sqrt{\lambda}}{4\pi} \qquad a = A - 2\pi$$

- From this one finds that the general topological correlators take the form of integrals

$$\langle :\tilde{\Phi}^{L_1} : :\tilde{\Phi}^{L_2} : \;\cdots :\tilde{\Phi}^{L_m} : \rangle = \oint d\mu \prod_{k=1}^{m} Q_{L_k}(x)$$

where $Q_L(x)$ are polynomials in $g(x-1/x)$, orthogonal with respect the given measure

- The functions $Q_L(x)$ turn out to coincide with analogous function $P_L(x)$ defined in the integrability solution of *Gromov, Sever et al*
- Later *(Gromov, Levkovich-Maslyuk '15)* these functions were shown to be directly related to the so-called "Q-functions" appearing in the Quantum Spectral Curve approach *(Gromov, Kazakov, Leurent, Volin '15)*
- In our localization result, we find that integrals of products of these "Q-functions" encode all correlators, in particular the structure constants of BPS insertions on the Wilson line
- A similar observation was made recently in the integrability context for the structure constants of Wilson lines with three cusps *(Cavagli, Gromov, Levkovich-Maslyuk '18)*
- It would be interesting to further explore the connection and interplay between localization and integrability approaches

Weak and Strong coupling results

- It is interesting to derive explicitly the weak and strong coupling expansion of the results.
- E.g., at weak coupling one finds that $Q_L(x)$ are related to Chebyshev polynomials, and the explicit result for 2 and 3-point function reads

$$\langle :\tilde{\Phi}^{L_1}::\tilde{\Phi}^{L_2}:\rangle\Big|_{O(g^0)} = (-g^2)^{L_1}\delta_{L_1,L_2}\,,$$

$$\langle :\tilde{\Phi}^{L_1}::\tilde{\Phi}^{L_2}::\tilde{\Phi}^{L_3}:\rangle\Big|_{O(g^0)} = (-g^2)^{\frac{L_{\text{tot}}}{2}} d_{L_1,L_2,L_3}$$

$$L_{\text{tot}} \equiv L_1 + L_2 + L_3$$

$$d_{L_1,L_2,L_3} = \begin{cases} 1 & (L_i + L_j \geq L_k) \wedge (\sum_{s=1}^{3} L_s : \text{even}) \\ 0 & \text{otherwise} \end{cases}$$

- This precisely matches the number of planar free Wick contractions of the matrices inserted on the loop
- The subleading one-loop term can also be seen to match the available results

Weak and Strong coupling results

- At strong coupling, one finds that $Q_L(x)$ are instead related to Hermite polynomials
- The explicit calculation gives

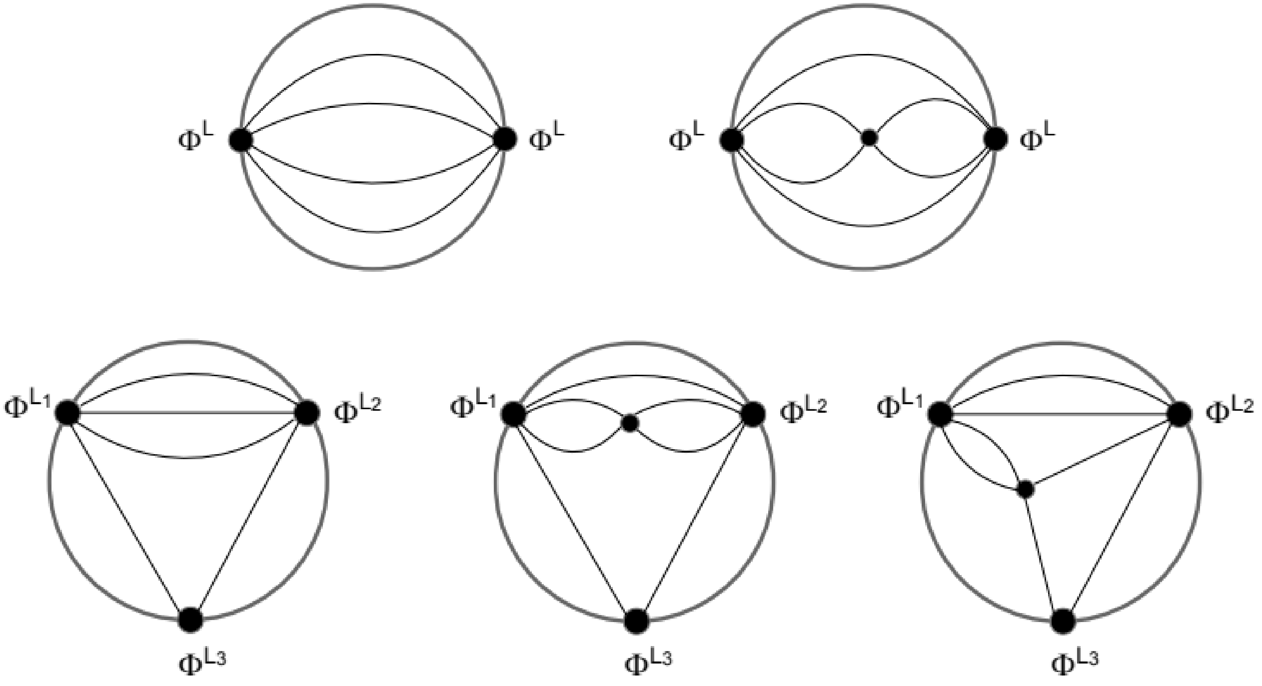
$$\langle\langle :\tilde{\Phi}^{L_1}::\tilde{\Phi}^{L_2}:\rangle\rangle\Big|_{g\to\infty} = \left(-\frac{g}{\pi}\right)^{L_1} L_1!\delta_{L_1L_2}\,,$$

$$\langle\langle :\tilde{\Phi}^{L_1}::\tilde{\Phi}^{L_2}::\tilde{\Phi}^{L_3}:\rangle\rangle\Big|_{g\to\infty} = \left(-\frac{g}{\pi}\right)^{\frac{L_{\rm tot}}{2}} \frac{L_1!L_2!L_3!\; d_{L_1,L_2,L_3}}{L_{12|3}!L_{23|1}!L_{31|2}!}$$

- Now, this instead precisely matches the number of Wick contractions of operators made of products of commuting free fields in AdS_2 !
- The subleading term at strong coupling can be computed as well, and we have also checked that it matches the string theory calculation in AdS_2

String theory calculation

- On the string theory side, we essentially need to compute correlation functions of products of S^5 fluctuations $(Y(\tau)\cdot y)^L = \tilde{y}^L$, bringing the insertion points of these operators to the boundary of AdS_2
- E.g. Witten diagrams for 2-point and 3-point functions



- With a bit of combinatorics, and using the 4-point function result in *SG, Roiban, Tseytlin* , they can be evaluated explicitly and match precisely the localization prediction

Conclusion

- Correlation functions on the 1/2-BPS Wilson loop have the structure of a d=1 conformal system living on the defect
- Correlators on the loop are dual to AdS_2 amplitudes for the transverse worldsheet fluctuations
  - Is there a manifestation of integrability in the AdS_2 amplitudes?
- Exact results may be obtained in a "topological subsector" of special operator insertions
  - Can these exact results for correlation functions be reproduced by integrability?
- More generally, can some of the non-protected data (e.g. the scaling dimension of Φ^6) be fixed exactly by integrability?
- Perhaps an exact solution of this d=1 system may be obtained from an interplay of localization, integrability and conformal bootstrap techniques

Conclusion

- Some other concrete directions
  - Loops in AdS_2
  - Non-planar corrections and Bulk-defect correlators
  - Wilson loop in higher representations and D3/D5 branes with AdS_2xS^2 and AdS_2xS^4 worldvolumes
  - Non-supersymmetric circular Wilson loop
    (Beccaria, SG, Tseytlin ‘17; Beccaria, Tseytlin ‘18)