Yangian Symmetry and Correlation Functions in Planar $\mathcal{N} = 4$ SYM

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Integrability in Gauge and String Theory
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work with A. Garus (1804.09110)
and with A. Garus, M. Rosso (1803.06310, 1701.09162).

Introduction and Overview

Aim:

Prove Yangian symmetry in integrable planar gauge theories.

Outline:

- Yangian Symmetry of Planar $\mathcal{N} = 4$ SYM
- Yangian Algebra and Gauge Transformations
- Correlation Functions

General Assumptions:

- $\mathcal{N} = 4$ supersymmetric Yang–Mills theory
- Planar limit
- Most results also apply to ABJM
  ($\mathcal{N} = 6$ supersymmetric Chern–Simons theory)

I. Yangian Symmetry of Planar $\mathcal{N} = 4$ SYM

AdS/CFT Integrability

Integrability: Curious feature of planar $\mathcal{N} = 4$ SYM (and related);
enables efficient computations:

- planar spectrum of anomalous dimensions (finite λ)
- correlation functions of local operators
- colour-ordered scattering amplitudes
- null polygon Wilson loops
- planar loop integrands (integrals?)
- ...

What (precisely) is integrability? How to prove it?

- several ansätze or definitions in particular situations
- ...
- hidden symmetry enhancement:

superconformal $\mathfrak{psu}(2,2|4) \to$ Yangian $\mathrm{Y}[\mathfrak{psu}(2,2|4)]$

Yangian Symmetry

"Symmetry" in what sense?

- Spectrum is not invariant (boundary conditions).
- Scattering amplitudes are IR divergent (massless particles).
- Null polygon Wilson loops are UV divergent.
- Smooth Maldacena–Wilson loops are finite and invariant.
- Symmetry for other observables less evident.
- Ordering principle, tools, ...

Invariance of the action!

Complications:

- representation non-linear in fields,
- cyclic boundary conditions,
- implementation of planar limit,
- non-local properties,
- quantum anomalies?

Yangian Algebra

Defined in terms of level-zero and level-one generators J^A, $\widehat{\mathrm{J}}^A$:

Algebra Relations:

$$[\mathrm{J}^A, \mathrm{J}^B] \sim f_C^{AB} \mathrm{J}^C,$$
$$[\mathrm{J}^A, \widehat{\mathrm{J}}^B] \sim f_C^{AB} \widehat{\mathrm{J}}^C,$$
$$\big[\widehat{\mathrm{J}}^A, [\widehat{\mathrm{J}}^B, \mathrm{J}^C]\big] + \text{cyclic} \approx \{\mathrm{J}, \mathrm{J}, \mathrm{J}\}.$$

$\widehat{\mathrm{J}}$ in adjoint; satisfies Serre relation.

Coproduct:

$$\Delta \mathrm{J}^C \sim \mathrm{J}^C \otimes 1 + 1 \otimes \mathrm{J}^C,$$
$$\Delta \widehat{\mathrm{J}}^C \sim \widehat{\mathrm{J}}^C \otimes 1 + 1 \otimes \widehat{\mathrm{J}}^C + f_{AB}^C \mathrm{J}^A \otimes \mathrm{J}^B.$$

$\mathrm{J}/\widehat{\mathrm{J}}$ acts locally/bi-locally.

Level-one momentum (dual conformal) $\widehat{\mathrm{P}}$ easiest:

$$\Delta \widehat{\mathrm{P}} \sim \widehat{\mathrm{P}} \otimes 1 + 1 \otimes \widehat{\mathrm{P}} + \mathrm{P} \wedge \mathrm{D} + \mathrm{P} \wedge \mathrm{L} + \mathrm{Q} \wedge \bar{\mathrm{Q}}.$$

- based on super-Poincaré $(\mathrm{P}, \mathrm{L}, \mathrm{Q})$ and dilatation (D);
- can be defined in many (other, related) models.

Field Polynomials

Consider field monomials:

$$Z_1 Z_2 \dots Z_n$$

- all (covariant) fields Z_k are $N \times N$ matrices;
- product monomial is (covariant) $N \times N$ matrix;
- ordering of fields matters (for sufficiently large N).

Field polynomials relevant for various objects and observables in QFT:

- local operators $\mathcal{O}(x) = \operatorname{tr} Z_1(x) \dots Z_n(x) + \dots$,
- Wilson lines $W = \mathrm{P} \exp \int A = 1 + \int A + \frac{1}{2} \iint A_1 A_2 + \dots$,
- colour-ordered correlators $F_n(x_1, \dots, x_n) = \langle \operatorname{tr} Z_1(x_1) \dots Z_n(x_n) \rangle$,
- action $\mathcal{S} = \int dx^4 \, \mathcal{L}(x) \sim \int dx^4 \, \operatorname{tr}(F^{\mu\nu} F_{\mu\nu}) + \dots$.

Yangian Bi-local Representation

Superconformal action (level-zero Yangian): local insertion

$$\mathrm{J}^C(Z_1 \ldots Z_n) = \sum_{k=1}^{n} Z_1 \ldots \mathrm{J}^C Z_k \ldots Z_n.$$

Level-one Yangian action: bi-local insertion follows coproduct

$$\widehat{\mathrm{J}}^C(Z_1 \ldots Z_n) = f^C_{AB} \sum_{k<l=1}^{n} Z_1 \ldots \mathrm{J}^A Z_k \ldots \mathrm{J}^B Z_l \ldots Z_n$$
$$+ \sum_{k=1}^{n} Z_1 \ldots \widehat{\mathrm{J}}^C Z_k \ldots Z_n.$$

Issues:

- local term $\widehat{\mathrm{J}} Z_k$ as completion of bi-local terms;
- non-linear action of $\mathrm{J} Z_k$ and $\widehat{\mathrm{J}} Z_k$.

Invariance of the Action

Aim: Show planar Yangian invariance of the action

$$\widehat{\mathrm{J}}\,\mathcal{S} = 0.$$

Essential features of the action $\mathcal{S}$:

- single-trace, conformal, finite (disc, level zero, no anomalies?);
- cyclic, integrated, non-homogeneous polynomial

Task: Reconcile non-linear, bi-local representation with cyclicity.

Found definition for $\widehat{\mathrm{J}}\,Z$ (local contribution) and "$\widehat{\mathrm{J}}\,\mathcal{S}$" such that:

- $\widehat{\mathrm{J}}\,\mathcal{S} = 0$ for $\mathcal{N} = 4$ SYM and other planar integrable models
- $\widehat{\mathrm{J}}\,\mathcal{S} \neq 0$ for non-integrable models (plain $\mathcal{N} < 4$ SYM)

Invariance of the action shown for $\widehat{\mathrm{P}}$ and others (~ 1000 terms).

Proper definition of integrability!

Potential Yangian Anomalies

More elegant proof: Consider classical anomaly term

$$\widehat{\mathcal{A}}^\mu := \widehat{\mathrm{P}}^\mu \mathcal{S} \overset{?}{=} 0.$$

From level-one algebra $[\mathrm{J}, \widehat{\mathrm{J}}] \sim \widehat{\mathrm{J}}$ a consistent anomaly requires:

$$\mathrm{P}\,\widehat{\mathcal{A}}^\mu = \mathrm{Q}\,\widehat{\mathcal{A}}^\mu = 0, \qquad \widehat{\mathcal{A}}^\mu \text{ is a vector of dimension 1.}$$

Therefore $\widehat{\mathcal{A}}^\mu = \int dx^4\,\widehat{\mathcal{O}}^\mu$ with local operator $\widehat{\mathcal{O}}^\mu$:

- dimension-5 vector operator $\widehat{\mathcal{O}}$
- top component of supermultiplet

However: top components of long multiplets at dimension ≥ 10.
No suitable short supermultiplets. No classical anomaly terms!

Even better: level-one bonus symmetry $\widehat{\mathrm{B}} \sim \mathrm{Q} \wedge \mathrm{S}$:

$$\widehat{\mathcal{B}} = \widehat{\mathrm{B}}\,\mathcal{S}; \qquad \mathrm{P}\,\widehat{\mathcal{B}} = \mathrm{L}\,\widehat{\mathcal{B}} = \mathrm{R}\,\widehat{\mathcal{B}} = \mathrm{D}\,\widehat{\mathcal{B}} = 0, \quad \Pi\,\widehat{\mathcal{B}} = -\widehat{\mathcal{B}}.$$

No Π-odd dimension-4 scalar operator $\widehat{\mathcal{O}}$ with $\widehat{\mathcal{B}} = \int dx^4\,\widehat{\mathcal{O}}$!

Yangian Symmetry in Quantum Theory

Yangian symmetry in classical action shown! Implications for QFT?
Noether: Conserved currents/charges? Bi-local representation?!

Consider general correlators of fields:

$$F_{1\ldots n}(x_1,\ldots,x_n) := \big\langle Z_1(x_1)\ldots Z_n(x_n)\big\rangle.$$

Ward–Takahashi identities for $F_{1\ldots n}(x_1,\ldots,x_n)$!

$$\begin{aligned}
\mathrm{J}\langle\ldots\rangle &= \sum\nolimits_k \big\langle Z_1(x_1)\ldots \mathrm{J}Z_k(x_k)\ldots Z_n(x_n)\big\rangle \stackrel{!}{=} 0,\\
\widehat{\mathrm{J}}\langle\ldots\rangle &= \sum\nolimits_{k<l} \big\langle Z_1(x_1)\ldots \mathrm{J}Z_k(x_k)\ldots \mathrm{J}Z_l(x_l)\ldots Z_n(x_n)\big\rangle\\
&\quad + \sum\nolimits_k \big\langle Z_1(x_1)\ldots \widehat{\mathrm{J}}Z_k(x_k)\ldots Z_n(x_n)\big\rangle \stackrel{!}{=} 0.
\end{aligned}$$

Complication: $\mathcal{N}=4$ SYM is gauge theory.
• gauge fixing • unphysical d.o.f. • Yangian closes onto gauge.

II. Yangian Algebra and Gauge Transformations

Gauge Ideal

Extended supersymmetry necessarily involves gauge transformations

$$\{\mathrm{Q}, \mathrm{Q}\} \sim \mathrm{G}[\Phi].$$

Supersymmetry closes onto translations: translations generate gauge

$$\{\mathrm{Q}, \bar{\mathrm{Q}}\} \sim \mathrm{P}, \qquad [\mathrm{P}_\mu, \mathrm{J}] = [\mathrm{P}_\mu, \mathrm{J}]_{\mathsf{alg}} + \mathrm{G}[\mathrm{J} A_\mu].$$

Now consider level-one momentum $\widehat{\mathrm{P}} =: \mathrm{P}^{(1)} \otimes \mathrm{P}^{(2)}$ (Sweedler)

$$[\mathrm{P}_\mu, \widehat{\mathrm{P}}] = \mathrm{G}[\mathrm{P}^{(1)} A_\mu] \wedge \mathrm{P}^{(2)}.$$

From $\mathrm{P}\mathcal{S} = \widehat{\mathrm{P}}\mathcal{S} = 0$ it follows

$$0 = [\mathrm{P}, \widehat{\mathrm{P}}]\,\mathcal{S} = \bigl(\mathrm{G}[\mathrm{P}^{(1)} A] \wedge \mathrm{P}^{(2)}\bigr)\,\mathcal{S}.$$

Additional bi-local symmetry: gauge but not ordinary local.

Bi-local Gauge Transformations

Significance of bi-local gauge transformations $\mathrm{G}[X] \wedge \mathrm{J}$?
Action of ordinary local gauge transformations $\mathrm{G}[X]$:

$$\mathrm{G}[X](Z_1 \ldots Z_n) \sim [X, Z_1 \ldots Z_n].$$

Action of bi-local gauge transformations $\mathrm{G}[X] \wedge \mathrm{J}$:

$$(\mathrm{G}[X] \wedge \mathrm{J})(Z_1 \ldots Z_n) \sim \big\{X, \mathrm{J}(Z_1 \ldots Z_n)\big\}.$$

Invariance of action follows from:

$$\mathrm{G}[X]\,\mathcal{S} = 0, \quad \mathrm{J}\,\mathcal{S} = 0 \qquad \text{and also} \quad \mathrm{J}X = 0 \text{ (for cyclicity)}.$$

Requirements hold for all superconformal gauge theories:

- bi-local gauge transformations form an ideal of Yangian algebra;
- gauge ideal less restrictive than full Yangian.

Gauge Fixing

Fix gauge by Faddeev–Popov method:

- introduce ghost and auxiliary fields $C, \bar{C}, B$;
- extra terms $\mathcal{S}_{\text{gf}}$ in action;

BRST symmetry Q (Q no longer supersymmetry)

$$\mathrm{Q}Z \sim \mathrm{G}[C], \quad \mathrm{Q}C \sim CC, \quad \mathrm{Q}\bar{C} \sim B, \quad \mathrm{Q}B = 0; \qquad \mathrm{Q}\mathrm{Q} = 0.$$

Consider BRST cohomology:

- action closed $\mathrm{Q}\,\mathcal{S} = 0$ (physical);
- gauge fixing terms exact $\mathcal{S}_{\text{gf}} = \mathrm{Q}\,\mathcal{K}_{\text{gf}}$ (irrelevant).

Extra terms needed for superconformal symmetry

$$\mathrm{J}\,\mathcal{S} = \mathrm{Q}\,\mathcal{K}[\mathrm{J}], \qquad \mathcal{K}[\mathrm{J}] := \mathrm{J}\,\mathcal{K}_{\text{gf}};$$

project out unphysical d.o.f. from invariance.

BRST and Yangian Symmetry

BRST is a residual gauge symmetry.
$\to$ additional bi-local BRST generators $\mathrm{Q} \wedge \mathrm{J}$ and $\mathrm{Q} \otimes \mathrm{Q}$.

Invariance of action requires further terms:

$$
\begin{aligned}
(\mathrm{Q} \otimes \mathrm{Q})\mathcal{S} &= \mathrm{Q}\,\mathcal{K}[\mathrm{Q} \otimes \mathrm{Q}],\\
(\mathrm{Q} \wedge \mathrm{J})\mathcal{S} &= \mathrm{Q}\,\mathcal{K}[\mathrm{Q} \wedge \mathrm{J}] + (\mathrm{Q} \otimes \mathrm{Q})\mathcal{K}[\mathrm{J}] + \mathrm{J}\,\mathcal{K}[\mathrm{Q} \otimes \mathrm{Q}],\\
\widehat{\mathrm{J}}\,\mathcal{S} &= \mathrm{Q}\,\mathcal{K}[\widehat{\mathrm{J}}] + (\mathrm{Q} \wedge \mathrm{J}^{(1)})\mathcal{K}[\mathrm{J}^{(2)}] + \mathrm{J}^{(1)}\,\mathcal{K}[\mathrm{Q} \wedge \mathrm{J}^{(2)}].
\end{aligned}
$$

Identities hold in gauge-fixed $\mathcal{N} = 4$ SYM (and ABJM).

- bi-local BRST generators needed for bi-local Yangian symmetry.

Slavnov–Taylor identities

Ward–Takahashi identities receive extra terms: Slavnov–Taylor identity

$$\big\langle \mathrm{J}\mathcal{O} + \mathcal{K}[\mathrm{J}]\,\mathrm{Q}\mathcal{O}\big\rangle = 0.$$

Holds by virtue of invariance of gauge-fixed action; variational identity.

Slavnov–Taylor identity for bi-local Yangian

$$\begin{aligned}0 &= \big\langle \widehat{\mathrm{J}}\mathcal{O}\big\rangle + \big\langle \mathcal{K}[\mathrm{J}^{(1)}]\,(\mathrm{Q}\wedge \mathrm{J}^{(2)})\mathcal{O}\big\rangle + \big\langle \mathcal{K}[\mathrm{J}^{(1)}]\,\mathcal{K}[\mathrm{J}^{(2)}]\,(\mathrm{Q}\otimes \mathrm{Q})\mathcal{O}\big\rangle \\ &\quad + \big\langle \big(\mathcal{K}[\widehat{\mathrm{J}}] + \mathcal{K}[\mathrm{Q}\wedge \mathrm{J}^{(1)}]\,\mathcal{K}[\mathrm{J}^{(2)}] + \mathcal{K}[\mathrm{Q}\otimes \mathrm{Q}]\,\mathcal{K}[\mathrm{J}^{(1)}]\,\mathcal{K}[\mathrm{J}^{(2)}]\big)\,\mathrm{Q}\mathcal{O}\big\rangle \\ &\quad + \big\langle \big(\mathcal{K}[\mathrm{Q}\wedge \mathrm{J}^{(1)}] + \mathcal{K}[\mathrm{Q}\otimes \mathrm{Q}]\,\mathcal{K}[\mathrm{J}^{(1)}]\big)\,\mathrm{J}^{(2)}\mathcal{O}\big\rangle.\end{aligned}$$

Note:

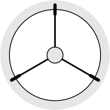
- analogous identities for bi-local BRST $\mathrm{Q}\otimes\mathrm{Q}$ and $\mathrm{Q}\wedge\mathrm{J}$.
- uses conjectural bi-local variational identity of planar path integral.

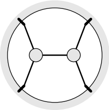
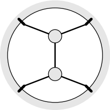
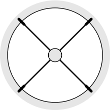
III. Correlation Functions

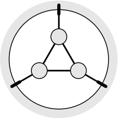
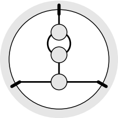
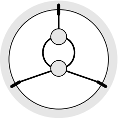
Correlators of Fields

Test Slavnov–Taylor identities for some correlators:

$\langle \mathrm{tr}\, Z_1 Z_2 \rangle =$  ,

$\langle \mathrm{tr}\, Z_1 Z_2 Z_3 \rangle = i$  ,

$\langle \mathrm{tr}\, Z_1 Z_2 Z_3 Z_4 \rangle = -$  $-$  $+\, i$  ,

$\langle \mathrm{tr}\, Z_1 Z_2 Z_3 \rangle_{(1)} = -i$  $-\, i$  $-$  .

- restrict to planar / colour-ordered contributions;
- off-shell: no complications due to mass shell condition;

Symmetries of Propagators

Conformal symmetry for propagators $\langle Z_1 Z_2 \rangle$

$\simeq 0.$

Invariance for matter fields

$$\mathrm{J}^C \big\langle Z_1\, Z_2 \big\rangle = \big\langle \mathrm{J}^C Z_1\, Z_2 \big\rangle + \big\langle Z_1\, \mathrm{J}^C Z_2 \big\rangle = 0;$$

invariance for gauge fields $\langle A_1 A_2 \rangle$

$$\mathrm{J}^C \big\langle A_1\, A_2 \big\rangle = \big\langle \mathrm{J}^C A_1\, A_2 \big\rangle + \big\langle A_1\, \mathrm{J}^C A_2 \big\rangle = d_1 H^C_{12} + d_2 H^C_{21}.$$

Yangian symmetry for propagator $\langle Z_1 Z_2 \rangle$

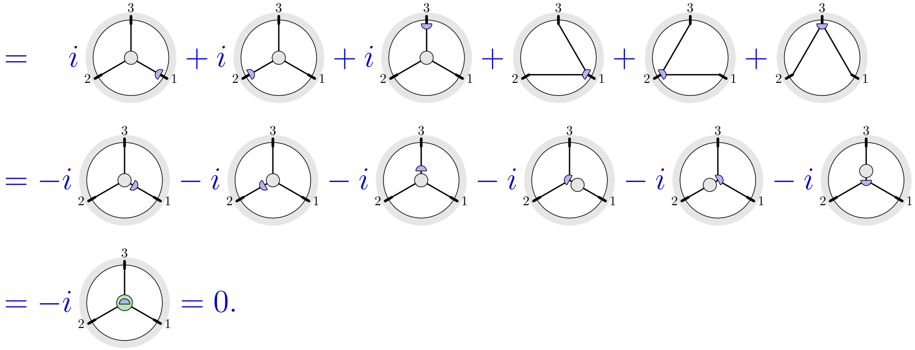
$$\widehat{\mathrm{J}}^C \big\langle Z_1\, Z_2 \big\rangle = f^C_{AB} \big\langle \mathrm{J}^A Z_1\, \mathrm{J}^B Z_2 \big\rangle = 0,$$
$$\widehat{\mathrm{J}}^C \big\langle A_1\, A_2 \big\rangle = d_1 d_2 \widehat{R}^C_{12}.$$

Level-one generators almost annihilate gauge propagator $\langle A_1 A_2 \rangle$.

Conformal Symmetry of 3-Point Function

Start simple: tree-level conformal invariance at 3 points

$$\mathrm{J}\big\langle \operatorname{tr} Z_1 Z_2 Z_3 \big\rangle$$

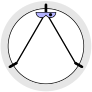
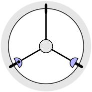
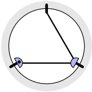
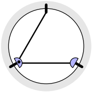


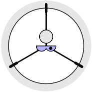
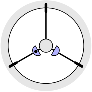
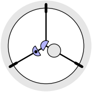
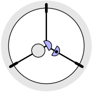
Invariance of action implies invariance of correlator.

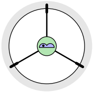
Also confirmed invariance for properly gauge-fixed correlator.

Yangian Symmetry of 3-Point Function

Yangian action on correlator of 3 fields at tree level

$\widehat{\mathrm{J}}\langle \mathrm{tr}\, Z_1 Z_2 Z_3 \rangle \simeq 3$  $+\, i$  $+$  $+$ 

$\simeq -3i$  $+\, i$  $+\, i$  $-\, i$ 

$\simeq -i$  $= 0.$

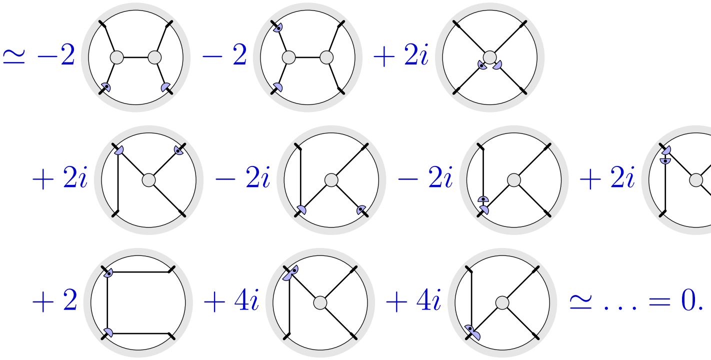
Invariance based on:

- conformal invariance of propagator and 3-vertex,
- Yangian invariance of 3-vertex.

Also showed $\mathrm{Q} \wedge \mathrm{J}$ invariance of gauge-fixed correlator.

Yangian Symmetry of 4-Point Function

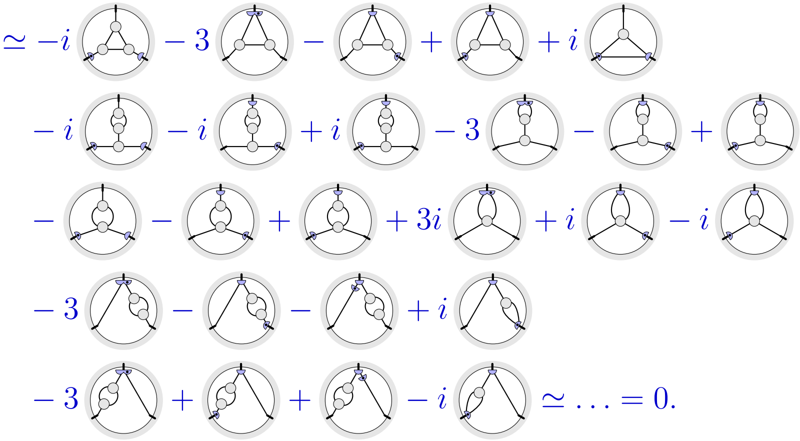
Yangian action on tree-level correlator of 4 fields $\widehat{\mathrm{J}}\langle \mathrm{tr}\, Z_1 Z_2 Z_3 Z_4 \rangle$



- conformal invariance of propagator, 3-vertex and 4-vertex,
- Yangian invariance of 3-vertex and 4-vertex,
- commutativity of constituents $[\mathrm{J}^{(1)}, \mathrm{J}^{(2)}] = 0$.

3-Function at One Loop

Yangian action on one-loop correlator of 3 fields $\widehat{\mathrm{J}}\langle \operatorname{tr} Z_1 Z_2 Z_3\rangle_{(1)}$



Invariance shown modulo gauge fixing and divergences.

Anomalies?

Classical symmetries may suffer from quantum anomalies:

- No established framework for anomalies of non-local symmetries (in colour-space not necessarily in spacetime).
- Violation of (non-local) current? Cohomological origin?

Potential anomaly terms:

- quantum analysis similar to classical one?
- consider gauge fixing ...
- consider regularisation ...

However:

- Not an issue for Wilson loop expectation value at one loop.
- Integrability "works" at finite coupling: no anomaly expected?

IV. Conclusions

Conclusions

Yangian Symmetry of Planar $\mathcal{N} = 4$ SYM:

- classical action of planar $\mathcal{N} = 4$ SYM Yangian invariant
- model classically integrable (same for ABJM)

Yangian Algebra and Gauge Transformations

- Yangian algebra produces non-local gauge transformations
- gauge transformations form an ideal
- after gauge fixing: bi-local and mixed BRST transformations
- additional terms to eliminate unphysical d.o.f.
- Yangian compatible with gauge fixing

Correlation Functions

- Ward–Takahashi/Slavnov–Taylor identities tested
- No quantum anomalies to be expected?!

Outlook: Apply to scattering amplitudes (LSZ), Wilson loops, ...
Derive algebraic integrability methods?!