Yangian Symmetry
and Correlation Functions
in Planar $\mathcal{N} = 4$ SYM

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Introduction and Overview

Aim:

Prove Yangian symmetry in integrable planar gauge theories.

Outline:

• Yangian Symmetry of Planar $\mathcal{N} = 4$ SYM
• Yangian Algebra and Gauge Transformations
• Correlation Functions

General Assumptions:

• $\mathcal{N} = 4$ supersymmetric Yang–Mills theory
• Planar limit
• Most results also apply to ABJM ($\mathcal{N} = 6$ supersymmetric Chern–Simons theory)
I. Yangian Symmetry of Planar $\mathcal{N} = 4$ SYM
AdS/CFT Integrability

**Integrability:** Curious feature of planar $\mathcal{N} = 4$ SYM (and related); enables efficient computations:

- planar spectrum of anomalous dimensions (finite $\lambda$)
- correlation functions of local operators
- colour-ordered scattering amplitudes
- null polygon Wilson loops
- planar loop integrands (integrals?)
- ... 

**What (precisely) is integrability?** How to prove it?

- several ansätze or definitions in particular situations
- ...
- hidden symmetry enhancement:
  
  superconformal $\mathfrak{psu}(2, 2|4) \rightarrow$ Yangian $Y[\mathfrak{psu}(2, 2|4)]$
Yangian Symmetry

“Symmetry” in what sense?
- Spectrum is not invariant (boundary conditions).
- Scattering amplitudes are IR divergent (massless particles).
- Null polygon Wilson loops are UV divergent.
- Smooth Maldacena–Wilson loops are finite and invariant.
- Symmetry for other observables less evident.
- Ordering principle, tools, . . .

Invariance of the action!
Complications:
- representation non-linear in fields,
- cyclic boundary conditions,
- implementation of planar limit,
- non-local properties,
- quantum anomalies?
Yangian Algebra

Defined in terms of level-zero and level-one generators $J^A$, $\hat{J}^A$:

**Algebra Relations:**

\[
\begin{align*}
[J^A, J^B] &\sim f^{AB}_C J^C , \\
[J^A, \hat{J}^B] &\sim f^{AB}_C \hat{J}^C , \\
[\hat{J}^A, [\hat{J}^B, J^C]] &+ \text{cyclic} \approx \{J, J, J\}.
\end{align*}
\]

$\hat{J}$ in adjoint; satisfies Serre relation.  $J/\hat{J}$ acts locally/bi-locally.

**Coproduct:**

\[
\begin{align*}
\Delta J^C &\sim J^C \otimes 1 + 1 \otimes J^C , \\
\Delta \hat{J}^C &\sim \hat{J}^C \otimes 1 + 1 \otimes \hat{J}^C \\
&+ f^{C}_{AB} J^A \otimes J^B.
\end{align*}
\]

Level-one momentum (dual conformal) $\hat{P}$ easiest:

\[
\Delta \hat{P} \sim \hat{P} \otimes 1 + 1 \otimes \hat{P} + P \wedge D + P \wedge L + Q \wedge \bar{Q}.
\]

- based on super-Poincaré $(P, L, Q)$ and dilatation $(D)$;
- can be defined in many (other, related) models.
Field Polynomials

Consider field monomials:

\[ Z_1 Z_2 \ldots Z_n \]

- all (covariant) fields \( Z_k \) are \( N \times N \) matrices;
- product monomial is (covariant) \( N \times N \) matrix;
- ordering of fields matters (for sufficiently large \( N \)).

Field polynomials relevant for various objects and observables in QFT:

- local operators \( \mathcal{O}(x) = \text{tr} Z_1(x) \ldots Z_n(x) + \ldots \),
- Wilson lines \( W = \text{P} \exp \int A = 1 + \int A + \frac{1}{2} \int \int A_1 A_2 + \ldots \),
- colour-ordered correlators \( F_n(x_1, \ldots, x_n) = \langle \text{tr} Z_1(x_1) \ldots Z_n(x_n) \rangle \),
- action \( S = \int dx^4 \mathcal{L}(x) \sim \int dx^4 \text{tr}(F^{\mu\nu} F_{\mu\nu}) + \ldots \).
Yangian Bi-local Representation

Superconformal action (level-zero Yangian): local insertion

\[ J^C(Z_1 \ldots Z_n) = \sum_{k=1}^{n} Z_1 \ldots J^C Z_k \ldots Z_n. \]

Level-one Yangian action: bi-local insertion follows coproduct

\[ \hat{J}^C(Z_1 \ldots Z_n) = f_{AB}^C \sum_{k<l=1}^{n} Z_1 \ldots J^A Z_k \ldots J^B Z_l \ldots Z_n \]

\[ + \sum_{k=1}^{n} Z_1 \ldots \hat{J}^C Z_k \ldots Z_n. \]

Issues:
- local term \( \hat{J}Z_k \) as completion of bi-local terms;
- non-linear action of \( JZ_k \) and \( \hat{J}Z_k \).
Invariance of the Action

**Aim:** Show planar Yangian invariance of the action

\[ \hat{\mathcal{J}} S = 0. \]

**Essential features of the action \( S \):**
- single-trace, conformal, finite (disc, level zero, no anomalies?);
- cyclic, integrated, non-homogeneous polynomial

**Task:** Reconcile non-linear, bi-local representation with cyclicity.

Found definition for \( \hat{\mathcal{J}} Z \) (local contribution) and “\( \hat{\mathcal{J}} S \)” such that:
- \( \hat{\mathcal{J}} S = 0 \) for \( \mathcal{N} = 4 \) SYM and other planar integrable models
- \( \hat{\mathcal{J}} S \neq 0 \) for non-integrable models (plain \( \mathcal{N} < 4 \) SYM)

Invariance of the action shown for \( \hat{P} \) and others (\( \sim 1000 \) terms).

Proper definition of integrability!
Potential Yangian Anomalies

More elegant proof: Consider classical anomaly term

\[ \hat{A}^\mu := \hat{P}^\mu S \equiv 0. \]

From level-one algebra \([J, \hat{J}] \sim \hat{J}\) a consistent anomaly requires:

\[ \hat{P} \hat{A}^\mu = Q \hat{A}^\mu = 0, \quad \hat{A}^\mu \text{ is a vector of dimension 1}. \]

Therefore \(\hat{A}^\mu = \int dx^4 \hat{O}^\mu\) with local operator \(\hat{O}^\mu\):
- dimension-5 vector operator \(\hat{O}\)
- top component of supermultiplet

However: top components of long multiplets at dimension \(\geq 10\).
No suitable short supermultiplets. No classical anomaly terms!

Even better: level-one bonus symmetry \(\hat{B} \sim Q \wedge S\):

\[ \hat{B} = \hat{B} S; \quad \hat{P} \hat{B} = L \hat{B} = R \hat{B} = D \hat{B} = 0, \quad \Pi \hat{B} = -\hat{B}. \]

No \(\Pi\)-odd dimension-4 scalar operator \(\hat{O}\) with \(\hat{B} = \int dx^4 \hat{O}\)!
Yangian Symmetry in Quantum Theory

Yangian symmetry in classical action shown! Implications for QFT? Noether: Conserved currents/charges? Bi-local representation?!

Consider general correlators of fields:

\[ F_{1\ldots n}(x_1, \ldots, x_n) := \langle Z_1(x_1) \ldots Z_n(x_n) \rangle. \]

Ward–Takahashi identities for \( F_{1\ldots n}(x_1, \ldots, x_n) \):

\[
\begin{align*}
J \langle \ldots \rangle &= \sum_k \langle Z_1(x_1) \ldots J Z_k(x_k) \ldots Z_n(x_n) \rangle \tag{1} = 0, \\
\hat{J} \langle \ldots \rangle &= \sum_{k<l} \langle Z_1(x_1) \ldots J Z_k(x_k) \ldots J Z_l(x_l) \ldots Z_n(x_n) \rangle \\
&\quad + \sum_k \langle Z_1(x_1) \ldots \hat{J} Z_k(x_k) \ldots Z_n(x_n) \rangle \tag{2} = 0.
\end{align*}
\]

Complication: \( \mathcal{N} = 4 \) SYM is gauge theory.

- gauge fixing  
- unphysical d.o.f.  
- Yangian closes onto gauge.
II. Yangian Algebra

and

Gauge Transformations
Extended supersymmetry necessarily involves gauge transformations

\[ \{Q, Q\} \sim G[\Phi]. \]

Supersymmetry closes onto translations: translations generate gauge

\[ \{Q, \bar{Q}\} \sim \mathbb{P}, \quad [P_\mu, J] = [P_\mu, J]_{\text{alg}} + G[J A_\mu]. \]

Now consider level-one momentum \( \hat{P} =: P^{(1)} \otimes P^{(2)} \) (Sweedler)

\[ [P_\mu, \hat{P}] = G[P^{(1)} A_\mu] \wedge P^{(2)}. \]

From \( PS = \hat{P}S = 0 \) it follows

\[ 0 = [P, \hat{P}] S = (G[P^{(1)} A] \wedge P^{(2)}) S. \]

Additional bi-local symmetry: gauge but not ordinary local.
Bi-local Gauge Transformations

Significance of bi-local gauge transformations $G[X] \wedge J$?

Action of ordinary local gauge transformations $G[X]$:

$$G[X](Z_1 \ldots Z_n) \sim [X, Z_1 \ldots Z_n].$$

Action of bi-local gauge transformations $G[X] \wedge J$:

$$(G[X] \wedge J)(Z_1 \ldots Z_n) \sim \{X, J(Z_1 \ldots Z_n)\}.$$

Invariance of action follows from:

$$G[X]S = 0, \quad JS = 0 \quad \text{and also} \quad JX = 0 \text{ (for cyclicity)}.$$

Requirements hold for all superconformal gauge theories:

- bi-local gauge transformations form an ideal of Yangian algebra;
- gauge ideal less restrictive than full Yangian.
Gauge Fixing

Fix gauge by Faddeev–Popov method:
- introduce ghost and auxiliary fields $C, \bar{C}, B$;
- extra terms $S_{gf}$ in action;

BRST symmetry $Q$ ($Q$ no longer supersymmetry)

\[ QZ \sim G[C], \quad QC \sim CC, \quad Q\bar{C} \sim B, \quad QB = 0; \quad QQ = 0. \]

Consider BRST cohomology:
- action closed $QS = 0$ (physical);
- gauge fixing terms exact $S_{gf} = Q\mathcal{K}_{gf}$ (irrelevant).

Extra terms needed for superconformal symmetry

\[ JS = Q\mathcal{K}[J], \quad \mathcal{K}[J] := JK_{gf}; \]

project out unphysical d.o.f. from invariance.
BRST and Yangian Symmetry

BRST is a residual gauge symmetry. 
→ additional bi-local BRST generators $Q \wedge J$ and $Q \otimes Q$.

Invariance of action requires further terms:

\[
(Q \otimes Q) S = Q \mathcal{K}[Q \otimes Q], \\
(Q \wedge J) S = Q \mathcal{K}[Q \wedge J] + (Q \otimes Q) \mathcal{K}[J] + J \mathcal{K}[Q \otimes Q], \\
\hat{J} S = Q \mathcal{K}[\hat{J}] + (Q \wedge J^{(1)}) \mathcal{K}[J^{(2)}] + J^{(1)} \mathcal{K}[Q \wedge J^{(2)}].
\]

Identities hold in gauge-fixed $\mathcal{N} = 4$ SYM (and ABJM).

- bi-local BRST generators needed for bi-local Yangian symmetry.
Slavnov–Taylor identities

Ward–Takahashi identities receive extra terms: Slavnov–Taylor identity

\[ \langle J \mathcal{O} + \mathcal{K}[J] Q \mathcal{O} \rangle = 0. \]

Holds by virtue of invariance of gauge-fixed action; variational identity.

Slavnov–Taylor identity for bi-local Yangian

\[ 0 = \langle \hat{J} \mathcal{O} \rangle + \langle \mathcal{K}[J^{(1)}] (Q \wedge J^{(2)}) \mathcal{O} \rangle + \langle \mathcal{K}[J^{(1)}] \mathcal{K}[J^{(2)}] (Q \otimes Q) \mathcal{O} \rangle \\
+ \langle (\mathcal{K}[J] + \mathcal{K}[Q \wedge J^{(1)}] \mathcal{K}[J^{(2)}] + \mathcal{K}[Q \otimes Q] \mathcal{K}[J^{(1)}] \mathcal{K}[J^{(2)}]) Q \mathcal{O} \rangle \\
+ \langle (\mathcal{K}[Q \wedge J^{(1)}] + \mathcal{K}[Q \otimes Q] \mathcal{K}[J^{(1)}]) J^{(2)} \mathcal{O} \rangle. \]

Note:
- analogous identities for bi-local BRST \( Q \otimes Q \) and \( Q \wedge J \).
- uses conjectural bi-local variational identity of planar path integral.
III. Correlation Functions
Correlators of Fields

Test Slavnov–Taylor identities for some correlators:

\[
\langle \text{tr } Z_1 Z_2 \rangle = 1,
\]

\[
\langle \text{tr } Z_1 Z_2 Z_3 \rangle = i,
\]

\[
\langle \text{tr } Z_1 Z_2 Z_3 Z_4 \rangle = -1 - 1 + i,
\]

\[
\langle \text{tr } Z_1 Z_2 Z_3 \rangle_{(1)} = -i - i - 1.
\]

- restrict to planar / colour-ordered contributions;
- off-shell: no complications due to mass shell condition;
Symmetries of Propagators

Conformal symmetry for propagators $\langle Z_1 Z_2 \rangle$

$$JZ \quad Z + Z \quad JZ \simeq 0.$$  

Invariance for matter fields

$$J^C \langle Z_1 Z_2 \rangle = \langle J^C Z_1 Z_2 \rangle + \langle Z_1 J^C Z_2 \rangle = 0;$$

invariance for gauge fields $\langle A_1 A_2 \rangle$

$$J^C \langle A_1 A_2 \rangle = \langle J^C A_1 A_2 \rangle + \langle A_1 J^C A_2 \rangle = d_1 H^C_1 + d_2 H^C_2.$$  

Yangian symmetry for propagator $\langle Z_1 Z_2 \rangle$

$$\hat{J}^C \langle Z_1 Z_2 \rangle = f^C_{AB} \langle J^A Z_1 J^B Z_2 \rangle = 0,$$

$$\hat{J}^C \langle A_1 A_2 \rangle = d_1 d_2 \hat{R}^C_{12}.$$  

Level-one generators almost annihilate gauge propagator $\langle A_1 A_2 \rangle$. 
Conformal Symmetry of 3-Point Function

Start simple: tree-level conformal invariance at 3 points

\[ J \langle \text{tr} \, Z_1 Z_2 Z_3 \rangle \]

\[ = i \begin{array}{c} 3 \\ 2 \end{array} + i \begin{array}{c} 3 \\ 1 \end{array} + i \begin{array}{c} 3 \\ 1 \end{array} + i \begin{array}{c} 3 \\ 1 \end{array} + i \begin{array}{c} 3 \\ 1 \end{array} + i \begin{array}{c} 3 \\ 2 \end{array} \]

\[ = -i \begin{array}{c} 3 \\ 2 \end{array} - i \begin{array}{c} 3 \\ 1 \end{array} - i \begin{array}{c} 3 \\ 1 \end{array} - i \begin{array}{c} 3 \\ 1 \end{array} - i \begin{array}{c} 3 \\ 1 \end{array} - i \begin{array}{c} 3 \\ 2 \end{array} \]

\[ = -i \begin{array}{c} 3 \\ 1 \end{array} = 0. \]

Invariance of action implies invariance of correlator.
Also confirmed invariance for properly gauge-fixed correlator.
**Yangian Symmetry of 3-Point Function**

Yangian action on correlator of 3 fields at tree level

\[ \hat{J} \langle \text{tr} Z_1 Z_2 Z_3 \rangle \simeq 3 + i \quad + \quad + \quad + \]

\[ \simeq -3i \quad + \quad i \quad + \quad i \quad - i \]

\[ \simeq -i = 0. \]

Invariance based on:
- conformal invariance of propagator and 3-vertex,
- Yangian invariance of 3-vertex.

Also showed \( Q \wedge J \) invariance of gauge-fixed correlator.
Yangian Symmetry of 4-Point Function

Yangian action on tree-level correlator of 4 fields $\hat{J}\langle \text{tr } Z_1 Z_2 Z_3 Z_4 \rangle$

\[ \simeq -2 \quad -2 \quad +2i \]
\[ +2i \quad -2i \quad -2i \quad +2i \]
\[ +2 \quad +4i \quad +4i \quad \simeq \ldots = 0. \]

- conformal invariance of propagator, 3-vertex and 4-vertex,
- Yangian invariance of 3-vertex and 4-vertex,
- commutativity of constituents $[J^{(1)}, J^{(2)}] = 0$. 

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3-Function at One Loop

Yangian action on one-loop correlator of 3 fields $\hat{J}\langle \text{tr } Z_1 Z_2 Z_3 \rangle_{(1)}$

\[ \simeq -i + 3 - i + i + i + i + i \]

\[ \simeq -i - i + i - 3 - i + i + 3i + i + i - i \]

\[ \simeq -3 - i - i + i + i + i - i \]

\[ \simeq -3 + i + i - i - i - i \]

\[ \simeq \ldots = 0. \]

Invariance shown modulo gauge fixing and divergences.
Anomalies?

Classical symmetries may suffer from quantum anomalies:
• No established framework for anomalies of non-local symmetries (in colour-space not necessarily in spacetime).
• Violation of (non-local) current? Cohomological origin?

Potential anomaly terms:
• quantum analysis similar to classical one?
• consider gauge fixing . . .
• consider regularisation . . .

However:
• Not an issue for Wilson loop expectation value at one loop.
• Integrability “works” at finite coupling: no anomaly expected?
IV. Conclusions
Conclusions

Yangian Symmetry of Planar $\mathcal{N} = 4$ SYM:
- classical action of planar $\mathcal{N} = 4$ SYM Yangian invariant
- model classically integrable (same for ABJM)

Yangian Algebra and Gauge Transformations
- Yangian algebra produces non-local gauge transformations
- gauge transformations form an ideal
- after gauge fixing: bi-local and mixed BRST transformations
- additional terms to eliminate unphysical d.o.f.
- Yangian compatible with gauge fixing

Correlation Functions
- Ward–Takahashi/Slavnov–Taylor identities tested
- No quantum anomalies to be expected?!

Outlook: Apply to scattering amplitudes (LSZ), Wilson loops, ... Derive algebraic integrability methods?!