Integrability in Gauge and String Theory 2018

Rationalizing Loop Integration

Jacob Bourjaily

Niels Bohr International Academy

based on work in collaboration with

Dixon, Dulat, Panzer; He, ***McLeod,*** *Spradlin,* ***von Hippel, Wilhelm***

The Niels Bohr International Academy

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Better ***Integrands*** *for Amplitudes*

- Loop ***integrands*** are becoming “easy” to construct —(exposing and) preserving much simplicity

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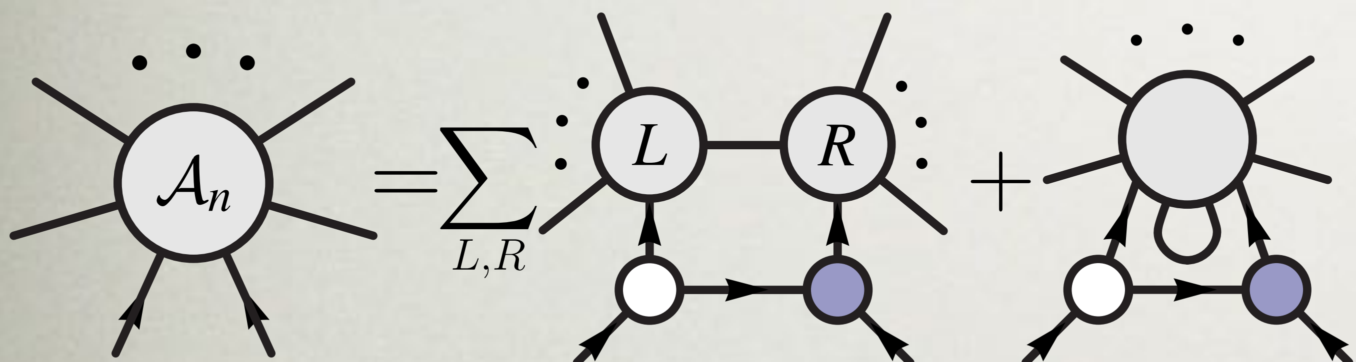
Recursion Relations



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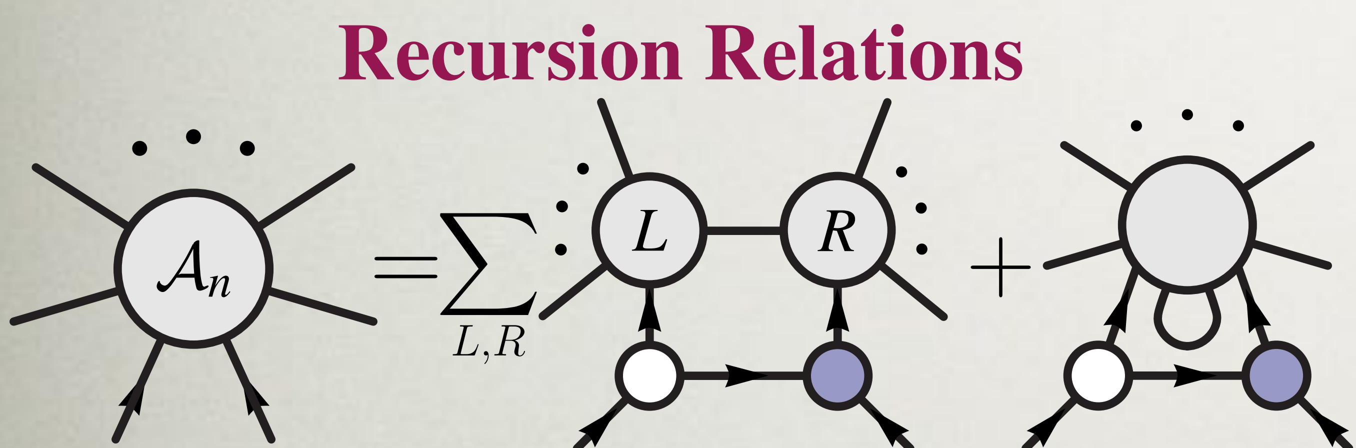

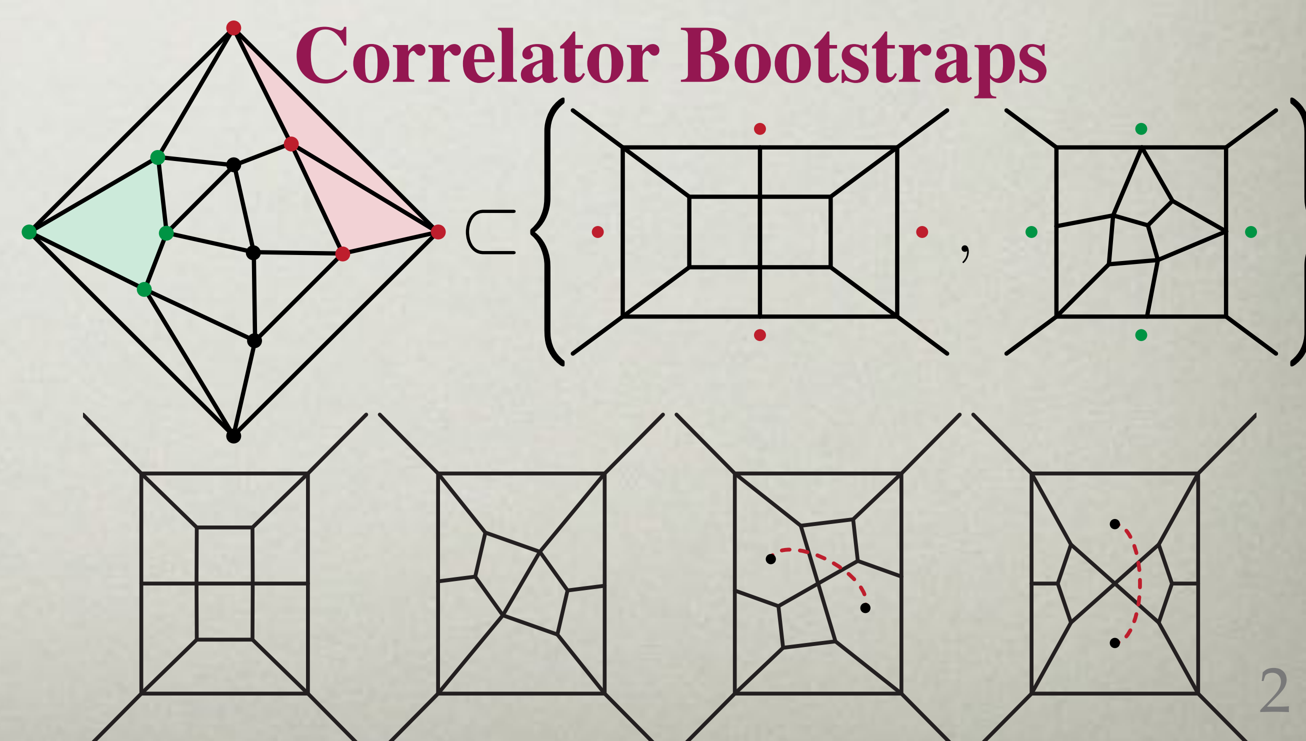

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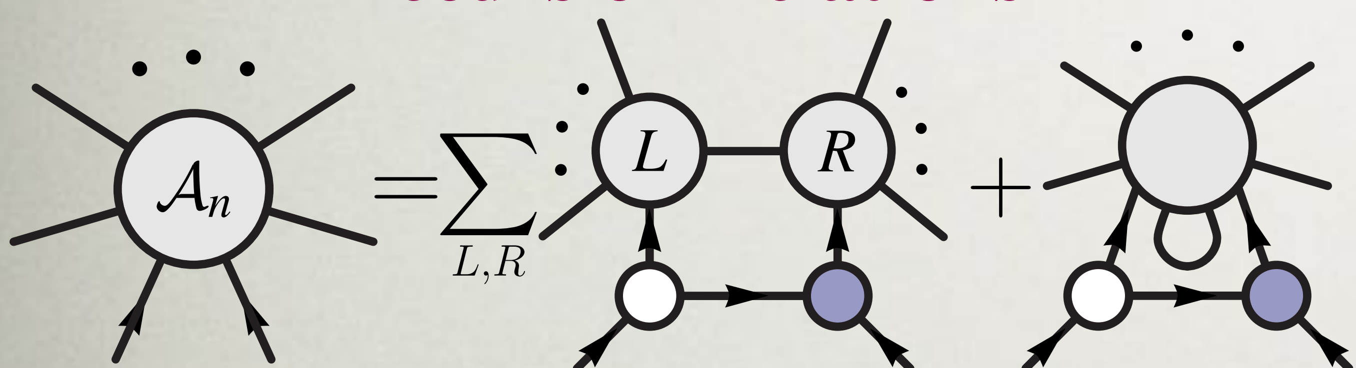

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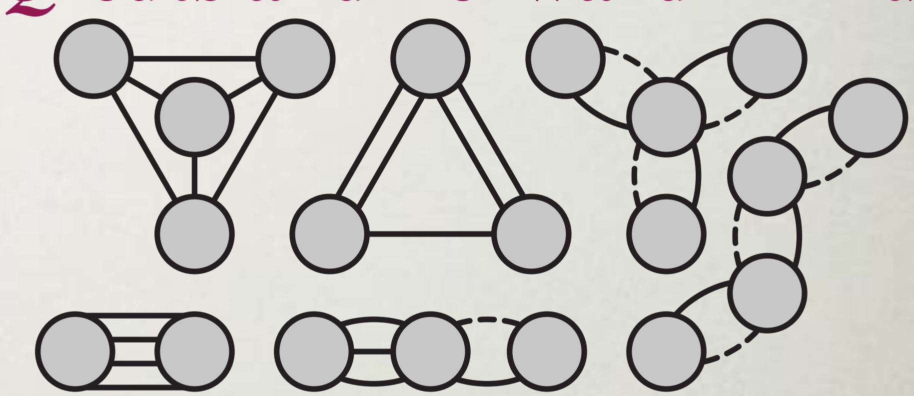

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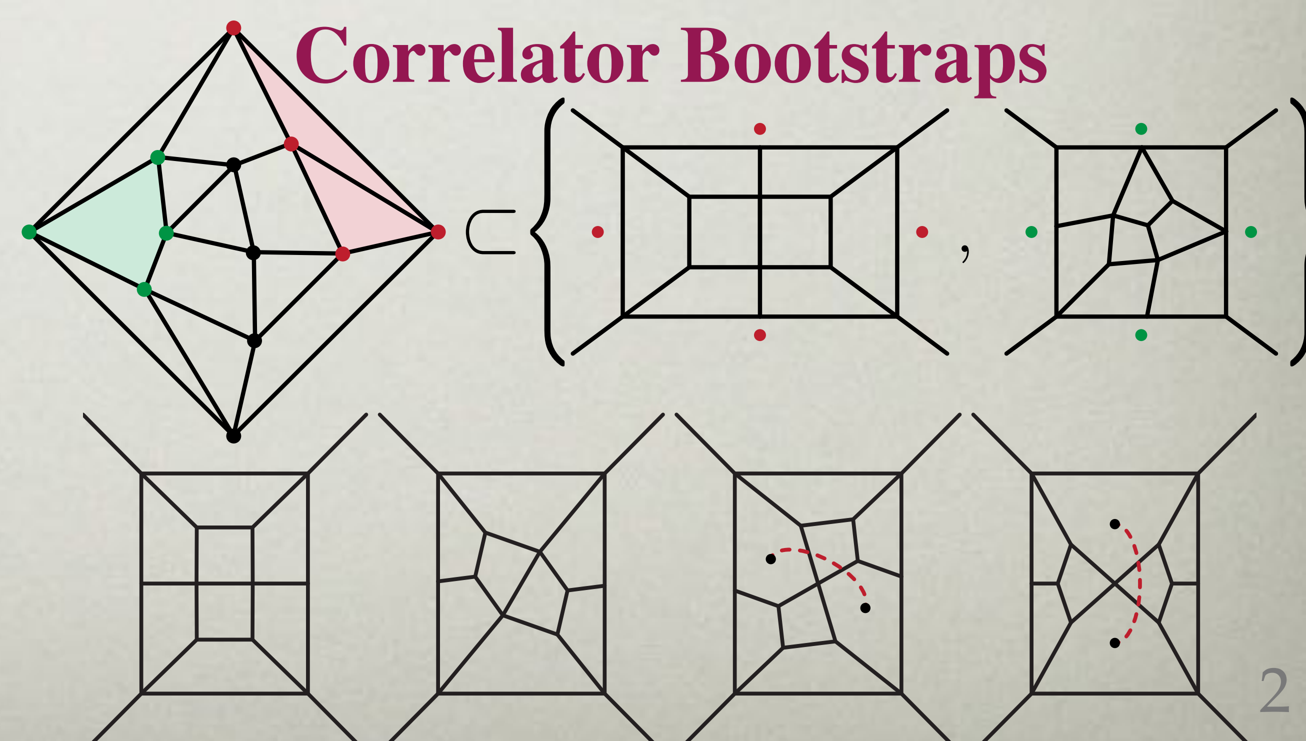

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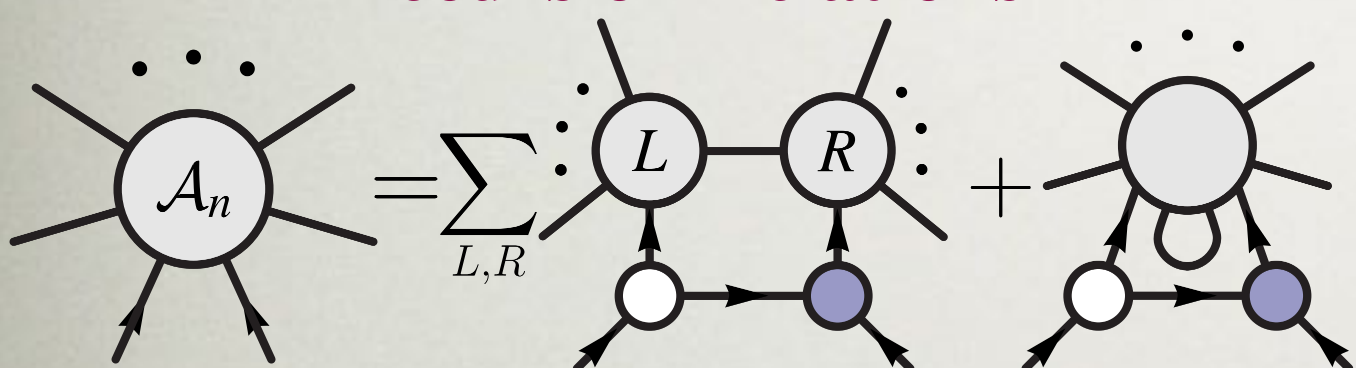

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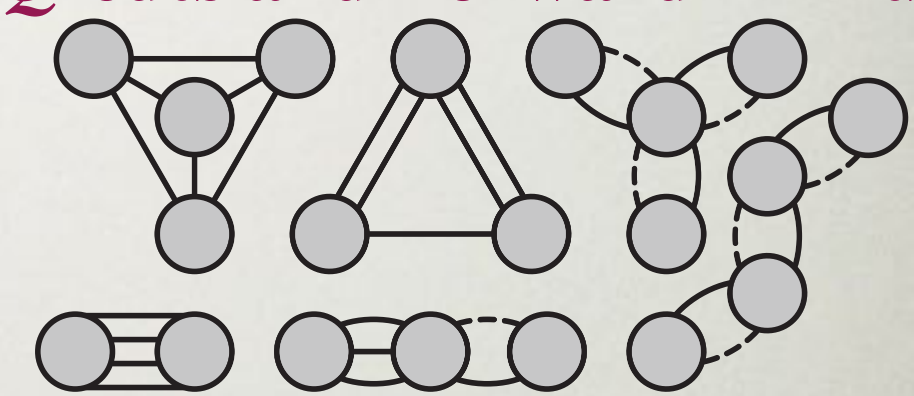

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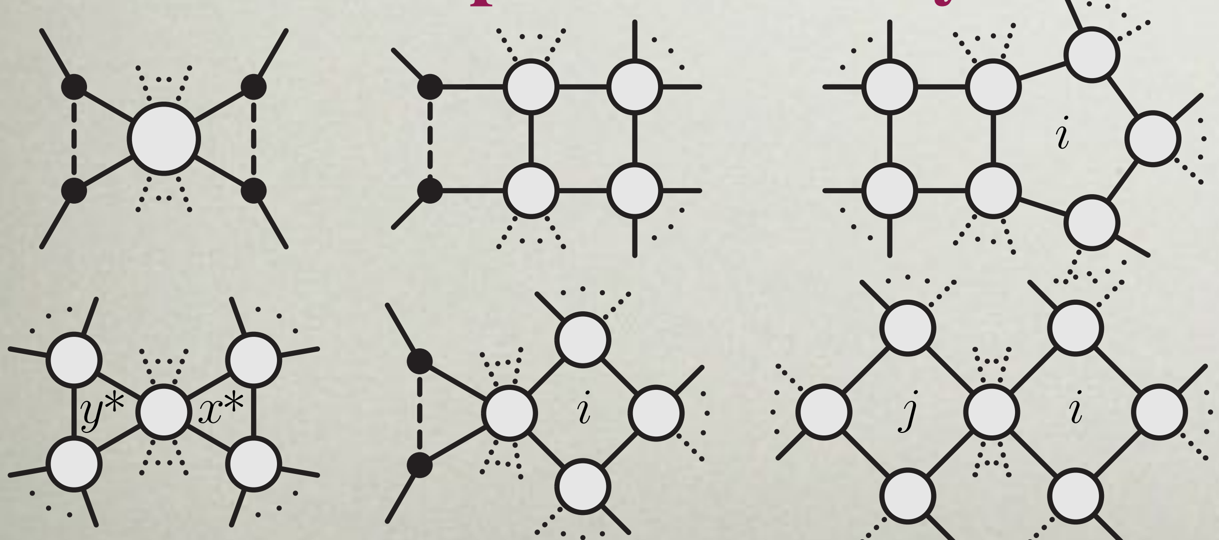

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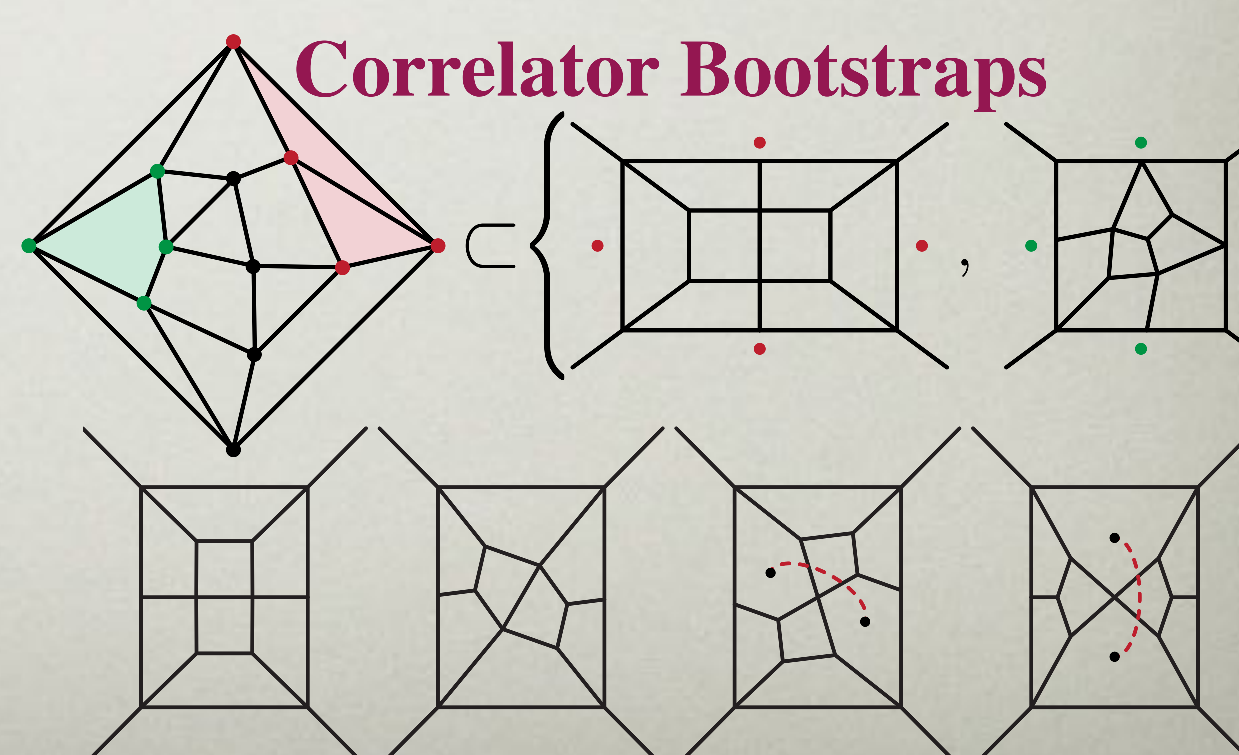

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- *Exempli gratia*: we now have closed formulae for all amplitude integrands in planar SYM through 3 loops:
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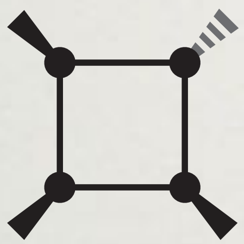

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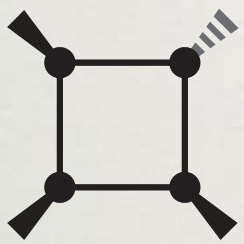

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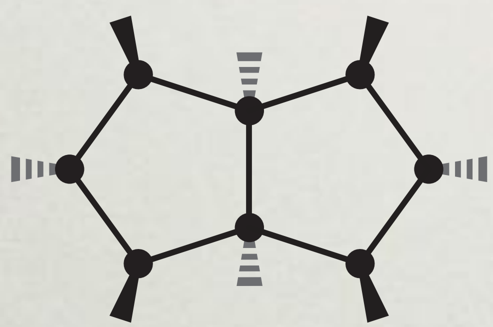

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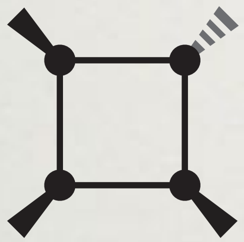
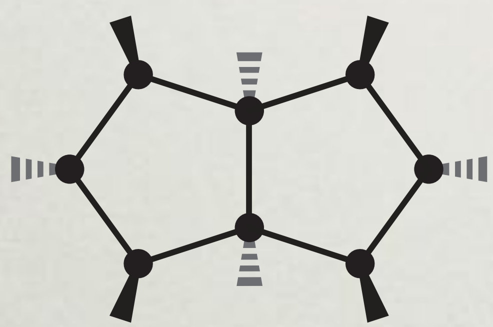
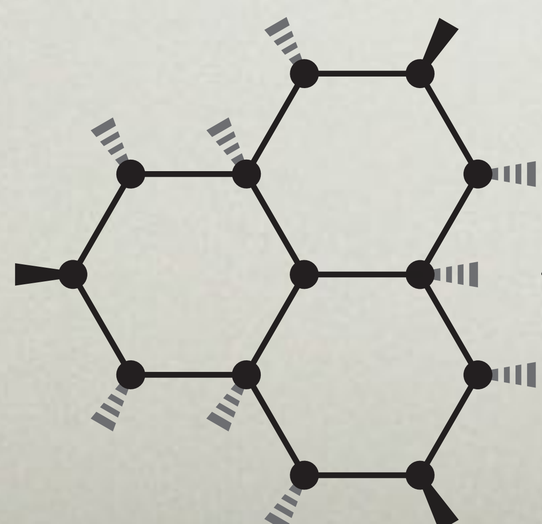
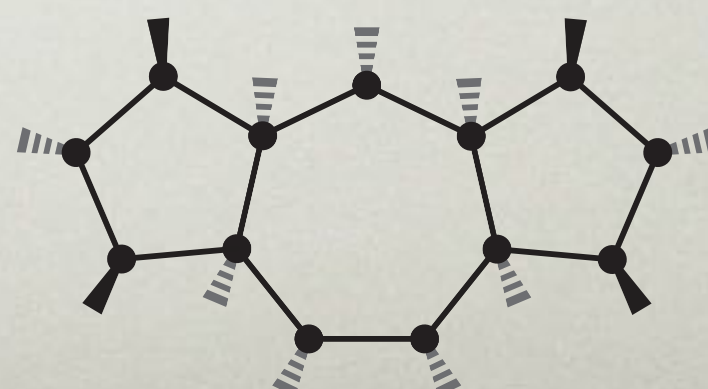
$$\mathcal{A}_n^{L=1} = \sum_{\mathcal{L}} f_{\mathcal{L}}$$ 

$$\mathcal{A}_n^{L=2} = \sum_{\mathcal{L}} f_{\mathcal{L}}$$ 

$$\mathcal{A}_n^{L=3} = \sum_{\mathcal{W}} f_{\mathcal{W}}$$  $$+ \sum_{\mathcal{L}} f_{\mathcal{L}}$$ 

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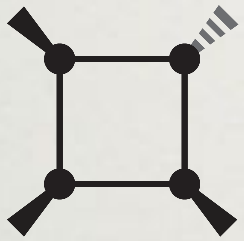
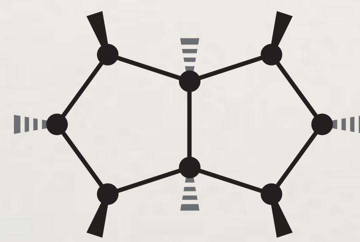
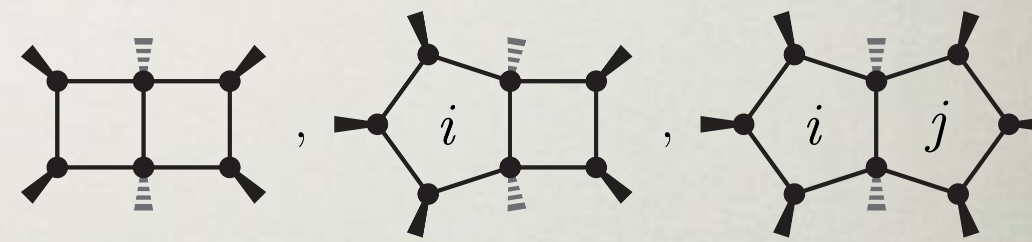
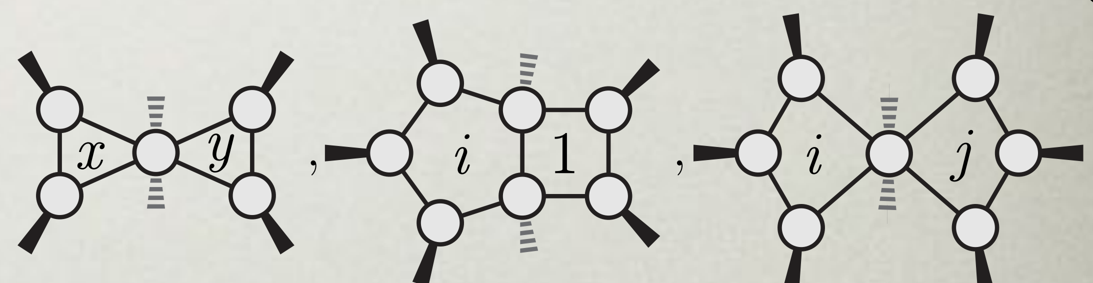
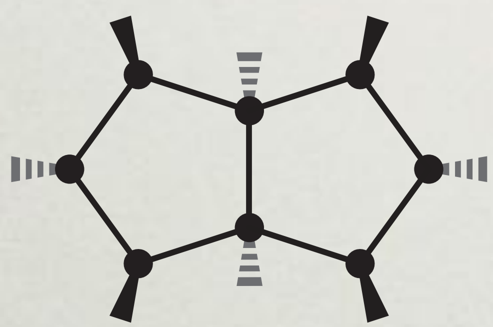
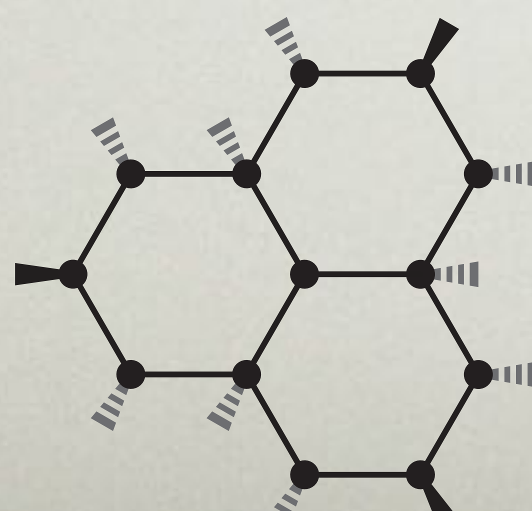
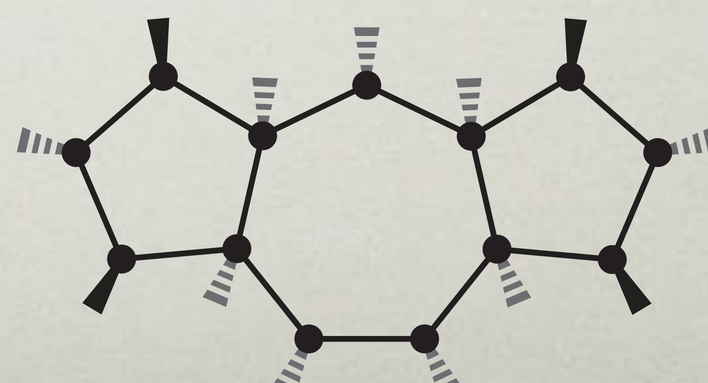
$$\mathcal{A}_n^{L=1} = \sum_{\mathcal{L}} f_{\mathcal{L}}$$ 

 $\in \{$  $\}$

$f_{\mathcal{L}} \in \{$  $\}$

$$\mathcal{A}_n^{L=2} = \sum_{\mathcal{L}} f_{\mathcal{L}}$$ 

$$\mathcal{A}_n^{L=3} = \sum_{\mathcal{W}} f_{\mathcal{W}}$$  $$+ \sum_{\mathcal{L}} f_{\mathcal{L}}$$ 

Roadmap: Polylogs to Traintracks

- ***Spiritus Movens*(/*Loop Integration Polemics*)**
  *When has an **integrand** been **integrated**?*

- **Integrating Loop Integrals *Rationally***
  - ***dual-conformal** sufficiency* [**JB**, Dixon, Dulat, Panzer (*to appear*)]
  - ***momentum twistor** reducibility*
    [**JB**, McLeod, von Hippel, Wilhelm (2018)]

- **Ubiquity of Non-Polylogarithmicity**
  - integrals beyond (even elliptic) polylogarithms
    [**JB**, McLeod, Spradlin, von Hippel, Wilhelm (2017)]
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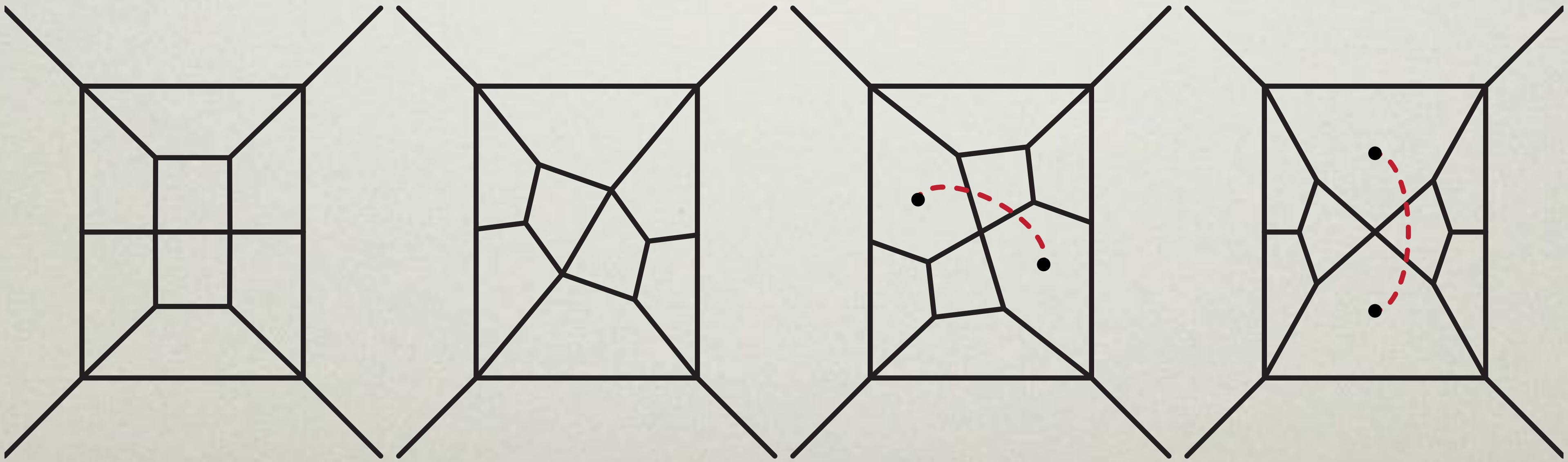

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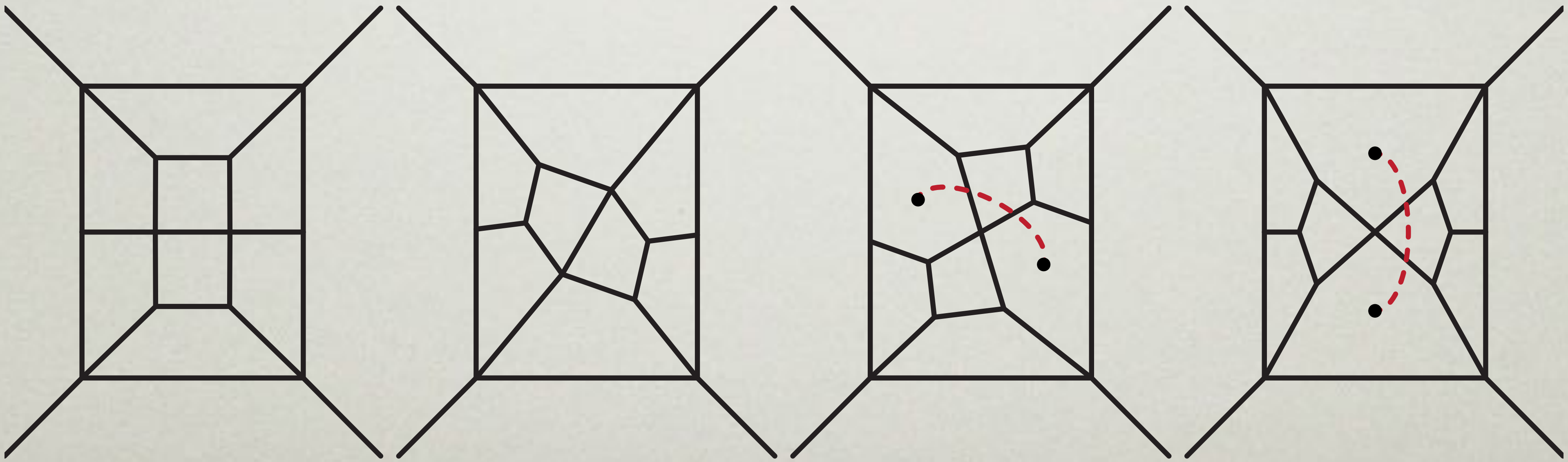

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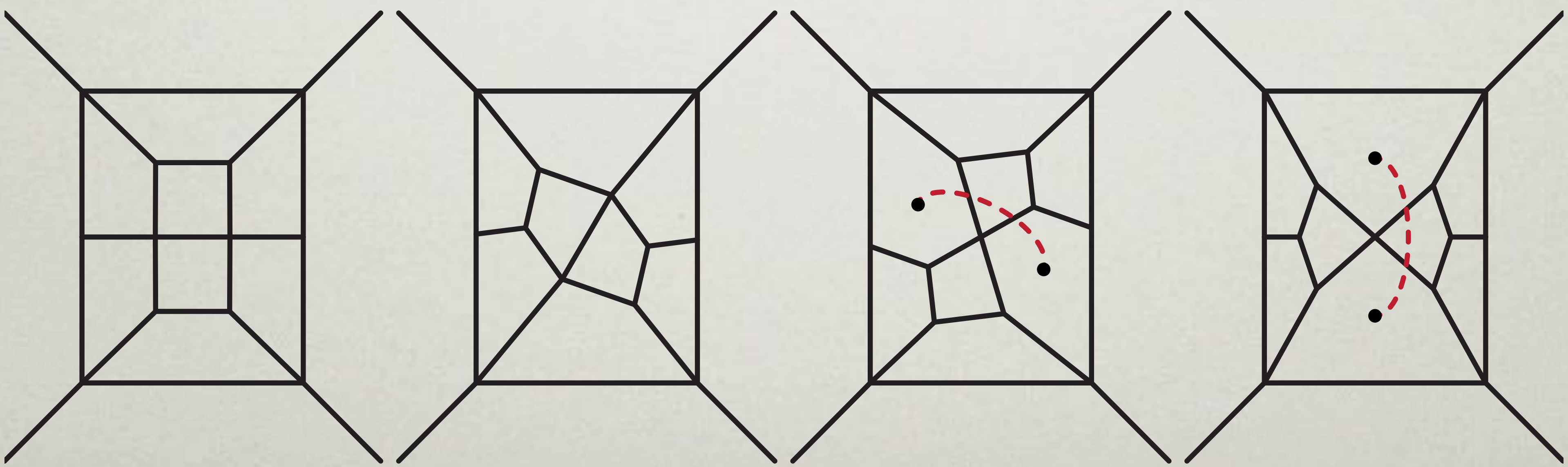

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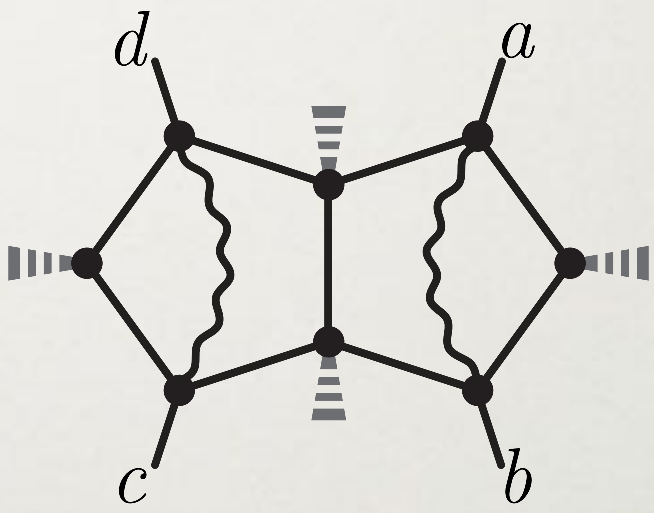

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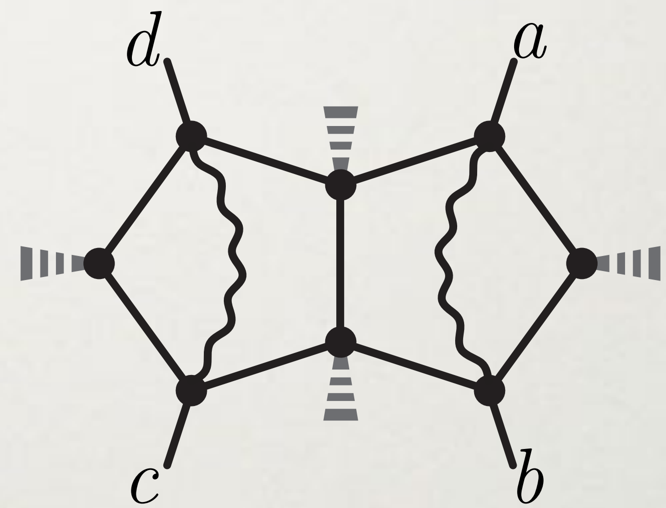

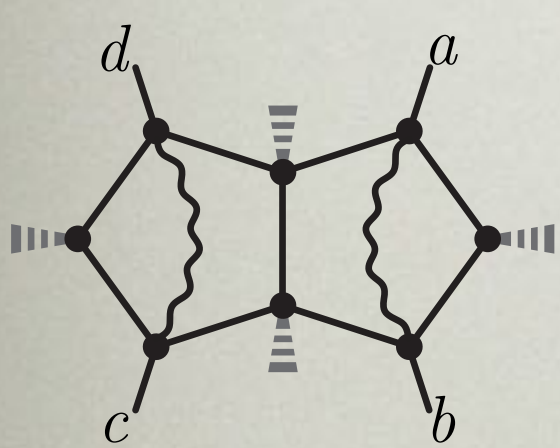

$$\equiv \frac{(\ell_1,N_1)(\ell_2,N_2)}{(\ell_1,a)(\ell_1,a{+}1)(\ell_1,b)(\ell_1,b{+}1)(\ell_1,\ell_2)(\ell_2,c)(\ell_2,c{+}1)(\ell_2,d)(\ell_2,d{+}1)}$$

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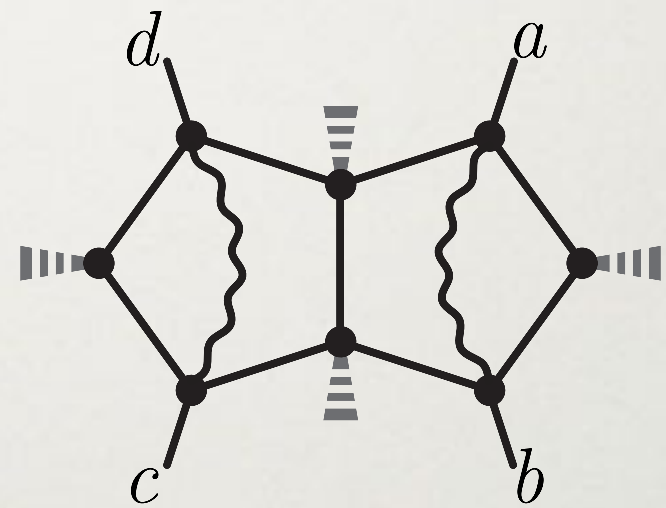

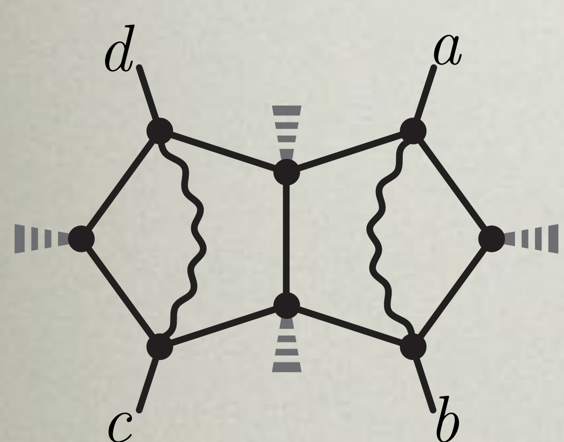

$$\equiv \int \frac{d^4\ell_1 d^4\ell_2 \qquad (\ell_1,N_1)(\ell_2,N_2)}{(\ell_1,a)(\ell_1,a{+}1)(\ell_1,b)(\ell_1,b{+}1)(\ell_1,\ell_2)(\ell_2,c)(\ell_2,c{+}1)(\ell_2,d)(\ell_2,d{+}1)}$$

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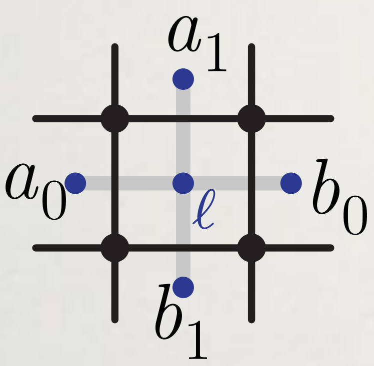

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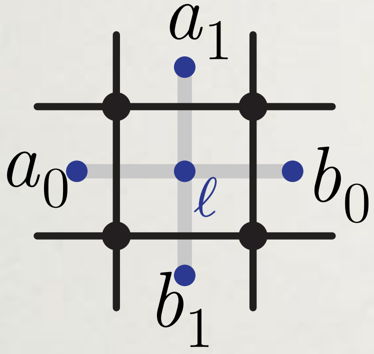

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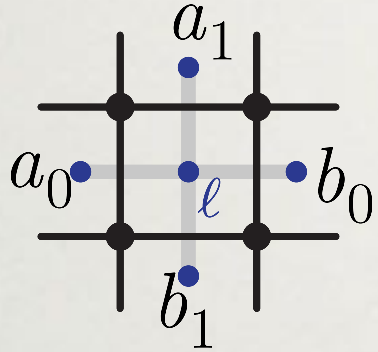

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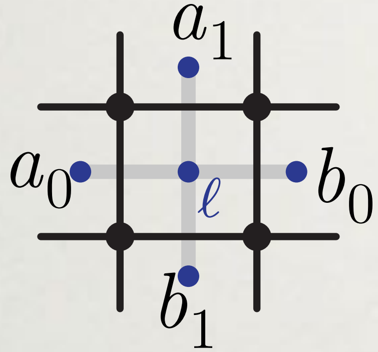

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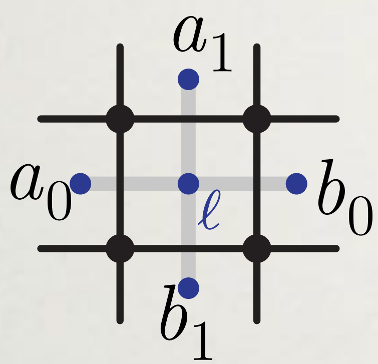

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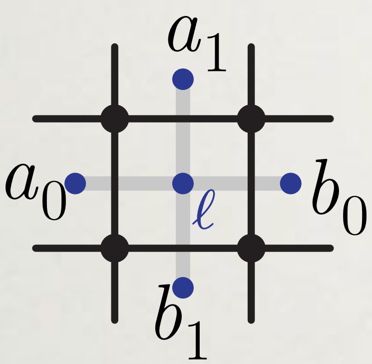

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[Symanzik (1972)]

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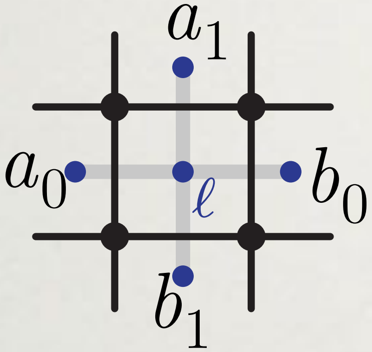

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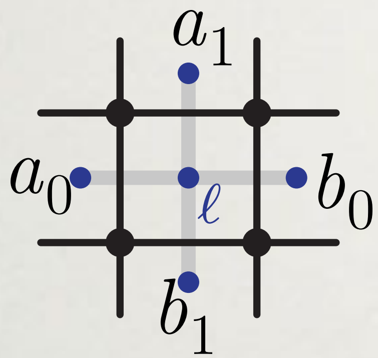

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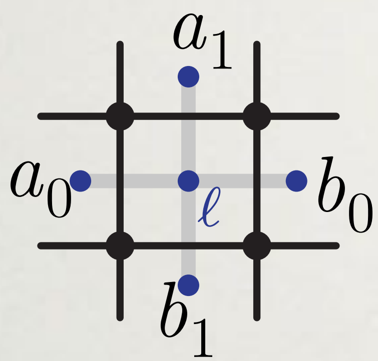

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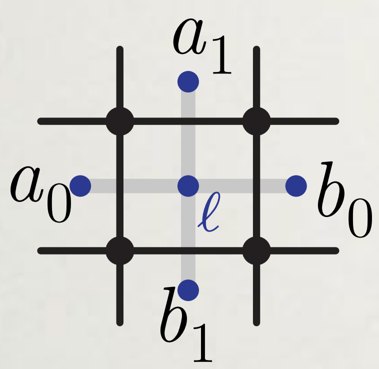

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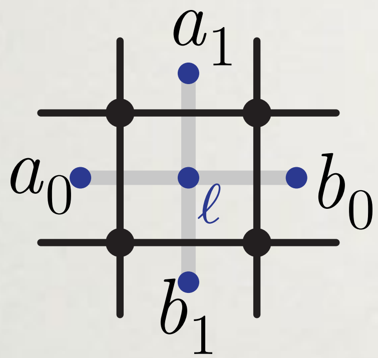

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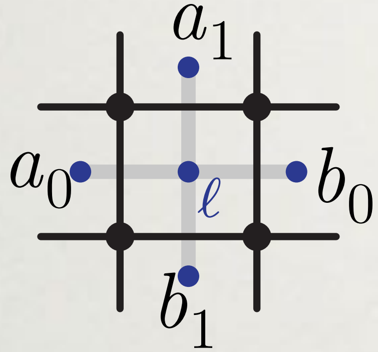

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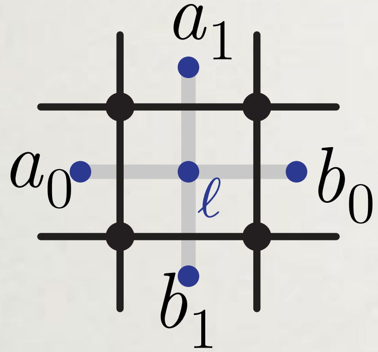

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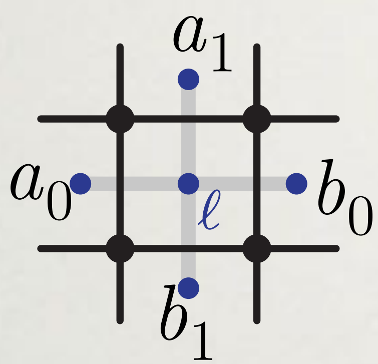

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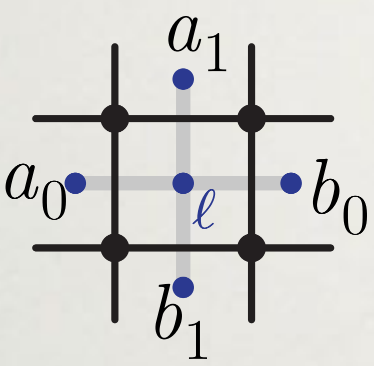

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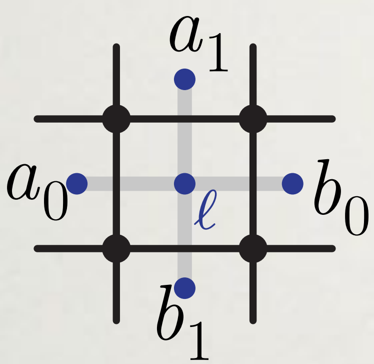

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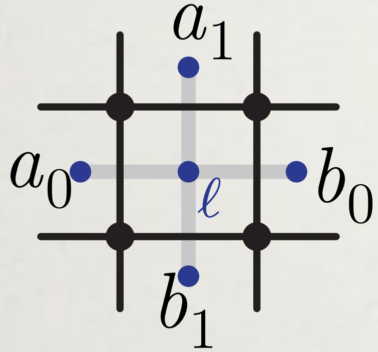

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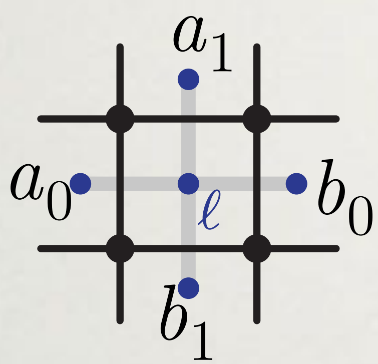

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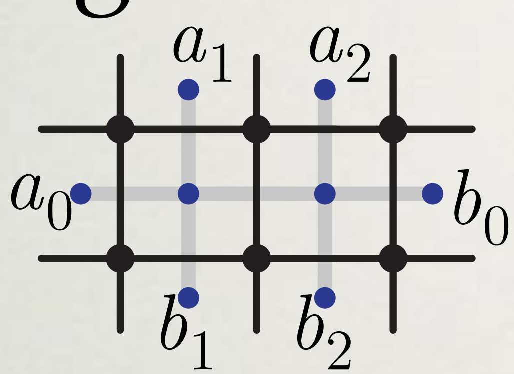
[**JB**, McLeod, Spradlin, von Hippel, Wilhelm (2017)]



$$\Rightarrow \int d^{8}\vec{\ell}\,\frac{(a_0,b_0)(a_1,b_1)(a_2,b_2)}{(\ell_1,a_0)(\ell_1,a_1)(\ell_1,b_1)(\ell_1,\ell_2)(\ell_2,a_2)(\ell_2,b_2)(\ell_2,b_0)}$$

$$= \int_0^\infty d^{6}\vec{\alpha}\;\frac{\mathscr{U}}{\mathscr{F}^3} \qquad = \int_{-i\infty}^{i\infty} \bar{d}^{7}\vec{z}\,\Big[\Gamma(-z_1)^2\cdots\Big]\Big[u_1^{z_1}\cdots u_7^{z_7}\Big]$$

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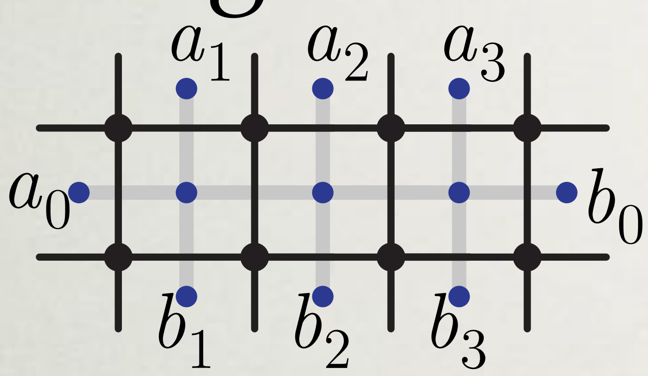

$$\Rightarrow \int d^{12}\vec{\ell}\,\frac{(a_0,b_0)(a_1,b_1)(a_2,b_2)(a_3,b_3)}{(\ell_1,a_0)(\ell_1,a_1)(\ell_1,b_1)(\ell_1,\ell_2)\cdots(\ell_3,b_0)}$$

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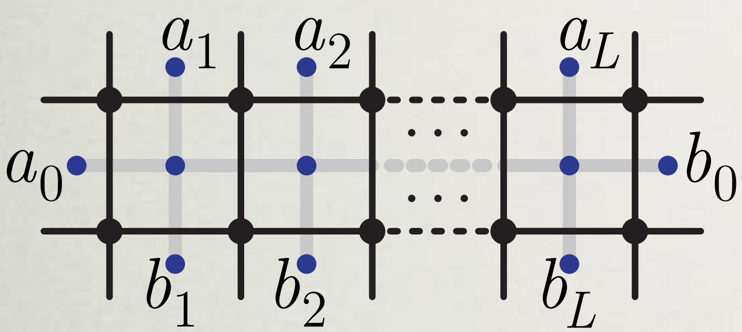

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$$= \int_0^\infty d^{3L}\vec{\alpha}\,\frac{\mathscr{U}^{L-1}}{\mathscr{F}^{L+1}} = \int_{-i\infty}^{i\infty} \bar{d}^{(2L^2-L+1)}\vec{z}\,\Big[\Gamma(-z_1)^2\cdots\Big]\prod_{i=1}^{2L^2-L+1} u_i^{z_i}$$

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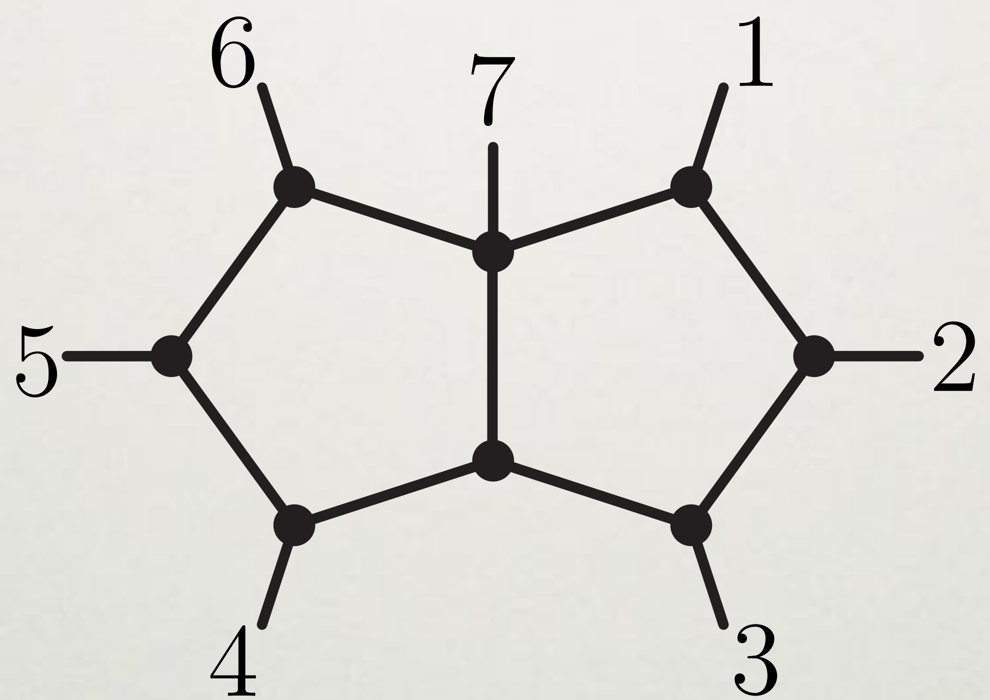

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- *Partial fraction to death* (e.g. use **HyperInt**) [Panzer (2014)]

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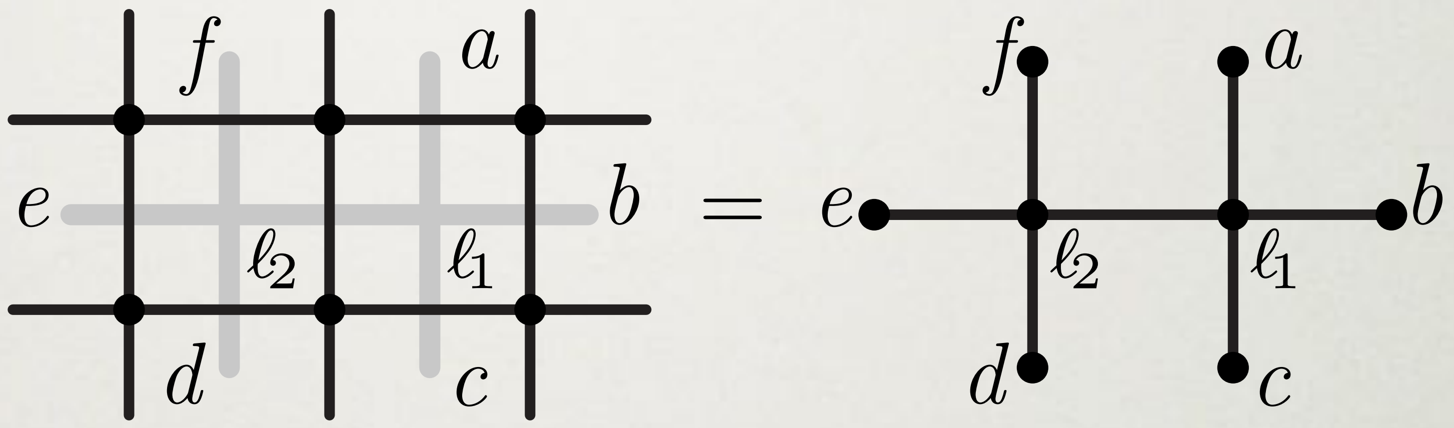

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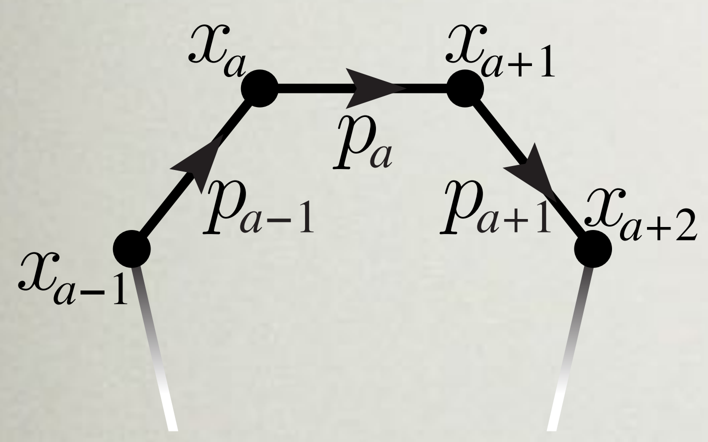

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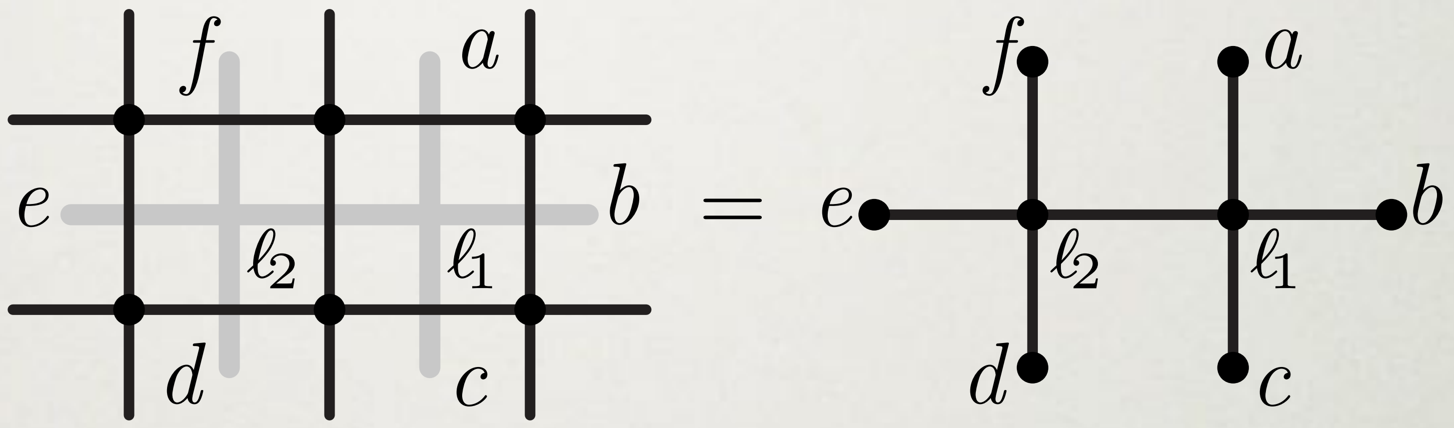

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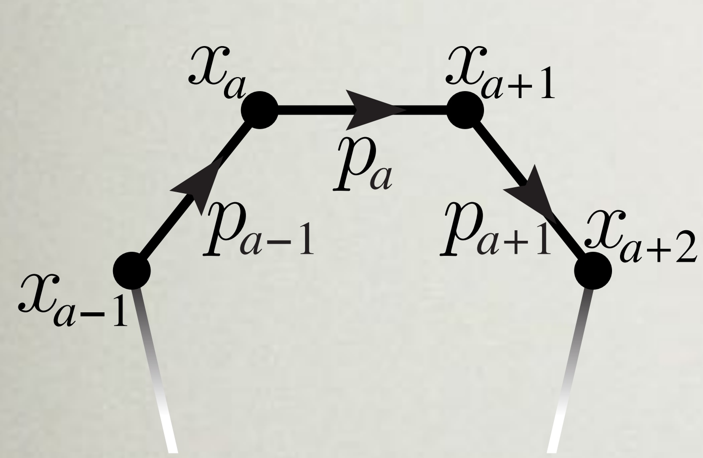

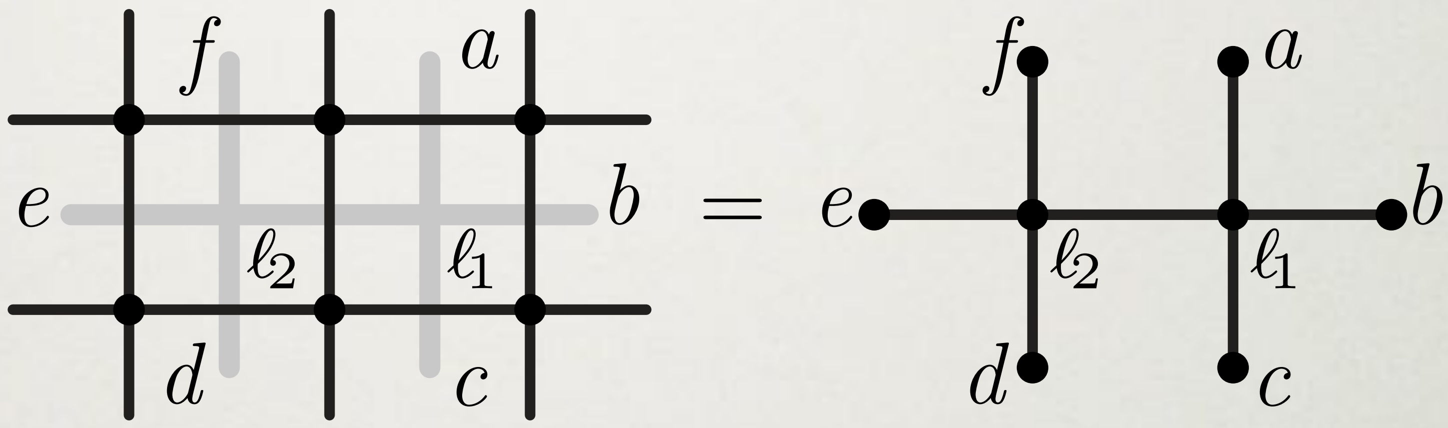

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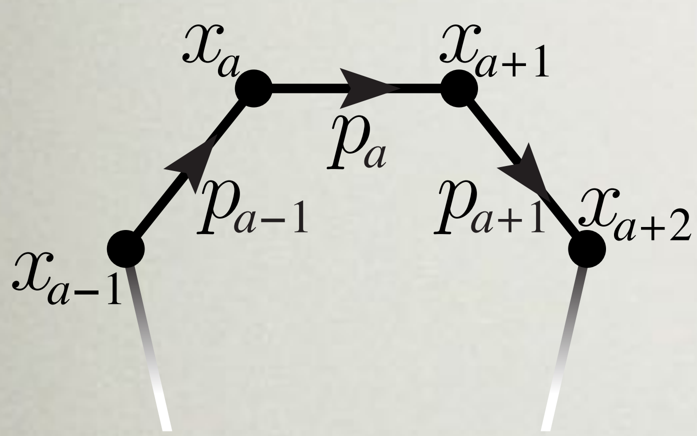

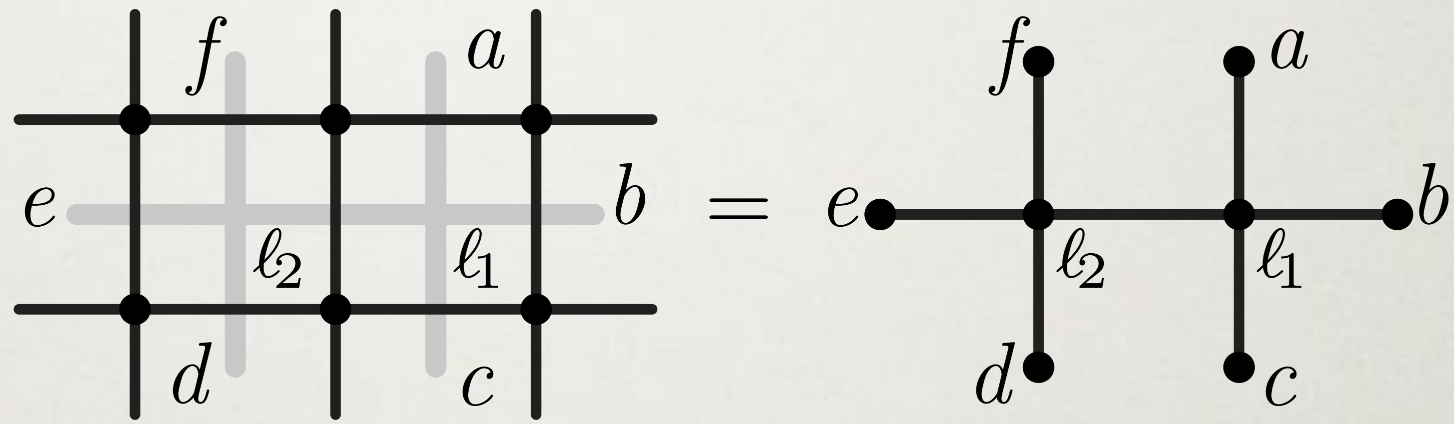

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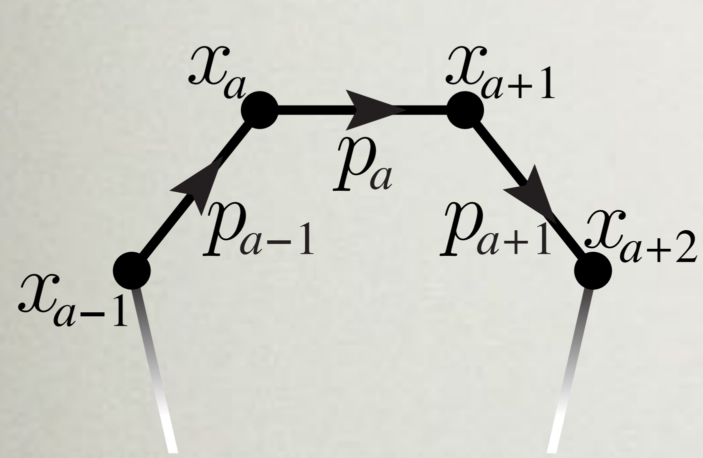

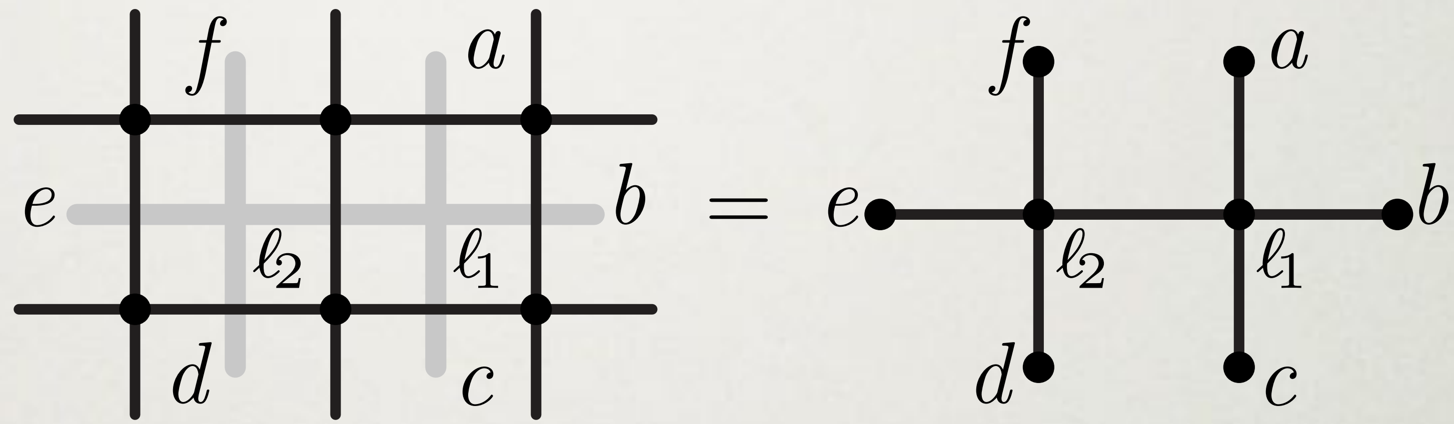

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- **Dual-Conformal Invariance** is conformality in x's

[Drummond, Henn, Smirnov, Sokatchev;
Drummond, Korchemsky, Henn; …]

Planarity & Dual-Conformality

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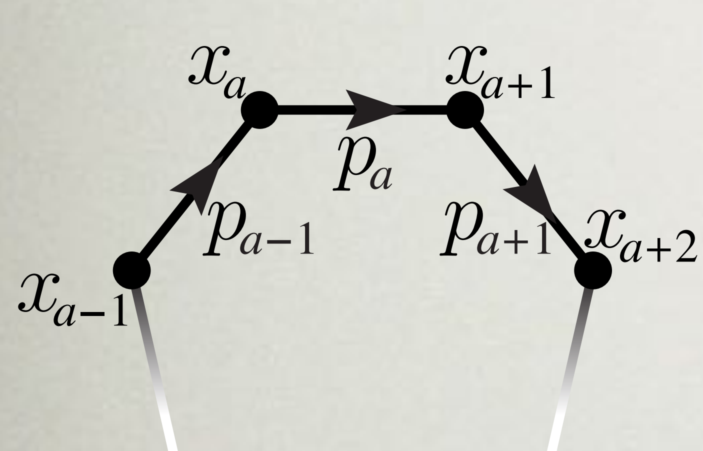

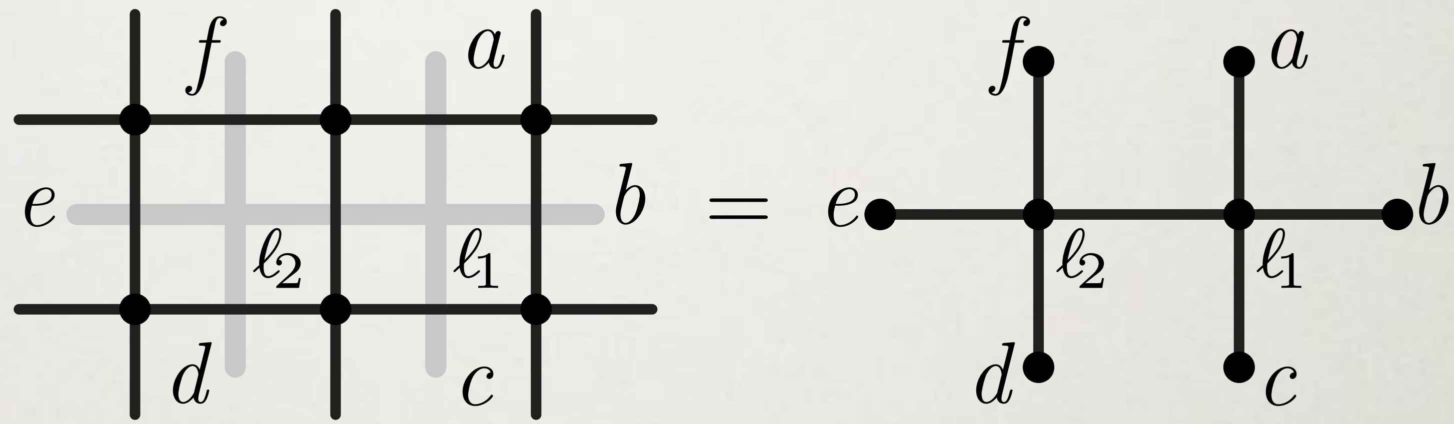

$$p_a \equiv (x_{a+1} - x_a) \qquad \equiv \int \frac{d^4\ell_1 d^4\ell_2 \qquad (a,c)(b,e)(d,f)}{(\ell_1,a)(\ell_1,b)(\ell_1,c)(\ell_1,\ell_2)(\ell_2,d)(\ell_2,e)(\ell_2,f)}$$

$$(a,b) = (b,a) \equiv (x_b - x_a)^2 = (p_a + \ldots + p_{b-1})^2 \equiv s_{a\cdots b-1} \quad \text{and} \quad (\ell,a) \equiv (x_\ell - x_a)^2$$

- **Dual-Conformal Invariance** is conformality in x's

$$(ab;cd) \equiv \frac{(a,b)(c,d)}{(a,c)(b,d)}$$

[Drummond, Henn, Smirnov, Sokatchev; Drummond, Korchemsky, Henn; …]

The Dual-Conformal Regulator

- The basic idea of the dual-conformal regulator is to give legs masses, but controlled by a parameter 'δ' that is ***dimensionless*** & ***has no conformal weight***

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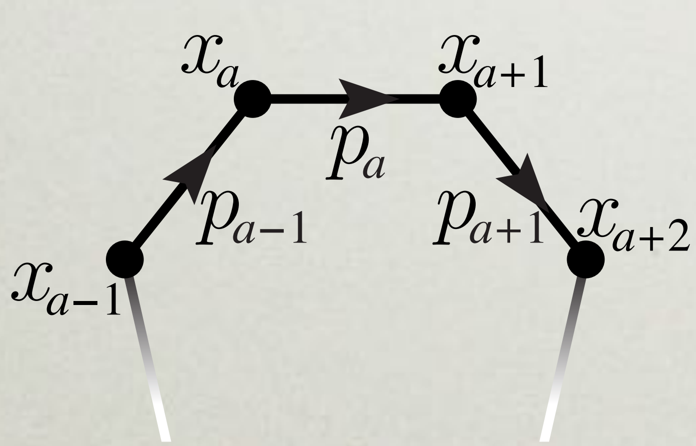

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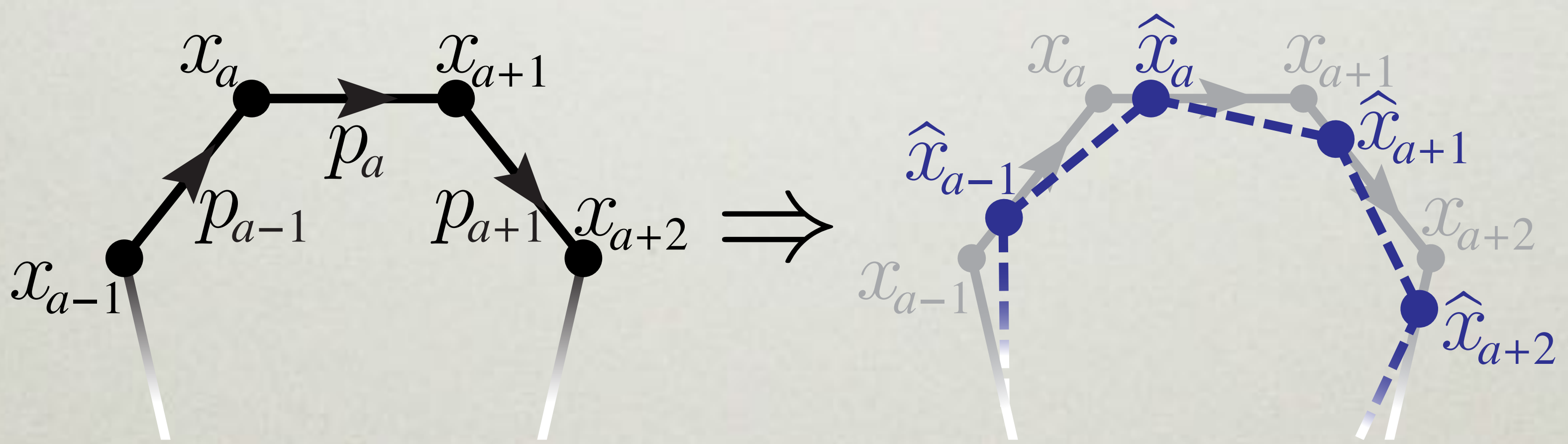

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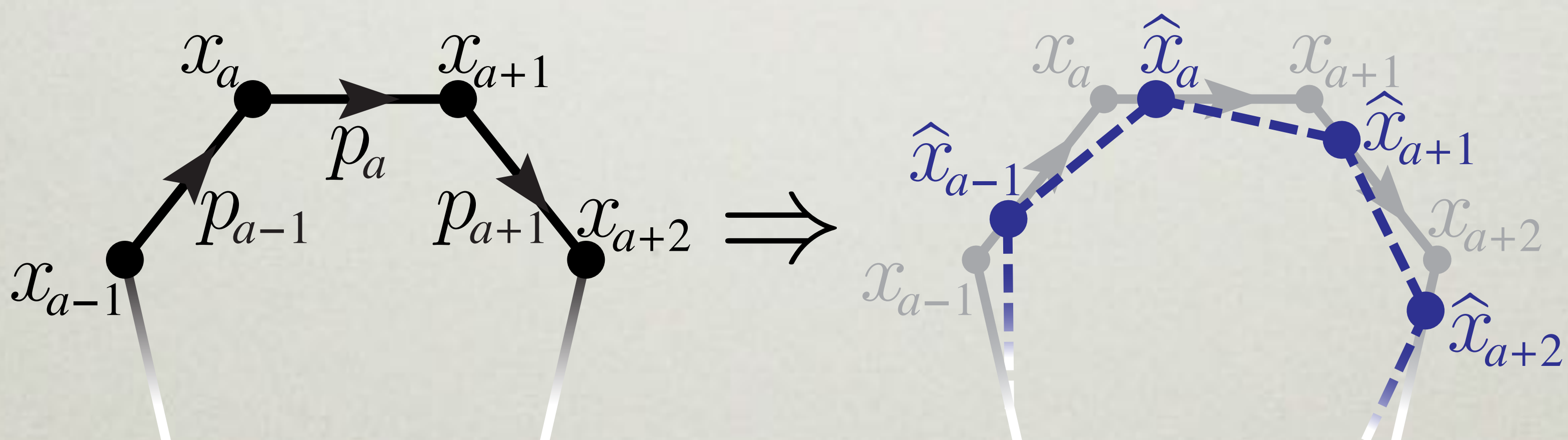

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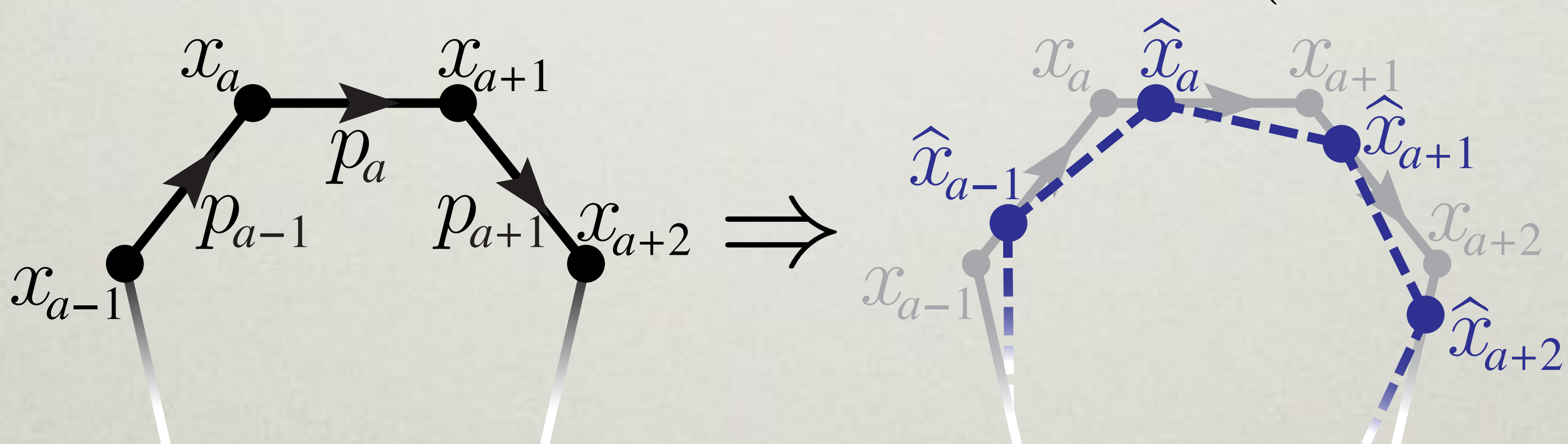

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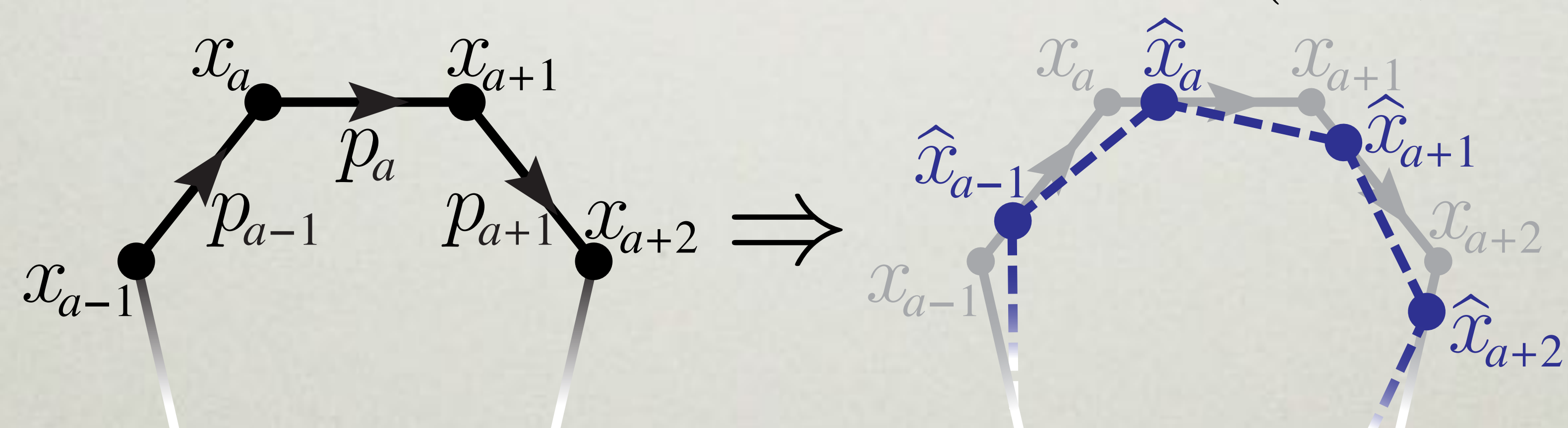

$$I = \int_{\mathbb{R}^{3,1}} \prod_{i=1}^{L} d^4\ell_i\, \mathcal{I} \;\mapsto\; I^\delta \equiv \int_{\mathbb{R}^{3,1}} \prod_{i=1}^{L}\left[d^4\ell_i\left(\prod_a \frac{(\ell_i,a)}{(\ell_i,\widehat{a})}\right)\right]\mathcal{I}$$

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- Using the dual-conformal regularization scheme,

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- Coefficients of each divergence can be obtained as *strictly finite* (Feynman-) parametric integrals—which can always be rendered *manifestly* DCI

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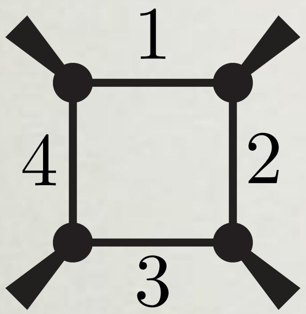

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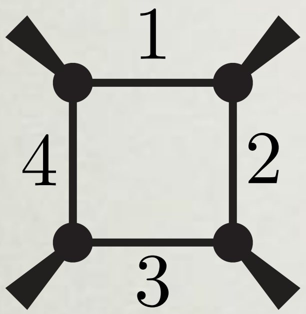

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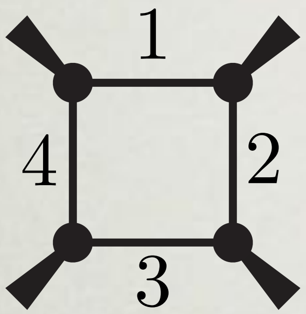

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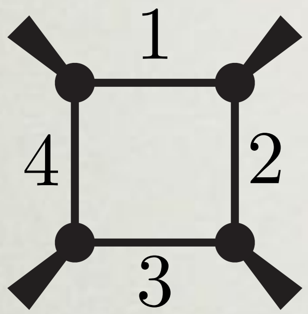

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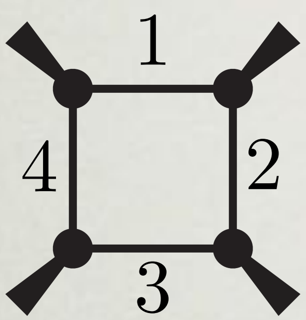

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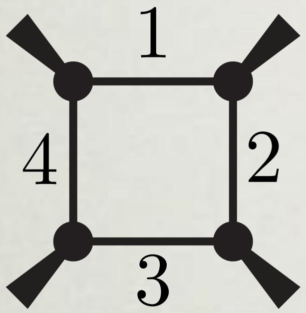

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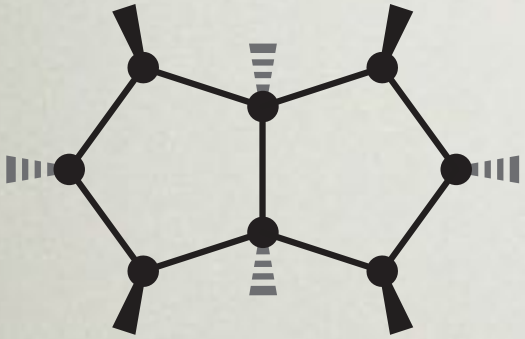

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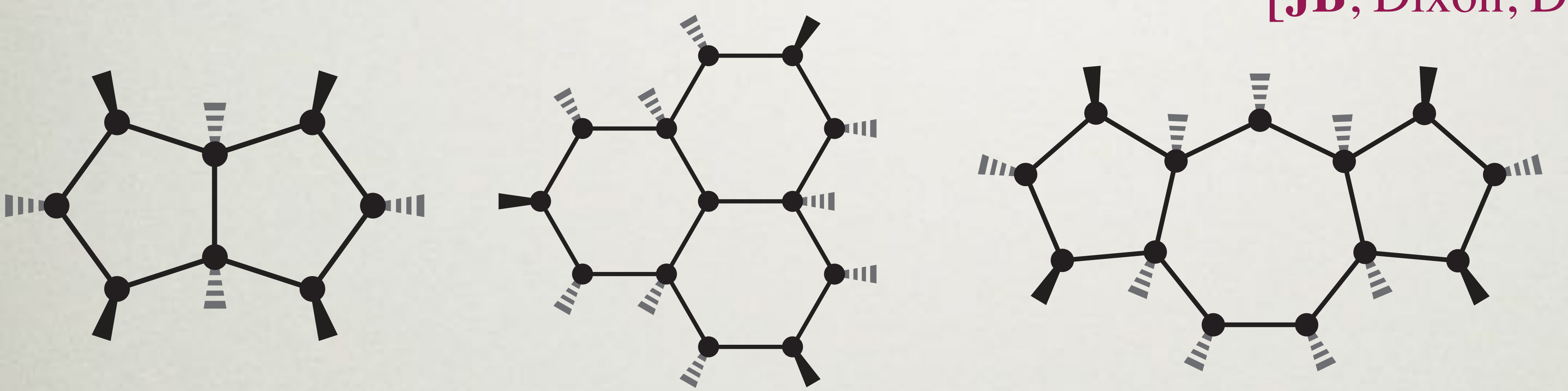

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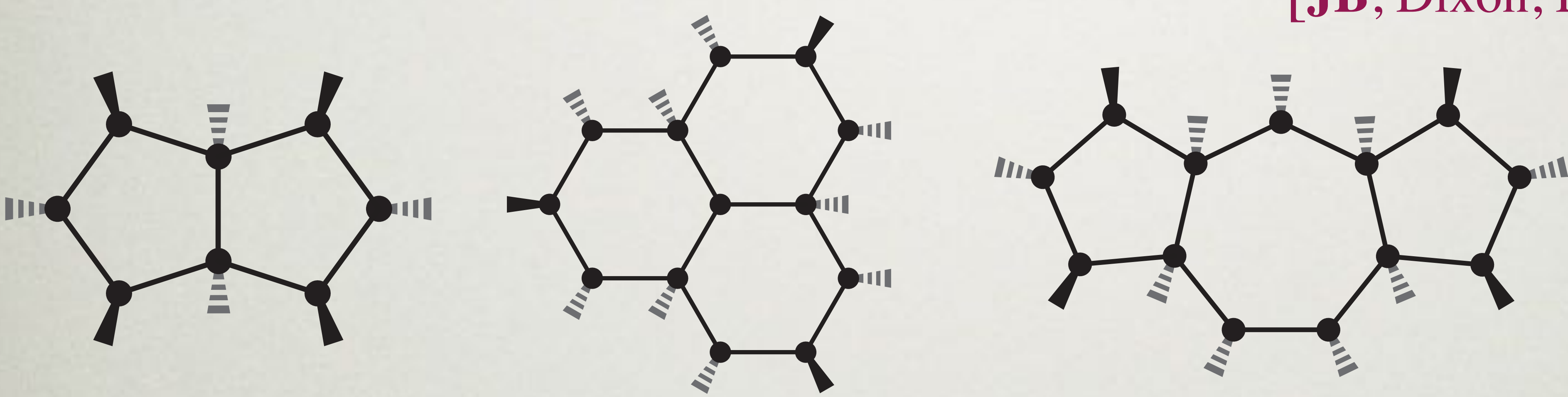

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where u's are parity-even cross-ratios: $(ab;cd) \equiv \dfrac{(a,b)(c,d)}{(a,c)(b,d)}$

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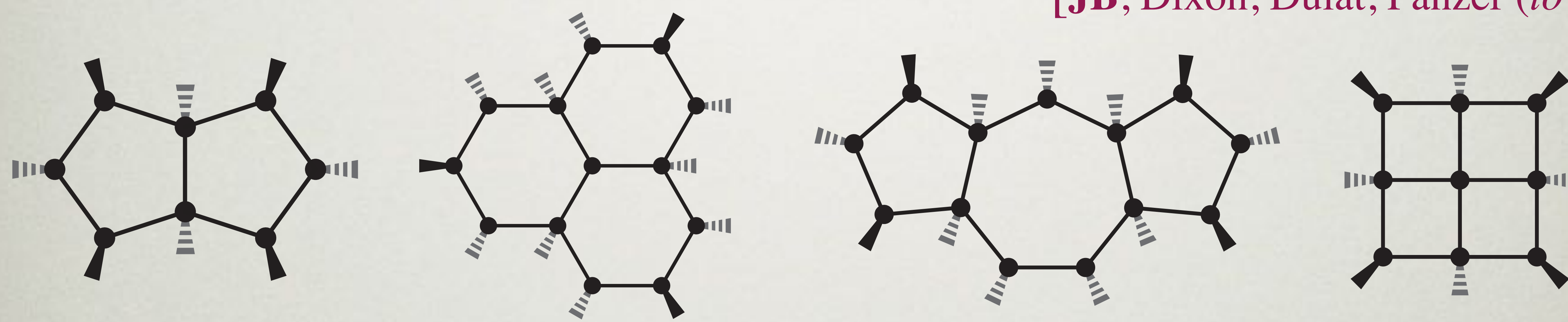

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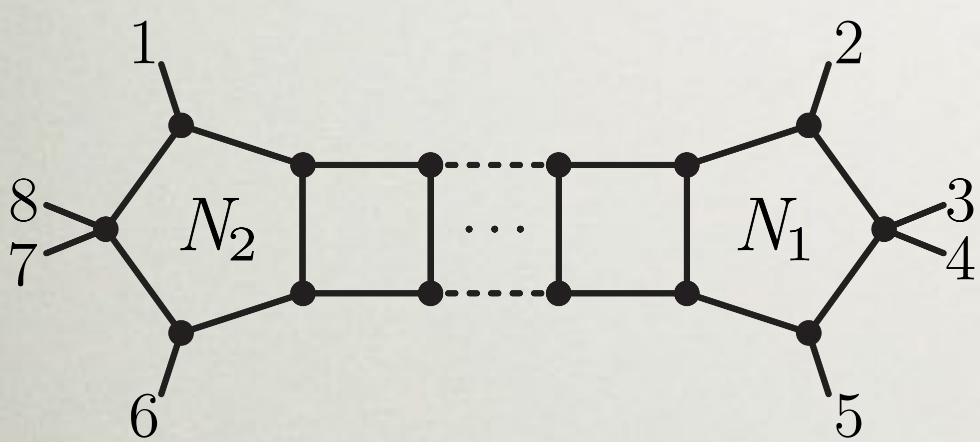

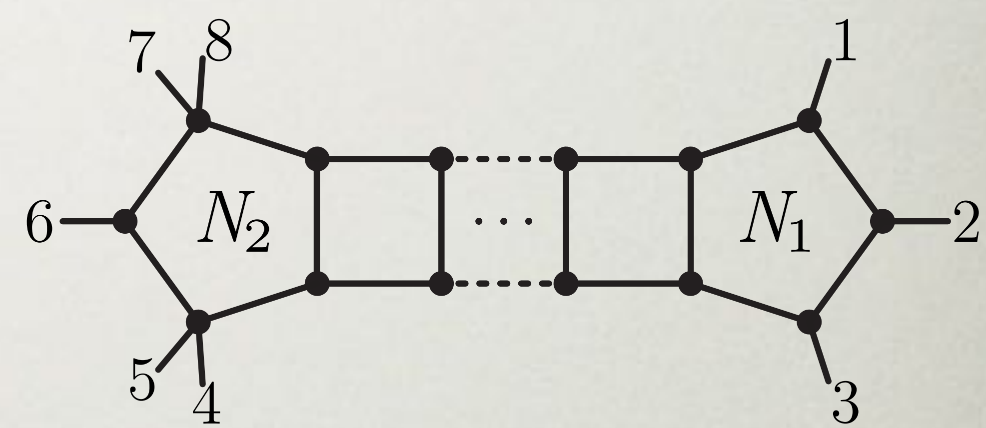

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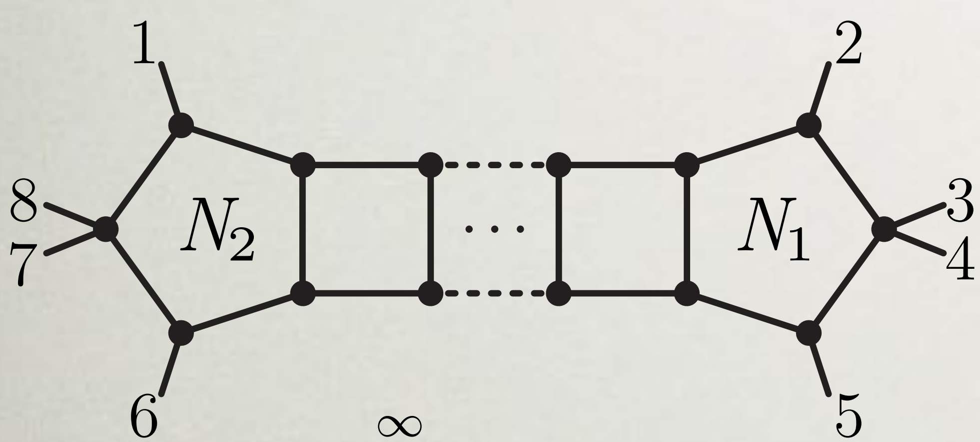

$$I_{8,\text{B}}^{(L)} \equiv \int_0^\infty d^{2L}\vec{\alpha}\,[d\beta]\,\frac{u_1}{(f_1\cdots f_{L-1})\,g_1\,g_2\,g_3}\left(\frac{\alpha_2^L(\beta_1 n_1^1+\beta_2 n_2^1)}{g_1}+\frac{\beta_1 n_1^2+\beta_2 n_2^2}{g_2}-1\right)$$

$$f_k \equiv (\alpha_1^1+\ldots+\alpha_1^k)\beta_2 u_2 + (\alpha_2^1+\ldots+\alpha_2^k)\beta_1 u_3 + \beta_1\beta_2 u_2 u_3 u_4 + \sum_{i,j=1}^{k}\alpha_1^i\alpha_2^j\,;$$

$$g_1 \equiv f_{L-1}+(\alpha_2^1+\ldots+\alpha_2^{L-1})(\alpha_1^L+\alpha_2^L)+\alpha_1^L\beta_2\,u_2+\alpha_2^L(\beta_1+\beta_2);$$

$$g_2 \equiv (\alpha_1^1+\ldots+\alpha_1^L)+\alpha_2^L u_5+\beta_1+\beta_2 u_1; \qquad g_3 \equiv (\alpha_1^1+\ldots+\alpha_1^L)+\alpha_2^L+\beta_1 u_3,$$

where *u's* are parity-even cross-ratios: $(ab;cd) \equiv \dfrac{(a,b)(c,d)}{(a,c)(b,d)}$

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- Although a good start, we haven’t yet eliminated ***all*** conformal redundancies—just the rescalings

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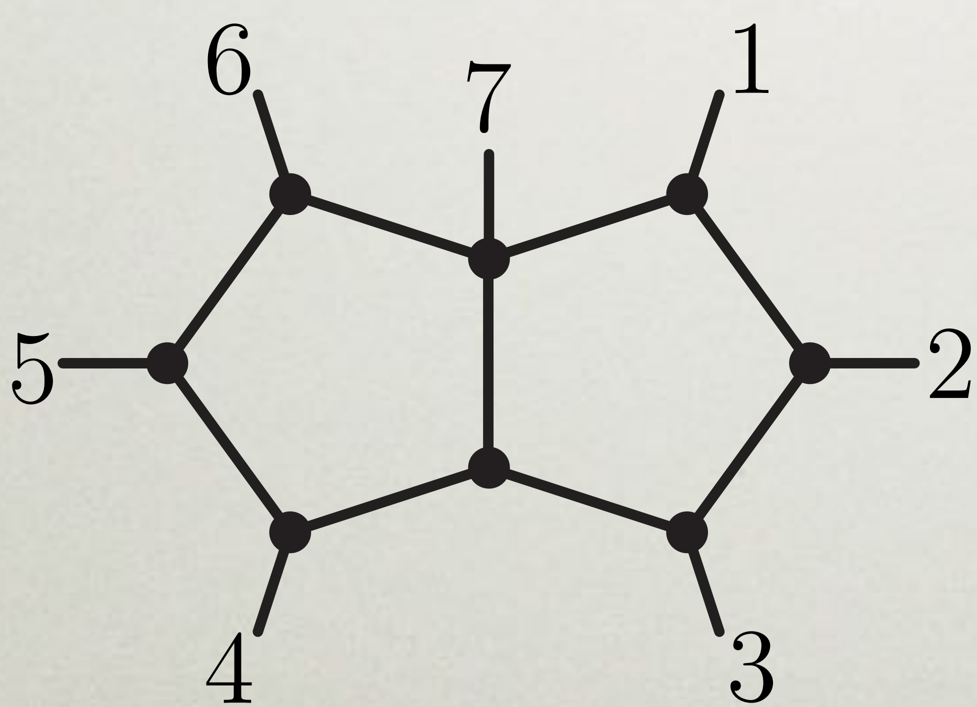

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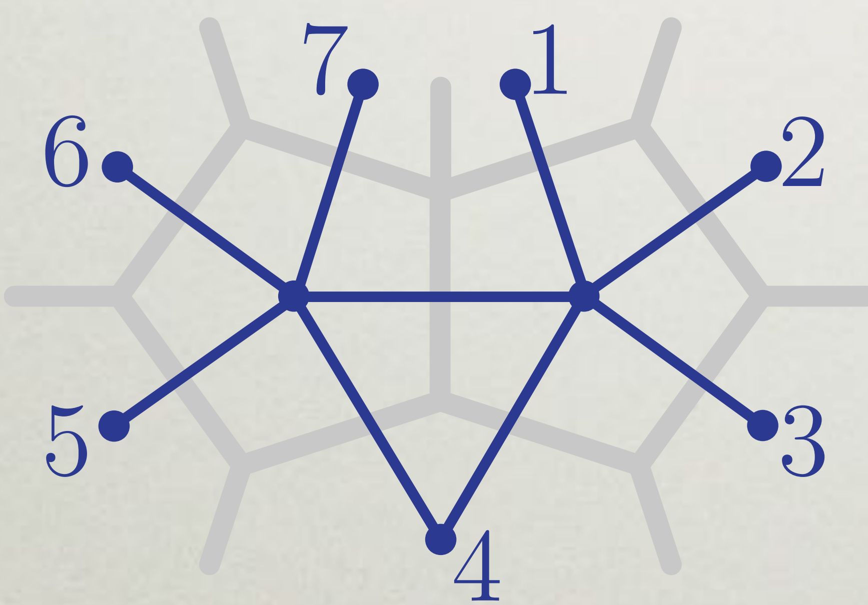

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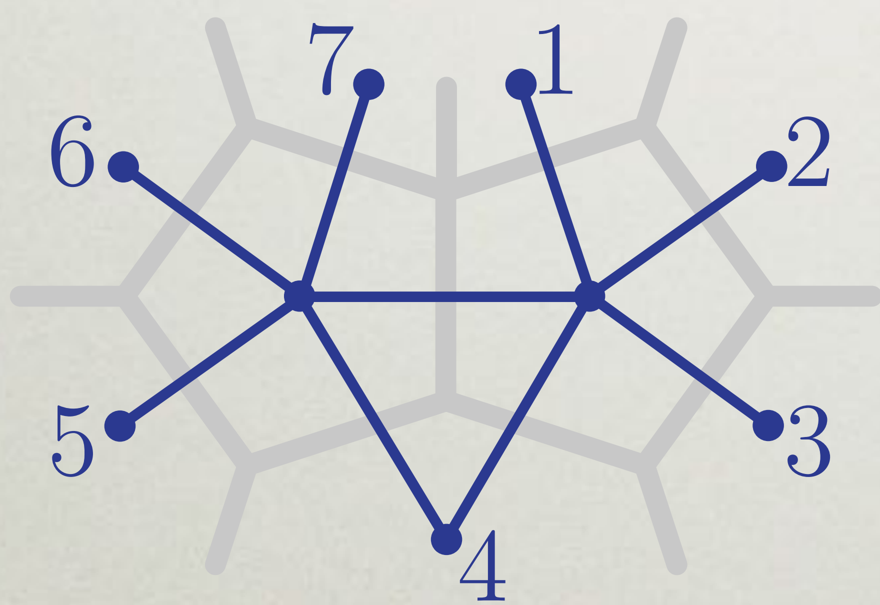

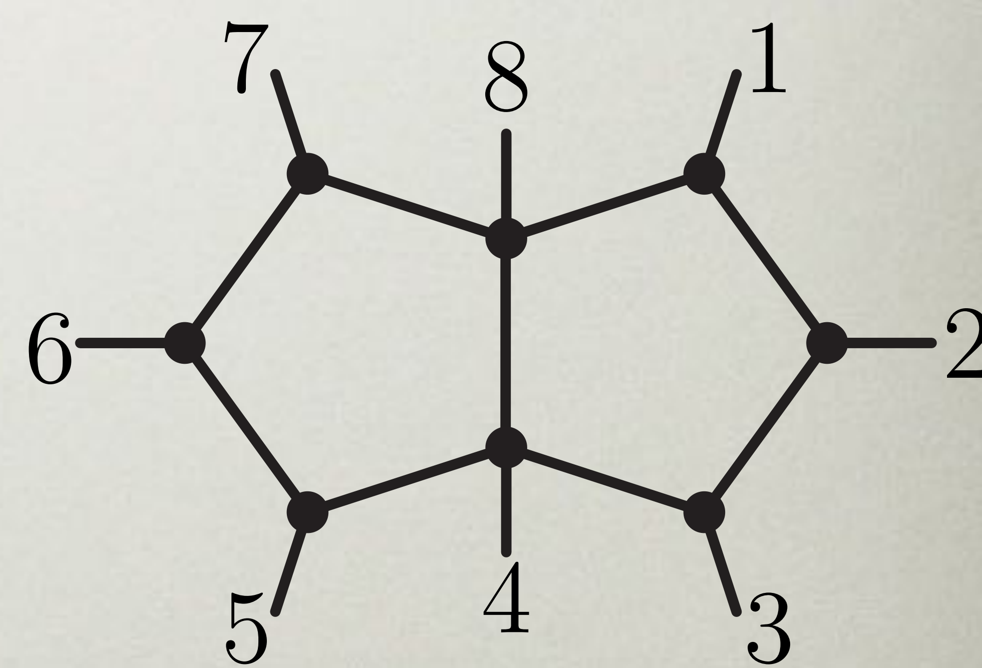

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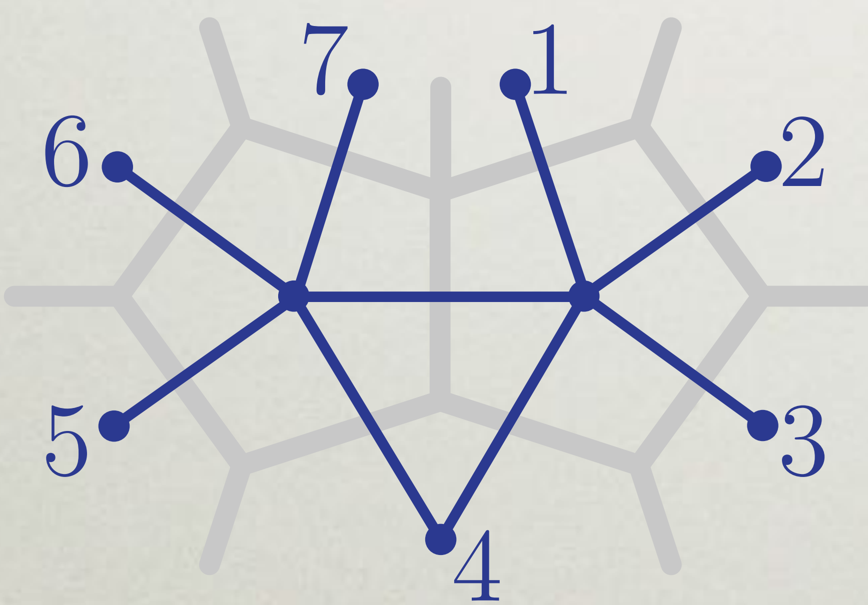

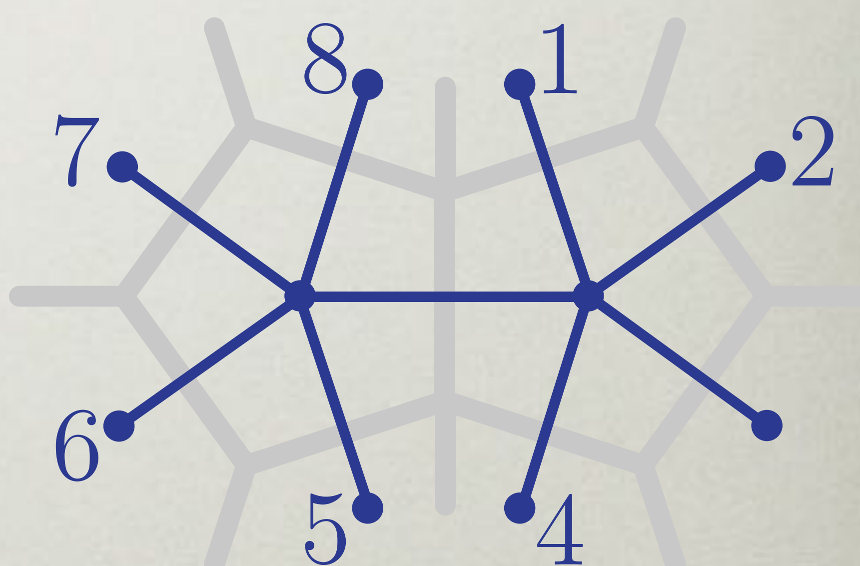

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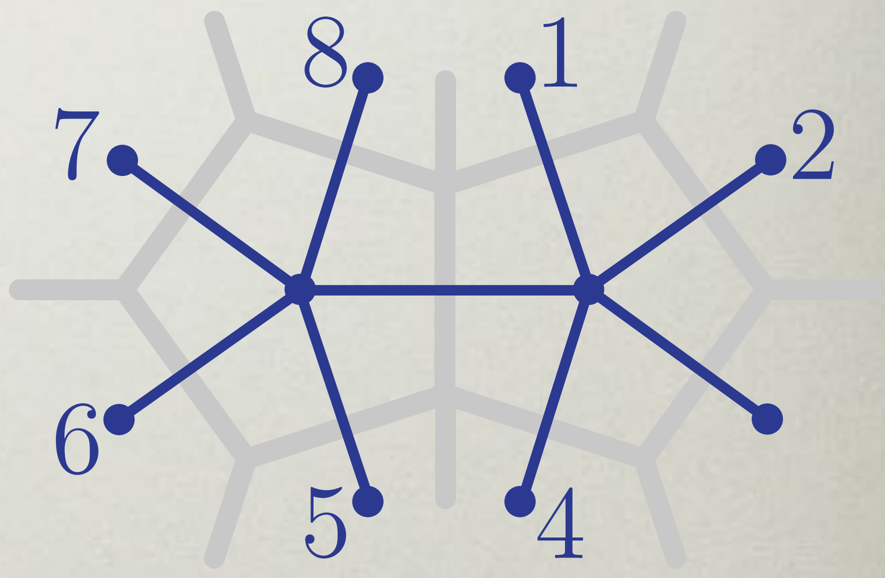

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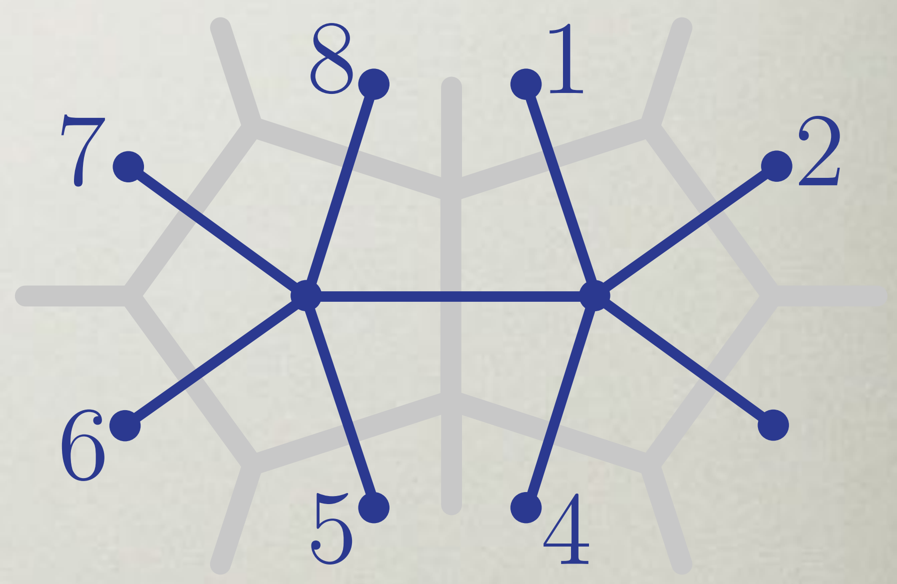

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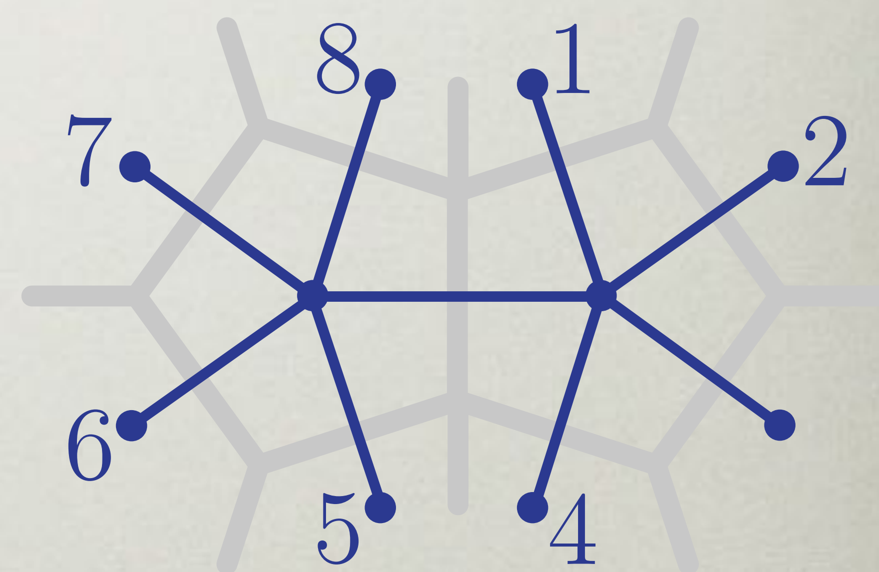

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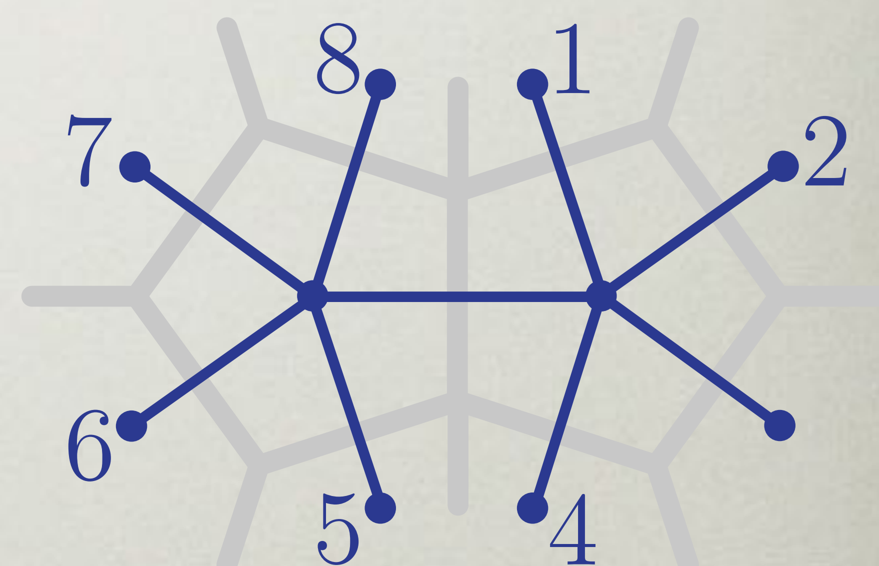

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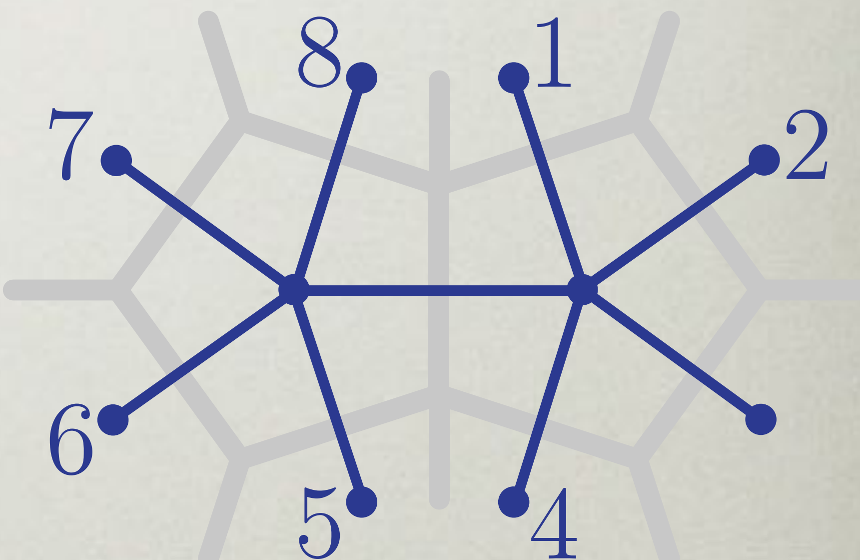

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[Hodges (2009)]

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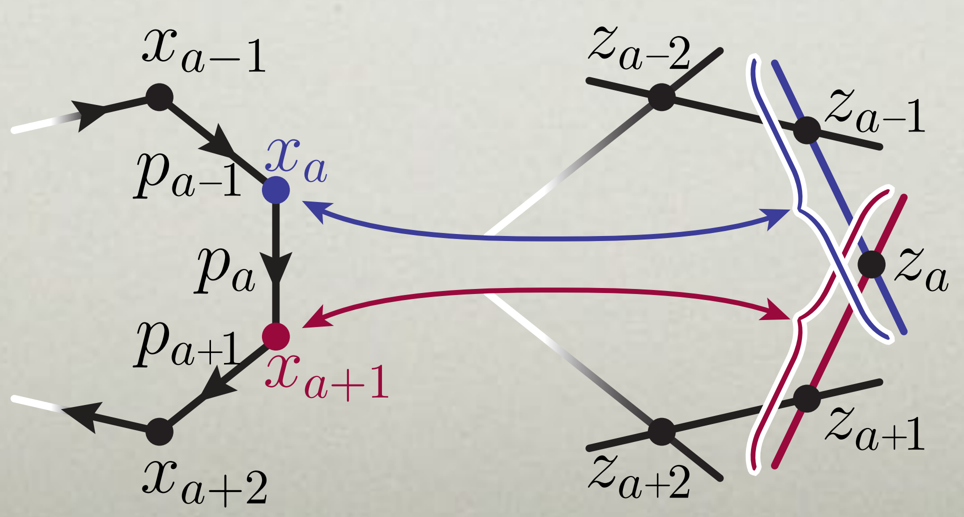

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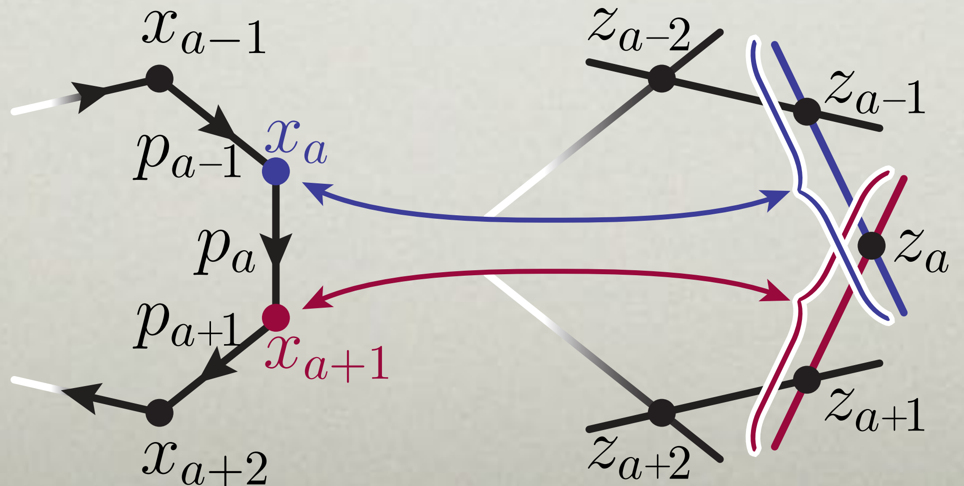

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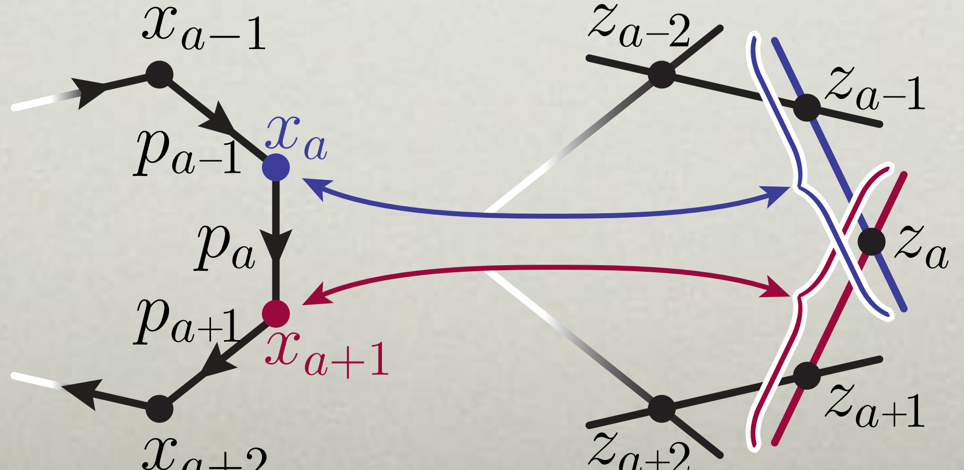

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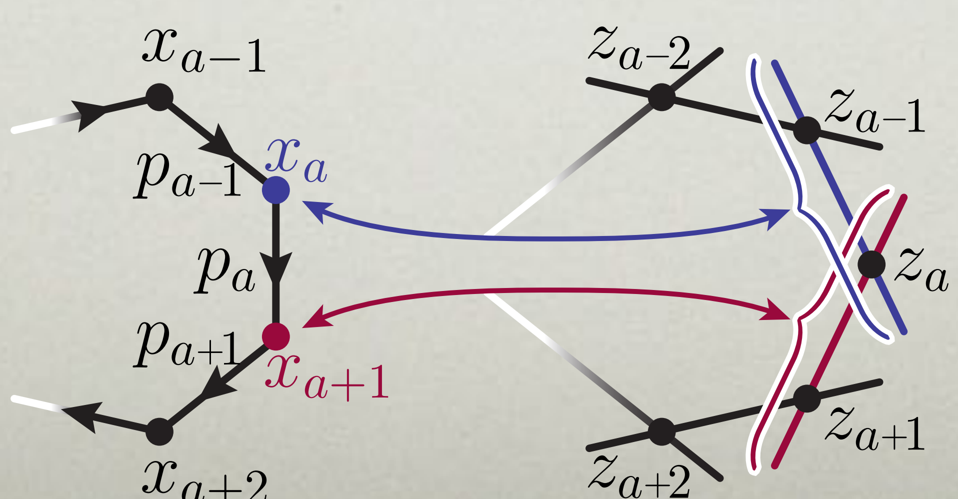

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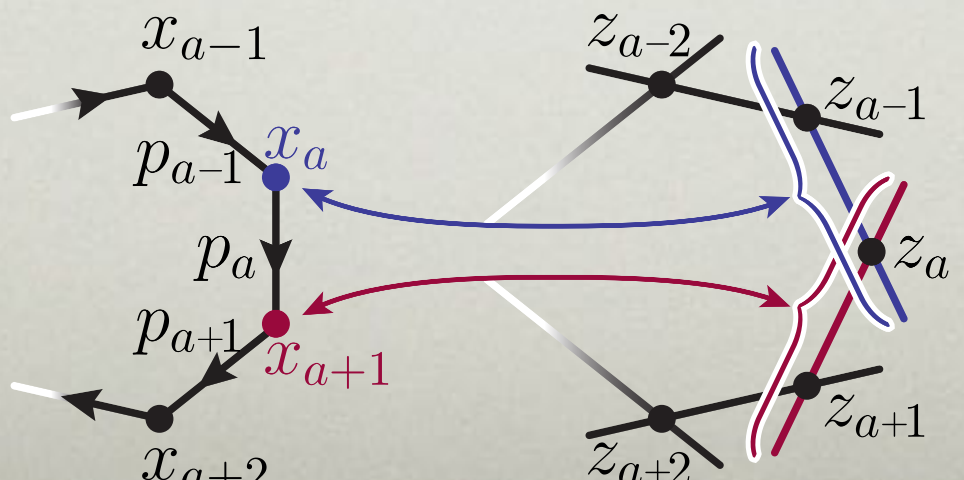

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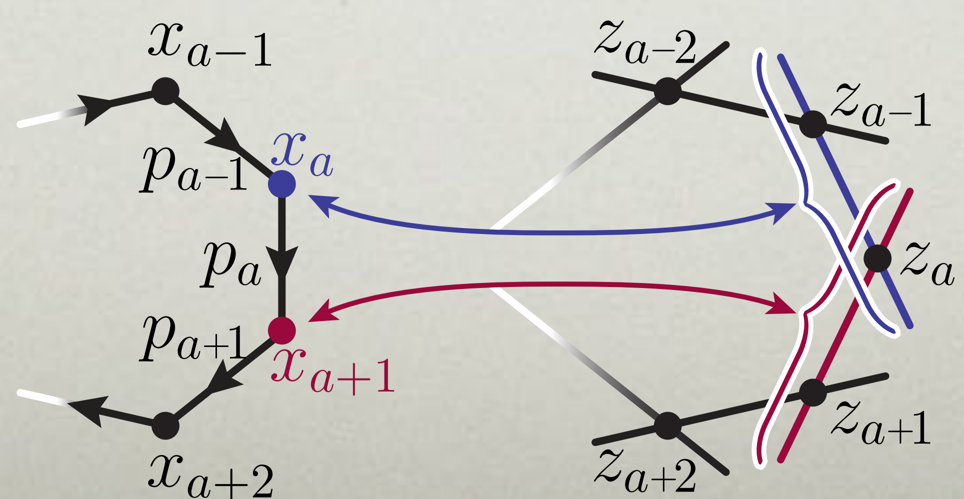

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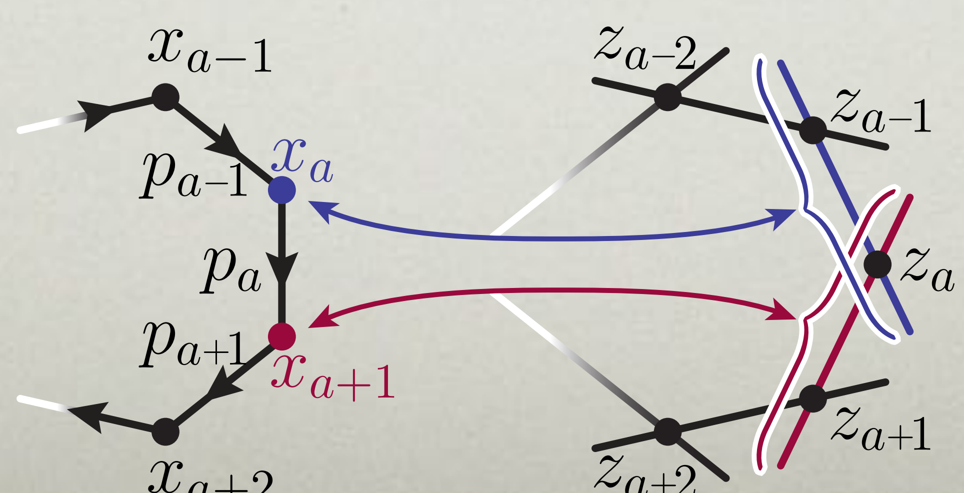

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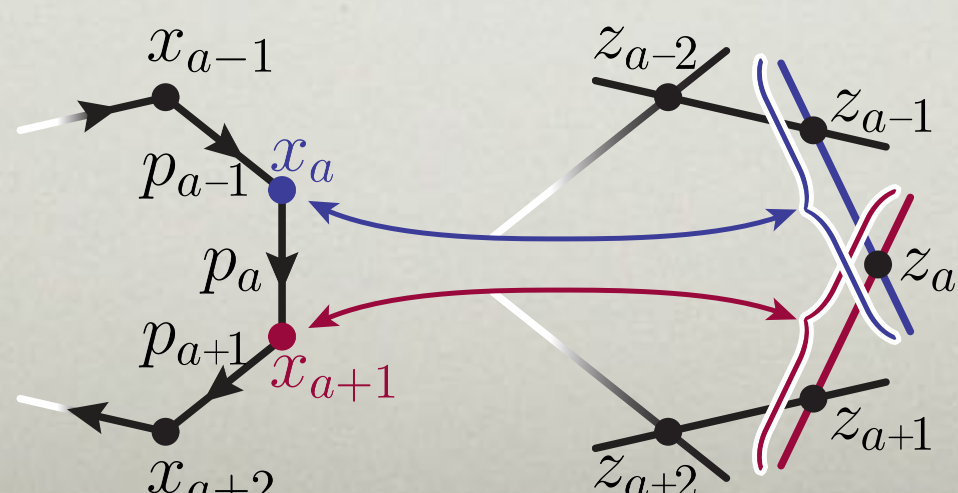

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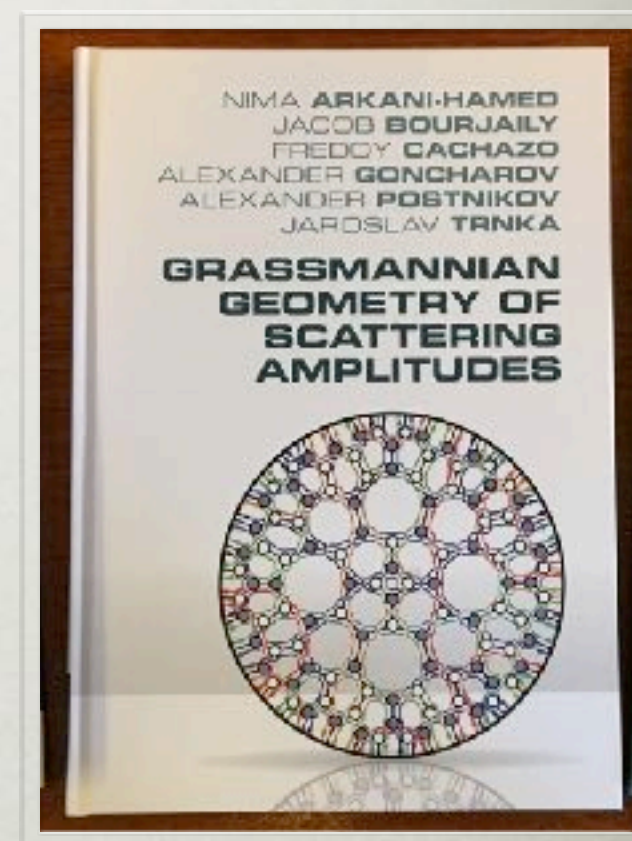

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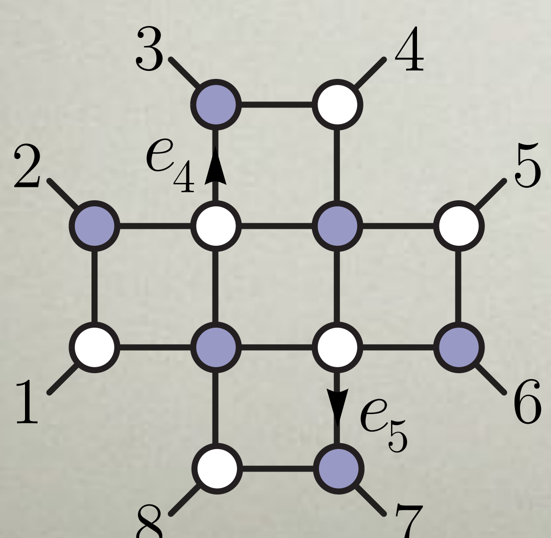

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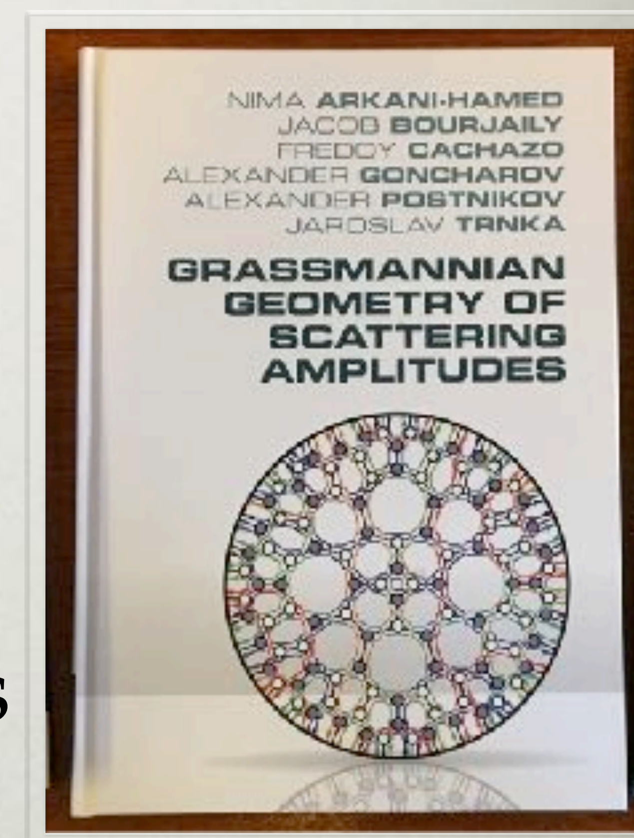

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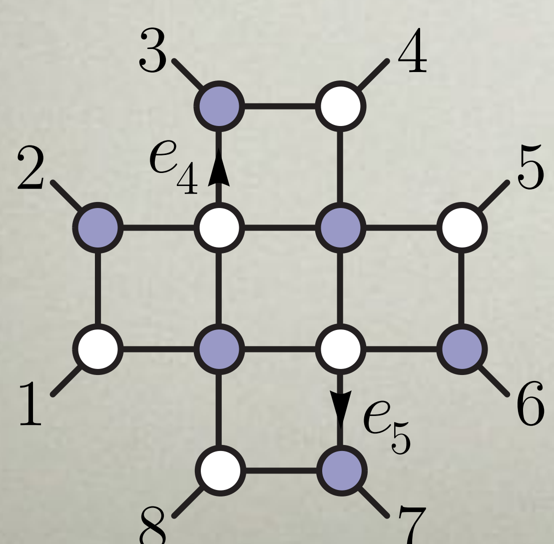

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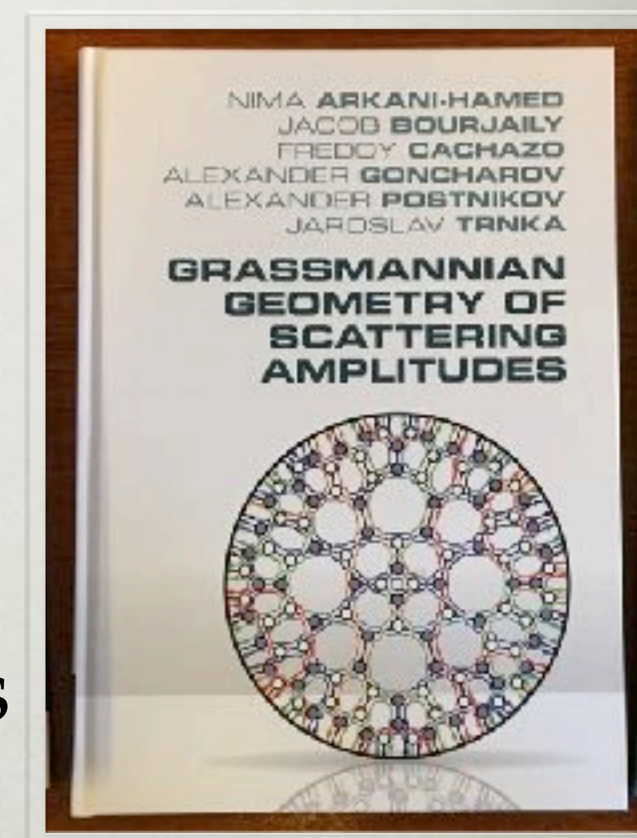

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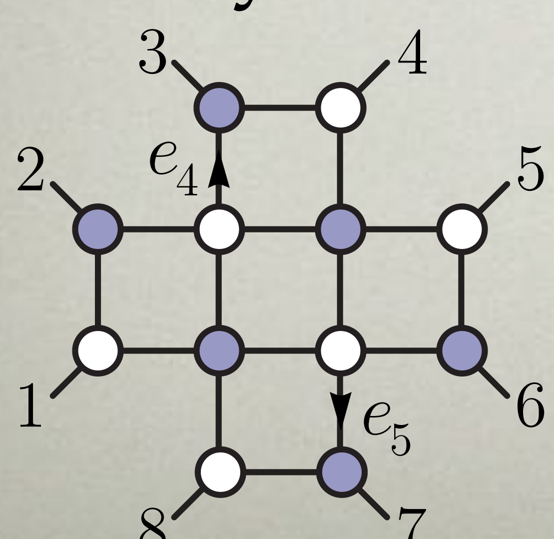

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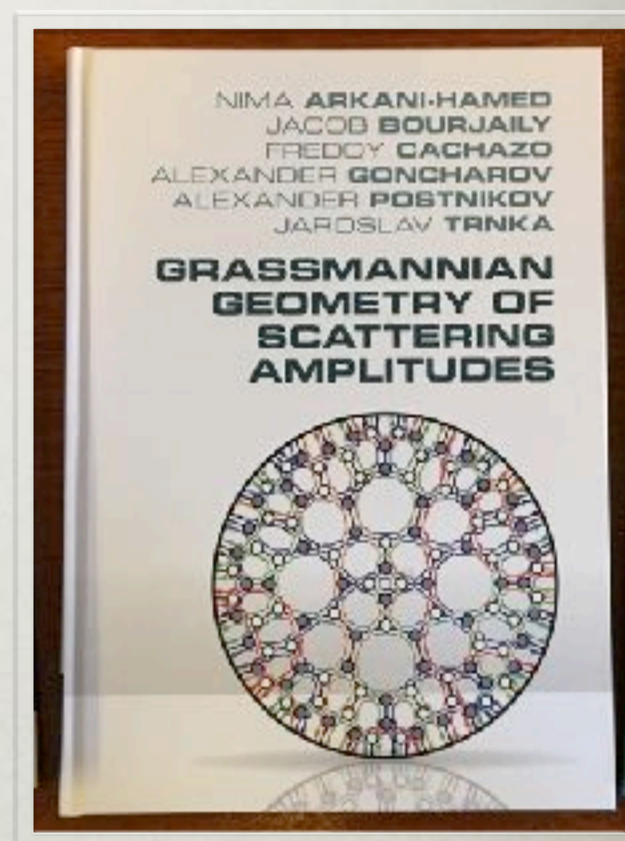

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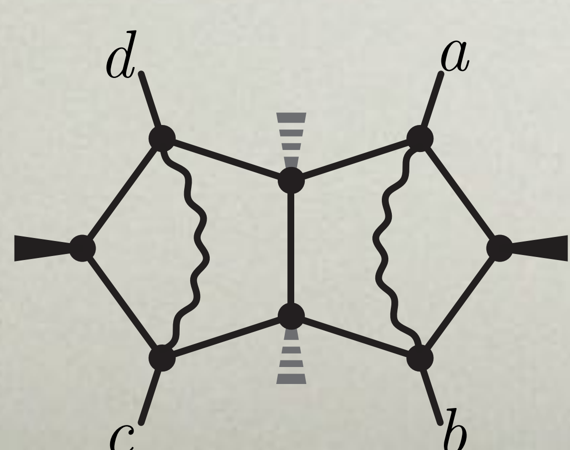

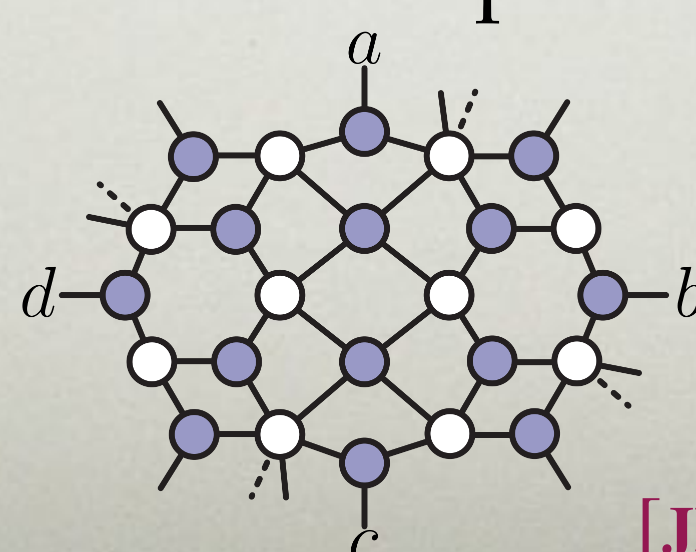

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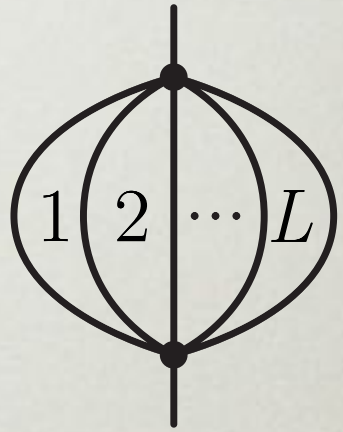

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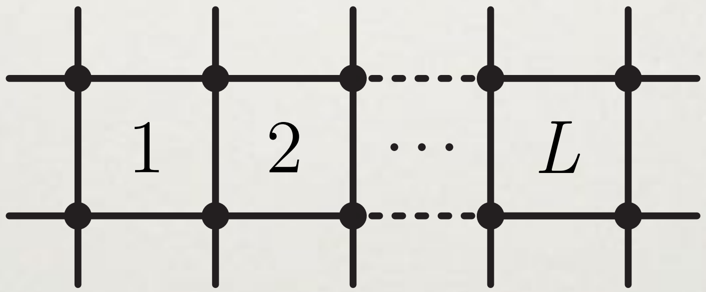

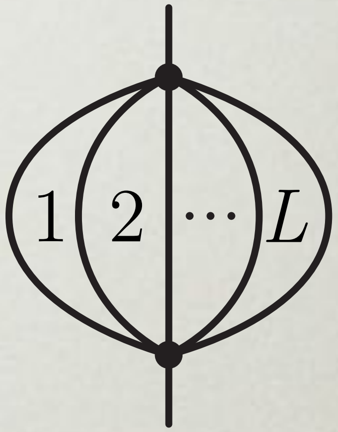

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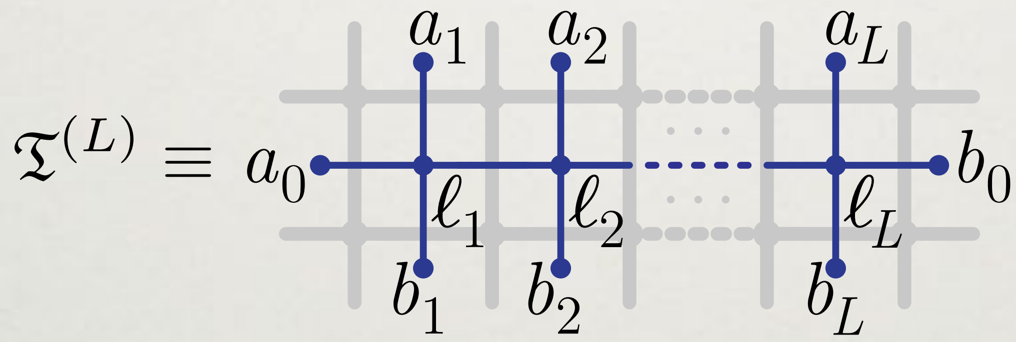

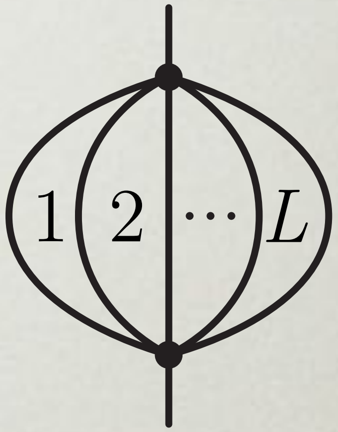

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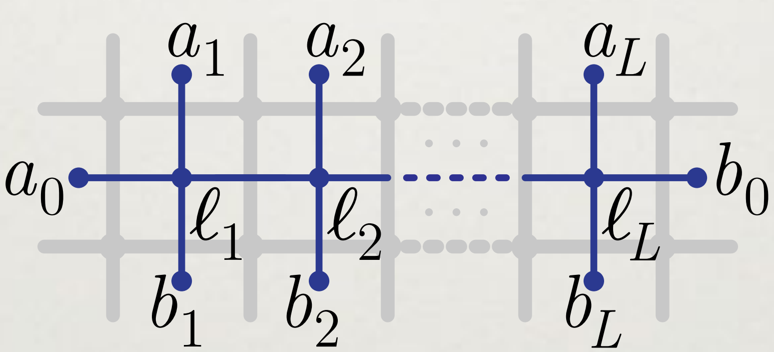
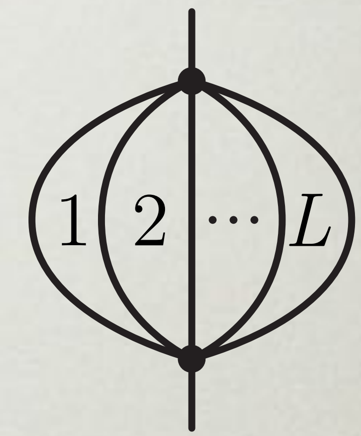
$\mathfrak{T}^{(L)} \equiv$ 



$$\mathfrak{T}^{(L)} = \int_0^\infty [d^L\vec{\alpha}]\, d^L\vec{\beta}\, \frac{1}{(f_1 \cdots f_L)\, g_L}$$

$$\begin{aligned} f_k \equiv\, & (a_0 a_{k-1}; a_k b_{k-1})(a_{k-1} b_k; b_{k-1} a_0)(a_k b_k; a_{k-1} b_{k-1})\, f_{k-1} \\ & + \alpha_0(\alpha_k + \beta_k) + \alpha_k \beta_k + \sum_{j=1}^{k-1} \Big[\alpha_j \alpha_k (b_j a_0; a_j a_k) \\ & + \alpha_j \beta_k (b_j a_0; a_j b_k) + \alpha_k \beta_j (a_0 a_j; a_k b_j) + \beta_j \beta_k (a_0 a_j; b_k b_j) \Big], \end{aligned}$$

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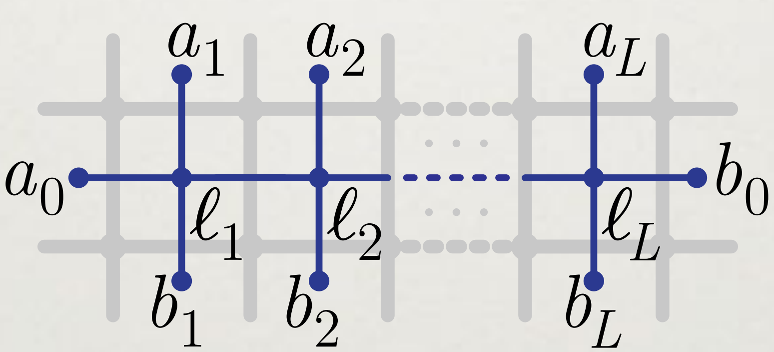
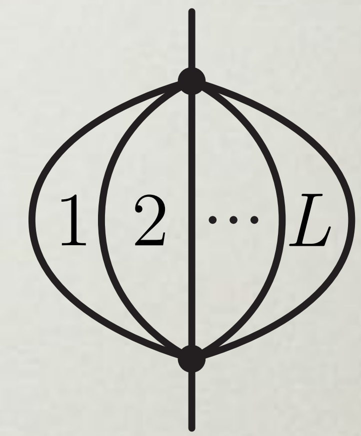
Traintracks Past Polylogarithms

- Despite their ubiquity at low multiplicity and low loop orders, iterated polylogarithms are far from the only class of integrals that are needed in QFT

[**JB**, He, McLeod, von Hippel, Wilhelm (2018)]
[Bloch, Kerr, Vanhove; Broadhurst;…]

$\mathfrak{T}^{(L)} \equiv$





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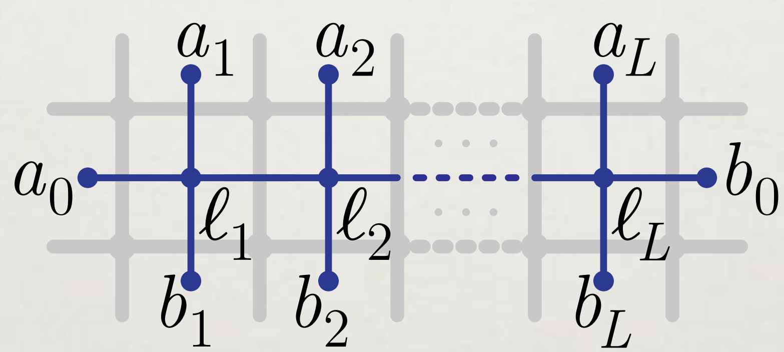
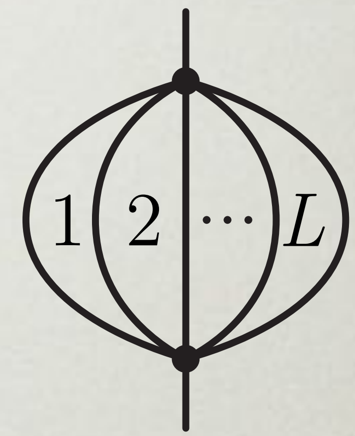
$$\mathcal{A}(\overbrace{\varphi_{12},\ldots,\varphi_{12}}^{L+1},\varphi_{13},\varphi_{13},\overbrace{\varphi_{34},\ldots,\varphi_{34}}^{L+1},\varphi_{24},\varphi_{24})$$

$\mathfrak{T}^{(L)} \equiv$





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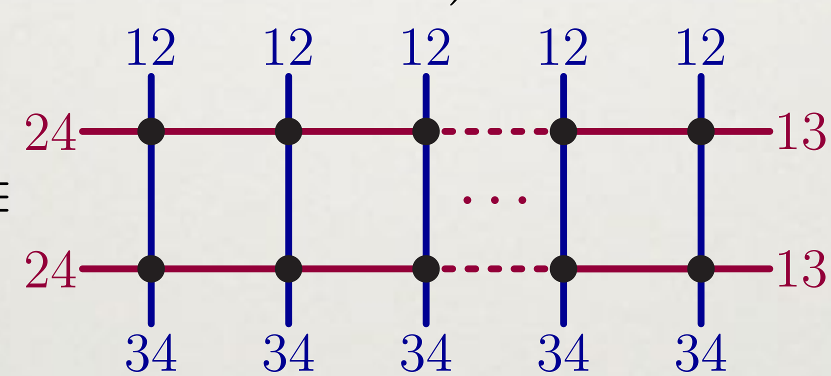
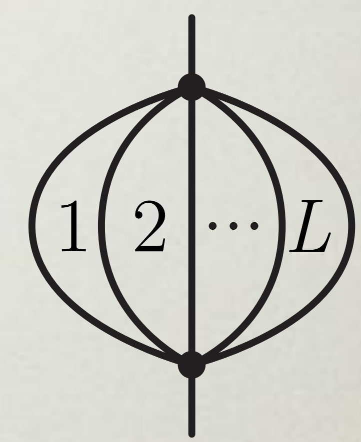
$\mathfrak{T}^{(L)} \equiv$ 



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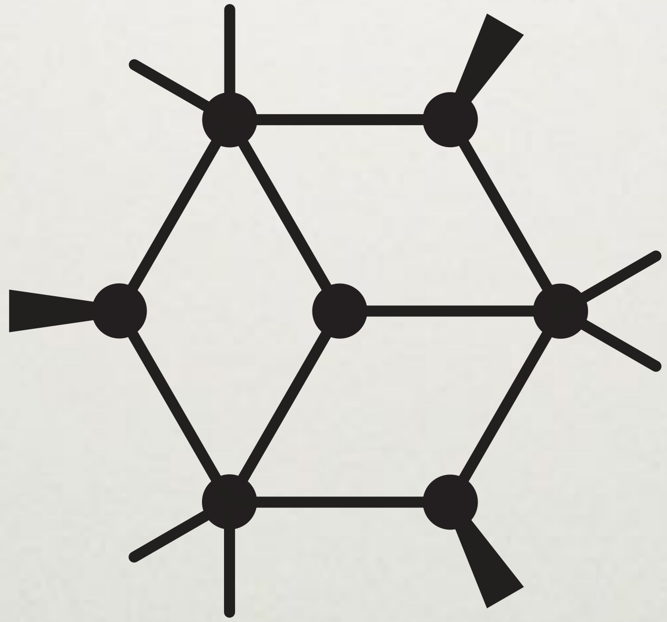
Further Novelties Found…

- It turns out that traintracks do not saturate functional complexity at fixed loop-order…



[**JB**, McLeod, von Hippel, Wilhelm (*in progress*)]

Questions?