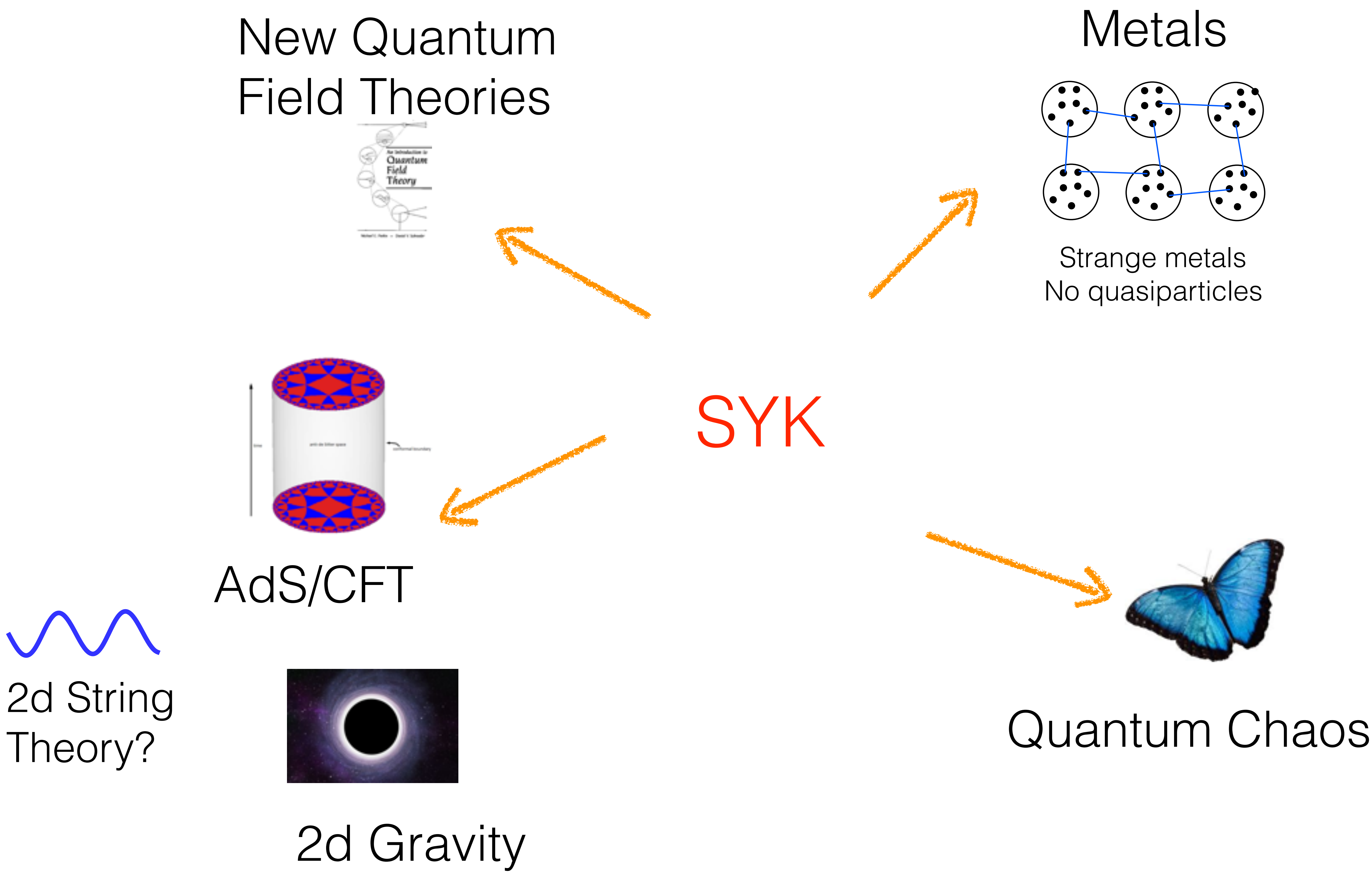
d-dimensional SYK, AdS Loops, and 6j Symbols

Vladimir Rosenhaus
KITP & IAS

J. Liu, E. Perlmutter, V.R., D. Simmons-Duffin, 1808.00612

D. Gross, V.R., 1710.08113

IGST, Copenhagen, Aug. 2018


New Quantum
Field Theories
Quantum
Field
Theory
Metals
Strange metals
No quasiparticles
SYK
AdS/CFT
2d String
Theory?
2d Gravity
Quantum Chaos

For the purposes of this talk the only thing that is relevant is:

For the purposes of this talk the only thing that is relevant is:

SYK in the infrared is a solvable large N CFT. One can compute all correlation functions, by summing the large N dominant Feynman diagrams.

For the purposes of this talk the only thing that is relevant is:

SYK in the infrared is a solvable large N CFT. One can compute all correlation functions, by summing the large N dominant Feynman diagrams.

One can also sum the same diagrams in d dimensions.

A key component of the correlation functions, arising from summing some particular Feynman diagrams, turns out to be a 6j symbol for the conformal group.

In fact, this talk will be centered around the 6j symbol.

A key component of the correlation functions, arising from summing some particular Feynman diagrams, turns out to be a 6j symbol for the conformal group.

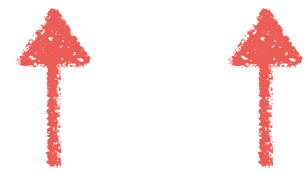
In fact, this talk will be centered around the 6j symbol.

We will discuss the appearance of the 6j symbol in CFT, as the crossing kernel, and the appearance of the 6j symbol in AdS loop diagrams. We will compute the 6j symbol.

Quantum Mechanics

Quantum Mechanics

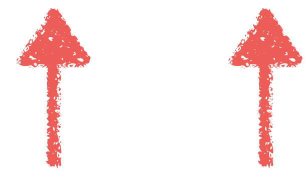
Addition of two spins



Clebsch-Gordan

Quantum Mechanics

Addition of two spins



Clebsch-Gordan

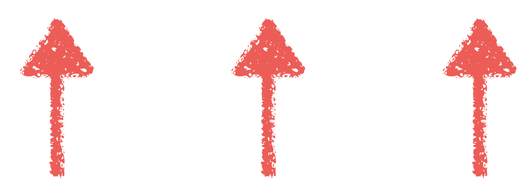
Addition of three spins

Quantum Mechanics

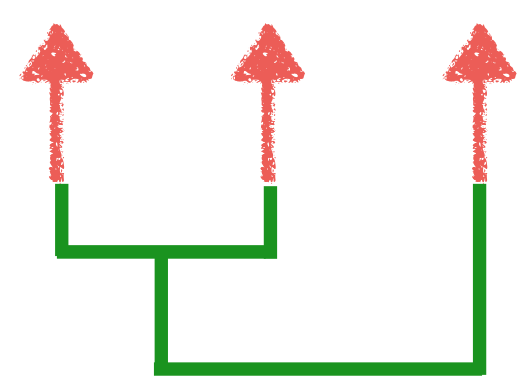
Addition of two spins

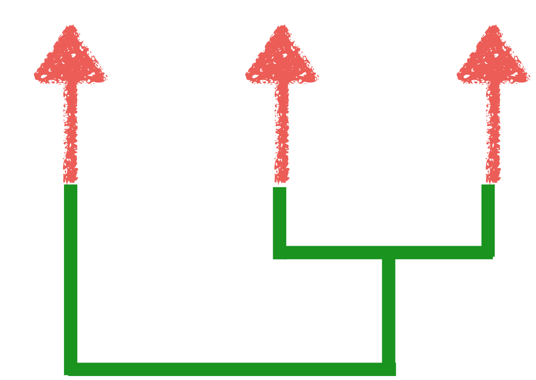
Clebsch-Gordan

Addition of three spins



Two ways of adding



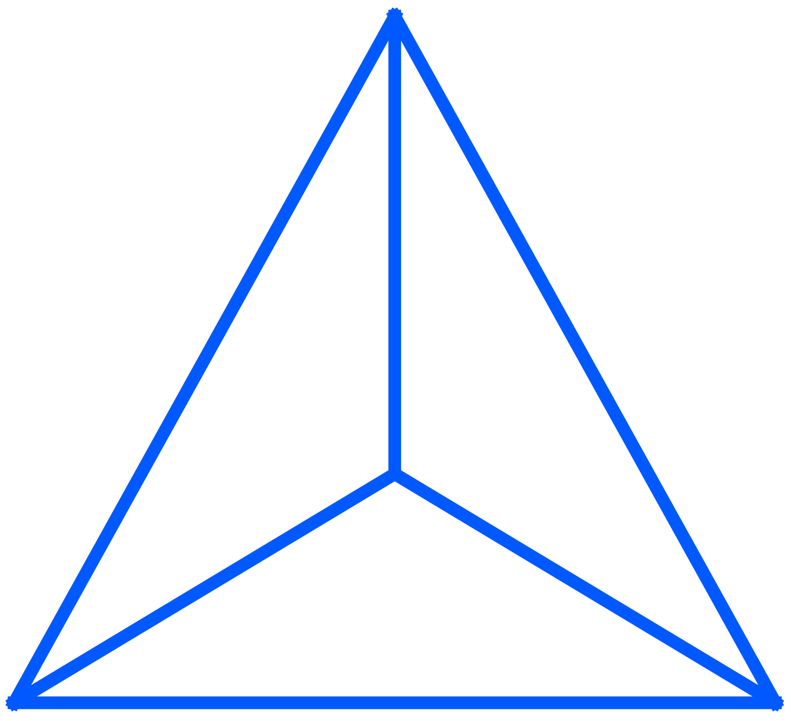


Addition of three spins

6j symbols: products of 4 Clebsch-Gordan coefficients, summed over m_i

edges: spins

vertex: Clebsch-Gordan



This was for SU(2). We would like to study the Euclidean conformal group, SO(d+1,1)

This was for SU(2). We would like to study the Euclidean conformal group, SO(d+1,1)

SU(2) SO(d+1,1)

This was for SU(2). We would like to study the Euclidean conformal group, SO(d+1,1)

SU(2)	SO(d+1,1)
angular momentum J	dimension Δ, spin J

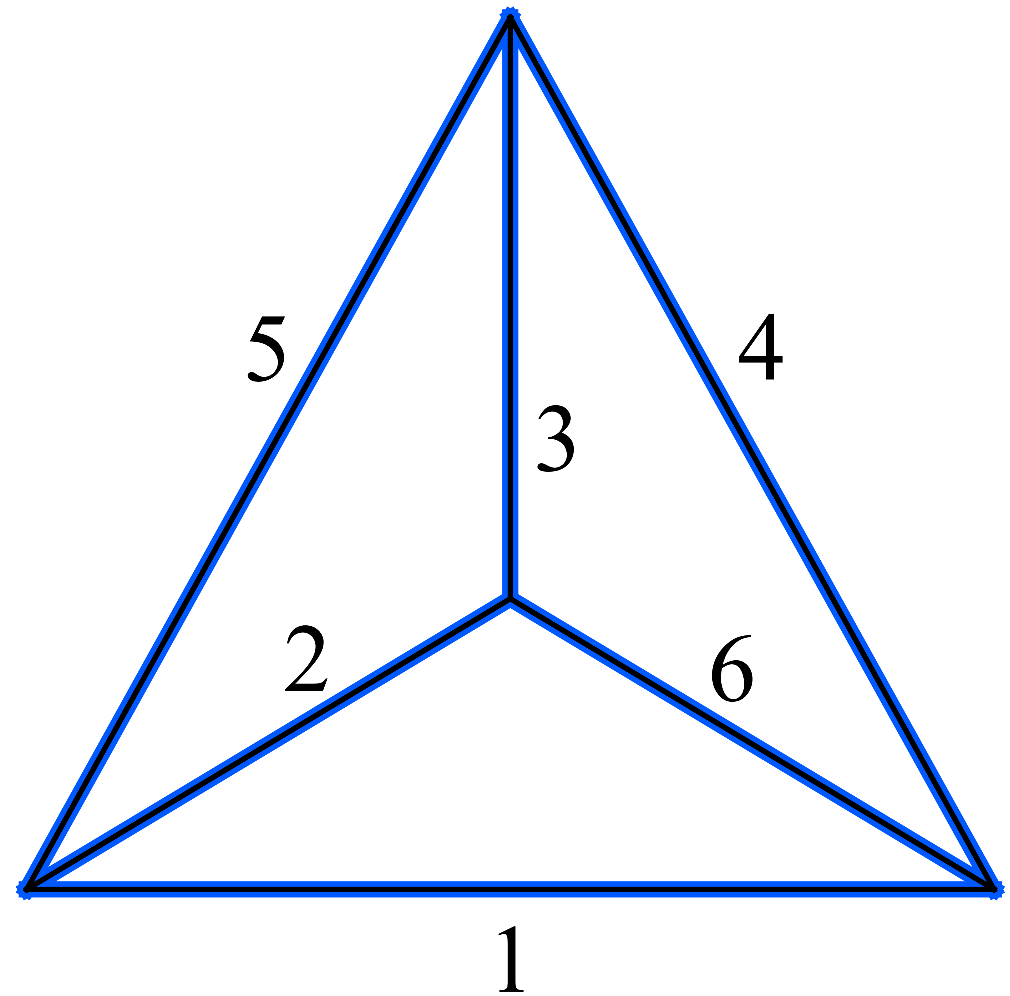
This was for SU(2). We would like to study the Euclidean conformal group, SO(d+1,1)

SU(2)	SO(d+1,1)
angular momentum J	dimension Δ, spin J
z-component angular momentum	position x

This was for SU(2). We would like to study the Euclidean conformal group, SO(d+1,1)

SU(2)	SO(d+1,1)
angular momentum J	dimension Δ, spin J
z-component angular momentum	position x
Clebsch-Gordan	3-pt function $\langle \mathcal{O}_{\Delta_1,J_1}(x_1)\mathcal{O}_{\Delta_2,J_2}(x_2)\mathcal{O}_{\Delta_3,J_3}(x_3)\rangle$

Conformal 6j symbol



$$= \int \frac{d^d x_1 \cdots d^d x_6}{\mathrm{vol}(\mathrm{SO}(d+1,1))} \langle \widetilde{\mathcal{O}}_1 \widetilde{\mathcal{O}}_2 \widetilde{\mathcal{O}}_5 \rangle^a \langle \mathcal{O}_5 \widetilde{\mathcal{O}}_3 \widetilde{\mathcal{O}}_4 \rangle^b \langle \mathcal{O}_3 \mathcal{O}_2 \mathcal{O}_6 \rangle^c \langle \widetilde{\mathcal{O}}_6 \mathcal{O}_1 \mathcal{O}_4 \rangle^d$$

Where does the 6j symbol appear?

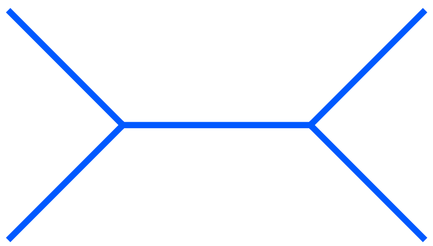
Where does the 6j symbol appear?

A CFT four-point function can be expanded in terms of conformal blocks.

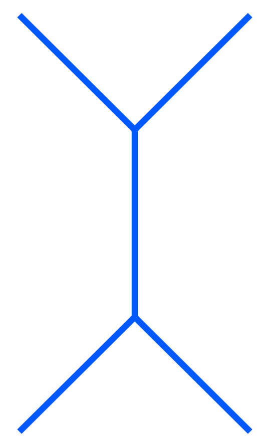
Where does the 6j symbol appear?

A CFT four-point function can be expanded in terms of conformal blocks.

One can expand in either the s-channel blocks



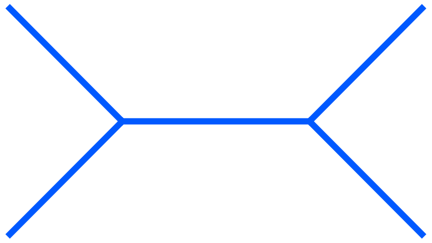
or the t-channel blocks

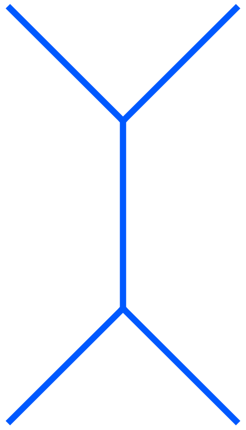


Where does the 6j symbol appear?

A CFT four-point function can be expanded in terms of conformal blocks.

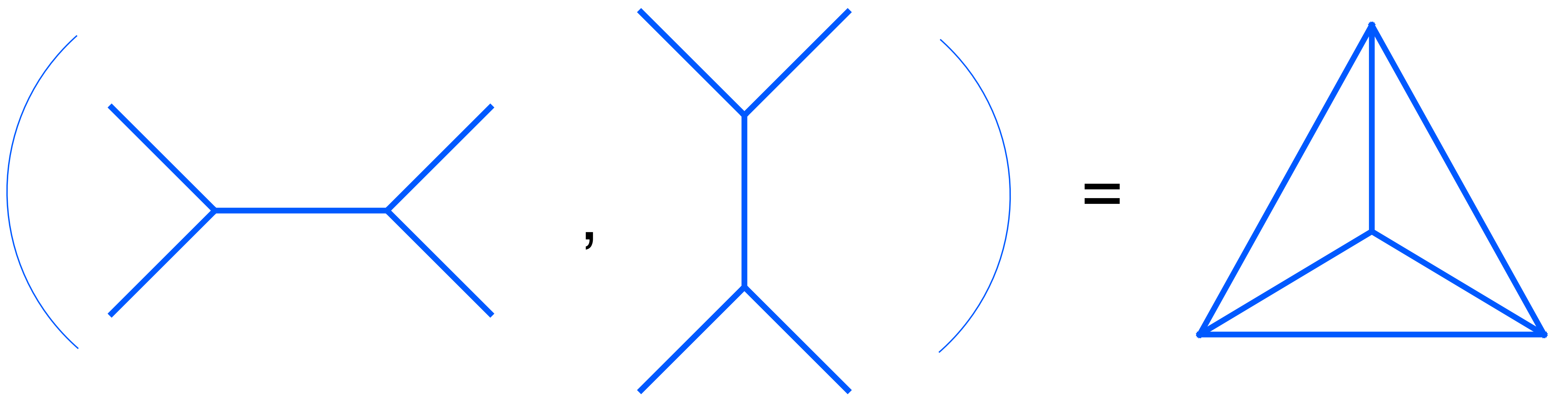
One can expand in either the s-channel blocks or the t-channel blocks



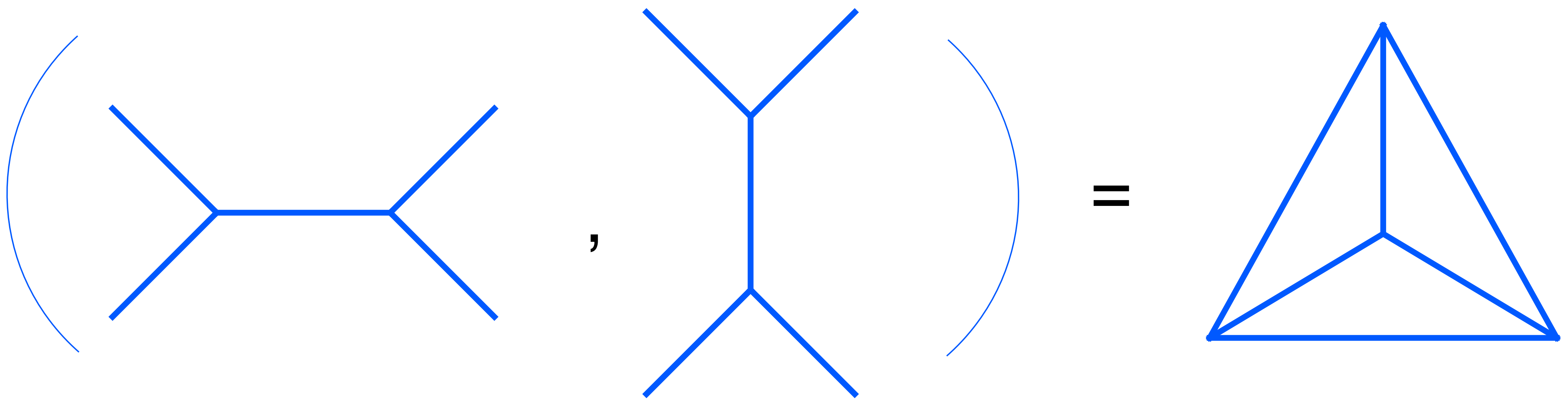


Conformal bootstrap: equality between the two expansions gives constraints on the OPE coefficients

The 6j symbol is the overlap between the s-channel and the t-channel conformal partial waves (crossing kernel)



The 6j symbol is the overlap between the s-channel and the t-channel conformal partial waves (crossing kernel)



$$\left(\Psi^{\widetilde{\Delta}_1,\widetilde{\Delta}_2,\widetilde{\Delta}_3,\widetilde{\Delta}_4}_{\widetilde{\Delta},J},\Psi^{\Delta_3,\Delta_2,\Delta_1,\Delta_4}_{\Delta',J'}\right)$$

$$=\int\frac{d^dx_1\cdots d^dx_4}{\mathrm{vol}(\mathrm{SO}(d+1,1))}\Psi^{\widetilde{\Delta}_1,\widetilde{\Delta}_2,\widetilde{\Delta}_3,\widetilde{\Delta}_4}_{\widetilde{\Delta},J}(x_1,x_2,x_3,x_4)\Psi^{\Delta_3,\Delta_2,\Delta_1,\Delta_4}_{\Delta',J'}(x_3,x_2,x_1,x_4)\ ,$$

$$\Psi^{\Delta_i}_{\Delta,J}(x_i)=\int d^dx_5\,\langle\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta,J}(x_5)^{\mu_1\cdots\mu_J}\rangle\,\langle\widetilde{\mathcal{O}}_{\Delta,J;\,\mu_1\cdots\mu_J}(x_5)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle$$

Summary:

The 6j symbol is a group-theoretic quantity.

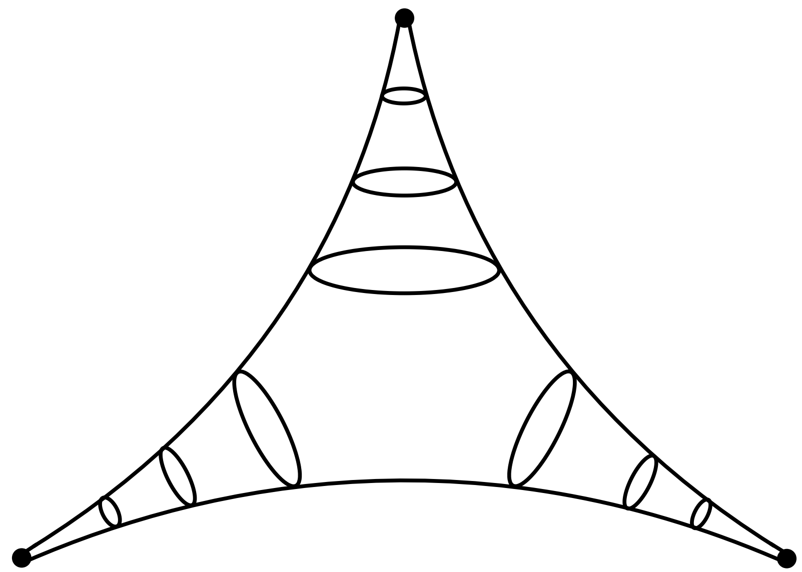
It is clearly important to know it.

It appears in the bootstrap as the crossing kernel.

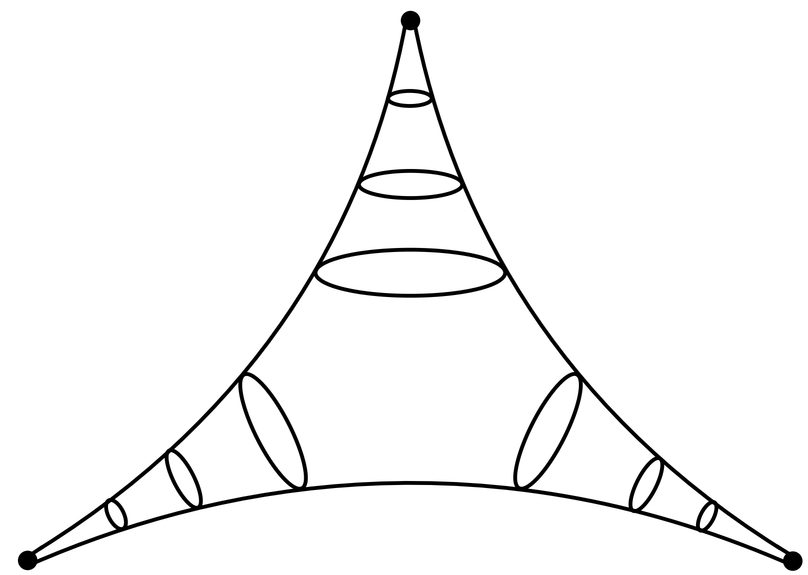
We compute the 6j symbol in dimensions 1,2,4,
for external scalars. I will describe the computation later.

In fact, the conformal 6j symbol also appears in an entirely different, and *dynamical* context.

Consider summing the following Feynman diagrams, in a CFT.

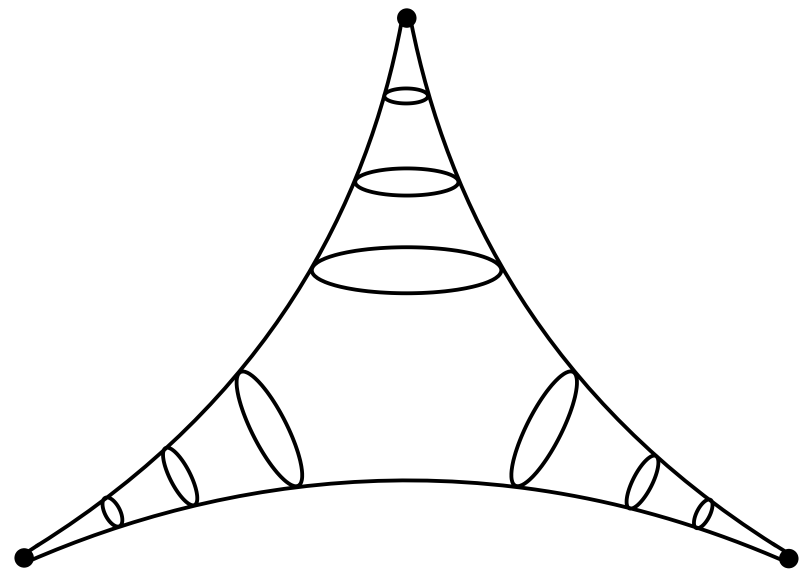


Consider summing the following Feynman diagrams, in a CFT.



These appear as the planar diagram contribution of the three-point function of bilinears in SYK.

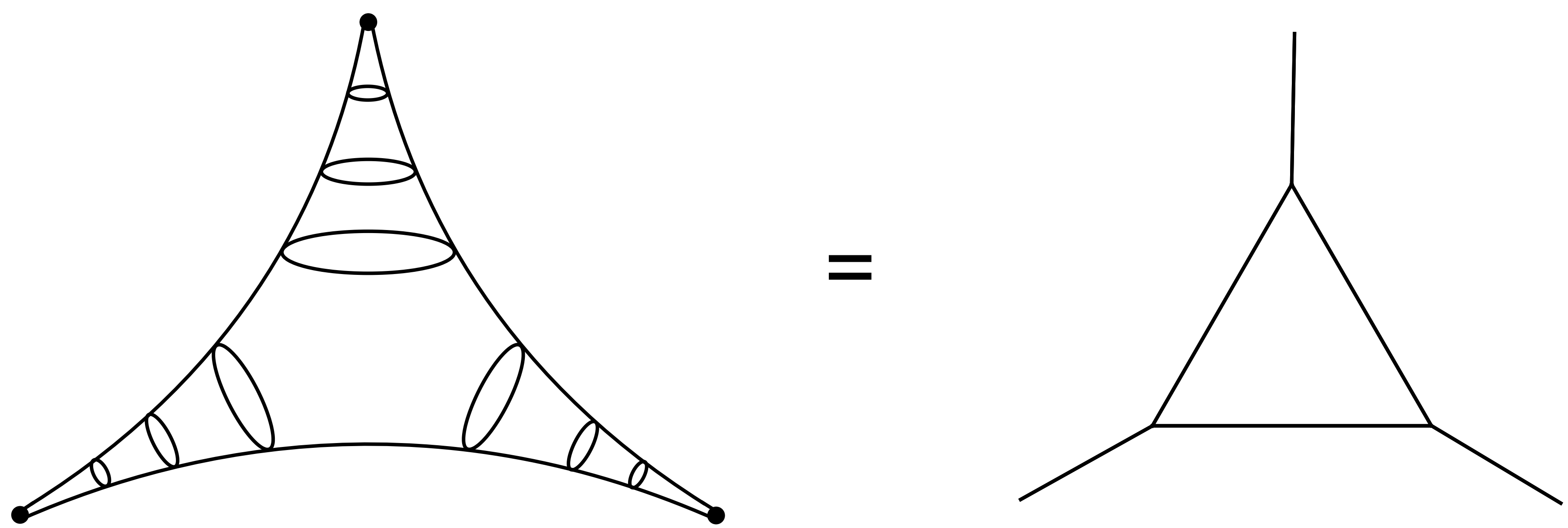
Consider summing the following Feynman diagrams, in a CFT.



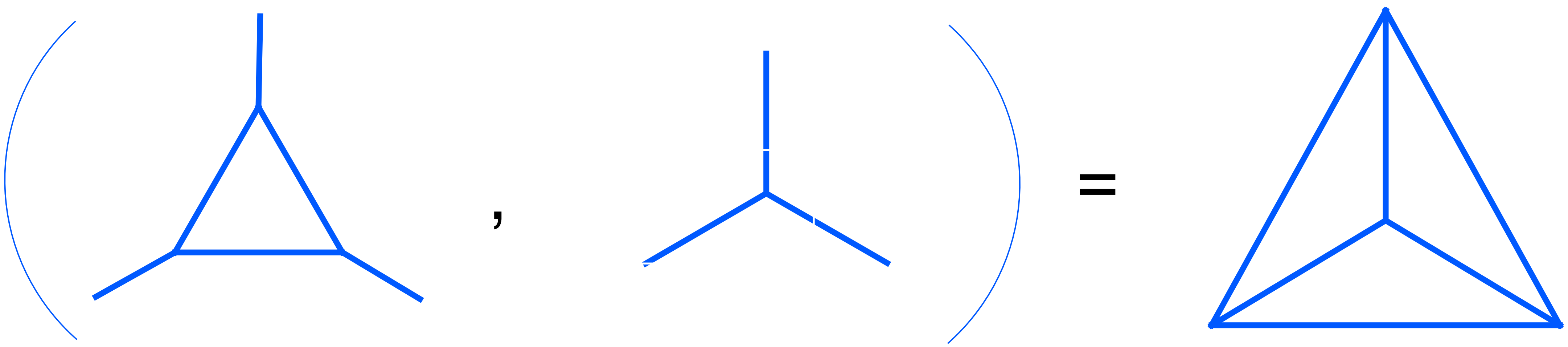
These appear as the planar diagram contribution of the three-point function of bilinears in SYK.

The melons are not important here. What is important is that we are gluing three 3-point functions.

In the notation from before, with each vertex denoting a three-point function,



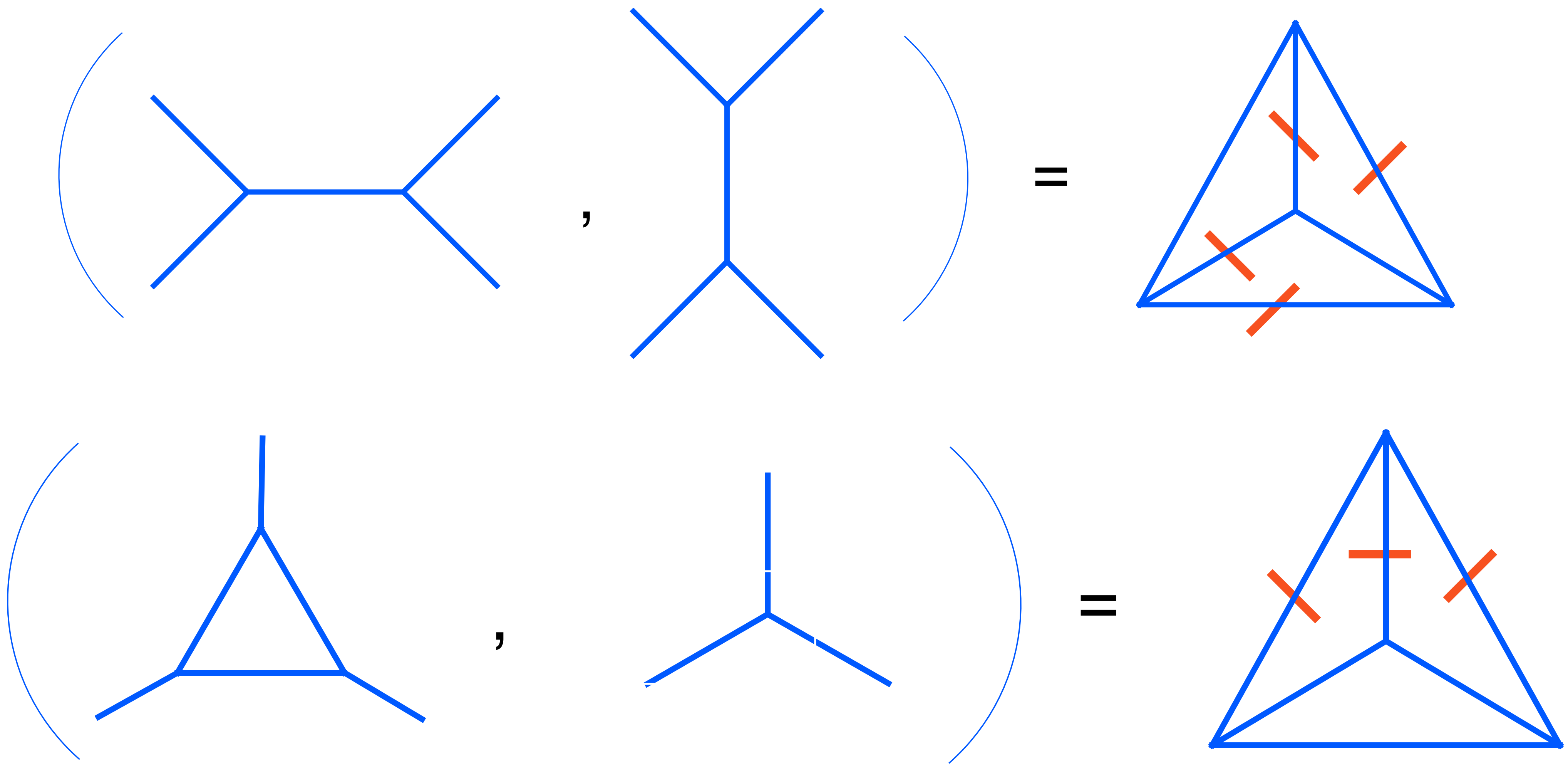
The functional form of a three-point function is fixed by conformal symmetry. We can extract the coefficient by contracting with a (bare) three-point function of shadow operators, The functional form of a three-point function is fixed by conformal symmetry. We can extract the coefficient by contracting with a (bare) three-point function of shadow operators,



This is a tetrahedron: a 6j symbol

The overlap of two partial waves - a group theoretic quantity- and the planar Feynman diagrams in an SYK correlation function - a dynamical quantity- are just two different ways of splitting a tetrahedron

The overlap of two partial waves - a group theoretic quantity- and the planar Feynman diagrams in an SYK correlation function - a dynamical quantity- are just two different ways of splitting a tetrahedron



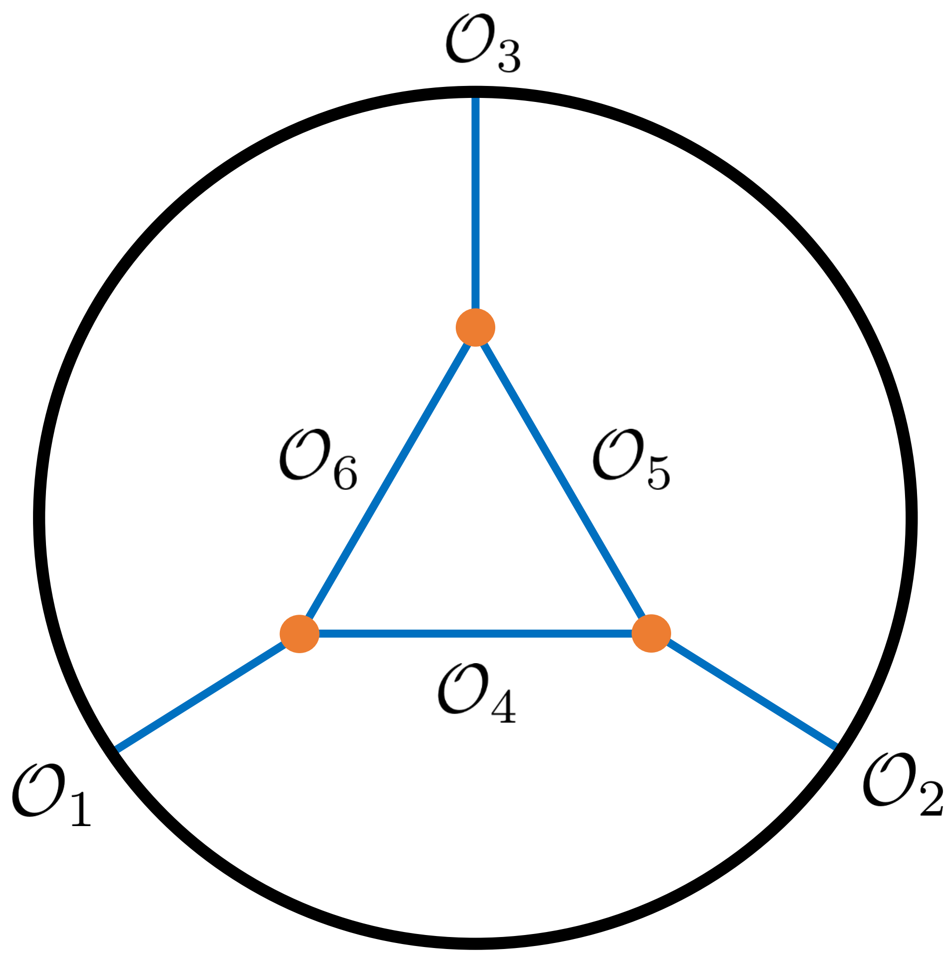
AdS

There is a third context in which the 6j symbol appears: loop diagrams in AdS.

AdS

There is a third context in which the 6j symbol appears: loop diagrams in AdS.

The preamplitude for the triangle diagram is a 6j symbol



Outline

1. SYK and SYK-like models, and all point correlation functions.

2. Computation of 6j symbols

3. AdS triangle diagram.

SYK

$$S = \int d\tau \left(\frac{1}{2} \chi_i \partial_\tau \chi_i + \frac{1}{4!} J_{ijkl}\, \chi_i \chi_j \chi_k \chi_l \right)$$

O(N)

$$\overline{J_{ijkl}^2} = 3! \frac{J^2}{N^3}$$

Sachdev & Ye
Kitaev

SYK

$$S = \int d\tau \left(\frac{1}{2} \chi_i \partial_\tau \chi_i + \frac{1}{4!} J_{ijkl}\, \chi_i \chi_j \chi_k \chi_l \right)$$

O(N)

$$\overline{J_{ijkl}^2} = 3! \frac{J^2}{N^3}$$

Sachdev & Ye
Kitaev

Tensor Model

Different symmetry,
same leading large N diagrams

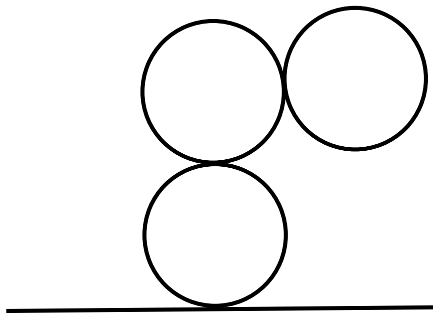
$$S = \int d\tau \left(\frac{1}{2} \psi_{abc} \partial_\tau \psi_{abc} + \frac{1}{4} g\, \psi_{abc} \psi_{ade} \psi_{fbe} \psi_{fdc} \right)$$

$O(N)^3$

Gurau; Witten; Carrozza & Tanasa
Klebanov & Tarnopolsky

Large N Models

- Vector model: ϕ_a $\qquad (\vec{\phi}\cdot\vec{\phi})^2 \equiv \phi_a\phi_a\phi_b\phi_b$

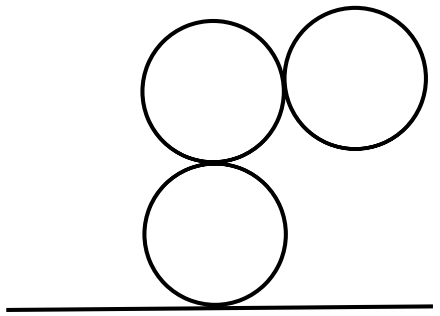


Easy

bubbles

Large N Models

- Vector model: ϕ_a $\qquad (\vec{\phi}\cdot\vec{\phi})^2 \equiv \phi_a\phi_a\phi_b\phi_b$



Easy

bubbles

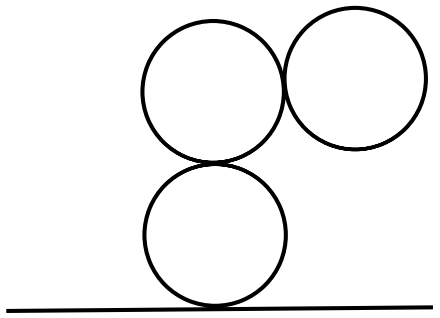
- Matrix model: ϕ_{ab} $\qquad \mathrm{Tr}(\phi^4) \equiv \phi_{ab}\phi_{bc}\phi_{cd}\phi_{da}$

Hard

all planar

Large N Models

- Vector model: ϕ_a $\quad (\vec{\phi}\cdot\vec{\phi})^2 \equiv \phi_a\phi_a\phi_b\phi_b$



Easy

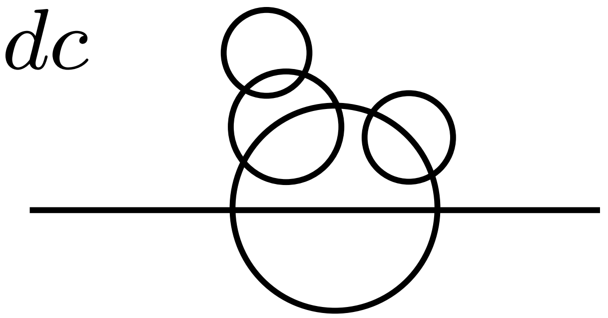
bubbles

- Matrix model: ϕ_{ab} $\quad \mathrm{Tr}(\phi^4) \equiv \phi_{ab}\phi_{bc}\phi_{cd}\phi_{da}$

Hard

all planar

- Tensor model: ϕ_{abc} $\quad \phi_{abc}\phi_{ade}\phi_{fbe}\phi_{fdc}$

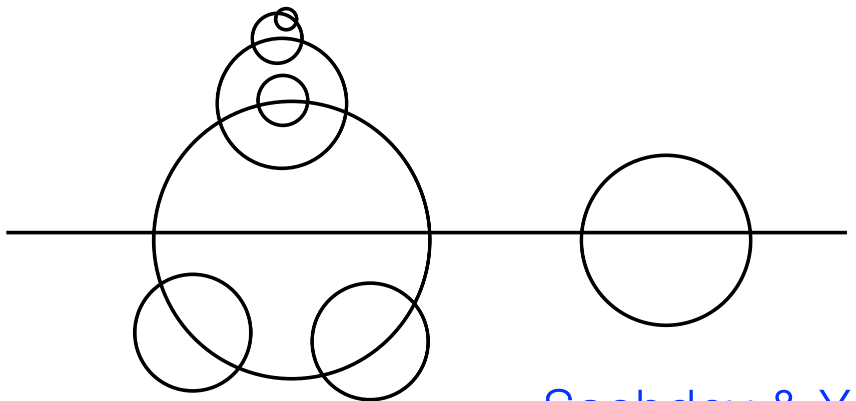


Medium

melons

We can compute all large N correlation functions in SYK by summing all Feynman diagrams.

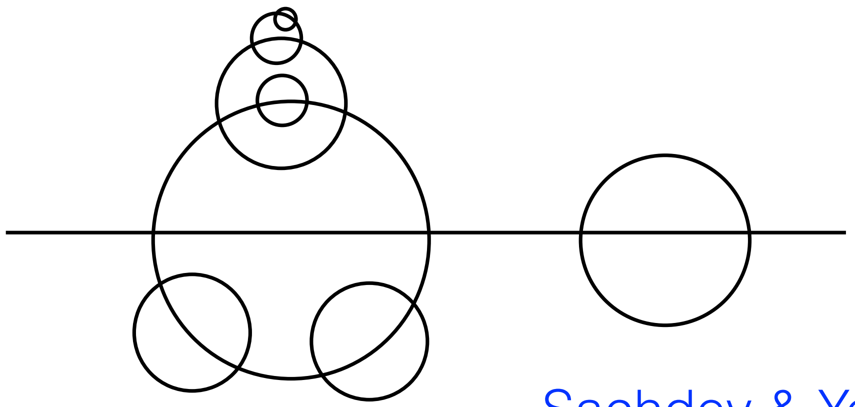
We can compute all large N correlation functions in SYK by summing all Feynman diagrams.



Sachdev & Ye

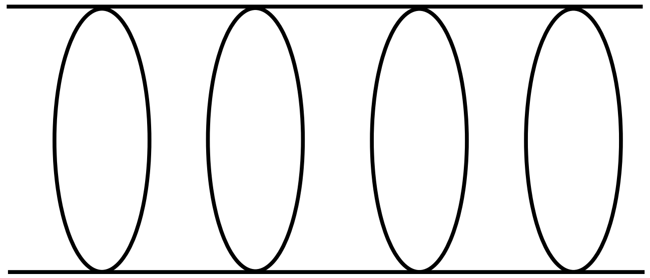
2-pt: Melons -> Conformal in IR

We can compute all large N correlation functions in SYK by summing all Feynman diagrams.



Sachdev & Ye

2-pt: Melons -> Conformal in IR

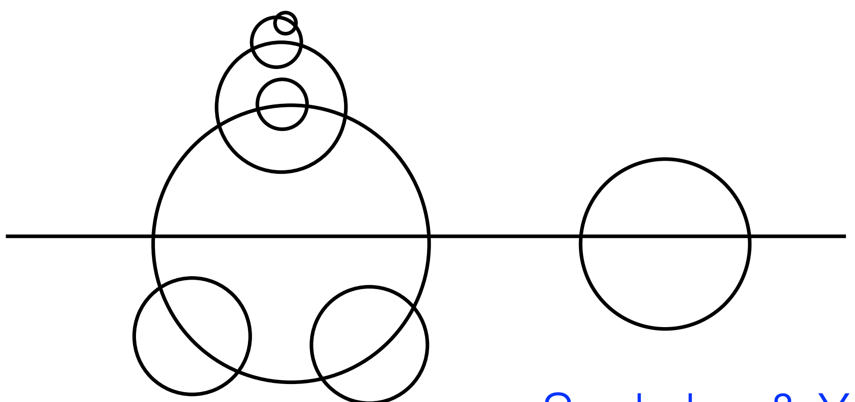


Kitaev; Polchisnki+ V.R.; Maldacena & Stanford

4-pt: Ladders: geometric sum

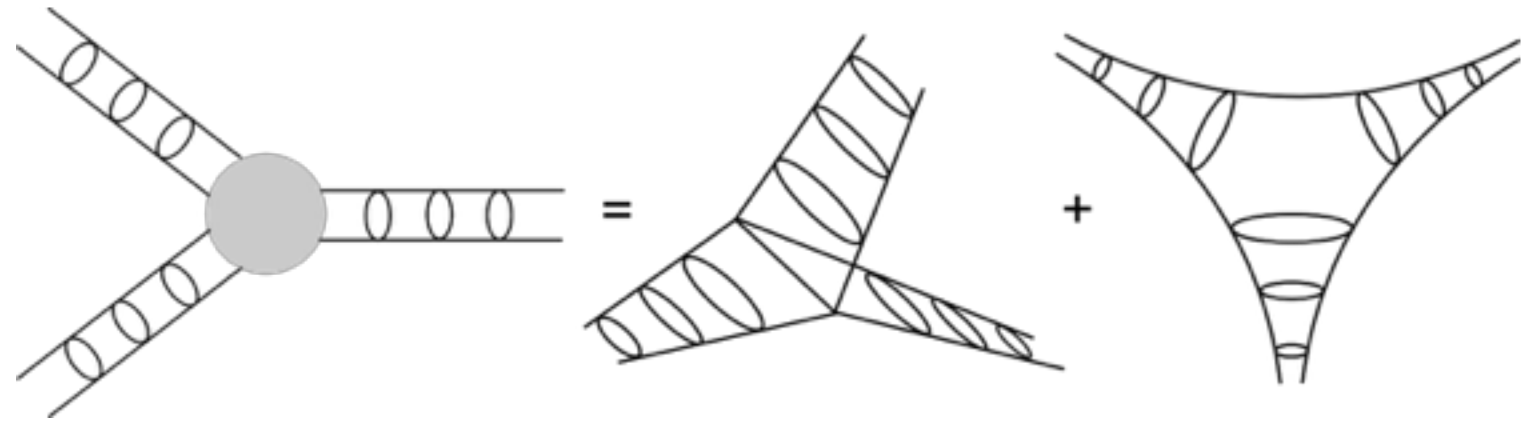
(the lines on the higher-point functions are really dressed propagators)

We can compute all large N correlation functions in SYK by summing all Feynman diagrams.



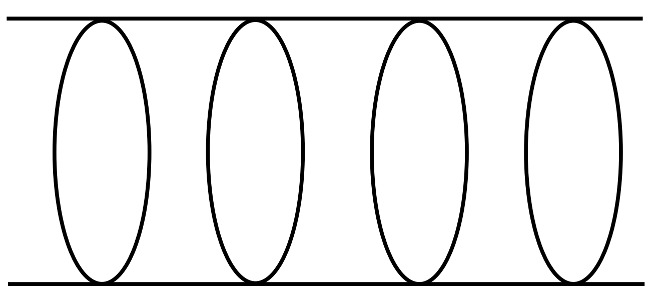
Sachdev & Ye

2-pt: Melons -> Conformal in IR



6-pt: glue three 4-pt functions

Gross & V.R.

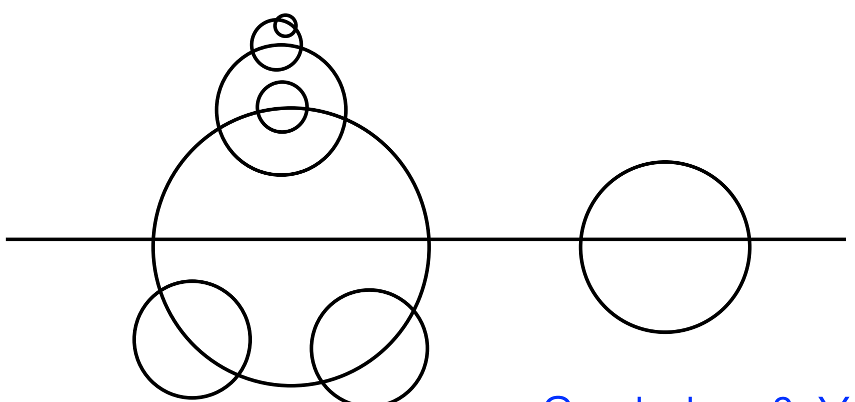


Kitaev; Polchisnki+ V.R.; Maldacena & Stanford

4-pt: Ladders: geometric sum

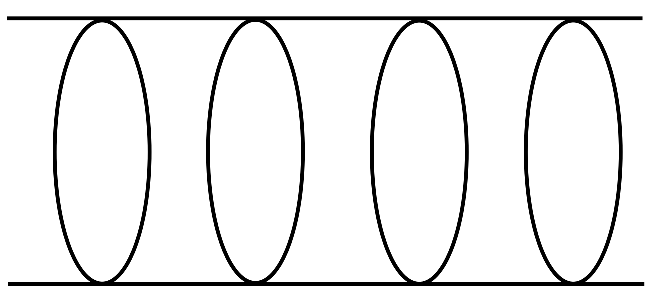
(the lines on the higher-point functions are really dressed propagators)

We can compute all large N correlation functions in SYK by summing all Feynman diagrams.



Sachdev & Ye

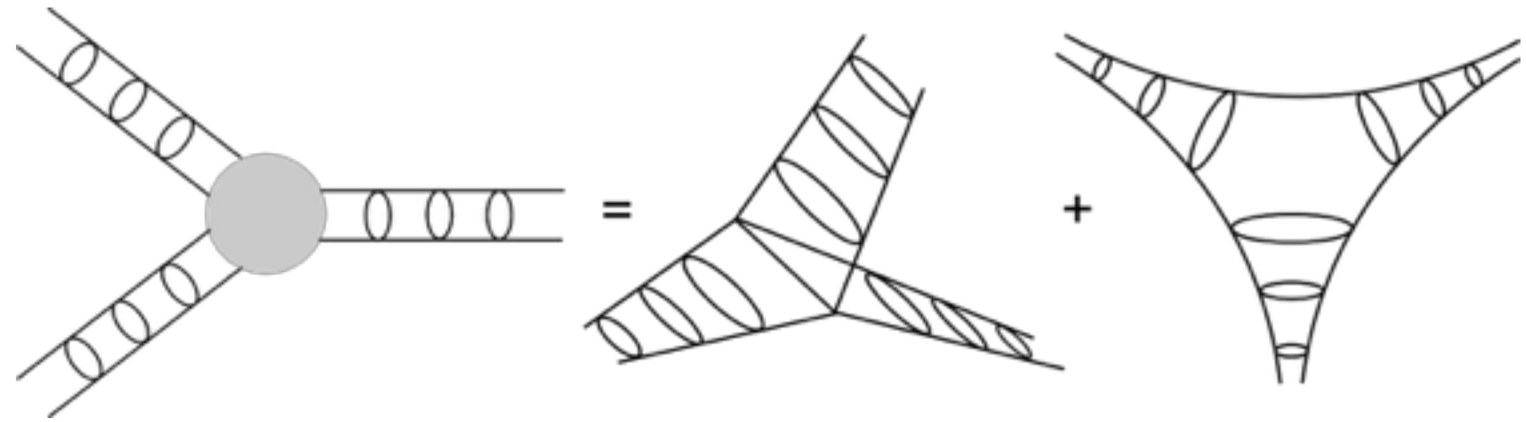
2-pt: Melons -> Conformal in IR



Kitaev; Polchisnki+ V.R.; Maldacena & Stanford

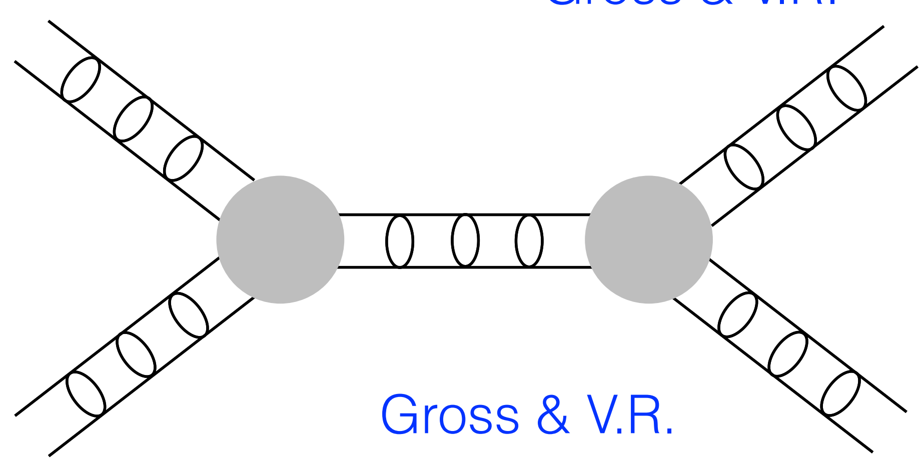
4-pt: Ladders: geometric sum

(the lines on the higher-point functions are really dressed propagators)



6-pt: glue three 4-pt functions

Gross & V.R.



Gross & V.R.

+ cross-channels
- no exchanged melons
+ 4-pt contact

8-pt: glue 4-pt functions

Higher Dimensions

$$\mathcal{L} = \frac{1}{2}(\partial\phi_i)^2 + \frac{1}{q!}J_{i_1 i_2 \cdots i_q}\phi_{i_1}\phi_{i_2}\cdots\phi_{i_q}$$

Bosonic d-dimensional model would give same diagrams, but has negative directions. It is only formally defined.

Higher Dimensions

$$\mathcal{L} = \frac{1}{2}(\partial\phi_i)^2 + \frac{1}{q!}J_{i_1 i_2 \cdots i_q}\phi_{i_1}\phi_{i_2}\cdots\phi_{i_q}$$

Bosonic d-dimensional model would give same diagrams, but has negative directions. It is only formally defined.

One can construct a well-defined two-dimensional supersymmetric SYK model. Murugan, Stanford, Witten

Higher Dimensions

$$\mathcal{L} = \frac{1}{2}(\partial\phi_i)^2 + \frac{1}{q!} J_{i_1 i_2 \cdots i_q} \phi_{i_1} \phi_{i_2} \cdots \phi_{i_q}$$

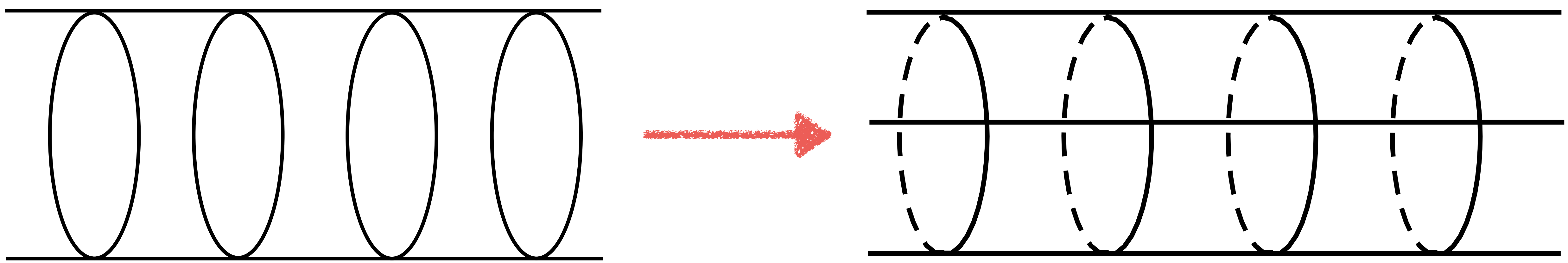
Bosonic d-dimensional model would give same diagrams, but has negative directions. It is only formally defined.

One can construct a well-defined two-dimensional supersymmetric SYK model. Murugan, Stanford, Witten

Perhaps a bosonic higher dimensional model will be found Giombi, Klebanov, Popov, Prakash, Tarnopolsky

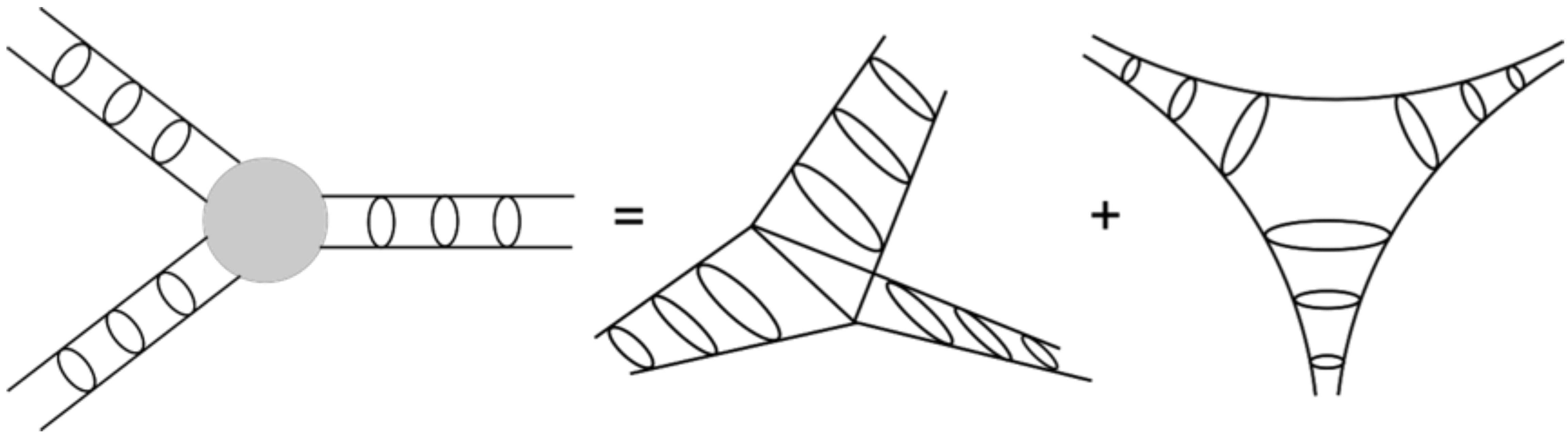
Fishnet theory

Talks by Basso, Gromov, Kazakov



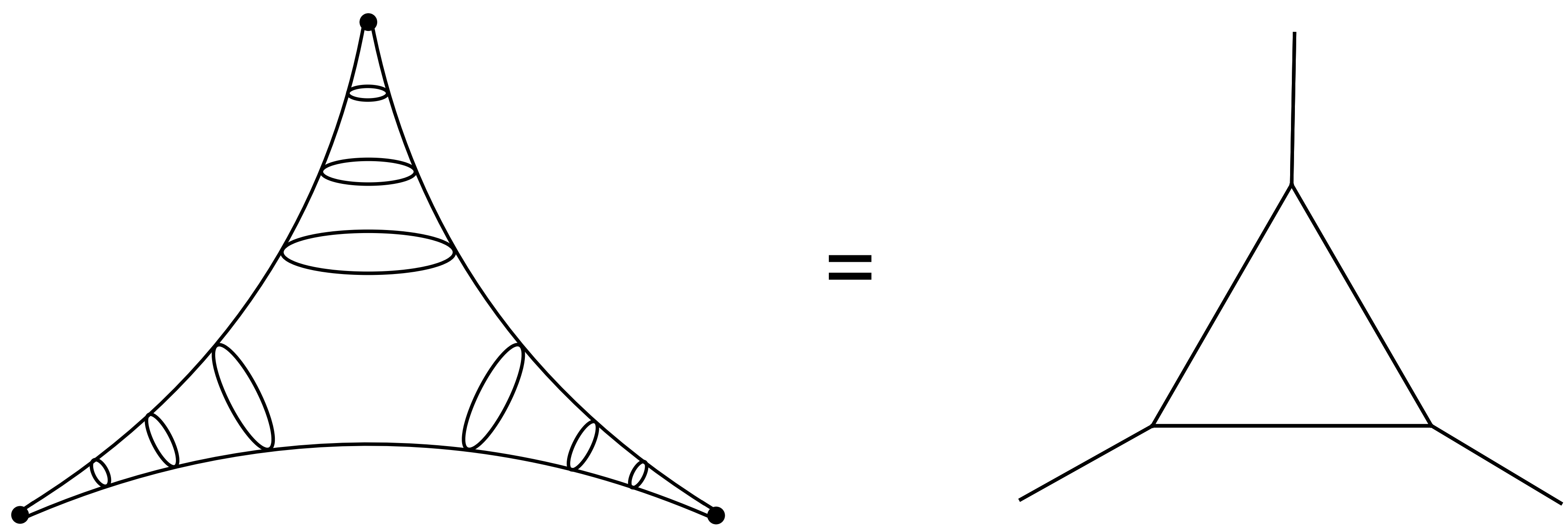
A generalization of SYK diagrams. Can have n parallel lines, corresponding to a two-point function of, schematically, tr(ϕ^n)

SYK 6-pt function of fundamentals (3-pt of bilinears)



The first diagram is what we call the contact diagram, and the second is planar.

Let us look at the planar diagram. As we said before, it corresponds to three 3-pt functions glued together.

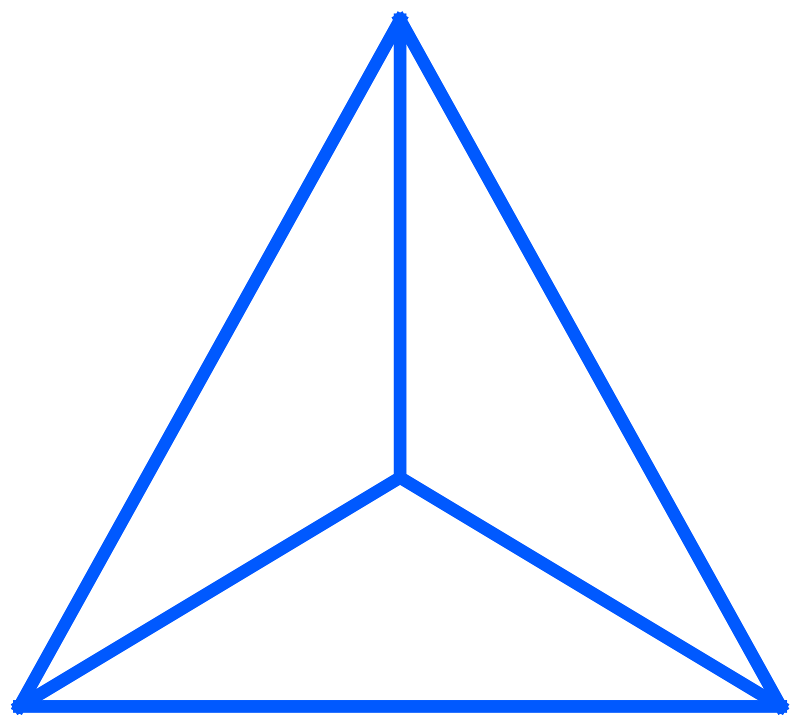


$$\langle \mathcal{O}_{\Delta_1,J_1}(x_1)\mathcal{O}_{\Delta_2,J_2}(x_2)\mathcal{O}_{\Delta_3,J_3}(x_3)\rangle_2 = \int d^d x_a d^d x_b d^d x_c \langle \mathcal{O}_{\Delta_1,J_1}(x_1)\phi(x_a)\phi(x_b)\rangle_{\rm amp}$$
$$\langle \mathcal{O}_{\Delta_2,J_2}(x_2)\phi(x_c)\phi(x_a)\rangle_{\rm amp}\langle \mathcal{O}_{\Delta_3,J_3}(x_3)\phi(x_b)\phi(x_c)\rangle_{\rm amp}$$

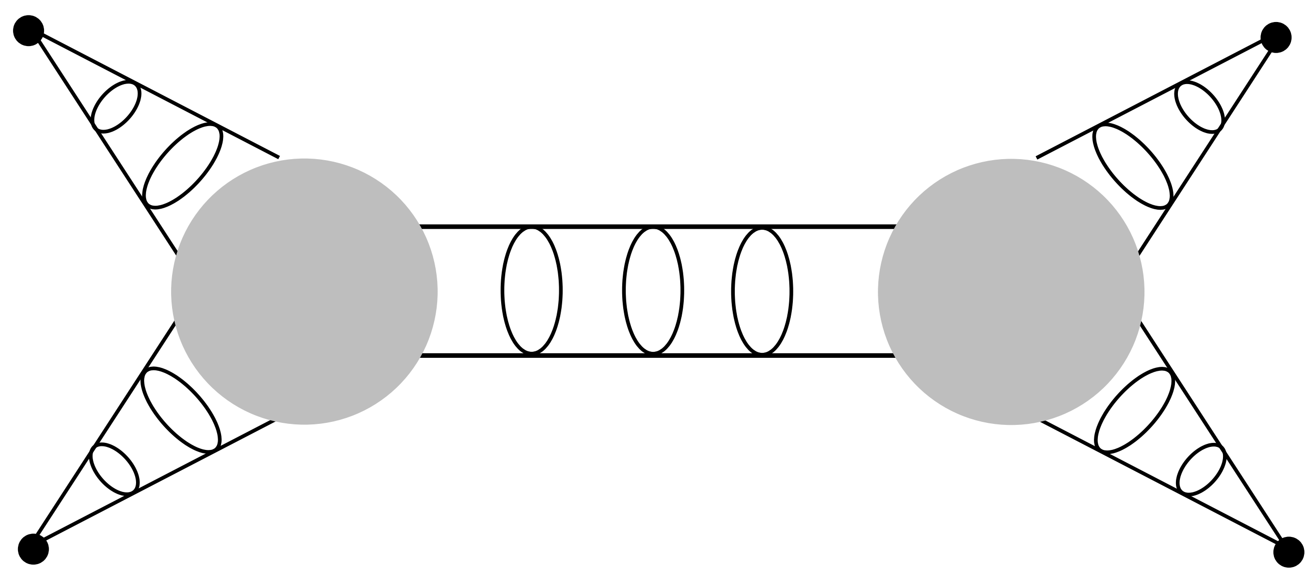
We extract the coefficient by contracting with the functional form of a shadow three-point function.

$$\int \frac{d^d x_1 d^d x_2 d^d x_3 \, d^d x_a d^d x_b d^d x_c}{\mathrm{vol}(\mathrm{SO}(d+1,1))} \langle \mathcal{O}_{\Delta_1,J_1}(x_1)\phi(x_a)\widetilde{\phi}(x_b)\rangle \langle \mathcal{O}_{\Delta_2,J_2}(x_2)\phi(x_c)\widetilde{\phi}(x_a)\rangle$$
$$\langle \mathcal{O}_{\Delta_3,J_3}(x_3)\phi(x_b)\widetilde{\phi}(x_c)\rangle \langle \widetilde{\mathcal{O}}_{\Delta_1,J_1}(x_1)\widetilde{\mathcal{O}}_{\Delta_2,J_2}(x_2)\widetilde{\mathcal{O}}_{\Delta_3,J_3}(x_3)\rangle^b \,.$$

A 6j symbol, as advertised earlier



One can derive a simple formula for the diagrams appearing in the higher point functions.



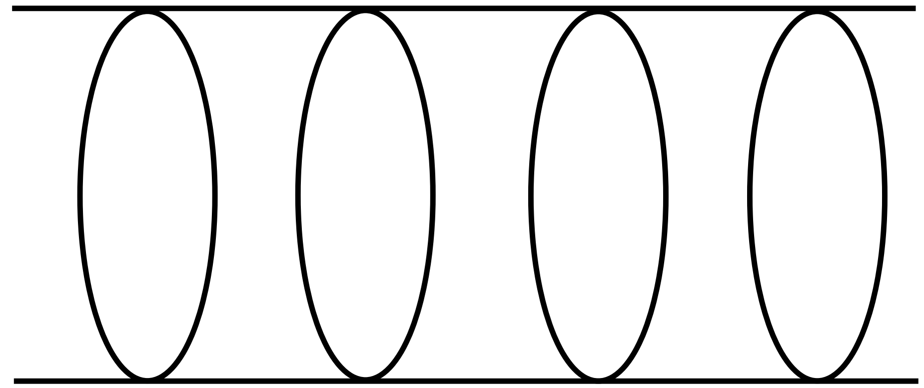
In 1d, h is dimension

$$\int_{\mathcal{C}} \frac{dh}{2\pi i} \, \widetilde{\rho}(h) \, c_{h_1 h_2 h} \, c_{h_3 h_4 h} \, \mathcal{F}^{h}_{h_i}(\tau_i)$$

4-pt function fundamentals (sum of ladders)

3-pt function bilinears

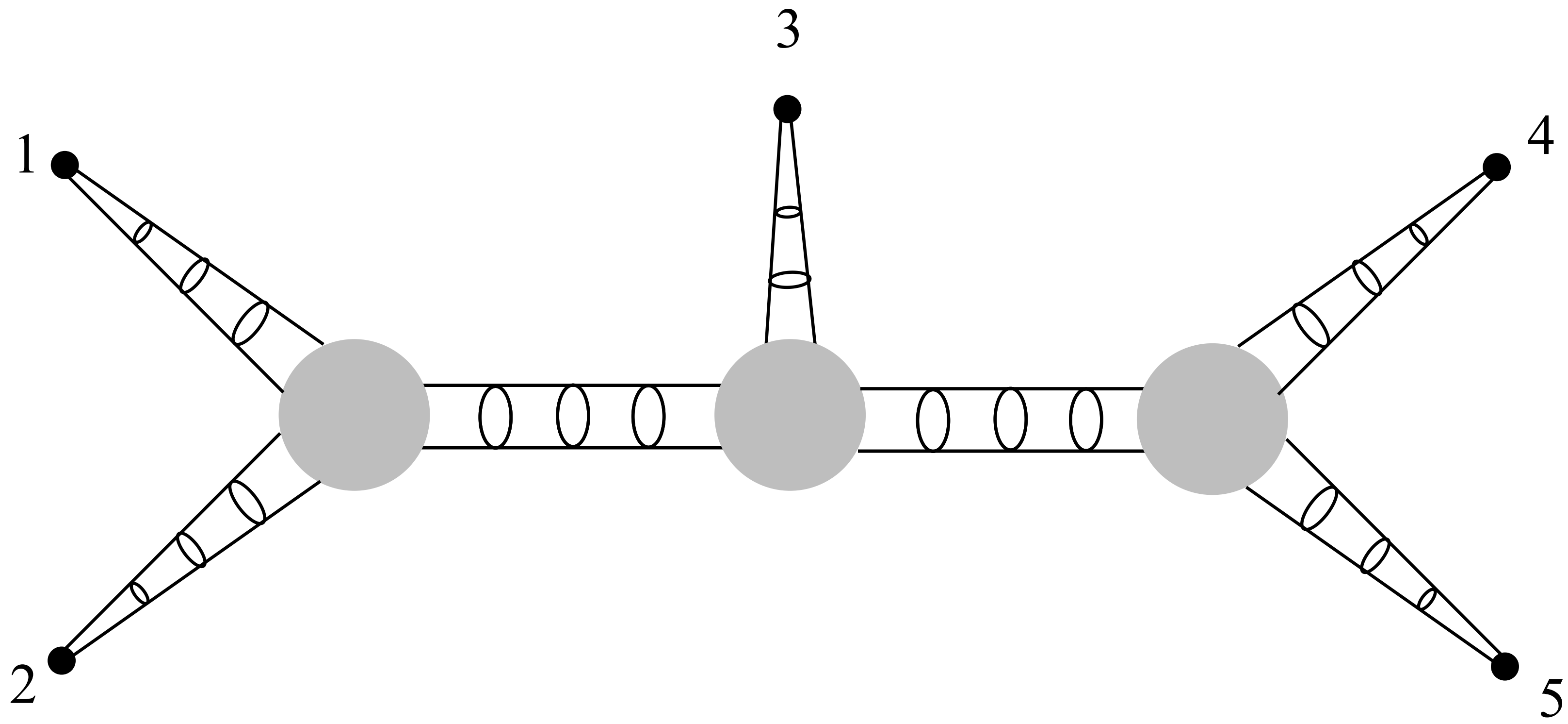
Conformal Block

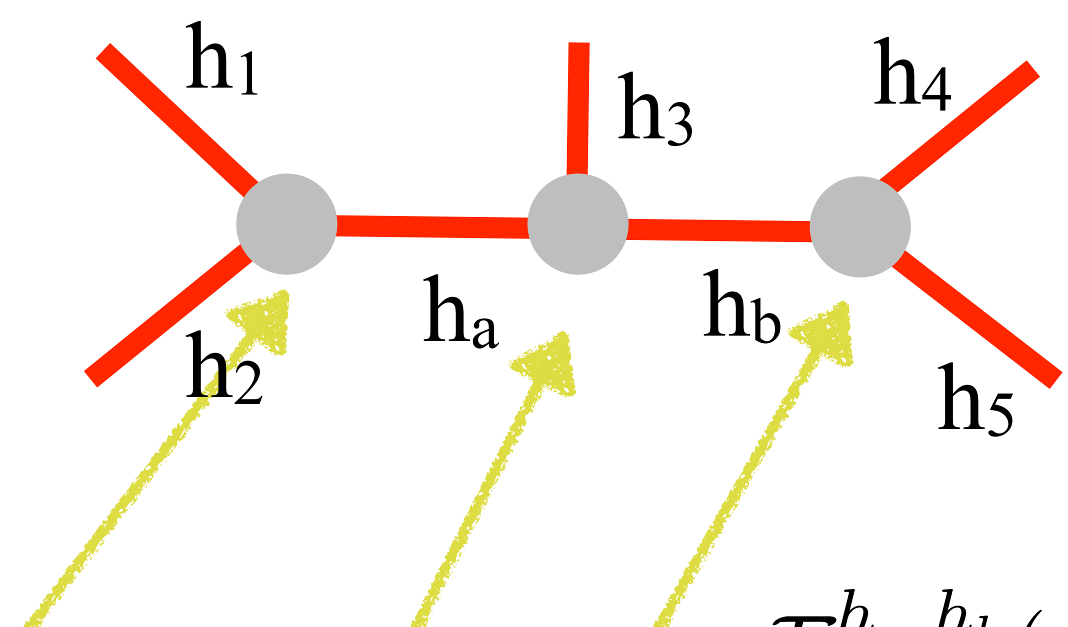


$$\int_{\mathcal{C}} \frac{dh}{2\pi i}\, \rho(h)\, \mathcal{F}^{h}_{\Delta}(\tau_i) \qquad\qquad \rho(h) \sim \frac{k(h)}{1-k(h)}$$

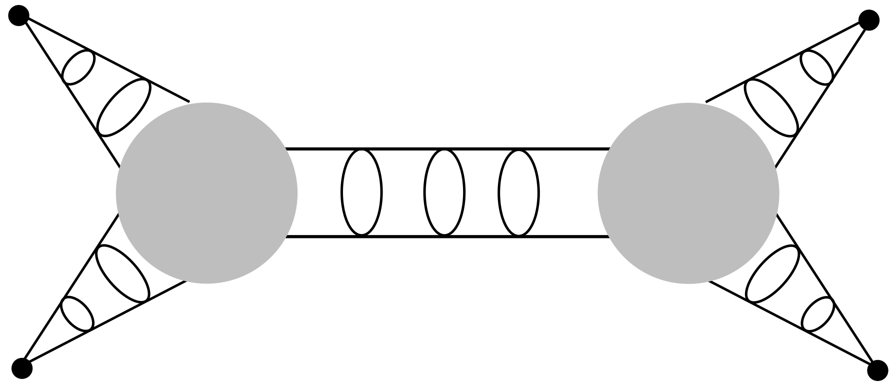
- h-space is to the conformal group, SL(2,R), what Fourier space is to translations

$$\int \frac{dp}{2\pi} f(p)\, e^{ipx}$$



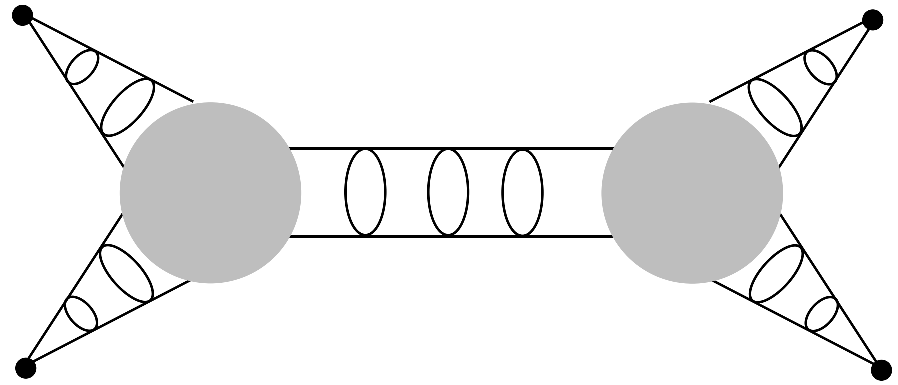


$$\int_{\mathcal{C}} \frac{dh_a}{2\pi i}\,\widetilde{\rho}(h_a)\;\int_{\mathcal{C}} \frac{dh_b}{2\pi i}\,\widetilde{\rho}(h_b)\; c_{h_1 h_2 h_a}\, c_{h_a h_3 h_b}\, c_{h_b h_4 h_5}\; \mathcal{F}_{h_i}^{h_a,h_b}(\tau_i)$$



$$\int_{\mathcal{C}} \frac{dh}{2\pi i}\, \widetilde{\rho}(h)\, c_{h_1 h_2 h}\, c_{h_3 h_4 h}\, \mathcal{F}^{h}_{h_i}(\tau_i)$$

- These are simple rules for summing an infinite number of diagrams. It doesn’t matter that the four-point function is made up of ladders. These apply to any four-point functions.

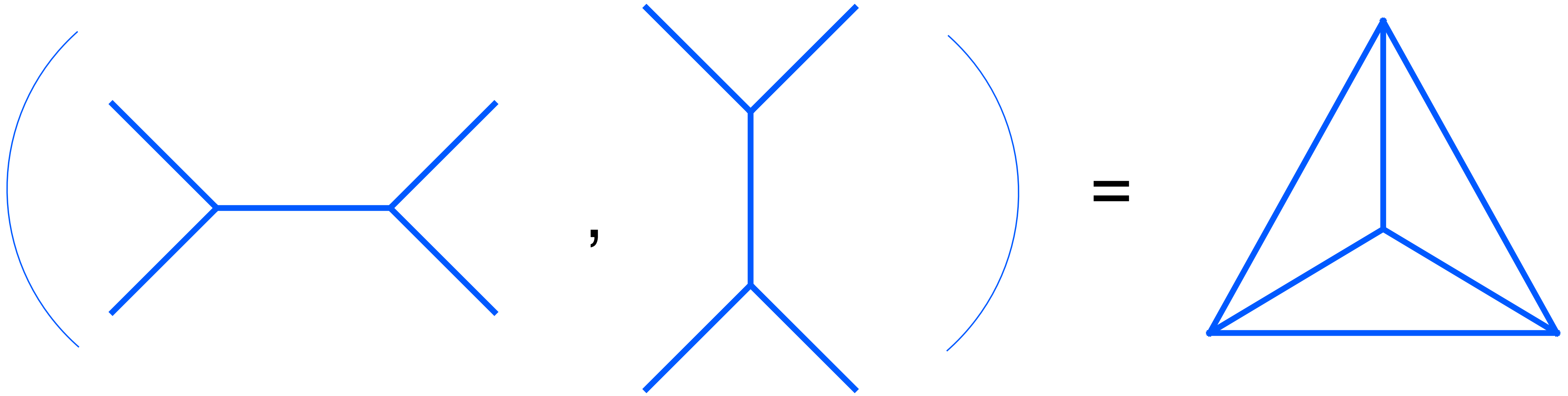


$$\int_{\mathcal{C}} \frac{dh}{2\pi i}\, \widetilde{\rho}(h)\, c_{h_1 h_2 h}\, c_{h_3 h_4 h}\, \mathcal{F}^{h}_{h_i}(\tau_i)$$

- These are simple rules for summing an infinite number of diagrams. It doesn't matter that the four-point function is made up of ladders. These apply to any four-point functions.

- This is not just an OPE expansion. The $c_{h_1 h_2 h_3}$ are the analytically extended OPE coefficients of the single-trace operators. The four-point function is a sum of conformal blocks of single-trace operators and double-trace operators. This emerges upon closing the contour.

Computing the 6j Symbol



$$\left(\Psi^{\widetilde{\Delta}_1,\widetilde{\Delta}_2,\widetilde{\Delta}_3,\widetilde{\Delta}_4}_{\widetilde{\Delta},J}, \Psi^{\Delta_3,\Delta_2,\Delta_1,\Delta_4}_{\Delta',J'}\right)$$

Need to evaluate integral

$$= \int \frac{d^d x_1 \cdots d^d x_4}{\mathrm{vol}(\mathrm{SO}(d+1,1))} \Psi^{\widetilde{\Delta}_1,\widetilde{\Delta}_2,\widetilde{\Delta}_3,\widetilde{\Delta}_4}_{\widetilde{\Delta},J}(x_1,x_2,x_3,x_4) \Psi^{\Delta_3,\Delta_2,\Delta_1,\Delta_4}_{\Delta',J'}(x_3,x_2,x_1,x_4) \ ,$$

$$\Psi^{\Delta_i}_{\Delta,J}(x_i) = \int d^d x_5 \, \langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta,J}(x_5)^{\mu_1\cdots\mu_J}\rangle^a \langle \widetilde{\mathcal{O}}_{\Delta,J;\,\mu_1\cdots\mu_J}(x_5)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle$$

The conformal partial wave is a sum of a conformal block and the shadow block

The conformal partial wave is a sum of a conformal block and the shadow block

In 1d, the conformal block is the hypergeometric function ${}_2F_1$ of a single cross ratio. One can evaluate the integral for the 6j symbol directly, to find a ${}_4F_3$

The conformal partial wave is a sum of a conformal block and the shadow block

In 1d, the conformal block is the hypergeometric function ${}_2F_1$ of a single cross ratio. One can evaluate the integral for the 6j symbol directly, to find a ${}_4F_3$

In higher dimensions, the integral is harder. One would like to somehow make the integral factorize, into a product of one-dimensional integrals. It turns out one can do this, by an appropriate analytic continuation of the contour into Lorentzian signature.

In fact, one can apply Caron-Huot's Lorentzian inversion formula to our integral, which is a special case, with a four-point function that is a conformal partial wave.

In fact, one can apply Caron-Huot's Lorentzian inversion formula to our integral, which is a special case, with a four-point function that is a conformal partial wave.

First, recall that one can expand any four-point function in terms of partial waves

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_{J=0}^{\infty}\int_{\frac{d}{2}}^{\frac{d}{2}+i\infty}\frac{d\Delta}{2\pi i}\frac{I_{\Delta,J}}{n_{\Delta,J}}\Psi^{\Delta_1,\Delta_2,\Delta_3,\Delta_4}_{\Delta,J}(x_i)$$

In fact, one can apply Caron-Huot's Lorentzian inversion formula to our integral, which is a special case, with a four-point function that is a conformal partial wave.

First, recall that one can expand any four-point function in terms of partial waves

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_{J=0}^{\infty}\int_{\frac{d}{2}}^{\frac{d}{2}+i\infty}\frac{d\Delta}{2\pi i}\frac{I_{\Delta,J}}{n_{\Delta,J}}\Psi^{\Delta_1,\Delta_2,\Delta_3,\Delta_4}_{\Delta,J}(x_i)$$

Trivially, using the orthogonality of the partial waves, one can invert this

$$I_{\Delta,J} = \left(\langle O_1\cdots O_4\rangle, \Psi^{\widetilde{\Delta}_i}_{\widetilde{\Delta},J}\right) = \int \frac{d^dx_1\cdots d^dx_4}{\mathrm{vol}(\mathrm{SO}(d{+}1,1))}\langle O_1\cdots O_4\rangle\Psi^{\widetilde{\Delta}_i}_{\widetilde{\Delta},J}(x_i)$$

Deforming the contour gives Caron Huot's Lorentzian inversion formula

$$I_{\Delta,J} = \alpha_{\Delta,J}\Bigg[(-1)^J \int_0^1 \int_0^1 \frac{d\chi d\overline{\chi}}{(\chi\overline{\chi})^d} |\chi - \overline{\chi}|^{d-2} G^{\widetilde{\Delta}_i}_{J+d-1,\Delta-d+1}(\chi,\overline{\chi}) \frac{\langle[\mathcal{O}_3,\mathcal{O}_2][\mathcal{O}_1,\mathcal{O}_4]\rangle}{T^{\Delta_i}}$$

$$+ \int_{-\infty}^0 \int_{-\infty}^0 \frac{d\chi d\overline{\chi}}{(\chi\overline{\chi})^d} |\chi - \overline{\chi}|^{d-2} \widehat{G}^{\widetilde{\Delta}_i}_{J+d-1,\Delta-d+1}(\chi,\overline{\chi}) \frac{\langle[\mathcal{O}_4,\mathcal{O}_2][\mathcal{O}_1,\mathcal{O}_3]\rangle}{T^{\Delta_i}}\Bigg]$$

Caron-Huot
Simmons-Duffin, Stanford, Witten

Deforming the contour gives Caron Huot's Lorentzian inversion formula

$$I_{\Delta,J} = \alpha_{\Delta,J}\Bigg[(-1)^J \int_0^1 \int_0^1 \frac{d\chi d\overline{\chi}}{(\chi\overline{\chi})^d}|\chi-\overline{\chi}|^{d-2} G^{\widetilde{\Delta}_i}_{J+d-1,\Delta-d+1}(\chi,\overline{\chi})\frac{\langle[\mathcal{O}_3,\mathcal{O}_2][\mathcal{O}_1,\mathcal{O}_4]\rangle}{T^{\Delta_i}} + \int_{-\infty}^0 \int_{-\infty}^0 \frac{d\chi d\overline{\chi}}{(\chi\overline{\chi})^d}|\chi-\overline{\chi}|^{d-2} \widehat{G}^{\widetilde{\Delta}_i}_{J+d-1,\Delta-d+1}(\chi,\overline{\chi})\frac{\langle[\mathcal{O}_4,\mathcal{O}_2][\mathcal{O}_1,\mathcal{O}_3]\rangle}{T^{\Delta_i}}\Bigg]$$

Caron-Huot
Simmons-Duffin, Stanford, Witten

Applying this we find the 6j symbols in 2d and 4d. It is expressed in terms of a product of two ${}_4F_3$'s

6j symbol, d=4

$$\mathcal{J}_4 = (-1)^J K^{\Delta_1,\Delta_4}_{\widetilde{\Delta}',J'} \alpha_{\Delta,J} \Bigg(\Theta(\Delta' + \widetilde{J}') \Omega^{\frac{\Delta_i}{2}}_{\frac{\Delta+J}{2},\frac{\Delta'+J'}{2},\frac{\Delta_2+\Delta_3}{2}-1} \Omega^{\frac{\Delta_i}{2}}_{\frac{\widetilde{\Delta}+J}{2},\frac{\Delta'+\widetilde{J}'}{2},\frac{\Delta_2+\Delta_3}{2}-1}$$

$$- (J' \leftrightarrow \widetilde{J}') - (\Delta \leftrightarrow \widetilde{\Delta}) + (J' \leftrightarrow \widetilde{J}', \Delta \leftrightarrow \widetilde{\Delta}) \Bigg)$$

$$+ (1 \leftrightarrow 3, 2 \leftrightarrow 4, \Delta \leftrightarrow \widetilde{\Delta}, \Delta' \leftrightarrow \widetilde{\Delta}'),$$

$$\widetilde{J} \equiv -2 - J.$$

$$\widetilde{\Delta}_i = d - \Delta_i$$

$$\Omega^{h_i}_{h,h',p} \equiv \int_0^1 \frac{d\chi}{\chi^2} \left(\frac{\chi}{1-\chi} \right)^p \chi^{h_{13}} k^{\widetilde{h}_1,\widetilde{h}_2,\widetilde{h}_3,\widetilde{h}_4}_{2h}(\chi) k^{h_3,h_2,h_1,h_4}_{2h'}(1-\chi)$$

$$\Omega^{h_i}_{h,h',p} = \frac{\Gamma(2h)\Gamma(h'-p+1)\Gamma(h'-h_{12}+h_{34}-p+1)\Gamma(-h'+h_{12}+h+p-1)}{\Gamma(h_{12}+h)\Gamma(h_{34}+h)\Gamma(h'-h_{12}+h-p+1)}$$
$$\times {}_4F_3 \left(\begin{matrix} h'+h_{23}, h'-h_{14}, h'-h_{12}+h_{34}-p+1, h'-p+1 \\ 2h', h'-h_{12}+h-p+1, h'-h_{12}-h-p+2 \end{matrix} ; 1 \right)$$
$$+ \frac{\Gamma(2h')\Gamma(h'-h_{12}-h-p+1)\Gamma(h_{13}+h+p-1)\Gamma(h_{42}+h+p-1)}{\Gamma(h'+h_{23})\Gamma(h'-h_{14})\Gamma(h'+h_{12}+h+p-1)}$$
$$\times {}_4F_3 \left(\begin{matrix} h_{13}+h+p-1, h_{42}+h+p-1, h_{34}+h, h_{12}+h \\ h'+h_{12}+h+p-1, 2h, -h'+h_{12}+h+p \end{matrix} ; 1 \right) .$$

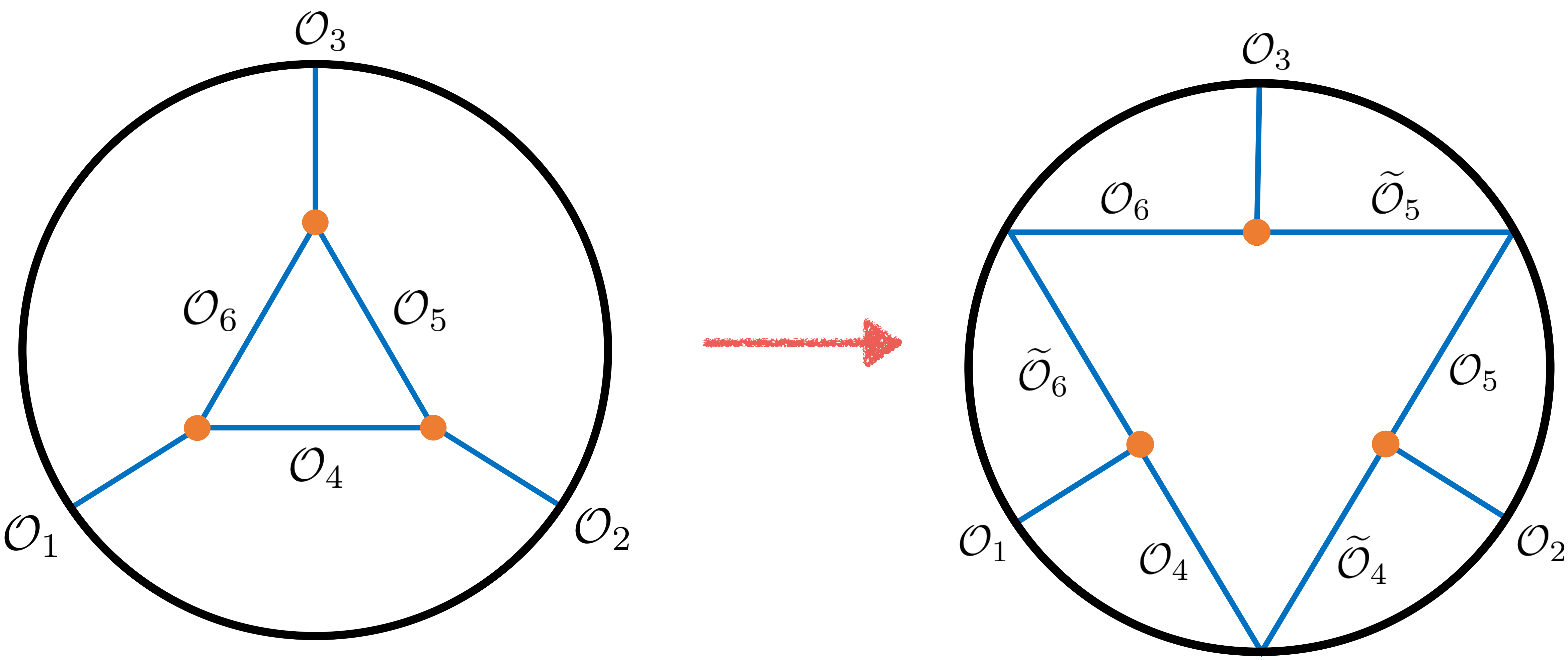
$$K^{\Delta_1,\Delta_2}_{[\Delta_3,J]} = (-\frac{1}{2})^J \frac{\pi^{\frac{d}{2}} \Gamma(\Delta_3 - \frac{d}{2}) \Gamma(\Delta_3 + J - 1) \Gamma(\frac{\widetilde{\Delta}_3+\Delta_1-\Delta_2+J}{2}) \Gamma(\frac{\widetilde{\Delta}_3+\Delta_2-\Delta_1+J}{2})}{\Gamma(\Delta_3 - 1)\Gamma(d-\Delta_3+J)\Gamma(\frac{\Delta_3+\Delta_1-\Delta_2+J}{2})\Gamma(\frac{\Delta_3+\Delta_2-\Delta_1+J}{2})} .$$

$$\Theta(x) \equiv \frac{4\pi^2}{\Gamma(\frac{\Delta_3+\Delta_2-x}{2})\Gamma(1-\frac{\Delta_3+\Delta_2-x}{2})\Gamma(\frac{\Delta_1+\Delta_4-x}{2})\Gamma(1-\frac{\Delta_1+\Delta_4-x}{2})}$$

$$\alpha_{\Delta,J} = -\frac{t_0}{2}(2\pi)^{d-2} \frac{\Gamma(J+1)}{\Gamma(J+\frac{d}{2})} \frac{\Gamma(\Delta - \frac{d}{2})}{\Gamma(\Delta-1)} \frac{\Gamma(\frac{\Delta_{12}+J+\Delta}{2})\Gamma(\frac{\Delta_{21}+J+\Delta}{2})\Gamma(\frac{\Delta_{34}+J+\widetilde{\Delta}}{2})\Gamma(\frac{\Delta_{43}+J+\widetilde{\Delta}}{2})}{\Gamma(J+\Delta)\Gamma(J+d-\Delta)}$$

Triangle Loop in AdS

Triangle Loop in AdS



$$\Omega_\nu(y_1, y_2) = \frac{i\nu}{2\pi}(G_\nu(y_1, y_2) - G_{-\nu}(y_1, y_2))$$

$$\Omega_\nu(y_1, y_2) = \frac{\nu^2}{\pi} \int dx_1 K_\nu(x_1, y_1) K_{-\nu}(x_1, y_2)$$

$$\Delta = \frac{d}{2} + i\nu$$

Finally, do the three bulk integrals. This leaves an integral involving three CFT 3-point functions.

Summary

- We gave three contexts in which the conformal 6j symbol appears: the crossing kernel, Feynman diagrams in SYK, and a Witten loop diagram in AdS
- We computed the 6j symbol in d=1, 2, 4
- We gave a simple formula for all-point correlation functions in SYK, by summing all Feynman diagrams.