The Chiral Algebra Program for $4d$ SCFTs

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Based on past work with Beem, Lemos, Liendo, Peelaers, van Rees
and upcoming work with Beem, Bonetti and Meneghelli.

IGST, Copenhagen
August 21 2018

To any $\mathcal{N} = 2$ SCFT, one can canonically associate a vertex operator algebra

$$\chi : \ \text{4d } \mathcal{N} = 2 \ \text{SCFT} \longrightarrow \text{VOA}$$

Beem Lemos Liendo Peelaers LR van Rees

The associated VOA $\mathcal{V}$ is a closed subalgebra of the full local operator algebra of the $4d$ theory. It has many uses:

- Exact results in 4d SCFTs.
- Predictions for VOAs.
- Organizing principle for the whole $\mathcal{N} = 2$ SCFT landscape.

The VOA is a very rich invariant of the $4d$ SCFT.
It appears to be deeply connected with the physics of the Higgs branch,
but precisely how has so far been elusive.

Lightening review of $\mathcal{N} = 2$ SCFTs

Conventional Lagrangian theories specified by the following data:

- Gauge group $G = G_1 \times G_2 \times \dots G_k$
  Vector multiplets $(\phi, \lambda_\alpha^{\mathcal{I}}, A_\mu)$ in Adj of G.
- A pseudoreal finite-dimensional representation ρ of G.
  Half-hypermultiplets (Q, ψ_α) in ρ of G.

For each simple factor G_i, must impose vanishing of the beta function,

$$\beta = -2h^\vee + C_2(\rho) = 0\,.$$

Complete combinatorial classification of possibilities by Bardhway Tachikawa.

Lagrangian uniquely fixed by $\mathcal{N} = 2$ superconformal symmetry,
up to the value of complexified gauge couplings $\{\tau_i\}$, $i = 1, \dots k$.
They parametrize the conformal manifold $\mathcal{U}$ of the $\mathcal{N} = 2$ SCFT.

We have learnt that the landscape of $\mathcal{N} = 2$ SCFTs is much bigger.

Exotic "matter" in the form of isolated SCFTs with no conventional Lagrangian description. Canonical examples: Gaiotto's T_N theories.

Coupling them to gauge fields (by gauging a subgroup G of their flavor symmetry group, such that $\beta_G = 0$) leads to large continuous families of SCFT.

In all known examples, exactly marginal couplings arise from weak gauging of isolated SCFTs. Perhaps this is a general fact.

We are interested in $\mathcal{N}=2$ $4d$ SCFTs from the viewpoint of the OPE algebra of their local operators,

$$\mathcal{O}_i(x)\mathcal{O}_j(y)=\sum_k \frac{c_{ij}^k}{|x-y|^{E_i+E_j-E_k}}\mathcal{O}_k(y)\,,\quad x\,,y\in\mathbb{R}^4$$

$\{\mathcal{O}_i(x)\}$ can be organized in representation of the $\mathfrak{su}(2,2|2)$ superalgebra.

Bosonic subalgebra: $\mathfrak{su}(2,2)\times\mathfrak{su}(2)_R\times\mathfrak{u}(1)_r$.
$\mathfrak{su}(2,2)\cong\mathfrak{s}o(4,2)$: conformal algebra in $d=4$,
$\mathfrak{su}(2)_R\times\mathfrak{u}(1)_r$ R-symmetry

Fermionic generators: Poincaré supercharges $\mathcal{Q}^{\mathcal{I}}_{\alpha}\,,\quad\tilde{\mathcal{Q}}_{\mathcal{I}\,\dot{\alpha}}$
Conformal supercharges $\mathcal{S}^{\alpha}_{\mathcal{I}}\,,\quad\tilde{\mathcal{S}}^{\mathcal{I}\,\dot{\alpha}}$

We label operators by the Cartan quantum numbers (E,R,r,j_1,j_2).

In our conventions:
ϕ complex scalar of the vector multiplet: $r=-1$, $R=0$
$Q^{\mathcal{I}}$ complex scalars of the hypermultiplet: $r=0$, $R=\pm1/2$.

While for operators in generic representations the OPE algebra is extremely complicated, simplifications occur for operators in shortened representations.

Some interesting *commutative* subalgebras are obtained by considering suitable cohomologies with respect to *Poincaré* supercharges:

- Coulomb chiral ring $\mathcal{R}_{\mathrm{C}}$, operators with $E = r$

  In a Lagrangian theory, $\mathcal{R}_{\mathrm{C}} =$ ring of G-invariant polynomials.
  E.g. $G = SU(N)$, $\mathcal{R}_{\mathrm{C}}$ generated by $\mathrm{Tr}\,\phi^2\,, \mathrm{Tr}\,\phi^3\,\ldots \mathrm{Tr}\,\phi^N$.

- Higgs chiral ring $\mathcal{R}_{\mathrm{H}}$, operators with $E = 2R$

  In a Lagrangian theory, $\mathcal{R}_{\mathrm{H}}$ spanned by gauge-invariants of $\mathcal{Q}$s.

- Hall-Littlewood chiral ring $\mathcal{R}_{\mathrm{HL}} \supset \mathcal{R}_{\mathrm{H}}$, operators with $E = 2R + j_2$

  In a Lagrangian theory, $\mathcal{R}_{\mathrm{HL}}$ spanned by gauge-invariants of $\mathcal{Q}$s and λs.

A much richer *non-commutative* subalgebra is defined by passing to the cohomology of $\mathbb{Q} = \mathcal{Q}^1_- + \tilde{\mathcal{S}}^{2\dot{-}}$.

It has the structure of a vertex operator algebra (VOA).

VOA basics

Informally, a VOA is a meromorphic OPE algebra in two dimensions,

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_k \frac{c_{12}^{\,k}\mathcal{O}_k(w)}{(z-w)^{h_1+h_2-h_k}}$$

Each operator can be expanded in a Laurent series,

$$\mathcal{O}(z) = \sum_{n=-\infty}^{\infty} z^{-n-h}\mathcal{O}_n\,, \qquad \mathcal{O}_n \in \mathrm{End}(\mathcal{V})$$

The modes are conserved charges acting on the state space $\mathcal{V}$.

$\mathcal{V}$ is identified with the space $\{\mathcal{O}_i(0)\}$ of local operators at the origin,

$$\partial^n\mathcal{O}(0) \Leftrightarrow \mathcal{O}_{-n-h}|\Omega\rangle\,.$$

The normal ordered product of operators a and b is defined at the origin as

$$\mathrm{NO}(a,b)(0) := a_{-h_a} b_{-h_b} |\Omega\rangle \,.$$

The bracket $\{,\}$ is defined by picking out the simple pole in the OPE

$$\{a,b\}(w) = \oint \frac{dz}{2\pi i} a(z) b(w)$$

A *strong generator* of a VOA is an $\mathfrak{sl}(2)$ primary that *cannot* be written as N.O. product. All operators of a VOA can then be written as N.O. products of the strong generators and their holomorphic derivatives.

Schur operators

Our state space $\mathcal{V}$ is the space of Schur operators of the SCFT, defined as the operators that satisfy

$$R = \frac{E - j_1 - j_2}{2}\,, \quad r = j_2 - j_1 \quad \Rightarrow L_0 = h = E - R\,,$$

where E is the conformal dimension and j_1, j_2 the Lorentz spins.

The Schur operators comprise:

- Higgs chiral ring operators: $E = 2R$, $r = 0 \Rightarrow h = R$.
- Hall-Littlewood (anti)-chiral ring operators: $h = R \pm r$.
- Generic Schur operators: semi-short operators with $h > R + |r|$.

(Coulomb ring operators not Schur.)

Vacuum character of VOA = Schur index of the $4d$ SCFT

$$q^{c_{4d}/2}\mathrm{STr}_{\mathcal{H}[S^3]}(q^{E-R}) = \mathrm{STr}_{\mathcal{V}}(q^{L_0 - c_{2d}/24})$$

Away from the origin, $\mathbb{Q}$-closed operators are "twisted-translated",

$$\mathcal{O}(z,\bar{z}) = e^{zL_{-1}+\bar{z}(\bar{L}_{-1}+\mathcal{R}^{-})}\,\mathcal{O}_{\mathrm{Sch}}(0)\,e^{-zL_{-1}-\bar{z}(\bar{L}_{-1}+\mathcal{R}^{-})}\ .$$

In terms of a basis $\mathcal{O}^{(I_1\ldots I_{2R})}$ of the $\mathfrak{su}(2)_R$ representation,

$$\mathcal{O}(z,\bar{z}) = \mathcal{O}^{(+\cdots+)}(z,\bar{z}) + \bar{z}\mathcal{O}^{(+\cdots-)}(z,\bar{z}) + \cdots + \bar{z}^{2R}\mathcal{O}^{(-\cdots-)}(z,\bar{z})\,.$$

The $\bar{z}$ dependence is in fact exact at the level of $\mathbb{Q}$-cohomology,
so correlators are meromorphic.

The holomorphic conformal weight is

$$h = E - R = R + j_1 + j_2$$

R grading is lost. OPE compatible with a filtration (R cannot increase).

VOAs that descend from $4d$ theories inherit the R-filtration.
However, the filtration is not intrinsic to $\mathcal{V}$.
Given an abstract presentation of $\mathcal{V}$, it is a priori unknown how to assign it.

Structural properties

$$\chi : \quad \text{4d } \mathcal{N} = 2 \text{ SCFT} \longrightarrow \text{VOA}$$

- $\chi[\mathcal{T}_u]$ is independent of exactly marginal couplings $u \in \mathcal{U}$.
- Virasoro enhancement of $\mathfrak{sl}(2)$, with $c_{2d} = -12\, c_{4d}$,
  where c_{4d} is one of the conformal anomaly coefficient.
- Affine enhancement of global flavor symmetry, with $k_{2d} = -\frac{k_{4d}}{2}$.
- Generators of $\mathcal{R}_{\rm HL}[\mathcal{T}]$ are strong generators of the VOA $\chi[\mathcal{T}]$.

Above properties often enough to determine $\chi[\mathcal{T}]$ from *minimal input* about $\mathcal{T}$.

- $\mathcal{M}_{\rm Higgs} = X_{\mathcal{V}}$ (still a conjecture) Beem LR

  $\mathcal{M}_{\rm Higgs}$ = Higgs branch of the $4d$ SCFT, $X_{\mathcal{V}}$ = associated variety of $\mathcal{V}$.

  In particular, the associated variety must be symplectic. This has deep consequences. *E.g.*, modular behavior of the Schur index.

The associated variety $X_{\mathcal{V}}$

Arakawa

There is a natural commutative algebra associated to $\mathcal{V}$,

$$\mathcal{R}_{\mathcal{V}} := \mathcal{V}/C_2[\mathcal{V}]$$

$$C_2[\mathcal{V}] := \mathrm{Span}\{a_{-h_a-1}b\,,\ a\,,b\in\mathcal{V}\}\,.$$

Normal ordering descends to a commutative product on $\mathcal{R}_{\mathcal{V}}$.

In words: to determine $\mathcal{R}_{\mathcal{V}}$, *remove* states containing holomorphic derivatives.
For example, for an affine current algebra at generic level k,

$$\mathcal{R}_{\mathcal{V}} = \mathrm{Span}\{J^{A_1}_{-1}\ldots J^{A_n}_{-1}|\Omega\rangle\}\,,$$

i.e., the polynomial ring $\mathbb{C}[\jmath^A]$ in the "zero-modes" $\jmath^A \equiv J^A_{-1}|\Omega\rangle$.

$\mathcal{R}_{\mathcal{V}}$ is also endowed with a Poisson structure induced by the bracket $\{,\}$.

$$X_{\mathcal{V}} := \mathrm{MaxSpec}\,\mathcal{R}_{\mathcal{V}}$$

What is the image of χ?

The category of general VOAs is unwieldy. No classification remotely in sight.

VOAs that arise from $4d$ SCFTs by our map χ are however very special.

In the rest of the talk, I'll describe a striking property true in many examples:

They admit free field constructions of their *simple quotient*.
Physically, these constructions can be understood in terms of the low energy degrees of freedom on the Higgs branch.

We believe this is universal fact about VOAs associated to $4d$ SCFT.
Combined with $4d$ unitarity, it should lead to a tractable classification program.

Example: H_1 Argyres-Douglas and $\mathfrak{sl}(2)_{-4/3}$

Beem Meneghelli LR, to appear

The H_1 SCFT can be realized by a single D3 brane probing the A_1 singularity. Its Higgs branch is the one-instanton moduli space for $\mathfrak{g} = \mathfrak{sl}(2)$,

$$\mathcal{M}_{\text{Higgs}} = \overline{\mathbb{O}_{\text{min}}}(\mathfrak{g}) := G\text{-orbit of the highest root of } \mathfrak{g}\,.$$

Writing

$$M = \begin{pmatrix} \jmath^3 & \jmath^+ \\ \jmath^- & -\jmath^3 \end{pmatrix} \in \mathfrak{sl}(2)\,, \qquad M^2 = \begin{pmatrix} (\jmath^3)^2 + \jmath^+ \jmath^- & 0 \\ 0 & (\jmath^3)^2 + \jmath^+ \jmath^- \end{pmatrix} \stackrel{!}{=} 0$$

we recognize

$$\mathcal{M}_{\text{Higgs}} = \text{Span}[\jmath^3, \jmath^+, \jmath^-]/\langle (\jmath^3)^2 + \jmath^+ \jmath^- \rangle \cong \mathbb{C}^2/\mathbb{Z}_2\,.$$

Symplectic holomorphic structure encoded in the brackets

$$\{\jmath^+, \jmath^-\} = 2\jmath^3\,, \quad \{\jmath^3, \jmath^+\} = \jmath^+\,, \quad \{\jmath^3, \jmath^-\} = -\jmath^-\,.$$

The VOA associated to the H_1 SCFT is the affine current algebra $\mathfrak{sl}(2)_{-4/3}$.

The generators are J^A, $A = 1, 2, 3$, with

$$J^A(z)J^B(w) \sim \frac{k\delta^{AB}}{(z-w)^2} + \frac{\epsilon^{ABC}J^C(w)}{z-w}, \quad k = -\frac{4}{3}$$

For $k = -4/3$, maximal ideal generated by level three singular vector

$$\mathcal{N}^A = J^A(J^1J^1 + J^2J^2 + J^3J^3)$$

and thus

$$\mathcal{R}_\mathcal{V} = \mathbb{C}[\jmath^1, \jmath^2, \jmath^3]/\langle \jmath^A(\jmath^1\jmath^1 + \jmath^2\jmath^2 + \jmath^3\jmath^3)\rangle .$$

Note that

$$\left(\sum_A \jmath^A\jmath^A\right)^2 = 0 \quad \text{in } \mathcal{R}_\mathcal{V} ,$$

so $\mathcal{R}_\mathcal{V}$ is *not* reduced. The nilradical is simply the quadratic Casimir, so

$$(\mathcal{R}_\mathcal{V})_{\text{red}} = \mathbb{C}[\jmath^1, \jmath^2, \jmath^3]/\langle \jmath^1\jmath^1 + \jmath^2\jmath^2 + \jmath^3\jmath^3\rangle .$$

Finally, the associated variety

$$X_\mathcal{V} := \text{MaxSpec}(\mathcal{R}_\mathcal{V}) = \text{Span}[\jmath^1, \jmath^2, \jmath^3]/\langle \jmath^1\jmath^1 + \jmath^2\jmath^2 + \jmath^3\jmath^3\rangle \cong \mathcal{M}_{\text{Higgs}} .$$

With hindsight, we can retrace the above steps and recover the VOA by an "affine uplift" of $X_{\mathcal{V}} \cong \mathcal{M}_{\rm Higgs}$.

The starting point is

$$\mathcal{M}_{\rm Higgs} = {\rm Span}[\jmath^3, \jmath^+, \jmath^-]/\langle (\jmath^3)^2 + \jmath^+ \jmath^- \rangle .$$

Let's solve the relation in the open patch $\jmath^+ \neq 0$,

$$\jmath^- = -\frac{(\jmath^3)^2}{\jmath^+} .$$

The symplectic structure is then encoded in the single bracket $\{\jmath^3, \jmath^+\} = \jmath^+$.

Define the affine uplift in terms of two chiral bosons φ and δ:

$$\jmath^3 \to J^3 = \frac{1}{\langle \varphi, \varphi \rangle} \partial \varphi \,, \quad \jmath^+ \to J^+ = e^{\delta + \varphi}$$

$$\{\jmath^3, \jmath^+\} = \jmath^+ \longrightarrow J^3(z) J^+(0) \sim \frac{J^+(0)}{z}$$

$$\{j^+, j^+\} = 0 \to J^+ J^+ \sim \text{reg} \Rightarrow \langle \delta, \delta \rangle = -\langle \varphi, \varphi \rangle$$

For $j^- \to J^-$, consider the ansatz

$$J^- = (c_1 (\partial \delta)^2 + c_2 (\partial \varphi)^2) \, e^{-\delta - \varphi}$$

c_1 and c_2 are uniquely fixed, and finally $J^- J^- \sim \text{reg}$ only if

$$k \in \{-2, -\frac{1}{2}, -\frac{4}{3}\} \, .$$

This is Adamovic's construction of $\mathfrak{sl}(2)_{-4/3}$.

Remarkably, this is a construction of the *simple quotient* of $\mathcal{V}$:
all singular vectors are automatically removed (they are identically zero).

Note that only *two* bosons are used, unlike in the standard Wakimoto construction of $\mathfrak{sl}(2)_k$, which needs *three*.

Physical interpretation: EFT on the Higgs branch

This is the first example of a seemingly universal phenomenon:
free field construction mimics the EFT description at a generic point of $\mathcal{M}_{\text{Higgs}}$.

In the H_1 SCFT, the low energy degrees of freedom on the Higgs branch consist of a single free hyper (= two half hypers), in correspondence with the two chiral bosons φ and δ.

We will now generalize this simple example in various directions, organized by the complexity of the EFT description:

- The EFT only includes hypers, but with more complicated geometry;
- The EFT includes vector multiplets in addition to hypermultiplets;
- Apart from free fields, an interacting SCFT (with trivial Higgs branch) survives at low energies.

Free field construction of Deligne-Cvitanović series

Beem Meneghelli LR, to appear

$\mathfrak{sl}(2)_{-4/3}$ generalizes to a remarkable sequence of AKM algebras $\hat{\mathfrak{g}}_k$,

with $\mathfrak{g} \in \{A_1\,, A_2\,, G_2\,, D_4\,, F_4\,, E_6\,, E_7\,, E_8\}$ and $k = -\frac{h^\vee}{6} - 1\,.$

With the exceptions of G_2 and F_4 (whose $4d$ interpretation in unclear), these are the VOAs are associated to the rank one SCFTs that arise on a single D3 brane probing the corresponding F-theory singularity.

$$\mathcal{M}_{\text{Higgs}} = \overline{\mathbb{O}_{\mathsf{min}}}(\mathfrak{g}) := \; G\text{-orbit of } e_\theta \text{ (highest root)}$$

A calculation shows that $X_{\mathcal{V}} = \overline{\mathbb{O}_{\mathsf{min}}}(\mathfrak{g}) \cong \mathcal{M}_{\text{Higgs}}$. Arakawa Moreau

Let $\mathfrak{sl}_\theta(2) = \langle e_\theta, f_\theta, h_\theta \rangle \subset \mathfrak{g}$ the $\mathfrak{sl}(2)$ triple associated to the highest root.

Decomposing $\mathfrak{g}$ with respect to $\mathfrak{sl}_\theta(2)$ and its commutant $\mathfrak{g}^\natural$,

$$\mathfrak{g} = \left(\mathfrak{g}^\natural \oplus \mathfrak{sl}_2\right) \oplus (\mathfrak{R}, 2) \; = \; \mathbb{C}f_\theta \oplus \mathfrak{R}^- \oplus \left(\mathfrak{g}^\natural \oplus \mathbb{C}h_\theta\right) \oplus \mathfrak{R}^+ \oplus \mathbb{C}e_\theta \, .$$

$\mathfrak{g}$	$\mathfrak{g}^\natural$	$\mathfrak{R}$	$h^\vee$	k
A_1	$-$	$-$	2	$-\frac{4}{3}$
A_2	$\mathbb{C}$	$\mathbf{1}_+ \oplus \mathbf{1}_-$	3	$-\frac{3}{2}$
G_2	A_1	$\mathbf{4}$	4	$-\frac{5}{3}$
D_4	$A_1 \oplus A_1 \oplus A_1$	$(\mathbf{2}, \mathbf{2}, \mathbf{2})$	6	-2
F_4	C_3	$\mathbf{14}'$	9	$-\frac{5}{2}$
E_6	A_5	$\mathbf{20}$	12	-3
E_7	D_6	$\mathbf{32}$	18	-4
E_8	E_7	$\mathbf{56}$	30	-6

"Broken" generators (that don't commute with e_θ): f_θ, h_θ, $\mathfrak{R}_-$
Hence $\dim_{\mathbb{C}}(\mathbb{O}_{\text{min}}) = 2 + \dim(\mathfrak{R}) = 2(h^\vee - 1)$.

$\mathfrak{R}$ is a pseudoreal representation of dimension $2(h^\vee - 2)$.
The commutant $\mathfrak{g}^\natural \subset \mathfrak{sp}(\mathfrak{R})$.

Basic idea: introduce δ, φ to construct $\mathfrak{sl}_\theta(2)$, as before

To capture Goldstones in $\mathfrak{R}$, introduce a system $\mathbb{V}_\xi$ of symplectic bosons,

$$\xi_A(z_1)\xi_B(z_2) \sim \frac{\Omega_{AB}}{z_{12}}\,, \quad A, B = 1, \dots h^\vee - 2\,.$$

As before, $h_\theta \sim \partial\varphi$.

Affine subalgebra $\widehat{\mathfrak{g}}^\natural_{k^\natural} \subset \widehat{\mathfrak{sp}(\mathfrak{R})}_{-\frac{1}{2}} \subset \mathbb{V}_\xi$.

"Easy" generators are the highest weights,

$$e_\theta = e^{\delta+\varphi}\,, \quad e_A = \xi_A\, e^{\frac{\delta+\varphi}{2}}\,.$$

Lowest weights more involved. For example,

$$f_\theta(z) = \Big(T^\natural - (\tfrac{k}{2}\,\partial\delta)^2 + \tfrac{k(k+1)}{2}\partial^2\delta\Big)e^{-(\delta+\varphi)}$$

where the extra term $T^\natural \in \mathbb{V}_\xi$.

This is again a free field construction of the *simple quotient.*

For $\mathfrak{g} \neq A_1$, the maximal ideal of the Deligne-Cvitanović VOAs are generated by level two singular vectors Arakawa Moreau.
Easy to see that they are identically zero in our construction.

Interpretation:
The low energy degrees of freedom on the Higgs branch consist purely of free massless hypers. In this class of examples, they are just the Goldstone bosons of the broken flavor symmetry.

In the (free) infrared SCFT, new assignment of R-charge, $\widetilde{R} = R + h_\theta$.
This explains the shifted scaling dimension (=1/2) of the ξ_A.

$\mathcal{N} = 4$ super Yang-Mills: unbroken $U(1)$s

The second class of generalizations is to theories whose Higgs branch EFT includes massless vector multiplets.

Canonical example is $\mathcal{N} = 4$ SYM with gauge algebra $\mathfrak{g}$.

$$\mathcal{M}_{\text{Higgs}} = \frac{\mathbb{C}^2 \otimes V_{\mathfrak{g}}}{\mathsf{Weyl}_{\mathfrak{g}}}\,, \qquad V_{\mathfrak{g}} \simeq \mathbb{R}^{\mathsf{rank}(\mathfrak{g})}\,.$$

At a generic point on the Higgs branch, an unbroken $U(1)^{\mathsf{rank}(\mathfrak{g})}$ gauge group.

The associated VOAs is a super $\mathcal{W}$-algebra with $\mathcal{N} = 4$ *small* superVirasoro. Besides the superVirasoro generators, there are additional strong generators, including one short superprimary for each fundamental invariant of $\mathsf{Weyl}_{\mathfrak{g}}$.
Beem Lemos Liendo Peelaers LR van Rees

For $\mathfrak{g} = \mathfrak{su}(2)$, the VOA is just $\mathrm{Vir}_{\mathcal{N}=4}$ with $c = -9$.

$$\mathcal{M}_{\mathrm{Higgs}} = \mathbb{C}^2/\mathbb{Z}_2 \,.$$

We expect, and find, a construction of the simple quotient in terms of:
the usual φ, δ $\oplus$ additional *fermionic* fields λ, $\tilde{\lambda}$ for the unbroken $U(1)$,

$$\lambda(z_1)\tilde{\lambda}(z_2) \sim \frac{1}{z_{12}^2}$$

The $\mathfrak{sl}(2)_{-3/2}$ subalgebra is realized as in the examples above, with $T^\natural = T_{\lambda\tilde{\lambda}}$, while the fermionic generators contain exponentials $e^{\pm\frac{1}{2}(\delta+\varphi)}$.

Closely related (by field redefinition) to a construction of Adamovic:

$\mathrm{Vir}_{\mathcal{N}=4}^{c=-9}$ realized in terms of $(\beta,\gamma)_{(1,0)}$ and $(b,c)_{(\frac{3}{2},-\frac{1}{2})}$.
All generators are written as N.O. products, with no exponentials.
Notably, $J^+ = \beta$, $G^+ = b$.

This latter construction generalizes easily to all gauge algebras, and beyond.
Bonetti Meneghelli LR, to appear

For general $\mathfrak{g}$, one uses rank($\mathfrak{g}$) $(\beta\gamma, bc)$ systems, of weights $h(\beta_\ell) = p_\ell/2$, $h(b_\ell) = (p_\ell + 1)/2$, where $\{p_\ell\}$ are the degrees of the fundamental invariants.

X	$p_1, \ldots, p_{\text{rank}}$	$-\frac{c}{3}$
A_{n-1}	$2, 3, \ldots, n$	$n^2 - 1$
B_n	$2, 4, \ldots, 2n$	$n(2n+1)$
D_n	$2, 4, \ldots, 2(n-1); n$	$n(2n-1)$
E_6	$2, 5, 6, 8, 9, 12$	78
E_7	$2, 6, 8, 10, 12, 14, 18$	133
E_8	$2, 8, 12, 14, 18, 20, 24, 30$	248
F_4	$2, 6, 8, 12$	52
H_3	$2, 6, 10$	33
H_4	$2, 12, 20, 30$	124
$I_2(p)$	$2, 2p$	$2(p+1)$

Curiously, the construction extends from Weyl to general Coxeter groups.
There is no obvious $4d$ interpretation of the non-crystallographic cases H_3, H_4, $I_2(p)$.

Residual Interacting SCFTs on the Higgs branch

Finally, generalization to SCFTs whose Higgs EFT cannot be described purely in terms of free fields.

The simplest examples are the Argyres-Doublas theories $\mathrm{AD}_{(A_1,D_{2n+1})}$.

$$\chi[\mathrm{AD}_{(A_1,D_{2n+1})}] = \mathfrak{sl}(2)_{\frac{-4n}{2n+1}}$$

For each n,

$$\mathcal{M}_{\text{Higgs}} = X_{\mathcal{V}} = \mathbb{C}^2/\mathbb{Z}_2 \,.$$

For $n = 1$, we recover the H_1 SCFT, studied in detail above.

For $n > 1$, the Higgs EFT includes, in addition to one free hyper, an interacting SCFT, $\mathrm{AD}_{(A_1,A_{2n-2})}$. One has $\chi[\mathrm{AD}_{(A_1,A_{2n-2})}] = \mathrm{Vir}_{2,2n+1}$.

We then expect to be able to construct $\mathfrak{sl}(2)_{\frac{-4n}{2n+1}}$ in terms of:
the usual δ, $\varphi \oplus$ an additional "abstract" $\mathrm{Vir}_{2,2n+1}$.
In particular, the correction term in f_θ should be $T^\natural = T_{\mathrm{Vir}}$.

Precisely such a construction was found by Adamovic.

A virtually inexhaustible list of additional examples!
Lagrangian theories, infinite sequences of AD theories, class $\mathcal{S}$, ...

All constructions mentioned so far can be viewed as "inversions" of the standard Drinfeld-Sokolov reduction.
Standard DS reduction imposes a quantum constrain $J_{\alpha_-} = 1$ on an affine current associated to a negative root.

We have also found examples of free field constructions that are "inverting" a more general reduction, where the operators set to 1 are *not* affine currents.

A nice example is $\mathcal{N} = 2$ Super QCD, *i.e.*, $SU(N)$ gauge theory with $2N$ fundamental hypers. The operators set to 1 are the baryons.

The R-filtration

All these free field constructions come equipped with a natural $\mathcal{R}$-filtration.
It appears to coincide with the R-filtration naively lost in the $4d \to 2d$ map.

This is natural. The free representations keeps track of what counts as "geometry" (the Higgs chiral ring operators, $E = 2R$) and what doesn't.

For example, for $\mathcal{N} = 4$ SYM, we assign the $\mathcal{R}$-weights

β_ℓ	γ_ℓ	b_ℓ	c_ℓ
$\frac{1}{2}\,p_\ell$	$1 - \frac{1}{2}\,p_\ell$	$\frac{1}{2}\,p_\ell$	$1 - \frac{1}{2}\,p_\ell$

$\mathcal{R}$ of a polynomial is the maximum $\mathcal{R}$ of its monomials.
In examples, refining the character by $\mathcal{R} \equiv R$ reproduces the Macdonald index.

General picture

Emerging picture:

The VOAs associated to $4d$ SCFTs can be "built" in terms of "irreducible" C_2-cofinite blocks $\oplus$ free fields.

This structure should arise in full generality due to the existence of an R-filtration and the constraints of $4d$ unitarity.

Even the irreducible C_2-cofinite blocks cannot be arbitrary, because $4d$ unitarity is very restrictive.

For example, if one assumes full VOA $=$ Virasoro, it appears that $\mathrm{Vir}(2, 2k+1)$ are the *only* models that can arise from unitary $4d$ theories!

Towards a classification program.