Unifying correlators in $\text{AdS}_5 \times S_5$ supergravity

Simon Caron-Huot
McGill University

1703.00278 ‘analyticity in spin in conformal theories’
on: 1711.02031 with Fernando Alday
1809.xxxxx with Anh-Khoi Trinh

Integrability in Gauge and String Theory 2018
August 20, Copenhagen
We’ll study 4-point correlators in N=4 SYM at strong ’t Hooft coupling

- Nonplanar effects (1/Nc) in a theory with nice planar limit. What remains of integrability?
- How does 10D supergravity on AdS5xS5 emerge from CFT?!?
  Get concrete data about string theory at low energies
- Eventual comparison with integrability results
General CFT framework:

1. Spectrum:
   - Protected single-traces:
     \[ \mathcal{O}^p \simeq \text{Tr}[\phi^{i_1} \cdots \phi^{i_p}] \quad \Delta = p \geq 2, \]
   - Unprotected single-traces \[\Delta \sim \lambda^{1/4} \gg 1 \quad \text{heavy}\]
   - Multi traces \[\mathcal{O}^p \partial^n \mathcal{O}^q \quad \Delta \approx p + q + n + \gamma/N_c^2\]

This talk’s focus: composites of various single-traces

p=2: stress tensor
p>2: \(S_5\) graviton spherical harmonics
General CFT framework:

2. Operator Product Expansion (OPE)

\[
\langle O^p O^q O^r O^s \rangle = \sum \left[ O^p + (O^p O^p)^{[n,\ell]} \right] \sim \frac{1}{N^2_c} \\
\sim \frac{1}{N^4_c} \\
+ \ldots
\]

Multi-traces encode supergravity loop expansion

\begin{align*}
\end{align*}
Main tool: recall Kramers-Kronig relation

\[ f(E) = f(\infty) + \int \frac{dE'}{2\pi(\text{Disc } f(E')) \left( E' - E - i0 \right)} \]

Ex: \( \text{Re}(f) \sim \) phase velocity of light
\( \text{Im}(f) \sim \) absorption by medium

Determines propagation from absorption

consequence of causality (analyticity at complex energies)
Reconstructs double-traces from single traces:

\[
\text{coeff. } \left[ \begin{array}{c} \ \ \ \\ \end{array} \right] = \sum \int \left[ \begin{array}{c} \ \ \ \\ \end{array} \right] + \left[ \begin{array}{c} \ \ \ \\ \end{array} \right]
\]

\text{finite sum } \text{`absorptive part'} \approx \text{single traces}

=CFT dispersion relation

[SCH '17]
Part I

- Lorentzian inversion formula
- Superconformal Ward identities & OPE
- Bootstrapping double-trace dimensions

Part II

An unexpected $SO(10,2)$ symmetry
Lorentzian inversion formula

\[ c(J, \Delta) = \int \frac{1}{\text{causal diamond}} \left[ G_{\Delta+1-d,j+d-1} \right] \times \left[ \text{dDisc} \ G \right] \]

\( s\)-channel OPE coefficients

\( J \) and \( \Delta \) exchanged

converges for \( J > 1 \)
(boundedness in Regge limit)

[see also: Simmons-Duffin, Stanford& Witten; Kravchuk& Simmons-Duffin ‘18]
Input: double commutator

\[ d \text{Disc } G \equiv -\frac{1}{2} \langle 0 | [\phi_4, \phi_1][\phi_2, \phi_3]|0 \rangle \]

Positive-definite and bounded object. Simplifications:

1. Large J \((z \to 1)\): low twists in cross-channel
   \[ \Rightarrow 1/J \text{ expansion} \quad \text{[cf Fernando Alday's talk!]} \]

2. Large \(N_c\): dominated by single-traces
Commutators kill double-traces:

$$\phi_1 \phi_4 \sim \sum_{j, \Delta} c_{j, \Delta}((x_1 - x_4)^2)^{\frac{\Delta - \Delta_1 - \Delta_4}{2}}$$

$$[\phi_1, \phi_4] \sim \sum_{j, \Delta} c_{j, \Delta} |(x_1 - x_4)^2|^{\frac{\Delta - \Delta_1 - \Delta_4}{2}} \sin(\pi \frac{\Delta - \Delta_1 - \Delta_4}{2}) \propto \gamma / N_c^2$$

dDisc $\sim$ imaginary part of Witten diagrams!
Nonperturbative picture:

Knowing just light single-traces, dispersion relation yields:

\[ \Rightarrow \text{full OPE data w/ controlled perturbative+nonperturbative corrections!} \]

[see also: Alday, Bissi & Perlmutter; Li, Meltzer & Poland]
Superconformal OPE

We study correlator of four half-BPS primaries in \([0, p, 0]\)

\[ O^p(x, y) \equiv y^{i_1} \cdots y^{i_p} \text{Tr}[\phi^{i_1} \cdots \phi^{i_p}] \quad (\text{multi traces}) \]

Null six-vectors \(y\) conveniently tracks R-symmetry indices

Correlator depends on cross-ratios:

\[ u = \frac{x_{12}^2 x_{34}^2}{x_{13} x_{24}^2} = z\bar{z}, \quad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2} = (1 - z)(1 - \bar{z}), \]

\[ \sigma = \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2} = \alpha\bar{\alpha}, \quad \tau = \frac{y_{23}^2 y_{14}^2}{y_{13}^2 y_{24}^2} = (1 - \alpha)(1 - \bar{\alpha}), \]
In principle we should use superconformal blocks

For four half-BPS ops, really only need Ward identity:

the $z$ dependence disappears when $\alpha = z$

[Nirschl & Osborn ’04]

General solution:

$$G_{\{p_i\}}(z, \bar{z}, \alpha, \bar{\alpha}) = k\chi(z, \alpha)\chi(\bar{z}, \bar{\alpha}) + \frac{(z - \alpha)(z - \bar{\alpha})(\bar{z} - \alpha)(\bar{z} - \bar{\alpha})}{(\alpha - \bar{\alpha})(\bar{\alpha} - \bar{\bar{\alpha}})(\alpha - \bar{\bar{\alpha}})(\bar{\bar{\alpha}} - \bar{\alpha})}$$

$$\times \left( -\frac{\chi(\bar{z}, \bar{\alpha})f(z, \alpha)}{\alpha z(z - \bar{\alpha})} + \frac{\chi(z, \alpha)f(z, \bar{\alpha})}{\bar{\alpha} z(\bar{\alpha} - \bar{\alpha})} + \frac{\chi(z, \bar{\alpha})f(\bar{z}, \alpha)}{\alpha \bar{z}(z - \bar{\alpha})} - \frac{\chi(z, \alpha)f(\bar{z}, \bar{\alpha})}{\bar{\alpha} \bar{z}(z - \bar{\alpha})} \right)$$

$$+ \frac{(z - \alpha)(z - \bar{\alpha})(\bar{z} - \alpha)(\bar{z} - \bar{\alpha})}{(z\bar{z})^2(\alpha\bar{\alpha})^2}H_{\{p_i\}}(z, \bar{z}, \alpha, \bar{\alpha}),$$

$$f(z, \alpha) = \text{protected} \quad H(z, \bar{z}, \alpha, \bar{\alpha}) = \text{dynamical}$$
Superconformal Casimir commutes with decomposition

\[ f_{\{p_i\}}(z, \alpha) = \sum_{j, m} b_{\{p_i\}}(j, m) k_{1+m+j}^1(z) k_{-m}^1(\alpha) \]

\[ H_{\{p_i\}}(z, \bar{z}, \alpha, \bar{\alpha}) = \sum_{j, \Delta, m, n} a_{\{p_i\}}(j, \Delta, m, n) G_{j, \Delta}(z, \bar{z}) Z_{m, n}(\alpha, \bar{\alpha}) \]

These are standard bosonic blocks:

\[ k \sim {}_2F_1, \quad G \sim \frac{z\bar{z}}{z - \bar{z}}(kk - kk), \quad Z \sim \text{similar} \]

[cf Bissi& Lukowski '15]

In practice, we won’t need ‘superconformal blocks’: 
\[ a(j,\Delta;m,n) \text{ and } b(j;m) \text{ contain all information!} \]
Disconnected correlator

At order $N^0$, correlator is very simple:

$$G_{pqqp}^{(0)} = \delta_{p,q} + \left( \frac{u}{\sigma} \right)^{\frac{p+q}{2}} \left[ \left( \frac{\tau}{v} \right)^q + \delta_{p,q} \right]$$

SUSY decomposition gives mess, however

Trick: magic differential operator which kills protected stuff
Supermultiplets which can appear in [0,p,0] correlators:

<table>
<thead>
<tr>
<th>Multiplet</th>
<th>Dynkin labels</th>
<th>Dimension (\Delta) and spin (\ell)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Half-BPS (B_{0,n})</td>
<td>[0, n, 0]</td>
<td>(\Delta = q, \ell = 0)</td>
</tr>
<tr>
<td>Quarter-BPS (B_{m,n})</td>
<td>[m, n-m, m]</td>
<td>(\Delta = m + n, \ell = 0, m \geq 1)</td>
</tr>
<tr>
<td>Semi-short (C_{\ell,m,n})</td>
<td>[m, n-m, m]</td>
<td>(\Delta = m + n + 2 + \ell)</td>
</tr>
<tr>
<td>Long (A_{\ell,\Delta,m,n})</td>
<td>[m, n-m, m]</td>
<td>(\Delta &gt; m + n + 2 + \ell)</td>
</tr>
</tbody>
</table>

All but longs are killed by a 8th order operator:

\[
\Delta^{(8)} H_{\{p_i\}} \equiv \frac{z\bar{z}\alpha\bar{\alpha}}{(z - \bar{z})(\alpha - \bar{\alpha})} \frac{1}{(D_z - D_\alpha)(D_z - D_{\bar{\alpha}})(D_{\bar{z}} - D_\alpha)(D_{\bar{z}} - D_{\bar{\alpha}})} \frac{(z - \bar{z})(\alpha - \bar{\alpha})}{z\bar{z}\alpha\bar{\alpha}} H_{\{p_i\}}
\]

\[
D_x \equiv x^2 \partial_x (1 - x) \partial_x - \frac{1}{2} (r + s) x^2 \partial_x - \frac{1}{4} r s x
\]

Physically, gives correlator of chiral Lagrangians (10D axidilaton)

\[\text{[Drummond, Gallot & Sokatchev '06]}\]

\[
\Delta^{(8)} H_{pqqp}^{(0)} = \frac{u^{p+q} + 2}{\sigma^{p+q} - 2} \left( \frac{\tau^{q-2}}{\nu^{q+2} + \delta_{p,q}} \right) \times C(p)C(q) \quad C(p) = p^2(p^2 - 1)
\]
Since \( \Delta^{(8)} H \) = generalized free field, OPE takes simple form:

\[
a_{\{p_i\}}(j, \Delta, m, n) = \frac{1}{\Delta^{(8)}_{m,n}(h, \bar{h})} \times (\Gamma\text{-functions})
\]

Here we have divided by the eigenvalue:

\[
\Delta^{(8)}_{m,n}(h, \bar{h}) = \left(h - \frac{n - m + 2}{2}\right) \left(h + \frac{n - m}{2}\right) \left(h - \frac{m + n + 4}{2}\right) \left(h + \frac{m + n + 2}{2}\right) \\
\times \left(\bar{h} - \frac{n - m + 2}{2}\right) \left(\bar{h} + \frac{n - m}{2}\right) \left(\bar{h} - \frac{m + n + 4}{2}\right) \left(\bar{h} + \frac{m + n + 2}{2}\right)
\]

\[
h = 1 + \frac{\Delta - j}{2}, \quad \bar{h} = 2 + \frac{\Delta + j}{2}
\]

[cf Drummond et al '18]
From this N\textsuperscript{0} OPE data, can derive formulas for various superconformal blocks

Ex: half-BPS block with [0,p+q,0]
= sum of all contributions with Casimir eigenvalue 0

\[
B_{0,\Delta}^{r,s} = \begin{cases} 
  k = 1, & f(z, \alpha) = \sum_{i=\max(|r|,|s|)}^{\Delta-2} k_{r,s}^{1+i/2} k_{-r,-s}^{1-i/2}, \\
  H(z, \bar{z}, \alpha, \bar{\alpha}) = \sum_{i=\max(|r|,|s|)}^{\Delta-4} \sum_{j=0}^{(\Delta-i)/2} G_{j,i+j+4}^{r,s}(z, \bar{z}) Z_{j,i+j}^{r,s}(\alpha, \bar{\alpha}).
\end{cases}
\]

Agrees with earlier formulas for superconformal blocks

[Dolan & Osborn, …]
At this order, dDisc saturated by single-traces (thus half-BPS)

\[ d\text{Disc } G = \sum \mathcal{O}_{p'} = \sum f_{qq'=f_{rr'=B_p'}} \]

Inversion formula (\textit{thank you SUSY!}) converges for \( J > -2 \)

In particular, we get a \textbf{crossing equation} for the half-BPS f’s
\[
\text{coeff. } \left[ \begin{array}{c}
\end{array} \right] = \sum \int \left[ \begin{array}{c}
\end{array} \right] + \left[ \begin{array}{c}
\end{array} \right]
\]

Ex:
\[
\begin{array}{c}
\end{array}_{\text{1/2-BPS}} = f_{233}^2 B_{0,3} = \int \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} = \left( \frac{1}{3} f_{222} f_{233} + \frac{7}{9} f_{233}^2 \right) B_{0,3} + \ldots
\]

From this we deduce: \( f_{233} = \frac{3}{2} f_{222} \). \checkmark

In this way we bootstrap all 3-point couplings!

Matches known @weak&strong coupling

\[
f_{pqr} = \sqrt{\frac{pqr}{4c}}
\]
Free theory

Having bootstrapped all 3-pt couplings, get full protected part \( f(z, \alpha) \) at order \( 1/N^2 \)

[1D inversion: Simmons-Duffin, Stanford& Witten]

Expect to match free theory = \( \text{poly}(u/\sigma, \tau/v) \)

In fact enough to fully reconstruct free theory!

\[
G_{\text{free}} = \sum \begin{array}{c}
\boxed{} \\
\boxed{+} \\
\boxed{+}
\end{array}
\]

where each line represents \( \geq 0 \) Wick contractions
Double-trace mixing
from the dDisc of the correlator, get all OPE data:

\[
\begin{align*}
c(\ell, \Delta, 0, 2)_{2442} &= \int_0^1 \frac{dz}{z^2} (1-z)^2 k_1(z)^{z^{-1}} \int_0^1 \frac{d\tilde{z}}{\tilde{z}^2} (1-\tilde{z})^2 \kappa(\tilde{h}) k_\tilde{h}(2) (\tilde{z})^{-1} \\
&\times \left[ 2 \left( \frac{z}{1-z} \right)^3 - 4 \left( \frac{z}{1-z} \right)^4 - 4 \left( \frac{z}{1-z} \right)^5 \log(z) \right] \text{dDisc} \left( \frac{\tilde{z}}{1-\tilde{z}} \right)^3 \\
&= \text{dDisc} \ G_{2442}
\end{align*}
\]

The log(z) term gives anomalous dimensions

All integrals give just bunch of \( \Gamma \)'s

\[
\int_0^1 \frac{d\tilde{z}}{\tilde{z}^2} (1-\tilde{z})^{1/2}(p_{21}+p_{34}) \tilde{h} \left( \frac{\tilde{h}}{2} \right) k_{\tilde{h}/2}^{p_{21},p_{34}}(\tilde{z}) \text{dDisc} \left[ \left( \frac{1-\tilde{z}}{\tilde{z}} \right)^\lambda \tilde{z}^{-p_{34}/2} \right]
\]

\[
= \frac{r_{\tilde{h}}^{p_{21},p_{34}}}{\Gamma(-\lambda) \Gamma(-\frac{p_{21}+p_{34}}{2}) \Gamma(\tilde{h} \lambda + p_{34}/2 + 1)}
\]
Anomalous dimension are really mixing matrices

For example, at twist 6 and R-symmetry rep \([0,2,0]\):

\[
\gamma^{(1)}_{6,0,2} = \begin{pmatrix}
\gamma^{(1)}_{24,24} & \gamma^{(1)}_{24,33} \\
\gamma^{(1)}_{33,24} & \gamma^{(1)}_{33,33}
\end{pmatrix}
\]

with:

\[
\gamma^{(1)}_{pq,rs} \equiv \frac{\langle a^{(0)} \gamma^{(1)} \rangle_{pqrs}}{\sqrt{\langle a^{(0)} \rangle_{ppqq} \langle a^{(0)} \rangle_{rrss}}}
\]

For even spins this evaluates to:

\[
\left( \gamma^{(1)} \right)_{6,0,2}^{+} = \frac{-60}{(\bar{h} - 4)(\bar{h} - 1)(\bar{h})(\bar{h} + 3)} \begin{pmatrix}
12 + \bar{h}(\bar{h} - 1) & 6\sqrt{\bar{h}(\bar{h} - 1)} \\
6\sqrt{\bar{h}(\bar{h} - 1)} & 6 + \bar{h}(\bar{h} - 1)
\end{pmatrix}
\]

Amazingly, the eigenvalues are simple:

\[
\left( \gamma^{(1)} \right)_{6,0,2}^{+} = \left\{ \frac{-\Delta^{(8)}_{0,2}(4, \bar{h})}{(\bar{h} - 4)_6}, \frac{-\Delta^{(8)}_{0,2}(4, \bar{h})}{(\bar{h} - 2)_6} \right\} \quad (\bar{h} = j + 5)
\]
Straightforward to look at other cases, ie twist=8 [0,2,0]:

\[
\begin{pmatrix}
(2442) & (2433) & (2435) & (2444) \\
(3342) & (3333) & (3335) & (3344) \\
(3542) & (3533) & (3553) & (3544) \\
(4442) & (4433) & (4453) & (4444)
\end{pmatrix}
\]

In all cases we reproduce a recent conjecture: [Aprile, Drummond, Heslop&Paul '18]

All eigenvalues take the form:

\[
\gamma^{(1)} = -\frac{1}{c} \frac{\Delta^{(8)}}{(j + 1 + \text{integer})_6}
\]
An unexpected $SO(10,2)$ symmetry
First, let us emphasize conjectured formula:

\[ \gamma = -\frac{\Delta^{(8)}}{c} \times \frac{1}{(j + 1 + m)_6} + O(1/c^2) \]

**Crazy** that complicated matrix has rational eigenvalues!

**Crazier:** Take flat space 10D dilation scattering:

Expand over 10D Legendre polynomials
First, let us emphasize conjectured formula:

\[
\gamma = - \frac{\Delta^{(8)}}{c} \times \frac{1}{(j + 1 + m)_6} + O(1/c^2)
\]

Crazy that complicated matrix has rational eigenvalues!

Crazier: Take flat space 10D dilation scattering:

\[
A^{(10)}(s, t) = 8\pi G_N s^4 \times \frac{1}{stu}
\]

Expand over 10D Legendre polynomials

\[
A_{\ell}^{(10)}(s) = - \frac{(L\sqrt{s}/2)^8}{c} \times \frac{1}{(j + 1)_6} + O(1/c^2)
\]

\[
\left( G_N = \frac{\pi^4 L^8}{8c} \right)
\]
\[ \gamma = -\frac{\Delta^{(8)}}{c} \frac{1}{(j + 1 + m)_6} \Leftrightarrow A_{\ell}^{(10)}(s) = -\frac{(L \sqrt{s}/2)^8}{c} \frac{1}{(j + 1)_6} \]

CFT correlator is just flat 10D amplitude!?!?!?!

\[(L \sqrt{s}/2)^8 \leftrightarrow \Delta^{(8)}\]
Our proposed explanation:

1. 10D supergravity [@4-pt] \(\cong\) CFT (coupling = \(G_{NS} s^4\))

\[
A^{(10)}(s, t) = 8\pi G_N s^4 \frac{1}{stu}
\]

2. AdS\(5\times S_5\) is conformal to flat space

Conjecture:

All 4-pt SYM correlators stem from a common 10D-conformal object

\[
SO(4, 2) \times SO(6)_R \subset SO(10, 2)
\]
Conjecture:

All 4-pt SYM correlators stem from a common 10D-conformal object

The object is ‘dilaton correlator’

\[ \langle \phi(w_1)\phi(w_2)\bar{\phi}(w_3)\bar{\phi}(w_4) \rangle_{10} \equiv \frac{G_{10}(u_{10}, v_{10})}{((x_{12}^2 - y_{12}^2)(x_{34}^2 - y_{34}^2))^4} \]

depends only on 10D distances \( x_{ij}^2 - y_{ij}^2 \):

\[ u_{10} = \frac{(x_{12}^2 - y_{12}^2)(x_{34}^2 - y_{34}^2)}{(x_{13}^2 - y_{13}^2)(x_{24}^2 - y_{24}^2)} \]
\[ v_{10} = \frac{(x_{23}^2 - y_{23}^2)(x_{14}^2 - y_{14}^2)}{(x_{13}^2 - y_{13}^2)(x_{24}^2 - y_{24}^2)} \]
To extract SYM correlator $H_{pqrs}$, series-expand $G_{10}$ in $y$'s and take term with correct weight

$$
\tilde{H}_{p_1 p_2 p_3 p_4}(u, v, \sigma, \tau) = \oint \prod_{i=1}^{4} \frac{da_i a_i^{1-p_i}}{2\pi i} \frac{(u/\sigma)^{p_1+p_2-2}}{(1 - \frac{\sigma}{u} a_1 a_2)^4(1 - a_3 a_4)^4} \times G_{10}\left(u \frac{(1 - \frac{\sigma}{u} a_1 a_2)(1 - a_3 a_4)}{(1 - a_1 a_3)(1 - a_2 a_4)}, v \frac{(1 - \frac{\tau}{v} a_2 a_3)(1 - a_1 a_4)}{(1 - a_1 a_3)(1 - a_2 a_4)}\right)
$$

SO(10,2) symmetry thus predicts differential relations:

$$
\tilde{H}_{pqrs} = D_{pqrs} \tilde{H}_{2222}
$$

\[D_{2222} = 1,\]
\[D_{2332} = -\frac{\sqrt{u}}{\sqrt{\sigma}} \tau \partial_v,\]
\[D_{2233} = 4 - u \partial_u,\]
\[D_{3333} = 16 - 8u \partial_u + \frac{u + \sigma}{\sigma} (u \partial_u)^2 + 2 \frac{u}{\sigma} u \partial_u v \partial_v + \frac{u(v + \tau)}{\sigma v} (v \partial_v)^2.\]

...
correct object is \( \tilde{H}^{(1)} = \frac{\Delta^{(8)} H}{\Delta^{(8)}} \equiv H^{(1)} - H^{(1), \text{free}} \)

1. Check against classic results:

\[
\begin{align*}
\tilde{H}_{2222}^{(1)} &= -u^4 \bar{D}_{2,4,2,2}(u, v) \\
\tilde{H}_{2332}^{(1)} &= -\frac{u^{9/2}}{\sqrt{\sigma}} \tau \bar{D}_{2,5,3,2}, \\
\tilde{H}_{2233}^{(1)} &= -u^4 (\bar{D}_{2,4,2,2} - \bar{D}_{2,4,3,3}), \\
\tilde{H}_{3333}^{(1)} &= \ldots
\end{align*}
\]

[D’Hoker, Freedman, Mathur, Matusis & Rastelli ’99; Arutyunov, Dolan, Osborn & Sokatchev ’02-; Berdichevsky & Naaijkens ’03; Dolan, Nirschl & Osborn ’06; Uruchurtu ’08-]

2. More generally: suffices to check the dDisc=pole terms

\[
\tilde{H}_{pqrs}^{(1)} \bigg|_{v-\text{poles}} \equiv D_{pqrs} \left[ -\frac{2u^4 \log u}{(1 - u)^3 v} - \frac{u^3(1 + u)}{(1 - u)^2 v} \right]
\]

easy to check to high order p,q,r,s\sim 10!!
Derive Rastelli-Zhou’s formula: write Mellin rep for $H_{2222}$

$$G_{10}(w_i) = \int dsdt \frac{\Gamma(2 - \frac{s}{2})^2 \Gamma(2 - \frac{s}{2})^2 \Gamma(\frac{s+t}{2})^2}{(s-4)(t-4)(s+t-2)} \times (x_{12}^2 - y_{12}^2)^{\frac{s}{2} - 2} \cdots$$

Expand in $y$: each « Gamma times power » just gets shifted!

$$\Gamma \left( -\frac{s}{2} \right) \left( x_{12}^2 - y_{12}^2 \right)^{\frac{s}{2}} = \sum_{p=0}^{\infty} \frac{(y_{12}^2)^p}{p!} \Gamma \left( -\frac{s}{2} - p \right) \left( x_{12}^2 - y_{12}^2 \right)^{\frac{s}{2} - p}$$

General Mellin space correlator = sum of shifted $1/(stu)$’s:

$$M_{pqrs}(s, t) = \sum \frac{\#}{(s-\#)(t-\#)(u-\#)}$$

[Rastelli&Zhou ’16] each coefficient is product of six $(1/p!)$
Rastelli & Zhou: all correlators in Mellin space

Aprile et al:
- double-trace mixing
- one loop

SO(10,2) symmetry

eigenvalues
		 inversion formula

this work

perfect agreement between different methods!
From trees to loops

\[
\sum_{p',p''} = \text{all one needs for the Kramers-Kronig relation}
\]

Trees predict the dDisc ($\sim \log^2 v$ terms)
At one-loop, studied 2222 correlator

Flat space limit perfectly matches 1-loop supergravity

\[ \mathcal{G}(z, \bar{z}) \]  

double-disc. \( \text{dDisc} [\mathcal{G}] \)  

inversion integral  

amplitude \( A_5(s, t) \)  

discontinuity \( \text{Disc} [A_5] \)  

flat space  

dispersion relation

[Allday & SCH, ’17]
For general correlators, really a matrix product:

\[ \sum_{p',p''} \]
Start with OPE decomposition of 10D free field:

\[
G^{(0)}_{10}(u, v) = 144 \left( u^4 + \frac{u^4}{v^4} \right) = \sum_{j=0, \text{even}}^{\infty} \frac{8\Gamma(j + 4)^2}{\Gamma(2j + 7)} (j + 1)_6 \cdot G^{(10D)}_{\ell, 8+\ell}(u, v)
\]

A single block for each per spin: 10D dilatons have \( \Delta = 4 \)

When reduced to 4D, 10D blocks are orthogonal!

[Sch, & Trinh, to appear]

⇒ Just need to add powers of \( 1/(j+1)_6 \) in the above!

Note: 10D extremal blocks extremely simple:

\[
G^{(d=10)}_{\ell, 8+\ell}(z, \bar{z}) = D_{(3)} \cdot \frac{120}{(j + 1)(j + 2)(j + 3)} \cdot z^{j+1} {2F1}(j + 1, j + 4, 2j + 8, z)
\]

[Trinh & Dolan '11]

\[
D_{(3)} f(z) \equiv \left[ \left( \frac{z\bar{z}}{\bar{z} - z} \right)^7 f(z) + \left( \frac{z\bar{z}}{\bar{z} - z} \right)^6 \frac{z^2}{2} \partial_z f(z) + \left( \frac{z\bar{z}}{\bar{z} - z} \right)^5 \frac{z^3}{10} \partial_z^2 (zf(z)) + \left( \frac{z\bar{z}}{\bar{z} - z} \right)^4 \frac{z^4}{120} \partial_z^3 (z^2 f(z)) \right]
\]
Explicit formula for leading-log at each loop order:

$$ \mathcal{H}_{pqrs}^{(k)}(z, \bar{z}, \alpha, \bar{\alpha}) \bigg|_{\log^k u} = \left[ \Delta^{(8)} \right]^{k-1} \cdot D_{pqrs} \cdot D_{(3)} \cdot h^{(k)}(z). $$

**Ex:** 

$$ h^{(2)}(z) = \frac{\text{Li}_2(z) - (1 - z)^5 \text{Li}_2(z/(1 - z))}{4z^5} - \frac{(1 - z)(2z^2 - 7z + 7) \log(1 - z)}{8z^4} + \frac{(z - 2)(1 - z)}{z^3} + \frac{235 z - 2}{576 z}. $$

gives one-loop log² terms for all correlators

matches 2222 from: [Alday & Bissi ’17, Aprile,Drummond,Heslop&Paul ‘17]

general formula:

$$ h^{(k)}(z) \equiv \frac{1}{k!} \left( -\frac{1}{2} \right)^k \sum_{\ell=0, \text{even}}^{\infty} \frac{960\Gamma(j+1)\Gamma(j+4)}{\Gamma(2j+7)} \frac{1}{[(\ell + 1)6]^{k-1}} z^{j+1} _2F_1(j+1, j+4, 2j+8, z). $$
Summary

- Studied double-trace mixing in strongly coupled N=4 SYM using Lorenzian inversion formula
- SO(10,2) symmetry: formula for all spherical harmonics!
- Leading logs to all orders in $1/N_c$

Further questions

- What more is true at higher loops/higher points?
  cf: [Loebbert, Mojaza& Plefka ‘18: hidden conformal symmetry]
  [cf Maldacena’ 11: Einstein vs conformal gravity]

- Other theories: 6D (2,0), ABJM?
‘Heavy’ part depends on nonperturbative UV completion.

It’s weighed by $\sim (\rho \bar{\rho})^{J/2}$. Use positivity + boundedness:

$$|c(j, \frac{d}{2} + i\nu)_{\text{heavy}}| \leq \frac{1}{c_T} \frac{\#}{(\Delta_{\text{gap}}^2)^{j-2}}$$

This establishes, from CFT, an EFT power-counting in AdS.