Non-planar Holographic Correlators from the Analytic Bootstrap

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Work with Aharony, Bissi, Caron-Huot, Perlmutter, Yacoby.

What will this talk be about?

Conformal Field Theories in $D > 2$ dimensions

- In general CFTs don't have a Lagrangian description.
- In a Lagrangian theory we can use Feynman diagrams:

$$A(g) = A^{(0)} + gA^{(1)} + \cdots$$

- But even in this case, things become very messy very quickly.
- Idea of the conformal bootstrap: resort to consistency conditions!
  - Conformal symmetry
  - Properties of the OPE
  - Unitarity
  - Crossing symmetry
- We will advocate an analytic version of the conformal bootstrap.

What will this talk be about?

Today: Operators with spin in a generic CFT

$$\varphi\partial_{\mu_1}\cdots\partial_{\mu_\ell}\varphi$$

- Study their dimension Δ for large values of the spin ℓ, as an expansion in $1/\ell$:

$$\Delta(\ell) = \ell + 2\Delta_\varphi + \frac{c_1}{\ell} + \frac{c_2}{\ell^2} + \cdots$$

- We will obtain analytic results to all orders in $1/\ell$!
- Our results will be even valid for finite values of the spin!

CFT - Basics

- Main ingredient: Conformal Primary local operators $\mathcal{O}_{\Delta,\ell}(x)$.
- Operators form an algebra (OPE)

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_{k \in \text{prim.}} C_{ijk}|x|^{\Delta_k-\Delta_i-\Delta_j}\left(\mathcal{O}_k(0) + \text{descendants}\right)$$

- The set Δ_i and C_{ijk} characterizes the CFT.
- Main observable: Correlation functions of primary operators.

Four-point function of identical operators:

$$\boxed{\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{\mathcal{G}(u,v)}{x_{12}^{2\Delta_{\mathcal{O}}}x_{34}^{2\Delta_{\mathcal{O}}}}}$$

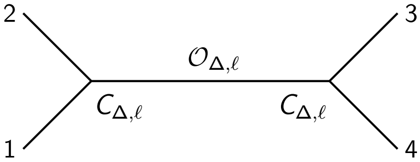
where $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$.

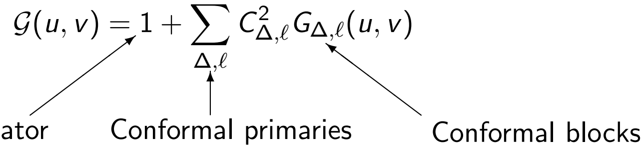
Four-point function - properties

Conformal partial wave decomposition

- OPE: $\mathcal{O} \times \mathcal{O} = \sum_i \mathcal{O}_i + \textit{descendants}$

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle = \sum_{\Delta,\ell}$$





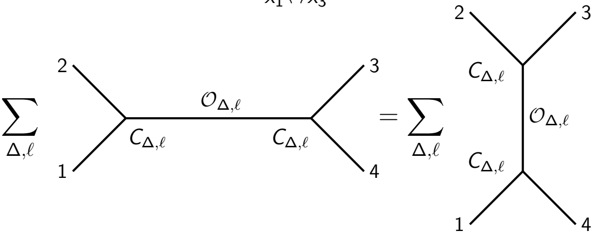
$$\mathcal{G}(u,v) = 1 + \sum_{\Delta,\ell} C^2_{\Delta,\ell} G_{\Delta,\ell}(u,v)$$

- Conformal blocks: For a given primary, take into account the contribution from all its descendants. Fully fixed function!

Conformal bootstrap

Crossing symmetry

$$v^{\Delta_{\mathcal{O}}}\mathcal{G}(u,v)\underbrace{=}_{x_1\leftrightarrow x_3}u^{\Delta_{\mathcal{O}}}\mathcal{G}(v,u)$$

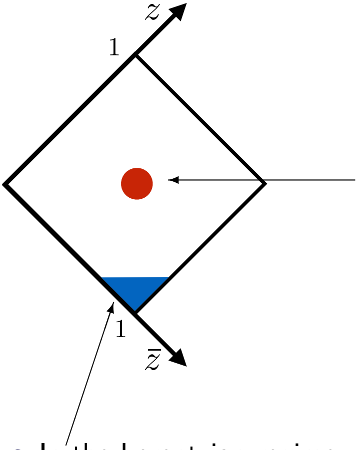


A remarkable...but hard equation!

$$\underbrace{v^{\Delta_{\mathcal{O}}}\left(1+\sum_{\Delta,\ell}C^2_{\Delta,\ell}G_{\Delta,\ell}(u,v)\right)}_{\text{Easy to expand around } u=0,\, v=1}=\underbrace{u^{\Delta_{\mathcal{O}}}\left(1+\sum_{\Delta,\ell}C^2_{\Delta,\ell}G_{\Delta,\ell}(v,u)\right)}_{\text{Easy to expand around } u=1,\, v=0}$$

Numerical vs Analytic bootstrap

Study this equation in different regions, $u = z\bar{z}, v = (1-z)(1-\bar{z})$



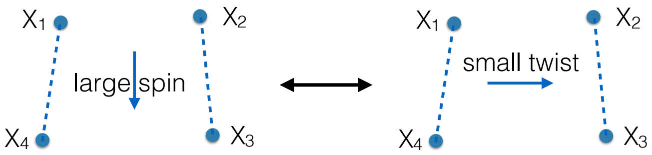
- In the Euclidean regime $\bar{z} = z^*$.
- We can study crossing around $u = v = \frac{1}{4}$
- Starting point of the numerical bootstrap.

- In the Lorentzian regime $z, \bar{z}$ are independent real variables and we can consider $u, v \to 0$.
- Starting point of the analytic (light-cone) bootstrap!

Analytic (light-cone) bootstrap

Why is this a good idea?

- In Minkowski space we can have $x_{23}^2 \to 0, x_{23} \neq 0$.
- When some operators become null-separated the correlator develops singularities.



Dominated by high spin operators $\Leftrightarrow$ Dominated by low twist operators

$$(\mathcal{O}\partial_{\mu_1} \cdots \partial_{\mu_\ell}\mathcal{O}) \qquad (1, T_{\mu,\nu}, \cdots)$$

Conformal blocks - technicalities

- Small u limit:

$$G_{\Delta,\ell}(u,v) \sim u^{\tau/2} f^{coll}_{\tau,\ell}(v), \quad \tau = \Delta - \ell$$

We will introduce the notation

$$G_{\Delta,\ell}(u,v) \equiv u^{\tau/2} f_{\tau,\ell}(u,v)$$

- Small v limit:

$$f_{\tau,\ell}(u,v) \sim \log v$$

Necessity of a large spin sector

- Consider the $v \ll 1$ limit of the crossing equation: $C^2_{\Delta,\ell} \to a_{\tau,\ell}$

$$v^{\Delta_{\mathcal{O}}} \left(1 + \sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) \right) = u^{\Delta_{\mathcal{O}}} \left(1 + \sum_{\tau,\ell} a_{\tau,\ell} v^{\tau/2} f_{\tau,\ell}(v,u) \right)$$

$$\Downarrow$$

$$1 + \sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) = \frac{u^{\Delta_{\mathcal{O}}}}{v^{\Delta_{\mathcal{O}}}} + \underbrace{\text{subleading terms}}_{\text{rest of operators sorted by twist}}$$

- The r.h.s. is power-law divergent as $v \to 0$.
- Each term on the l.h.s. diverges as $f_{\tau,\ell}(u,v) \sim \log v$.
- In order to reproduce the divergence on the right, we need infinite operators, with large spin and whose twist approaches $\tau = 2\Delta_{\mathcal{O}}$ (actually $\tau_n = 2\Delta_{\mathcal{O}} + 2n$)

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Example: Generalised free fields

- Simplest solution: Large N CFTs - Generalised free fields

$$\mathcal{G}^{(0)}(u,v) = 1 + \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} + u^{\Delta_{\mathcal{O}}}$$

- Intermediate ops: Double trace operators: $\mathcal{O}\Box^n\partial_{\mu_1}\cdots\partial_{\mu_\ell}\mathcal{O}$

$$\begin{aligned} \tau_{n,\ell} &= 2\Delta_{\mathcal{O}} + 2n \\ a_{n,\ell} &= a^{(0)}_{n,\ell} \end{aligned}$$

- Their OPE coefficients are such that the divergence of a single conformal block ($\sim \log v$) , as $v \to 0$, is enhanced!

$$1 + \sum_{\tau,\ell} a^{(0)}_{\tau_n,\ell} u^{\tau_n/2} f_{\tau_n,\ell}(u,v) = 1 + \underbrace{\left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}}}_{} + u^{\Delta_{\mathcal{O}}}$$

But this divergence is quite universal!

Large spin sector

Additivity property [Fitzpatrick, Kaplan, Poland, Simmons-Duffin; Komargodski, Zhiboedov; F.A. ,Maldacena]

In any CFT with $\mathcal{O}$ in the spectrum, crossing symmetry implies the existence of double trace operators with arbitrarily large spin and

$$\begin{aligned} \tau_{n,\ell} &= 2\Delta_{\mathcal{O}} + 2n + \mathcal{O}\left(1/\ell\right) \\ a_{n,\ell} &= a^{(0)}_{n,\ell}\left(1+\mathcal{O}\left(1/\ell\right)\right) \end{aligned}$$

- All CFTs have a large spin sector, where operators become "free"!
- Can we do perturbations around large spin? YES! Exploit the following idea:

$$\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) = \frac{u^{\Delta_{\mathcal{O}}}}{v^{\Delta_{\mathcal{O}}}} + \cdots$$

Behaviour at large spin $\Leftrightarrow$ Enhanced divergences as $v \to 0$

Large spin perturbation theory

Given $Sing(u,v)$, find $a_{\tau,\ell}$ such that

$$\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u,v) = Sing(u,v)$$

$$\uparrow$$

$$v^{-\Delta}, v^{n} \log^{2} v, v^{1/2}, \cdots$$

Idea

1. Construct a complete basis of functions with specific CFT-data and prescribed singularities.
2. Express $Sing(u,v)$ in that basis.

Large spin perturbation theory

$$Sing(u,v) \to a_{\tau,\ell} = a^{(0)}_{\tau,\ell}\left(1 + \frac{c_1}{\ell} + \frac{c_2}{\ell^2} + \cdots\right)$$

- LSPT fixes the CFT data to all orders in $1/\ell$, just from $Sing(u,v)$!

Inversion formula

- Is the CFT-data analytic in the spin? does LSPT give the full result?
- The answer appeared to be affirmative in all examples!
- Reformulation in terms of an inversion formula [Caron-Huot]

$$a_{\tau,\ell} \sim \int dudv\, K(u,v,\tau,\ell)\, Sing(u,v)$$

- Equivalent to LSPT but explicitly analytic in the spin!
- It can be extrapolated down to small spin as a consequence of the Regge behaviour!

LSPT: Strategy for CFTs with small paramaters

A nice machinery for theories with small parameters

Strategy

1. Use crossing symmetry to determine the enhanced singularities

$$\mathcal{G}(u,v) \leftarrow \mathcal{G}(u,v)|_{en.sing.} = \left(\frac{u}{v}\right)^{\Delta_0} \mathcal{G}(v,u)\Big|_{en.sing.}$$

   In theories with small parameters the latter follows from CFT-data at lower orders! (maybe including other correlators)

2. Use LSPT to reconstruct the CFT-data from the enhanced singularities.
3. Go to next order and repeat.

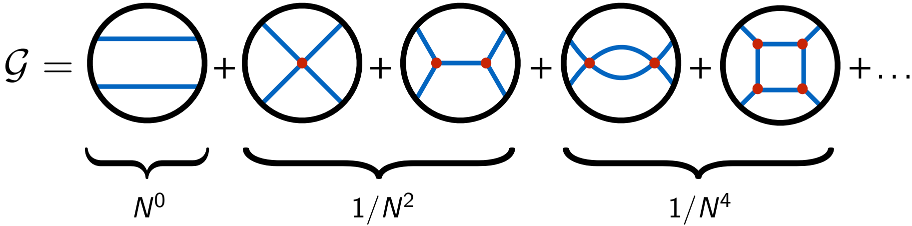
▷ Let's apply LSPT to find $1/N$ corrections to GFF!

Large N CFTs

AdS/CFT

Large N CFT in D-dimensions (GFF + corrections) $\Leftrightarrow$ Gravitational theory in AdS_{D+1}

$\frac{1}{N^2}$ expansion in CFT $\leftrightarrow$ loops in AdS/powers of G_N.



- Diagrams in AdS are hard to compute...Use crossing for the CFT!

Large N holographic CFTs

$$\mathcal{G}(u,v) = \mathcal{G}^{(0)}(u,v) + \frac{1}{N^2}\mathcal{G}^{(1)}(u,v) + \cdots$$

Two Sources of corrections

1. Double trace operators will acquire corrections:

$$\begin{aligned} \tau_{n,\ell} &= 2\Delta_{\mathcal{O}} + 2n + \frac{1}{N^2}\gamma^{(1)}_{n,\ell} + \cdots \\ a_{n,\ell} &= a^{(0)}_{n,\ell} + \frac{1}{N^2}a^{(1)}_{n,\ell} + \cdots \end{aligned}$$

2. We can also have new intermediate operators at order $1/N^2$.

Which corrections are consistent with crossing symmetry?

$$\mathcal{G}^{(1)}(u,v) = \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \mathcal{G}^{(1)}(v,u)$$

Large N holographic CFTs

Case 1: New single-trace operators at order $1/N^2$:

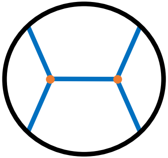
$$\mathcal{O}\times\mathcal{O} = 1 + [\mathcal{O},\mathcal{O}]_{n,\ell} + \frac{1}{N^2}\mathcal{O}_{st}$$

- This produces an enhanced divergence:

$$\mathcal{G}^{(1)}(u,v) = \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}}\mathcal{G}^{(1)}(v,u) \supset \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} v^{\Delta_{\mathcal{O}_{st}}/2} f_{\mathcal{O}_{st}}(v,u)$$

- The enhanced divergences fix $\gamma^{(1)}_{n,\ell}, a^{(1)}_{n,\ell}$ to all orders in $1/\ell$!
- Non-truncated solutions correspond to AdS exchanges

$$\mathcal{G}^{(1)}_{non-tr}(u,v) \ \sim$$



Large N holographic CFTs

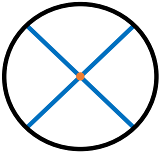
Case 2: No new operators at order $1/N^2$

- Double-trace operators don't produce enhanced divergences!

$$\mathcal{G}^{(1)}(u,v) = \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \underbrace{\mathcal{G}^{(1)}(v,u)}_{f_{DT}(v,u)\sim v^{\Delta\mathcal{O}+n}}$$

- $\gamma^{(1)}_{n,\ell}, a^{(1)}_{n,\ell}$ vanish to all orders in $1/\ell$!
- We can have truncated solutions ↔ local interactions in the bulk.

$$\mathcal{G}^{(1)}_{trunc}(u,v) \sim$$



Note: to order $1/N^4$ double-trace operators produce enh.-singularities!

$$\left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} \mathcal{G}(v,u) \sim \left(\frac{u}{v}\right)^{\Delta_{\mathcal{O}}} a_\ell v^{\Delta_{\mathcal{O}}+\frac{1}{N^2}\frac{\gamma^{(1)}}{2}+\cdots} f_\ell(v,u) \sim \frac{(\gamma^{(1)})^2}{N^4}\log^2 v$$

Non-planar $\mathcal{N} = 4$ SYM

Susy theory with $SU(4)$ R-charge

- Simplest correlator $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle$ [see Caron-Huot's talk]

$$\mathcal{G}(u,v) = \underbrace{\mathcal{G}^{short}(u,v)}_{\text{fixed by susy}} + \underbrace{\mathcal{H}(u,v)}_{\text{non-protected contribution}}$$

- Correlator in a $1/N$ expansion in the large λ regime:

$$\mathcal{H}(u,v) = \underbrace{\mathcal{H}^{(0)}(u,v)}_{\text{equivalent of GFF}} + \frac{1}{N^2}\mathcal{H}^{(1)}(u,v) + \frac{1}{N^4}\mathcal{H}^{(2)}(u,v) + \cdots$$

- To order $1/N^2$ the enhanced singularities arise only from protected single-trace operators.
- We can fully fix $\mathcal{H}^{(1)}(u,v)$ (the supergravity result)
- At large λ there is no ambiguity due to truncated solutions!

Non-planar $\mathcal{N} = 4$ SYM

- The enh.-sing. of $\mathcal{H}^{(2)}(u,v)$ can be computed after solving a mixing problem!

$$\mathcal{H}^{(2)}(z,\bar{z})\Big|_{\log^2 v} = R_1(z,\bar{z})Li_2(1-z) + \cdots + R_0(z,\bar{z})$$

e.g. Twist four, spin two operator for $\mathcal{N} = 4$ SYM

$$\Delta_{0,2} = 6 - \frac{4}{N^2} - \frac{45}{N^4} + \cdots$$

- Also $\langle(\gamma^{(1)}_{n,\ell})^2\rangle, \gamma^{(2)}_{n,\ell} \sim n^{11}$, Curious since $\gamma^{(sugra)}_{n,\ell} \sim n^3$.
- Thanks to mixing $3+3 \to 3+3+5$: we see the S^5!
- In the flat space limit we recover the 10D box function! (highly non-trivial!)
- Clear structure of UV divergences.

3d example: Large N Chern-Simons Vector Models

Spectrum at large N

- Scalar operator $J^{(0)}$ of dimension 1 or 2.
- Tower of HS conserved currents $J^{(s)}$, $s = 1, 2, \cdots$ of twist 1.
- Multitrace operators $[J^{(s)}, J^{(s')}]_{n,\ell}$.

$1/N$ corrections

Study crossing constraints on the correlator

$$\langle J^{(0)} J^{(0)} J^{(0)} J^{(0)} \rangle$$

- To zero order order only double trace operators $[J^{(0)}, J^{(0)}]_{n,\ell}$
- To order $1/N$ all the currents $J^{(s)}$ appear! their OPE is fixed by crossing.
- $\langle J^{(0)} J^{(0)} J^{(s)} \rangle$ agrees with Maldacena and Zhiboedov!
- Result to order $1/N^2$ includes elliptic integrals!!

$$G^{(2)}(z, \bar{z}) = \log^2 u \log^2 v K(z) + \cdots$$

Conclusions

- Generic CFTs have a large spin sector which becomes essentially free and we have shown how to perform a perturbation around that sector.
- The method applies to vast families of CFTs and is based on symmetries and consistency conditions.
- Efficient machinery to study $1/N$ corrections to holographic correlators in various dimensions!
- It would be great to connect this to other approaches, for instance integrability.
- Start asking quantitative questions about quantum (super)-gravity!