Non-planar Holographic Correlators from the Analytic Bootstrap

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Work with Aharony, Bissi, Caron-Huot, Perlmutter, Yacoby.
What will this talk be about?

Conformal Field Theories in $D > 2$ dimensions

- In general CFTs don’t have a Lagrangian description.
- In a Lagrangian theory we can use Feynman diagrams:
  \[ A(g) = A^{(0)} + gA^{(1)} + \ldots \]
- But even in this case, things become very messy very quickly.
- Idea of the conformal bootstrap: resort to consistency conditions!
  - Conformal symmetry
  - Properties of the OPE
  - Unitarity
  - Crossing symmetry
- We will advocate an analytic version of the conformal bootstrap.
Today: Operators with spin in a generic CFT

\[ \varphi \partial_{\mu_1} \cdots \partial_{\mu_\ell} \varphi \]

- Study their dimension \( \Delta \) for large values of the spin \( \ell \), as an expansion in \( 1/\ell \):

\[
\Delta(\ell) = \ell + 2\Delta_\varphi + \frac{c_1}{\ell} + \frac{c_2}{\ell^2} + \cdots
\]

- We will obtain analytic results to all orders in \( 1/\ell \)!
- Our results will be even valid for finite values of the spin!
Main ingredient: Conformal Primary local operators $\mathcal{O}_{\Delta, \ell}(x)$.

Operators form an algebra (OPE)

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_{k \in \text{prim.}} C_{ijk} x^{\Delta_k - \Delta_i - \Delta_j} (\mathcal{O}_k(0) + \text{descendants})$$

The set $\Delta_i$ and $C_{ijk}$ characterizes the CFT.

Main observable: Correlation functions of primary operators.

Four-point function of identical operators:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = \frac{g(u, v)}{x_{12}^{2\Delta_\mathcal{O}} x_{34}^{2\Delta_\mathcal{O}}}$$

where $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$.
Conformal partial wave decomposition

- OPE: $\mathcal{O} \times \mathcal{O} = \sum_i \mathcal{O}_i + \text{descendants}$

$$\langle \mathcal{O} \mathcal{O} \mathcal{O} \mathcal{O} \mathcal{O} \rangle = \sum_{\Delta, \ell} C_{\Delta, \ell} \mathcal{O}_{\Delta, \ell}$$

$$G(u, v) = 1 + \sum_{\Delta, \ell} C_{\Delta, \ell}^2 G_{\Delta, \ell}(u, v)$$

- **Conformal blocks**: For a given primary, take into account the contribution from all its descendants. Fully fixed function!
Conformal bootstrap

Crossing symmetry

\[ \nu^{\Delta_0} G(u, \nu) = u^{\Delta_0} G(\nu, u) \]

\[ \sum_{\Delta, \ell} \mathcal{C}_{\Delta, \ell} \mathcal{O}_{\Delta, \ell} = \sum_{\Delta, \ell} \mathcal{C}_{\Delta, \ell} \mathcal{O}_{\Delta, \ell} \]

A remarkable...but hard equation!

\[ \nu^{\Delta_0} \left( 1 + \sum_{\Delta, \ell} C_{\Delta, \ell}^2 G_{\Delta, \ell}(u, \nu) \right) = u^{\Delta_0} \left( 1 + \sum_{\Delta, \ell} C_{\Delta, \ell}^2 G_{\Delta, \ell}(\nu, u) \right) \]

Easy to expand around \( u = 0, \nu = 1 \)

Easy to expand around \( u = 1, \nu = 0 \)
Study this equation in different regions, \( u = z\bar{z}, \ v = (1 - z)(1 - \bar{z}) \)

- In the Euclidean regime \( \bar{z} = z^* \).
- We can study crossing around \( u = v = \frac{1}{4} \).
- Starting point of the numerical bootstrap.

- In the Lorentzian regime \( z, \bar{z} \) are independent real variables and we can consider \( u, v \to 0 \).
- Starting point of the analytic (light-cone) bootstrap!
Analytic (light-cone) bootstrap

Why is this a good idea?

- In Minkowski space we can have $x_{23}^2 \to 0, x_{23} \neq 0$.
- When some operators become null-separated the correlator develops singularities.

Dominated by high spin operators $\Leftrightarrow$ Dominated by low twist operators

\[(\mathcal{O} \partial_{\mu_1} \cdots \partial_{\mu_\ell} \mathcal{O}) \quad \Leftrightarrow \quad (1, T_{\mu,\nu}, \cdots)\]
Conformal blocks - technicalities

- Small $u$ limit:

\[ G_{\Delta,\ell}(u, v) \sim u^{\tau/2} f_{\tau,\ell}(v), \quad \tau = \Delta - \ell \]

We will introduce the notation

\[ G_{\Delta,\ell}(u, v) \equiv u^{\tau/2} f_{\tau,\ell}(u, v) \]

- Small $v$ limit:

\[ f_{\tau,\ell}(u, v) \sim \log v \]
Necessity of a large spin sector

- Consider the $v \ll 1$ limit of the crossing equation: $C_{\Delta, \ell}^2 \rightarrow a_{\tau, \ell}$

\[ v^{\Delta_\mathcal{O}} \left(1 + \sum_{\tau, \ell} a_{\tau, \ell} u^{\tau/2} f_{\tau, \ell}(u, v) \right) = u^{\Delta_\mathcal{O}} \left(1 + \sum_{\tau, \ell} a_{\tau, \ell} v^{\tau/2} f_{\tau, \ell}(v, u) \right) \]

\[ \Downarrow \]

\[ 1 + \sum_{\tau, \ell} a_{\tau, \ell} u^{\tau/2} f_{\tau, \ell}(u, v) = \frac{u^{\Delta_\mathcal{O}}}{v^{\Delta_\mathcal{O}}} + \text{subleading terms} \]

- The r.h.s. is power-law divergent as $v \rightarrow 0$.
- Each term on the l.h.s. diverges as $f_{\tau, \ell}(u, v) \sim \log v$.
- In order to reproduce the divergence on the right, we need infinite operators, with large spin and whose twist approaches $\tau = 2\Delta_\mathcal{O}$ (actually $\tau_n = 2\Delta_\mathcal{O} + 2n$)
Necessity of a large spin sector

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$$\Downarrow$$

$$1 + \sum_{\tau, \ell} a_{\tau, \ell} u^{\tau/2} f_{\tau, \ell}(u, v) = \frac{u^{\Delta_{\mathcal{O}}}}{v^{\Delta_{\mathcal{O}}}} + \text{subleading terms} \quad \text{rest of operators sorted by twist}$$

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Example: Generalised free fields

- Simplest solution: Large N CFTs - Generalised free fields

\[ G^{(0)}(u, v) = 1 + \left( \frac{u}{v} \right)^{\Delta_{\mathcal{O}}} + u^{\Delta_{\mathcal{O}}} \]

- Intermediate ops: Double trace operators:
  \[ \mathcal{O} \square^n \partial_{\mu_1} \cdots \partial_{\mu_\ell} \mathcal{O} \]
  \[ \tau_{n, \ell} = 2\Delta_{\mathcal{O}} + 2n \]
  \[ a_{n, \ell} = a_{n, \ell}^{(0)} \]

- Their OPE coefficients are such that the divergence of a single conformal block (\( \sim \log v \)), as \( v \to 0 \), is enhanced!

\[ 1 + \sum_{\tau, \ell} a_{\tau n, \ell}^{(0)} u^{\tau n/2} f_{\tau n, \ell}(u, v) = 1 + \left( \frac{u}{v} \right)^{\Delta_{\mathcal{O}}} + u^{\Delta_{\mathcal{O}}} \]

But this divergence is quite universal!
In any CFT with $\mathcal{O}$ in the spectrum, crossing symmetry implies the existence of double trace operators with arbitrarily large spin and

\[
\tau_{n,\ell} = 2\Delta_{\mathcal{O}} + 2n + \mathcal{O}(1/\ell)
\]
\[
a_{n,\ell} = a_{n,\ell}^{(0)} (1 + \mathcal{O}(1/\ell))
\]

- All CFTs have a large spin sector, where operators become "free"!
- Can we do perturbations around large spin? YES! Exploit the following idea:

\[
\sum_{\tau,\ell} a_{\tau,\ell} u^{\tau/2} f_{\tau,\ell}(u, v) = \frac{u^{\Delta_{\mathcal{O}}}}{v^{\Delta_{\mathcal{O}}}} + \cdots
\]

Behaviour at large spin $\Leftrightarrow$ Enhanced divergences as $v \to 0$
Given $\text{Sing}(u, v)$, find $a_{\tau, \ell}$ such that

$$\sum_{\tau, \ell} a_{\tau, \ell} u^{\tau/2} f_{\tau, \ell}(u, v) = \text{Sing}(u, v)$$

$$v^{-\Delta}, v^n \log^2 v, v^{1/2}, \ldots$$

Idea

1. Construct a complete basis of functions with specific CFT-data and prescribed singularities.
2. Express $\text{Sing}(u, v)$ in that basis.

Large spin perturbation theory

$$\text{Sing}(u, v) \rightarrow a_{\tau, \ell} = a_{\tau, \ell}^{(0)} (1 + \frac{c_1}{\ell} + \frac{c_2}{\ell^2} + \cdots)$$

- LSPT fixes the CFT data to all orders in $1/\ell$, just from $\text{Sing}(u, v)$!
Is the CFT-data analytic in the spin? does LSPT give the full result?
The answer appeared to be affirmative in all examples!
Reformulation in terms of an inversion formula [Caron-Huot]

$$a_{\tau,\ell} \sim \int dudv \, K(u, v, \tau, \ell) \, Sing(u, v)$$

Equivalent to LSPT but explicitly analytic in the spin!
It can be extrapolated down to small spin as a consequence of the Regge behaviour!
A nice machinery for theories with small parameters

**Strategy**

1. Use crossing symmetry to determine the enhanced singularities

   \[ G(u, v) \leftarrow G(u, v)|_{en.sing.} = \left( \frac{u}{v} \right)^{\Delta_0} G(v, u)|_{en.sing.} \]

   In theories with small parameters the latter follows from CFT-data at lower orders! (maybe including other correlators)

2. Use LSPT to reconstruct the CFT-data from the enhanced singularities.

3. Go to next order and repeat.

   - Let’s apply LSPT to find $1/N$ corrections to GFF!
Large $N$ CFTs

**AdS/CFT**

Large $N$ CFT in D-dimensions (GFF + corrections) $\iff$ Gravitational theory in $AdS_{D+1}$

\[
\frac{1}{N^2} \text{ expansion in CFT} \iff \text{loops in } AdS/\text{powers of } G_N.
\]

\[G = \underbrace{N^0} + \underbrace{\frac{1}{N^2}} + \underbrace{\frac{1}{N^4}} + \ldots\]

- Diagrams in $AdS$ are hard to compute...Use crossing for the CFT!
Large $N$ holographic CFTs

\[ G(u, v) = G^{(0)}(u, v) + \frac{1}{\mathcal{N}^2} G^{(1)}(u, v) + \cdots \]

Two Sources of corrections

1. Double trace operators will acquire corrections:

\[ \tau_{n,\ell} = 2\Delta\phi + 2n + \frac{1}{\mathcal{N}^2} \gamma_{n,\ell}^{(1)} + \cdots \]

\[ a_{n,\ell} = a_{n,\ell}^{(0)} + \frac{1}{\mathcal{N}^2} a_{n,\ell}^{(1)} + \cdots \]

2. We can also have new intermediate operators at order $1/\mathcal{N}^2$.

Which corrections are consistent with crossing symmetry?

\[ G^{(1)}(u, v) = \left( \frac{u}{v} \right)^{\Delta\phi} G^{(1)}(v, u) \]
Large $N$ holographic CFTs

**Case 1**: New single-trace operators at order $1/N^2$:

$$\mathcal{O} \times \mathcal{O} = 1 + [\mathcal{O}, \mathcal{O}]_{n,\ell} + \frac{1}{N^2} \mathcal{O}_{st}$$

- This produces an enhanced divergence:
  $$\mathcal{G}^{(1)}(u, v) = \left( \frac{u}{v} \right)^{\Delta_{\mathcal{O}}} \mathcal{G}^{(1)}(v, u) \supset \left( \frac{u}{v} \right)^{\Delta_{\mathcal{O}}} v^{\Delta_{\mathcal{O}_{st}}/2} f_{\mathcal{O}_{st}}(v, u)$$

- The enhanced divergences fix $\gamma^{(1)}_{n,\ell}, a^{(1)}_{n,\ell}$ to all orders in $1/\ell$!
- Non-truncated solutions correspond to $AdS$ exchanges

$$\mathcal{G}_{non-tr}^{(1)}(u, v) \sim$$
Case 2: No new operators at order $1/N^2$

- Double-trace operators don’t produce enhanced divergences!

$$G^{(1)}(u, v) = \left( \frac{u}{v} \right)^{\Delta \phi} G^{(1)}(v, u)$$

$$f_{DT}(v, u) \sim v^{\Delta \phi + n}$$

- $\gamma^{(1)}_{n, \ell}, a^{(1)}_{n, \ell}$ vanish to all orders in $1/\ell$!
- We can have truncated solutions $\leftrightarrow$ local interactions in the bulk.

$$G^{(1)}_{trunc}(u, v) \sim$$

Note: to order $1/N^4$ double-trace operators produce enh.-singularities!

$$\left( \frac{u}{v} \right)^{\Delta \phi} G(v, u) \sim \left( \frac{u}{v} \right)^{\Delta \phi} \frac{a_{\ell} v^{\Delta \phi + 1}}{N^2} \gamma^{(1)}_{2} + \cdots \sim \frac{\gamma^{(1)}_{2}^{2}}{N^4} \log^2 v$$
Non-planar $\mathcal{N} = 4$ SYM

Susy theory with $SU(4)$ R-charge

- Simplest correlator $\langle O_2 O_2 O_2 O_2 \rangle$ [see Caron-Huot’s talk]

\[ G(u, v) = G^{\text{short}}(u, v) + \underbrace{H(u, v)}_{\text{non-protected contribution}} \]

fixed by susy

- Correlator in a $1/N$ expansion in the large $\lambda$ regime:

\[ \mathcal{H}(u, v) = \mathcal{H}^{(0)}(u, v) + \frac{1}{N^2} \mathcal{H}^{(1)}(u, v) + \frac{1}{N^4} \mathcal{H}^{(2)}(u, v) + \cdots \]

equivalent of GFF

- To order $1/N^2$ the enhanced singularities arise only from protected single-trace operators.

- We can fully fix $\mathcal{H}^{(1)}(u, v)$ (the supergravity result)

- At large $\lambda$ there is no ambiguity due to truncated solutions!
Non-planar $\mathcal{N} = 4$ SYM

The enh.-sing. of $\mathcal{H}^{(2)}(u, \nu)$ can be computed after solving a mixing problem!

$$
\mathcal{H}^{(2)}(z, \bar{z})\bigg|_{\log^2 \nu} = R_1(z, \bar{z}) Li_2(1 - z) + \cdots + R_0(z, \bar{z})
$$

e.g. Twist four, spin two operator for $\mathcal{N} = 4$ SYM

$$
\Delta_{0,2} = 6 - \frac{4}{N^2} - \frac{45}{N^4} + \cdots
$$

Also $\langle (\gamma_{n,\ell}^{(1)})^2, \gamma_{n,\ell}^{(2)} \rangle \sim n^{11}$, Curious since $\gamma_{n,\ell}^{(\text{sugra})} \sim n^3$.

Thanks to mixing $3 + 3 \rightarrow 3 + 3 + 5$: we see the $S^5$!

In the flat space limit we recover the 10D box function! (highly non-trivial!)

Clear structure of UV divergences.
3d example: Large N Chern-Simons Vector Models

Spectrum at large $N$

- Scalar operator $J^{(0)}$ of dimension 1 or 2.
- Tower of HS conserved currents $J^{(s)}$, $s = 1, 2, \cdots$ of twist 1.
- Multitrace operators $[J^{(s)}, J^{(s')}]_{n,\ell}$.

1/$N$ corrections

Study crossing constraints on the correlator

$$\langle J^{(0)} J^{(0)} J^{(0)} J^{(0)} \rangle$$

- To zero order only double trace operators $[J^{(0)}, J^{(0)}]_{n,\ell}$
- To order 1/$N$ all the currents $J^{(s)}$ appear! their OPE is fixed by crossing.
- $\langle J^{(0)} J^{(0)} J^{(s)} \rangle$ agrees with Maldacena and Zhiboedov!
- Result to order 1/$N^2$ includes elliptic integrals!!

$$G^{(2)}(z, \bar{z}) = \log^2 u \log^2 v K(z) + \cdots$$
Conclusions

- Generic CFTs have a large spin sector which becomes essentially free and we have shown how to perform a perturbation around that sector.
- The method applies to vast families of CFTs and is based on symmetries and consistency conditions.
- Efficient machinery to study $1/N$ corrections to holographic correlators in various dimensions!
- It would be great to connect this to other approaches, for instance integrability.
- Start asking quantitative questions about quantum (super)-gravity!