Continuum limit of fishnet graphs and AdS sigma model

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based on work with De-liang Zhong

Motivation

Understand dynamics of planar graphs and its relation to sigma models

['t Hooft]
[Polyakov]

Best possible starting point: N=4 SYM

[Maldacena'97]

1. String dual is (believed to be) known
2. Theory is (believed to be) integrable
   = we have methods for re-summing planar graphs

Use solution to gain knowledge about other models by deforming / twisting the theory $\longrightarrow$ partial re-summations
Reduce complexity, but maintain as many important properties as possible: conformal symmetry, integrability, etc

Fishnet theory

Baby version of N = 4 SYM
A theory for matrix scalar fields with quartic coupling

[Gurdogan,Kazakov'15]
[Caetano,Gurdogan,Kazakov'16]

$$\mathcal{L}_{\text{fishnet}} = N\text{tr}\left[\partial_\mu \phi_1 \partial_\mu \phi_1^* + \partial_\mu \phi_2 \partial_\mu \phi_2^* + (4\pi g)^2 \phi_1 \phi_2 \phi_1^* \phi_2^*\right]$$

It can be obtained by twisting N=4 SYM theory, so-called γ deformation, sending the deformation parameter to i-infinity while taking YM coupling to zero

[Frolov'05]
[Lunin,Maldacena'05]
[Grabner,Gromov,Kazakov,Korchemsky'17]
[Sieg,Wilhelm'16]

1. Gluons and gauginos decouple
2. Gauge group becomes a flavour group, no SUSY
3. Conformal symmetry is preserved for any coupling (at least in planar limit and for fine-tuned double-trace couplings)
4. Integrability is retained

Fishnet theory

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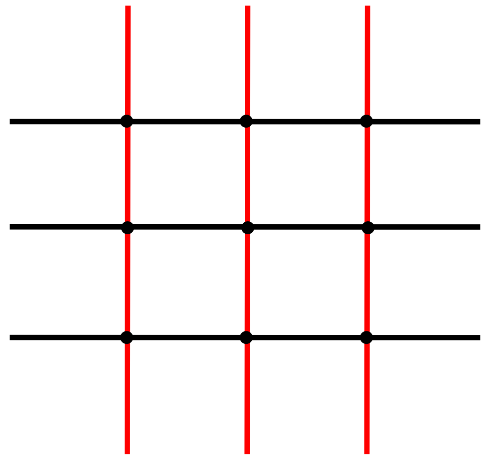
$$\mathcal{L}_{\text{fishnet}} = N\text{tr}\left[\partial_\mu\phi_1\partial_\mu\phi_1^* + \partial_\mu\phi_2\partial_\mu\phi_2^* + (4\pi g)^2\phi_1\phi_2\phi_1^*\phi_2^*\right]$$

Planar graphs all look the same



bulk graph

Integrability much less mysterious and links directly to properties of the quartic vertex in d=4

[Zamolodchikov'80]
[Isaev'03]
[Gromov,Kazakov,Korchemsky,Negro,Sizov'17]
[Chicherin,Kazakov,Loebbert,Muller,Zhong'16]

Win: simplicity (very few graphs)

Lose: unitarity

Continuum limit & string?

What about duality to string in AdS?

Extremal twisting procedure forces the YM coupling to be small
⟶ *string in highly curved AdS?*

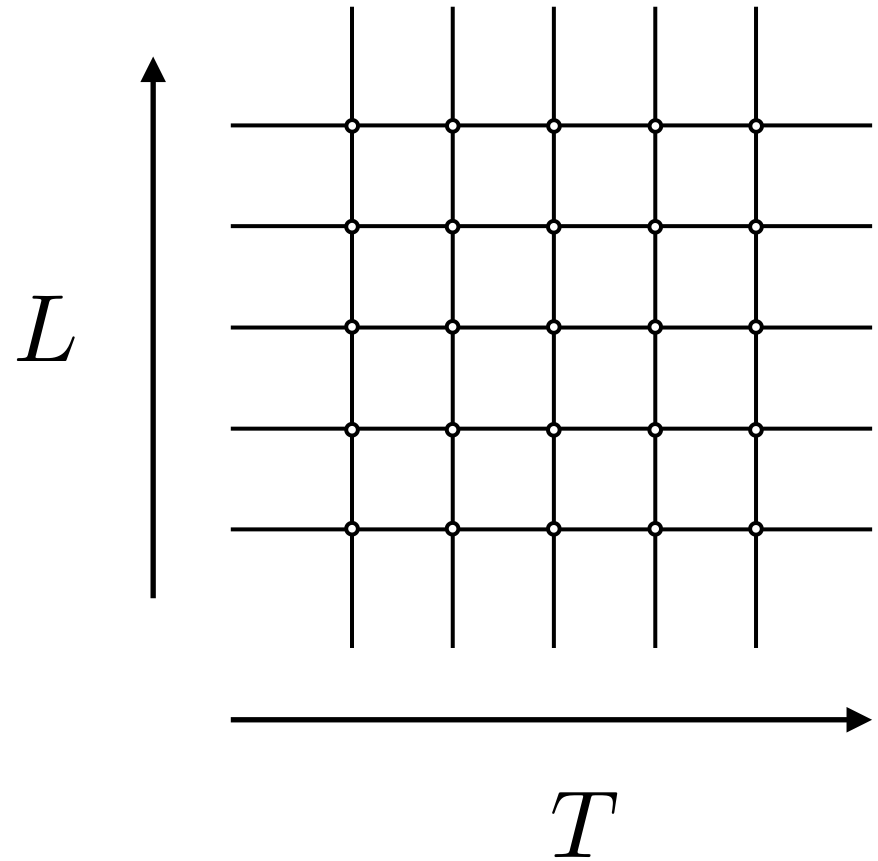
Related question: continuum limit of fishnet graphs?

Important observation concerning large order behaviour [Zamolodchikov'80]

Zamolodchikov's thermodynamical scaling $L, T \to \infty$

$$\log Z_{L,T} = -L \times T \log g_{cr}^2$$

$$g_{cr} = \frac{\Gamma(3/4)}{\sqrt{\pi}\Gamma(5/4)} = 0.7...$$



Critical coupling :
graphs become dense, continuum limit

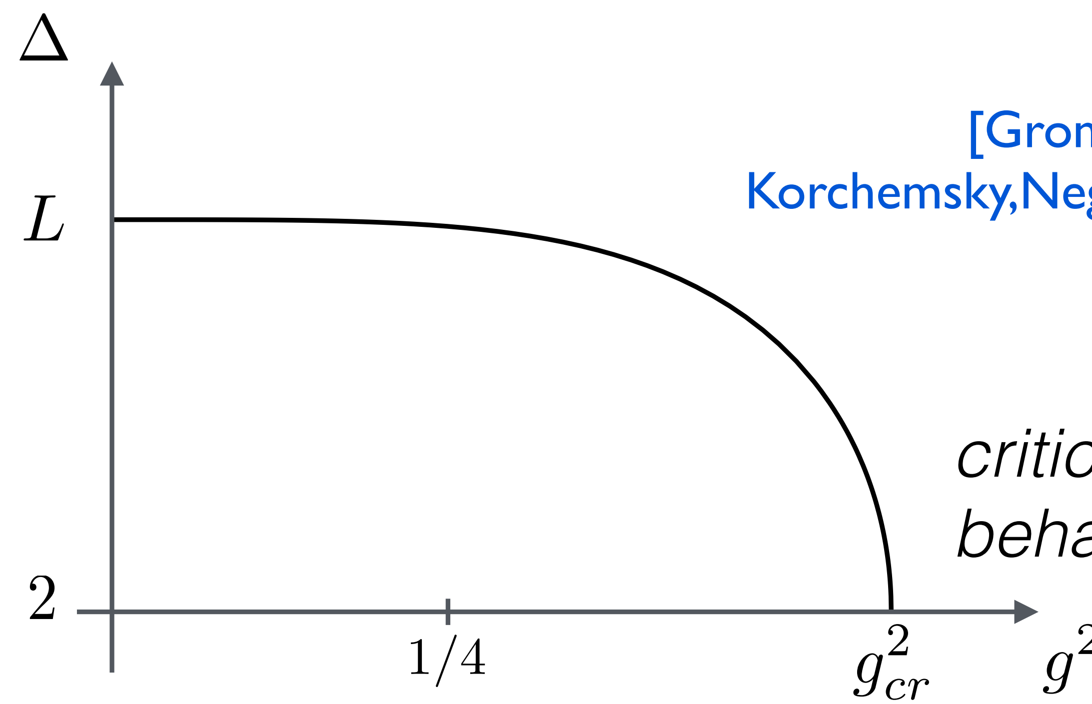
Plan

Study continuum limit using N=4 SYM integrable techniques

Probe: scaling dimension Δ of BMN vacuum operator $\mathcal{O} = \mathrm{tr}\,\phi_1^L$

Qualitative picture of Δ *as function of coupling* g^2 *when* $L \to \infty$

Thermodynamical scaling $\Delta \sim L f(g)$



see also [Gromov,Kazakov, Korchemsky,Negro,Sizov'17]

Analysis close to critical point: fishnet graph = 2d AdS5 sigma model

Dictionary:

BMN operator = tachyon $\quad \mathrm{tr}\,\phi_1^L \leftrightarrow V_\Delta \sim e^{-i\Delta t}$

4d coupling = worldsheet energy $\quad \log g^{2L} = \log g_{cr}^{2L} + E_{2d}(\Delta, L)$

Mapping graphs and magnons

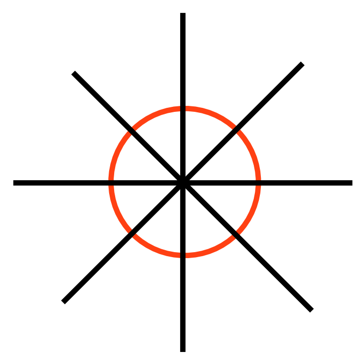
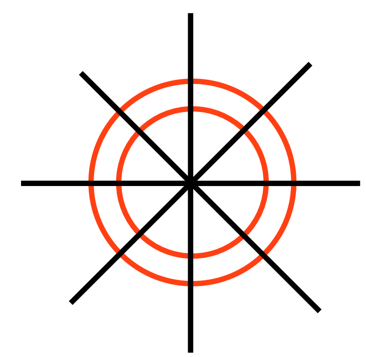
Computation of anomalous dimension

[Gurdogan,Kazakov'15]
[Caetano,Gurdogan,Kazakov'16]
[Gromov,Kazakov,
Korchemsky,Negro,Sizov'17]

BMN vacuum $\mathrm{tr}\,\phi_1^L$
(not protected)

$$\Delta = \Delta_L(g) = L + \gamma$$

Graphs: loop corrections come from the wheel diagrams

$Z = 1 +$  $+$  $+ \cdots$

wave-function
renormalization

1 wheel $\sim g^{2L}$

2 wheels $\sim g^{4L}$

Depends on cut off

$$R \sim \log \Lambda_{UV}$$

Anomalous dimension controls the logarithmic dependence on cut off

$$\log Z \sim -\gamma \times R$$

Mapping graphs and magnons

Computation of anomalous dimension

[Gurdogan,Kazakov'15]
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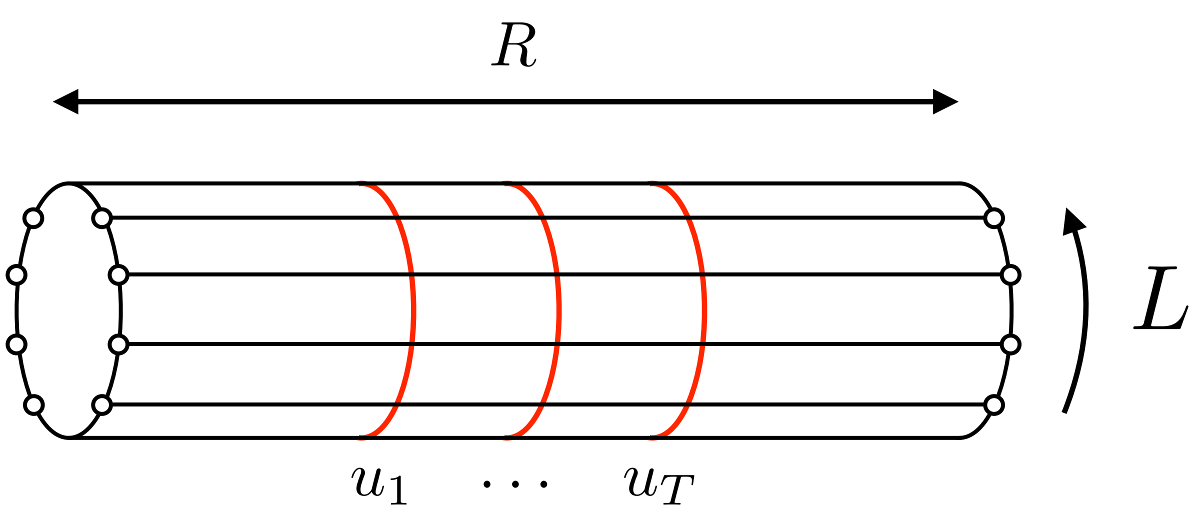
BMN vacuum (not protected) $\operatorname{tr}\phi_1^L$

$$\Delta = \Delta_L(g) = L + \gamma$$

Integrability: quantum mechanical interpretation of the graphs

1+1d picture: partition function on $\mathbb{R} \times S_L$

$$\mathcal{Z}_{L,R} = \sum_{T \geqslant 0} g^{2LT} \times$$



dof = magnon = wheel

magnon carries a rapidity "u", momentum along euclidean time direction, and a discrete label "a", for harmonics on 3-sphere

scaling dimension = free energy of a gas of magnon

$$\log \mathcal{Z}_{L,R} = -\Delta_L(g) R + O(R^0)$$

at temperature $1/L$ and chemical potential $h = \log g^2$

Thermodynamical Bethe Ansatz

Factorized scattering allows us to obtain free energy from TBA

$$\Delta = L - 2\sum_{a\geqslant 1}\int \frac{du}{2\pi}\mathbf{Y}_a(u)$$
$$-\sum_{a\geqslant 1}\int \frac{du}{2\pi}\mathbf{Y}_a^2(u) - 2\sum_{a,b\geqslant 1}\int \frac{dudv}{(2\pi)^2}\mathbf{Y}_a(u)\mathcal{K}_{a,b}(u,v)\mathbf{Y}_b(v) + O(\mathbf{Y}^3)$$

Boltzmann weight: $\mathbf{Y}_a(u) = a^2 e^{Lh - L\epsilon_a(u)} \ll 1$

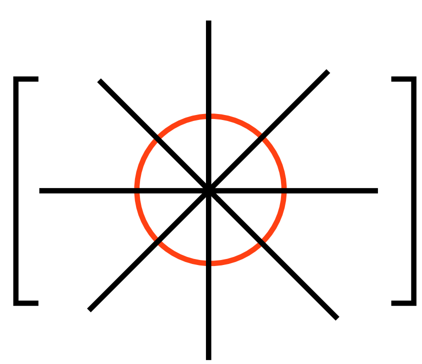
Magnon energy: $\epsilon_a(u) = \log\left(u^2 + a^2/4\right)$

Scattering kernel: $\mathcal{K}_{a,b}(u,v) = \frac{\partial}{i\partial u}\log S_{a,b}(u,v) + \text{matrix part}$

Coupling constant: fugacity for the magnons (wheels) $g^2 = e^h$

[Gurdogan,Kazakov'15]

ex. free wheel:

div  $= -\sum_{a\geqslant 1} a^2 \int \frac{du}{\pi}\frac{g^{2L}}{(u^2+a^2/4)^L} \propto g^{2L}\zeta(2L-3)$

Thermodynamic limit

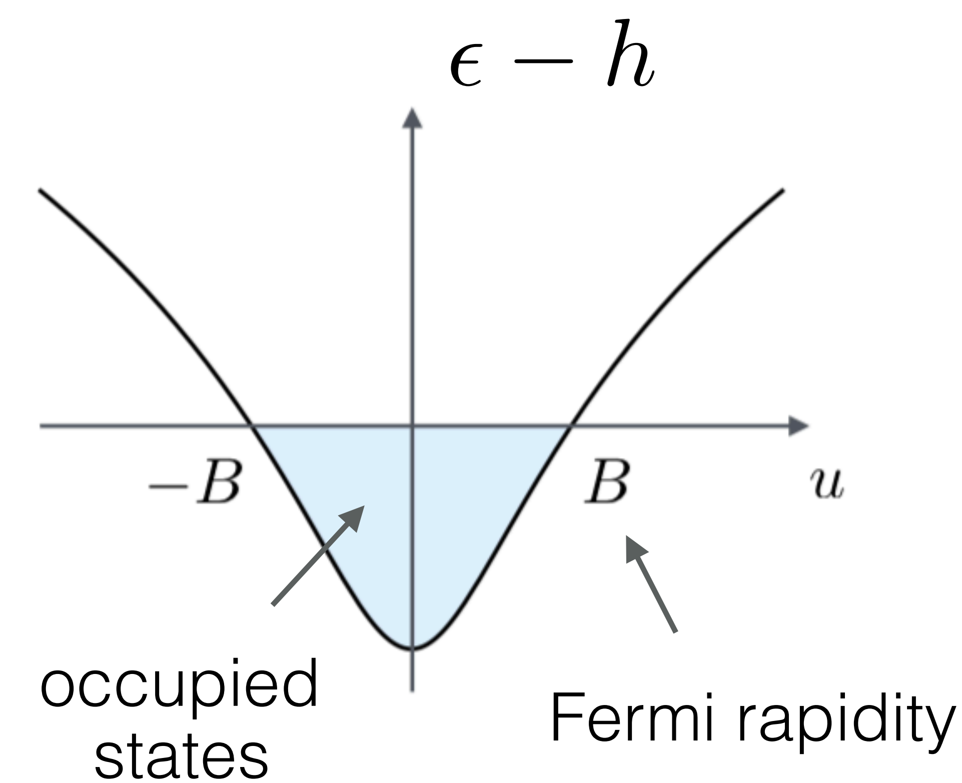
Thermodynamic limit $L \to \infty$

Interesting when chemical potential gets bigger than mass of lightest magnon

$$h = \epsilon(u = 0) = \log 1/4$$

that is for

$$g > 1/2$$



A Fermi sea forms

All states below the Fermi rapidity are filled

Increasing coupling amounts to increasing B

Comment: only the s-wave (lightest) magnons condense
(higher Lorentz harmonics decouple)

Linear integral equation

In the thermodynamic limit the TBA linearizes

Integral equation for the rapidity distribution of energy levels

$$\chi(u) = C - \epsilon(u) + \int_{-B}^{B} \frac{du}{2\pi} \mathcal{K}(u-v)\chi(v)$$

BC: $\chi(u = \pm B) = 0$

Effective chemical potential: $C = \log g^2 - \int_{-B}^{B} \frac{du}{2\pi} k(u)\chi(u)$

Kernel: $\mathcal{K}(u) = 2\psi(1+iu) + 2\psi(1-iu) + \frac{2}{1+u^2}$

Scaling dimension: $\Delta/L = 1 - \int_{-B}^{B} \frac{du}{\pi}\chi(u)$

Critical regime

Small B : dilute gas, density of magnons is small, free regime

$$j = -df/dh \sim 0 \qquad \varepsilon = f + hj \sim 1 \qquad f \sim 1$$

Critical regime : dense gas, large magnon density, $B \to \infty$

$$\varepsilon \sim j \log g_{cr}^2 \quad \longleftrightarrow \quad \log Z_{L,T} \sim -LT \log g_{cr}^2$$

(Zamolodchikov's microcanonical scaling)

Characteristics : all energy levels are filled, distribution covers real axis
(equation is immediately solved in Fourier space)

$$\chi_{cr}(u) = C_{cr} - \epsilon(u) + \int_{-\infty}^{\infty} \frac{dv}{2\pi} \mathcal{K}(u-v)\chi_{cr}(v) \quad \Rightarrow \quad \chi_{cr} = \log \frac{\sqrt{2}\cosh\theta + 1}{\sqrt{2}\cosh\theta - 1}$$

with $\theta = \pi u/2$

One finds:

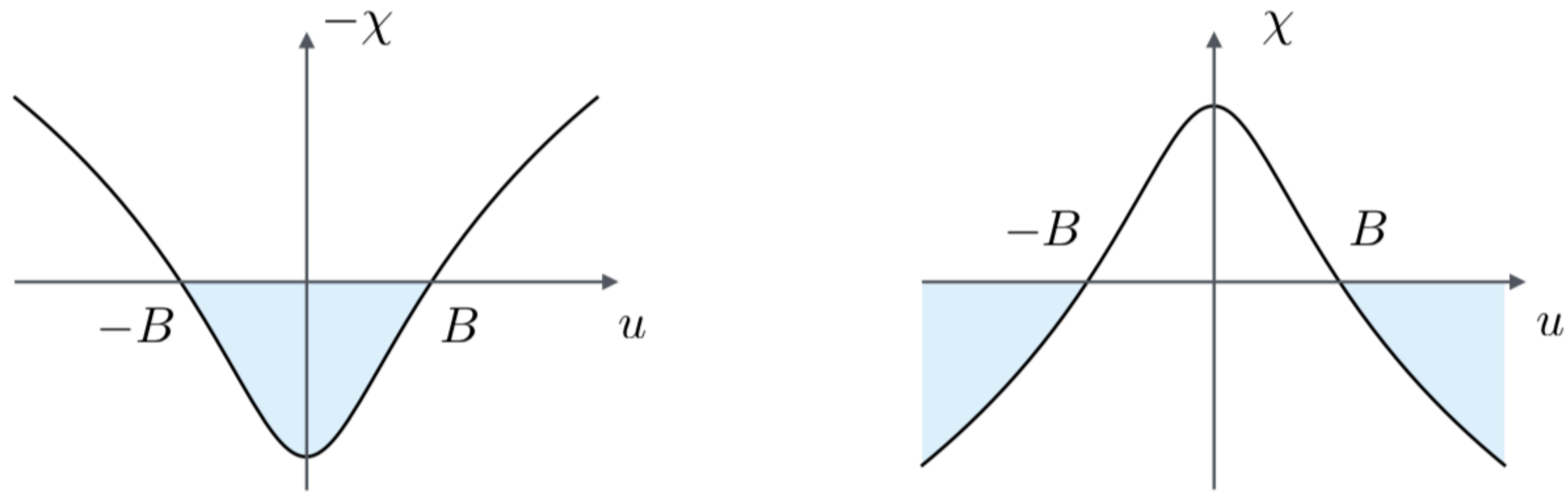
(i) Scaling function vanishes $f = \Delta/L \to 0$

(ii) Chemical potential (coupling) approaches predicted value

$$\chi_{cr} \sim e^{-|\theta|} \quad \Rightarrow \quad C_{cr} = 0 \quad \Rightarrow \quad g_{cr} = \Gamma(3/4)/\sqrt{\pi}\Gamma(5/4)$$

Near-critical regime

Particle-hole transformation:



Fermi sea of magnons

dual Fermi sea

Equation for dual excitations:

1) dualize kernel

$$K = -\frac{\mathcal{K}}{1 - \mathcal{K}*} = -\mathcal{K} - \mathcal{K} * \mathcal{K} - \ldots$$

2) act on both sides of the equation with $1 - K*$

Dual equations

Dual equation: $$\chi(\theta) = E(\theta) + \int\limits_{\theta^2 > B^2} \frac{d\theta'}{2\pi} K(\theta - \theta')\chi(\theta')$$

Dual energy formula: $$\log g^2 = \log g_{cr}^2 + \int\limits_{\theta^2 > B^2} \frac{d\theta}{2\pi} P'(\theta)\chi(\theta)$$

No chemical potential but extra BC (at ∞) $\chi(\theta) \sim -2\rho \log\theta$ $\rho = \Delta/L =$ charge density

1) Dual kernel:

$$K(\theta) = \frac{\partial}{i\partial\theta} \log \frac{\Gamma(1 + \frac{i\theta}{2\pi})\Gamma(\frac{1}{2} - \frac{i\theta}{2\pi})\Gamma(\frac{3}{4} + \frac{i\theta}{2\pi})\Gamma(\frac{1}{4} - \frac{i\theta}{2\pi})}{\Gamma(1 - \frac{i\theta}{2\pi})\Gamma(\frac{1}{2} + \frac{i\theta}{2\pi})\Gamma(\frac{3}{4} - \frac{i\theta}{2\pi})\Gamma(\frac{1}{4} + \frac{i\theta}{2\pi})}$$

2) Dual dispersion relation:

$$E(\theta) = \chi_{cr}(\theta) = \log\left[\frac{\sqrt{2}\cosh\theta + 1}{\sqrt{2}\cosh\theta - 1}\right] \sim \frac{m}{2} e^{-|\theta|}$$

$$P(\theta) = -iE(\theta + i\frac{\pi}{2}) = i\log\left[\frac{\sqrt{2}\sinh\theta + i}{\sqrt{2}\sinh\theta - i}\right] \sim \mp\frac{m}{2} e^{-|\theta|}$$

here $m = 4\sqrt{2}$ acts like a mass scale

Interpretation

What is the dual system describing?

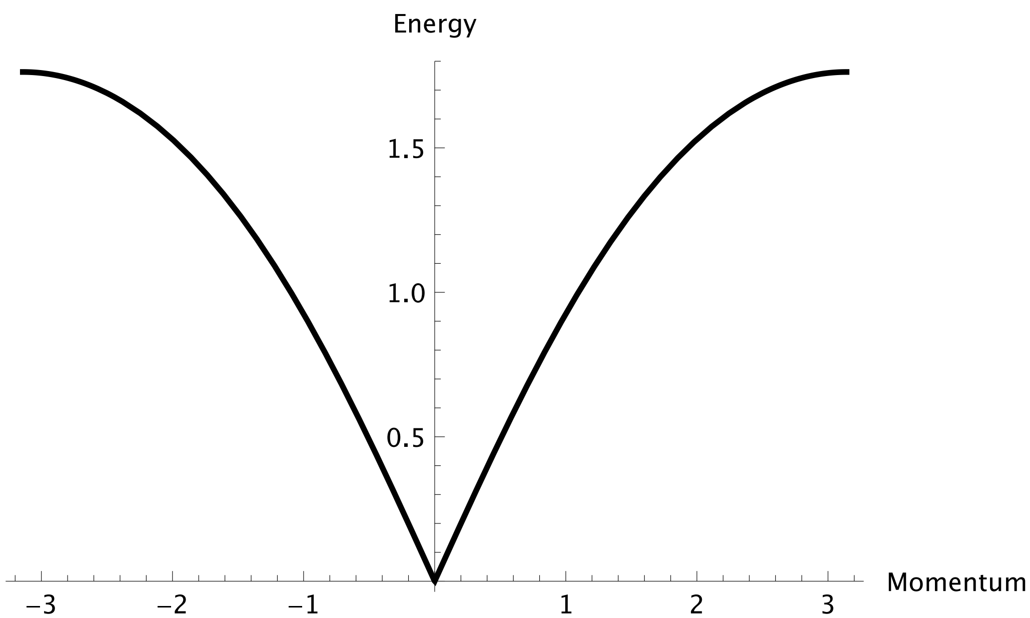
1) Kernel: $$K = -i\partial_\theta \log S_{O(6)}$$

Excitations scatter as particles in 2d O(6) non-linear sigma model

[Zamolodchikov&Zamolodchikov]

2) Dispersion relation:

$$\sinh^2\left(\frac{1}{2}E\right) = \sin^2\left(\frac{1}{2}P\right)$$



Gapless excitations (unlike O(6) model)

Besides, E decreases when θ increases
Support is non-compact and density is not normalizable (cannot count excitations)

No mass gap + continuous spectrum
Suggest: sigma model with ***non-compact*** target space
Proposal: integrable lattice completion of AdS_5 sigma model

Dual model: hyperbolic model

2d sigma model with curved target space

$$\mathcal{L} = -\frac{1}{2e^2} G^{AB} \partial_a Y_A \partial^a Y_B$$

Weak coupling (large radius of curvature) : $e^2 \ll 1$

beta function related to Ricci scalar

For AdS_{d+1} : $-Y_0^2 + Y_\perp^2 - Y_{d+1}^2 = -1$

it gives a positive beta function $\mu \frac{\partial}{\partial \mu} e^2(\mu) = \frac{d}{2\pi} e^4(\mu) + \ldots$

Alternatively, the coupling grows with the energy $\frac{1}{e^2(\mu)} = \frac{d}{2\pi} \log \left(\Lambda / \mu \right)$

1. Theory is weakly coupled in IR
2. There is no mass gap
3. There is no isolated vacuum

Dual state: tachyon

Consider sigma model on cylinder of radius L

Interested in 2d "ground state" energy: tachyon

(best candidate for extremum of energy at given charge = global time energy)

$$V_\Delta \sim e^{-i\Delta t}$$

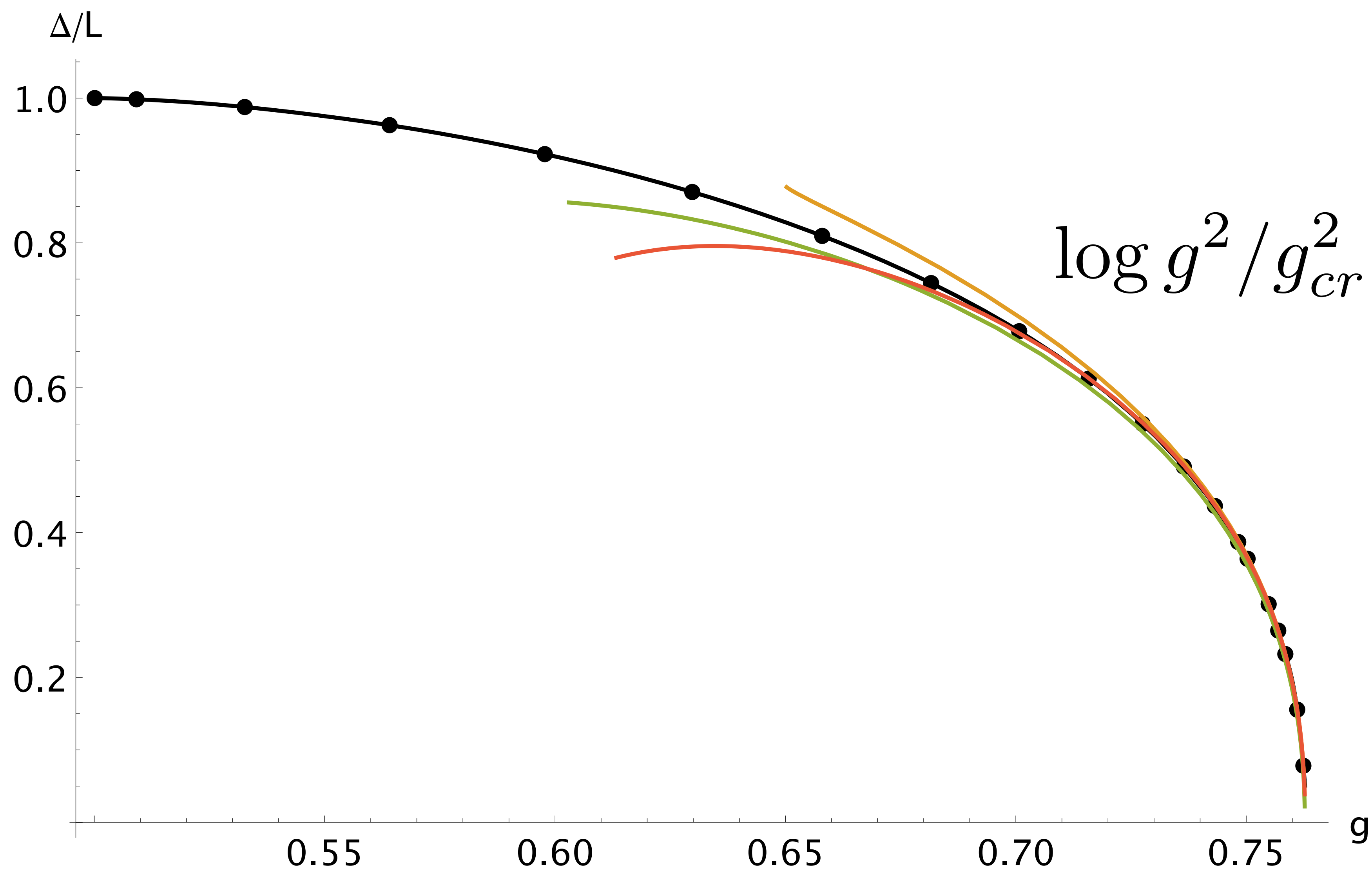
Classically, it corresponds to solution $$Y^0 \pm iY^{d+1} = e^{\pm iH\tau}$$ $$Y_\perp = 0$$

Classical (c-o-m) energy $$E = -\frac{e^2\Delta^2}{2L}$$

Same as in O(d+2) model if not for the sign of the coupling $e^2 \leftrightarrow -e^2$

Numerics

"Quadratic Casimir" scaling near critical point fits numerical sol. of linear eq.



$$\log g^2/g_{cr}^2 = -\frac{e^2\Delta^2}{2L^2}$$

WS PT
at 1st, 2nd
and 3rd
order

$$g_{\rm thr} = 0.5$$

$$g_{\rm cr} = 0.76276$$

thermodynamical window

All order perturbation theory

Argument that dual integral equation describes the tachyon of the AdS model to *all* orders in perturbation theory

Solutions to eqs for sphere and hyperboloid are the same to any order in $1/B$ if we formally flip the sign of the Fermi rapidity $B \to -B$ [Volin'09]

Since the Fermi rapidity plays role of the inverse running coupling

$$B \sim 1/e^2 \sim \log\left(L/\Delta\right) \gg 1$$

at energy scale $\sim \rho = \Delta/L$

flipping its sign has the same effect as flipping the curvature of the space

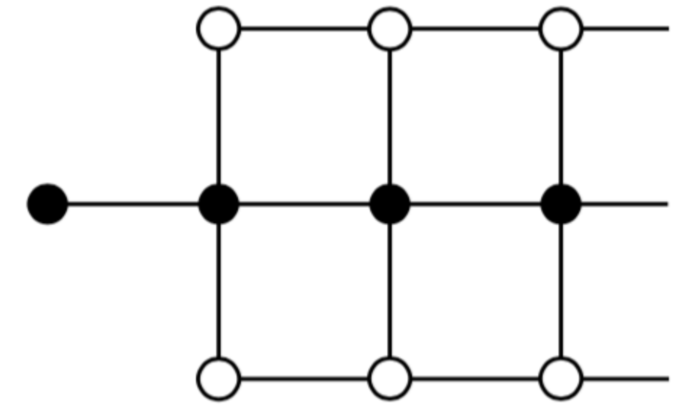
Two descriptions co-exist

Original magnon TBA (massive + chemical potential)

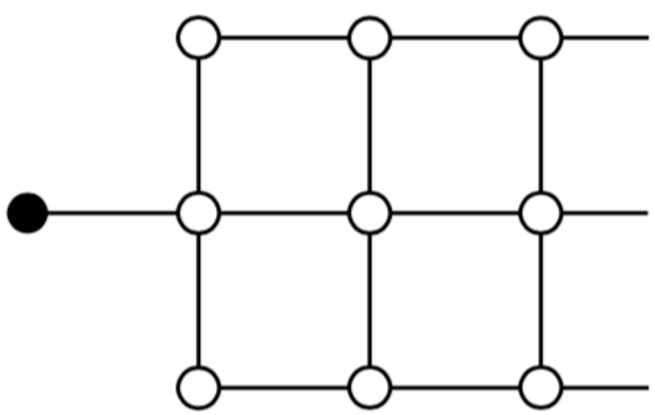
$$\log Y_1 = L\log g^2 - L\epsilon + \mathcal{K} * \log(1+Y_1) + \ldots$$

$$\Delta = L - \sum_{a=1}^{\infty}\int \frac{du}{\pi}\log(1+Y_a)$$

input : coupling
output : scaling dimension



massive TBA
(blacks nodes = energy carriers)



massless TBA
(only 1 momentum carrier)

Dual TBA (massless + no chemical potential)

$$\log Y_1 = LE - K_{O(6)} * \log(1+1/Y_1) + \ldots$$

$$\log g^{2L}/g_{cr}^{2L} = -\int \frac{d\theta}{2\pi} P'(\theta)\log(1+1/Y_1)$$

Δ = input (label tachyon rep)
coupling = output (sigma model energy)

Y system

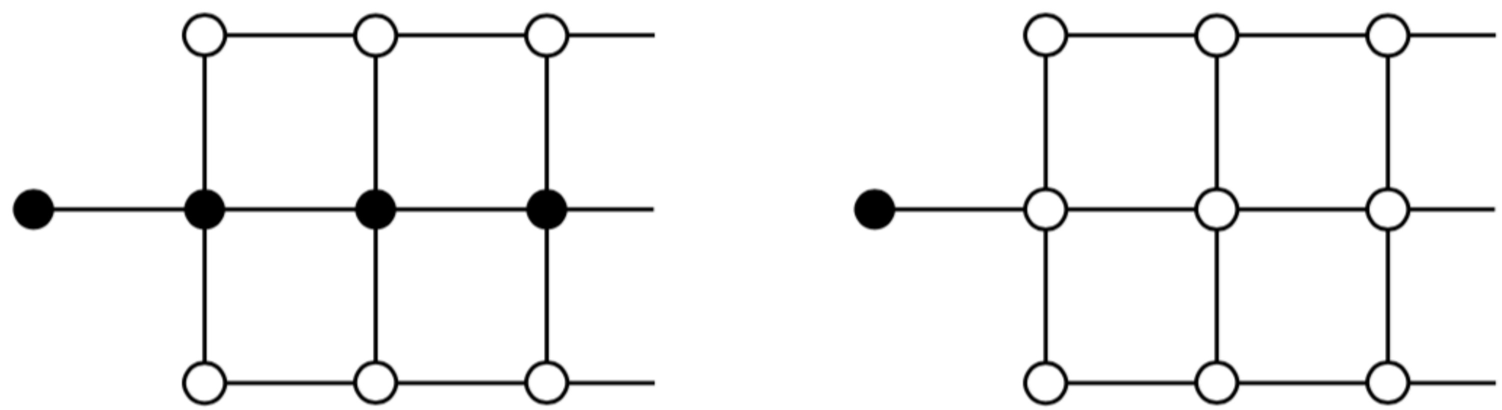
Y system equations stay the same (same form as for compact model)

[Balog,Hegedus'04]
[Fendley'99]

$$\frac{Y^+_{a,s}Y^-_{a,s}}{Y_{a+1,s}Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

$$\frac{1}{Y_1^{++}Y_1^{--}} = (1+\frac{1}{Y_{2,1}})(1+\frac{1}{Y^+_{2,0}})(1+\frac{1}{Y^-_{2,0}})(1+\frac{1}{Y_{2,-1}})$$

regardless of the phase



Finite size effects : central charge

TBA analysis in CFT limit (1/L effect = Casimir energy)

[Zamolodchikov'90s]
[Klassen-Melzer'90s]

$$L \gg 1 \qquad \Delta = O(1)$$

Problem split into left and right movers = scale-invariant (kink) solution
Kink is characterized by its asymptotic values on far left and far right

Standard dilog analysis yields TBA central charge $c = c_0 - c_\infty$

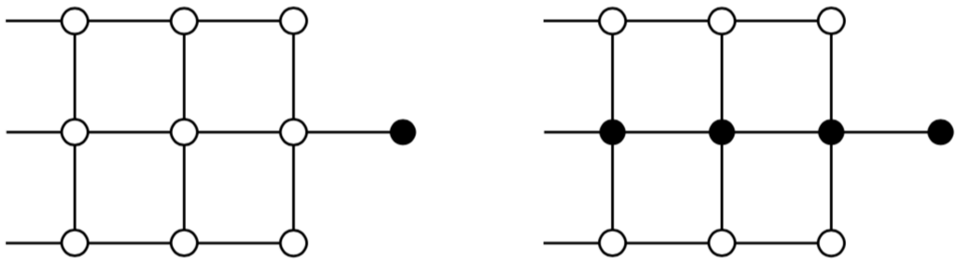
where $c_\star = \sum_i \mathcal{L}(\frac{Y_i^\star}{1+Y_i^\star})$

with Rogers dilogarithm $\mathcal{L}(x) = \frac{6}{\pi^2}(\mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x))$

Stationary solutions to Y-system are known

see e.g.
[Balog,Hegedus'04]

symmetric phase
O(6)
$c_0 = 7$



broken phase
O(4)
$c_\infty = 2$

TBA central charge: $c = 5$

Finite size effects : central charge

CFT analysis : close to IR fixed point, i.e. large L, the 2d CFT gives information about the behaviour of the energy levels

Operator-state correspondence: energy maps to 2d anomalous dimension of vertex operator (here tachyon)

$$V_\Delta \sim e^{-i\Delta t} \qquad E_{2d} = -\frac{\pi c_{eff}(L)}{6L} - \frac{e^2\Delta(\Delta - d)}{2L} + O(e^4)$$

running coupling at distance L: $e^2 \sim \frac{2\pi}{d \log L} \ll 1$

effective central charge at distance L: $c_{eff}(L) = d + 1 + O(e^2)$

(count the number of Goldstone bosons = dimensions of AdS_{d+1})

Agreement with TBA (for d=4 ie AdS5)

Spinning the wheels

Operators with spin $\mathrm{tr}\,\partial^M \phi_1^L$

Scaling dimension at weak coupling $\Delta = L + M + O(g^{2L})$

Conformal primaries map to solutions of Bethe equations for non-compact spin chain

$$1 = \left(\frac{v_k - i/2}{v_k + i/2}\right)^L \prod_{j\neq k}^{M} \frac{v_k - v_j - i}{v_k - v_j + i} \times e^{i\Phi_k}$$

dressing factor = long-range corrections from wheels

Anomalous dimension is obtained as before

$$\gamma = -\sum_{a\geqslant 1} \int \frac{du}{\pi} \log\left(1 + Y_a(u)\right)$$

(with the Y's solving TBA eqs with extra source terms from the v's)

We can repeat the same game as before and dualize the equations

Spinning the wheels

Dual energy formula

$$L \log g^2/g_{cr}^2 = \sum_{i=1}^{M} E(\theta_i) - \int \frac{d\theta}{2\pi} P'(\theta) \log\left(1 + 1/Y_1(\theta)\right)$$

mechanical energy of transverse excitations + "vacuum" or "center-of-mass" energy

Transverse energy is positive, while vacuum energy is negative, in agreement with signature of (Minkowskian) AdS space

Dual Bethe equations (neglecting effects triggered by tachyon background)

$$e^{iP(\theta_k)L} \prod_{j \neq k}^{M} S_{O(6)}(\theta_k - \theta_j) = 1$$

Same equations as for O(6) if not for the momentum $P(\theta) = \mp \frac{m}{2} e^{-|\theta|}$

(low momentum = large rapidity)

Spinning the wheels

Here again we find relation to the compact sigma model. The only difference is that the momentum decreases at large rapidity in our case. This again has the effect of flipping the sign of the coupling.

Hence, to any order in PT we expect agreement with sigma model analysis, at least as long as BAEs are applicable

In the sigma mode these states should correspond to vertex operators of the type

$$V_{\Delta,M,N} \sim \partial^N (Y_1 + iY_2)^M \times e^{-i\Delta t}$$

with number of 2d light-cone derivatives mapping to the level N

Summary

Conformal fishnet theory: laboratory for amplitudes, correlators, integrability, … techniques

Dual proposal: fishnet graphs define an integrable lattice regularization of the 2d AdS5 sigma model

Dual sigma model description is weakly coupled when fishnet length scales are large

i.e. large L and “small” quantum numbers = low worldsheet energy

String?

String worldsheet or not?

Marginality condition of sort $0 = L\mu + E_{2d}(L)$

with cosmological constant $\mu = \log g_{cr}^2 / g^2$

On-shell condition comes from the geometric sum over the wheels

$$\sum_{T \geqslant 0} (g/g_{cr})^{2LT} e^{-TE_{2d}(L,\Delta)} = \frac{1}{1 - (g/g_{cr})^{2L} e^{-E_{2d}(L,\Delta)}}$$

with T acting as a discrete proper time (Schwinger parameter)

Non-critical string with a tunable intercept exists in flat space,
at least classically

Could the fishnet be an AdS version of it?

THANK YOU!