Bethe ansatz and Algebraic geometry

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Based on the works

Y. Jiang and Y. Zhang, *JHEP* 1803 (2018) 084, arXiv: 1710.04693

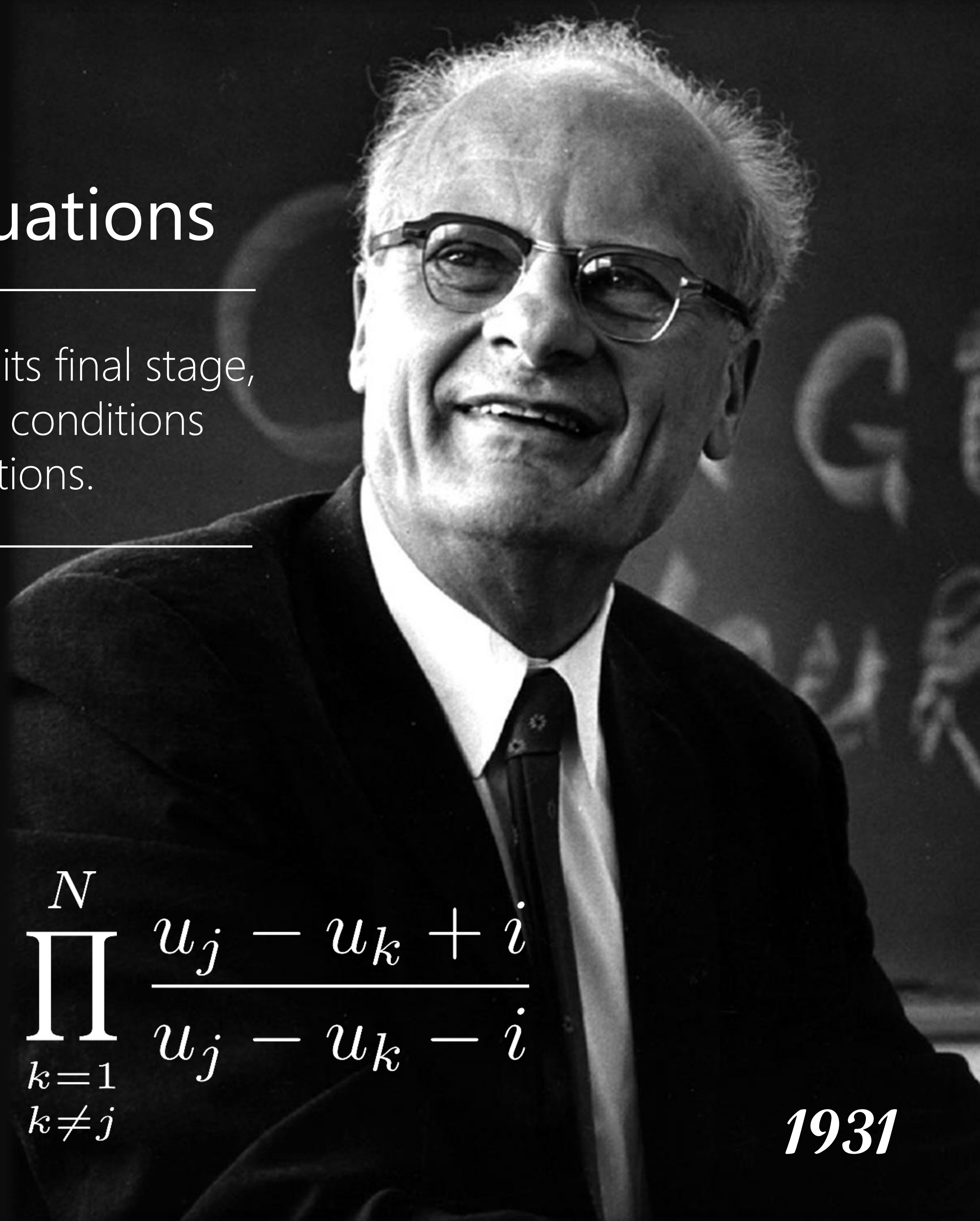
J. Jacobsen, ***Y. Jiang***, Y. Zhang, *to appear*

Y. Jiang, R. Nepomechie, Y. Zhang, *to appear*

Y. Jiang, Y. Zhang, *to appear*

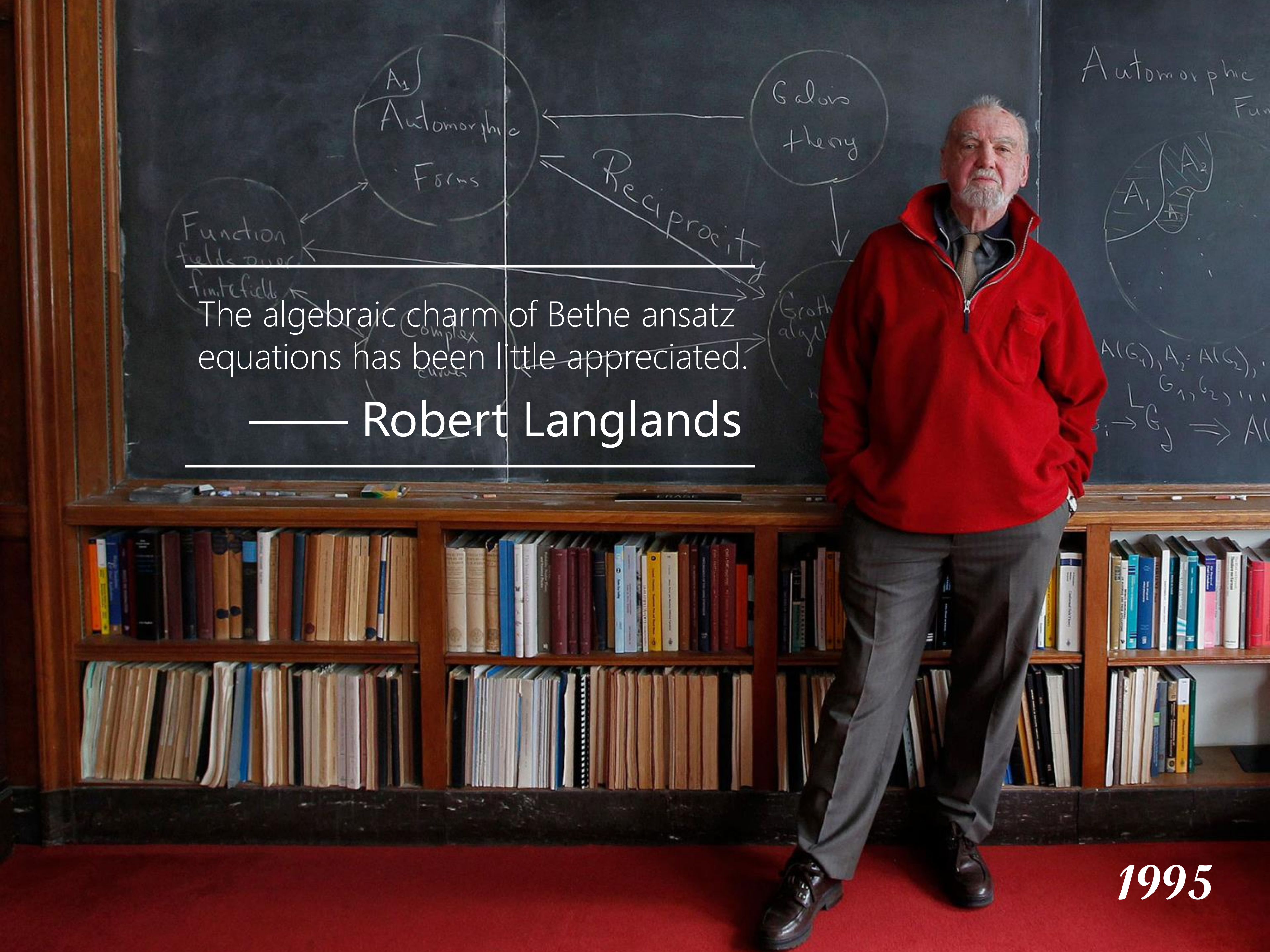
Bethe Ansatz Equations

All Bethe ansatz methods, at its final stage, leads to a set of quantization conditions called the Bethe ansatz equations.



$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{\substack{k=1 \\ k\neq j}}^{N} \frac{u_j - u_k + i}{u_j - u_k - i}$$

1931


The algebraic charm of Bethe ansatz equations has been little appreciated.
—— Robert Langlands
1995
Automorphic Forms
Galois theory
Reciprocity

Introduction

A baby problem

Equation

$$q(x) = x^5 - 5x^4 + 7x^3 + 5x^2 - 21x + 7$$

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Function

$$F(x) = \frac{x^6}{15} - 6x^5 + \frac{7}{6}x^3 + 3x^2 - \frac{x}{13} + \frac{3}{7}$$

Questions

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- **How many** solutions does the equation have ?

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Questions

- **How many** solutions does the equation have ?
- Compute the **sum** of the function **over all solutions** ?

$$\mathrm{F} = \sum_{\text{sol } q(x)=0} F(x)$$

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Computational algebraic Geometry

Although the questions we ask are somewhat trivial to solve for a single variable. They become highly non-trivial in the multi-variable cases and are among the main themes of modern computational algebraic geometry.

Numerical Method

Solution

1. By fundamental theorem of algebra, there are 5 solutions
2. Solve the equation numerically (up to 25 digits)

$$x_1 = -1.428817701781382219822436$$

$$x_2 = 0.3819660112501051517954132$$

$$x_3 = 2.618033988749894848204587$$

$$x_4 = 1.714408850890691109911218 - 1.399984900087945731206127i$$

$$x_5 = 1.714408850890691109911218 + 1.399984900087945731206127i$$

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$$F(x_1) = 39.55735720635546668510040$$

$$F(x_2) = 0.8533229627576063484431720$$

$$F(x_3) = -674.7602826697173133308150$$

$$F(x_4) = 299.0372554627563325085640 - 107.8373055698453223160120i$$

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$$F(x_4) = 299.0372554627563325085 64 - 107.8373055698453223160 12i$$

$$F(x_5) = 299.0372554627563325085 64 + 107.8373055698453223160 12i$$

$$F(x_1) + \cdots + F(x_5) = -36.2750915750915750915 8 \approx -\frac{99031}{2730}$$

Rational number !

Analytical Method

- **Linear space** spanned by

$$e_1 = x^4, \quad e_2 = x^3, \quad e_3 = x^2, \quad e_4 = x, \quad e_5 = 1$$

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$$e_1 = x^4, \quad e_2 = x^3, \quad e_3 = x^2, \quad e_4 = x, \quad e_5 = 1$$

- Divide $F(x)$ by $q(x)$, find the **remainder**

$$F(x) = a(x)q(x) + r(x)$$

$$r(x) = -\frac{144}{5}x^4 + \frac{81}{2}x^3 + \frac{491}{15}x^2 - \frac{23311}{195}x + \frac{842}{21}$$

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- Construct a matrix of the remainder in the linear space

$r(x)e_i = a_i(x)q(x) + r_i(x)$	$r_i(x) = M_{ij}e_j$

This matrix is called the **companion matrix** of the function

Analytical Method

$$M_F = \begin{pmatrix} -\frac{1910212}{1365} & \frac{801854}{195} & -\frac{24539}{13} & -\frac{4688677}{390} & \frac{303429}{65} \\ -\frac{43347}{65} & \frac{203171}{105} & -\frac{8341}{15} & -5222 & \frac{11893}{6} \\ -\frac{1699}{6} & \frac{292093}{390} & -\frac{9913}{210} & -\frac{19719}{10} & \frac{1449}{2} \\ -\frac{207}{2} & \frac{703}{3} & \frac{4769}{195} & -\frac{59294}{105} & \frac{1008}{5} \\ -\frac{144}{5} & \frac{81}{2} & \frac{491}{15} & -\frac{23311}{195} & \frac{842}{21} \end{pmatrix}$$

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Remarks

1. The result is exact, no need to solve equations
2. It is clear that the final result should be a rational number.

Questions

- **How many** solutions does the equation have ?
- Compute the **sum** of the function **over all solutions** ?

$$\mathrm{F} = \sum_{\mathrm{sol}\ q(x)=0} F(x)$$

A third question

Can we decompose the space of solutions

$$\mathrm{sol} = \mathrm{sol}_1 \cup \mathrm{sol}_2$$

such that

$$\sum_{\mathrm{sol}_i} F(x) \in \mathbb{Q}$$

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such that

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The answer is **YES** for our case

$$q(x) = (x^3 - 2x^2 + 7)(x^2 - 3x + 1)$$

We can consider each equation separately

Primary decomposition

$$M_F = \begin{pmatrix} \frac{868253}{2730} & \frac{154}{15} & -\frac{36001}{65} & 0 & 0 \\ \frac{5143}{65} & \frac{33557}{210} & \frac{154}{15} & 0 & 0 \\ -\frac{22}{15} & \frac{16001}{195} & \frac{33557}{210} & 0 & 0 \\ 0 & 0 & 0 & -\frac{2157161}{2730} & \frac{58918}{195} \\ 0 & 0 & 0 & -\frac{58918}{195} & \frac{4883}{42} \end{pmatrix}$$

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Remarks

1. We probe the internal structure of solution space 5=2+3
2. We can deal with smaller matrices by decomposition

Notions of algebraic geometry

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Polynomial ring $\mathbb{C}[x]$

All polynomials in x
with complex coefficients

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Ideal $\mathrm{I}_q = \langle q(x) \rangle$

All polynomials of the
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Quotient ring

A finite dimensional linear space $\mathrm{Q}_q = \mathbb{C}[x]/\mathrm{I}_q$

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Standard basis

All monomials that cannot
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Key results from AG

- Polynomial $F(x) \mapsto M_F$ is mapped to a **numerical matrix**
- **Dimension** of Q_q = number of solutions of $q(x) = 0$

Content

Count
number of solutions

1. Quotient ring
2. Completeness of Bethe ansatz

Sum
over all solutions

1. Companion matrix
2. Toroidal partition function

Decompose
Space of solutions

1. Primary decomposition
2. Probing structure of solution space

I. Count number of solutions

Completeness, Groebner basis, quotient ring

Completeness

Heisenberg spin chain

$$H_{\rm XXX} = \frac{1}{4}\sum_{j=1}^{L}(\vec{\sigma}_j \cdot \vec{\sigma}_{j+1} - 1)$$

Eigenstates

$$|\mathbf{u}\rangle = B(u_1)\cdots B(u_N)|\uparrow^L\rangle$$

Eigenvalues

$$E(\mathbf{u}) = -\sum_{j=1}^{N}\frac{1}{u_j^2 + 1/4}$$

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Completeness of Bethe ansatz

- Are **all eigenstates** of the Hamiltonian given by the Bethe ansatz ?
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- How to count the **number of solutions** ?

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Special Solutions

Coinciding rapidities

$$\{u, u, u_3, u_4, \cdots, u_N\}$$

Commonly believed to be **unphysical**

The eigenstate **violets** the **Pauli principle**

Special Solutions

Singular solutions

$$\{i/2, -i/2, u_3, \cdots, u_N\}$$

The eigenvalue and eigenstates are **singular** at these solutions

Careful **regularization** is needed

Some of them are **physical**, some of them are not

Singular Solutions

Physical solutions

$$\prod_{k=3}^{N}\left(\frac{u_k+i/2}{u_k-i/2}\right)^L=(-1)^L$$

Singular Solutions

Conjecture

[Nepomechie, Wang 2013]

$$\mathcal{N}_{L,N} - \mathcal{N}^{\mathrm{s}}_{L,N} + \mathcal{N}^{\mathrm{sphys}}_{L,N} = \binom{L}{N} - \binom{L}{N-1}$$

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[Nepomechie, Wang 2013]

$$\underbrace{\boxed{\mathcal{N}_{L,N}}}_{\text{pairwise distinct}} - \underbrace{\boxed{\mathcal{N}^{\mathrm{s}}_{L,N}}}_{\text{singular solution}} + \overbrace{\boxed{\mathcal{N}^{\mathrm{sphys}}_{L,N}}}^{\text{singular physical solution}} = \binom{L}{N} - \binom{L}{N-1}$$

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pairwise distinct

singular solution

singular physical solution

Physical solutions

$$\prod_{k=3}^{N}\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = (-1)^L$$

Check conjecture

given L, N, determine

$\mathcal{N}_{L,N}, \mathcal{N}^{\mathrm{s}}_{L,N}, \mathcal{N}^{\mathrm{sphys}}_{L,N}$

Comparison

Numerical

- Solve BAE numerically
- Homotopy continuation
- Use clusters to check up to

$$L = 14,\ N = 7$$

[Hao, Nepomechie, Sommese 2013]

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Numerical

- Solve BAE numerically
- Homotopy continuation
- Use clusters to check up to

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[Hao, Nepomechie, Sommese 2013]

Analytical

- No need to solve equations
- Groebner basis
- Use laptop to check up to

$$L = 14,\ N = 7$$

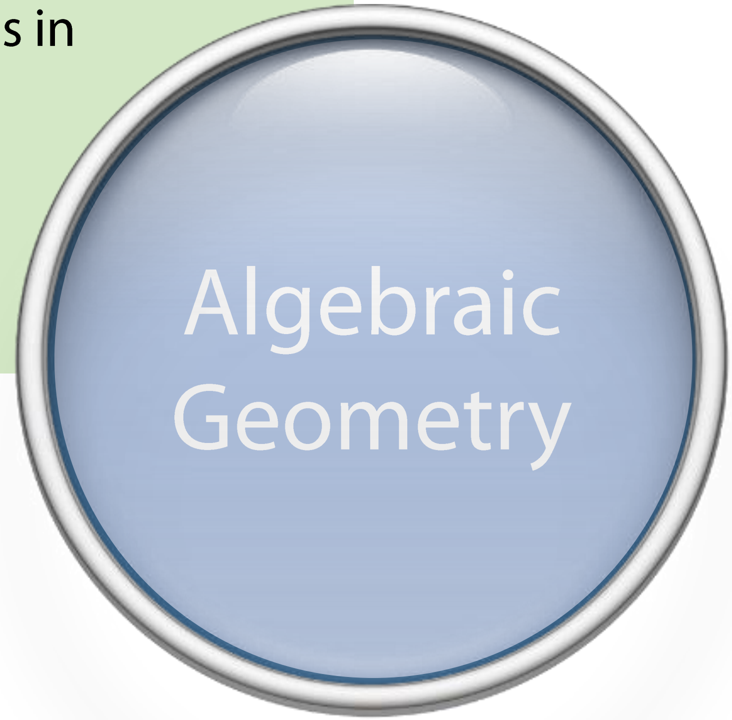
Algebraic Geometry

Polynomial ring

$\mathbb{C}[u_1, \cdots, u_N]$

All polynomials in

$\{u_1, \cdots, u_N\}$



Polynomial ring

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All polynomials in

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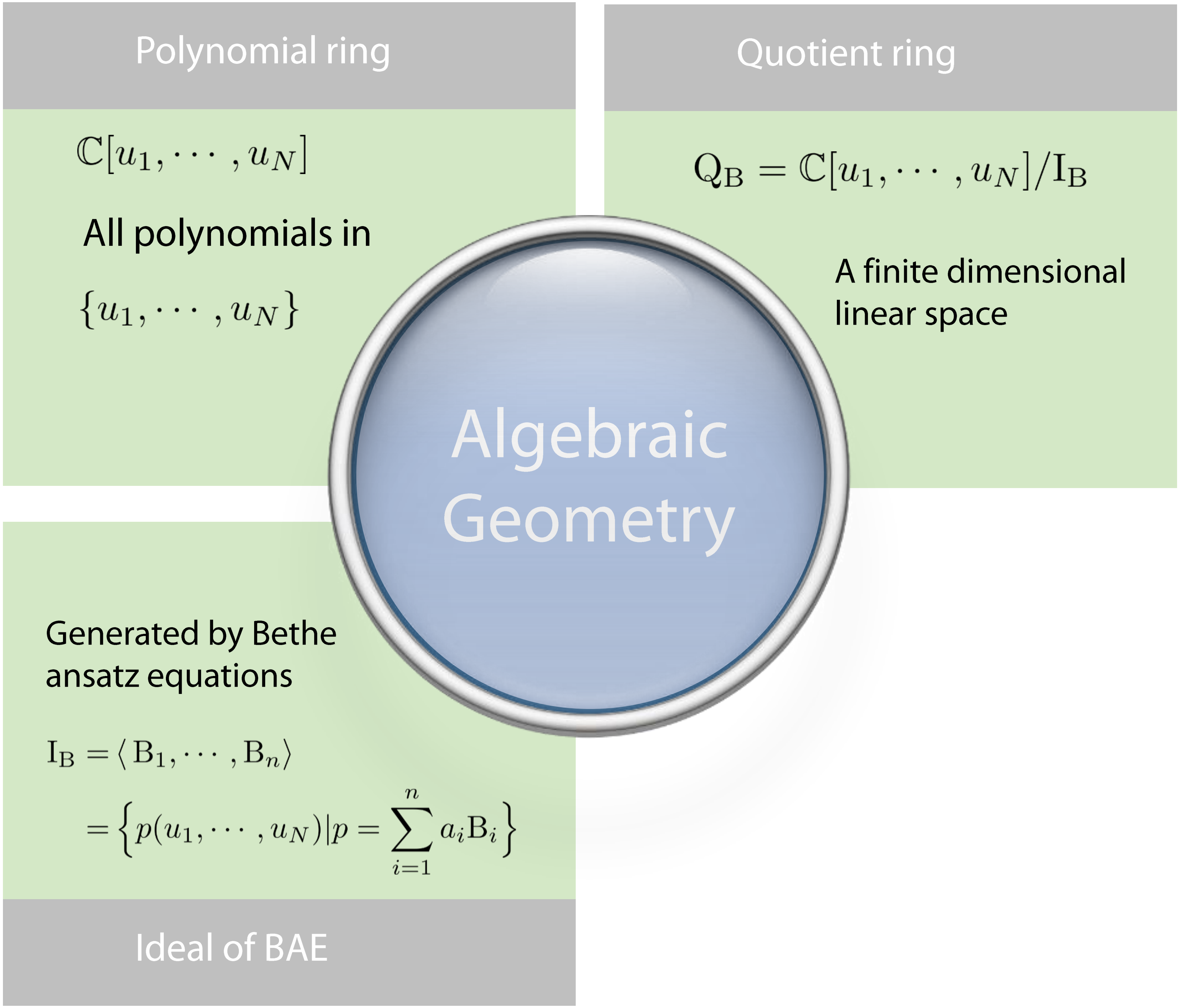
Algebraic Geometry

Generated by Bethe ansatz equations

$$\begin{aligned}\mathrm{I}_{\mathrm{B}} &= \langle \mathrm{B}_1, \cdots, \mathrm{B}_n \rangle \\ &= \left\{ p(u_1, \cdots, u_N) | p = \sum_{i=1}^{n} a_i \mathrm{B}_i \right\}\end{aligned}$$

Ideal of BAE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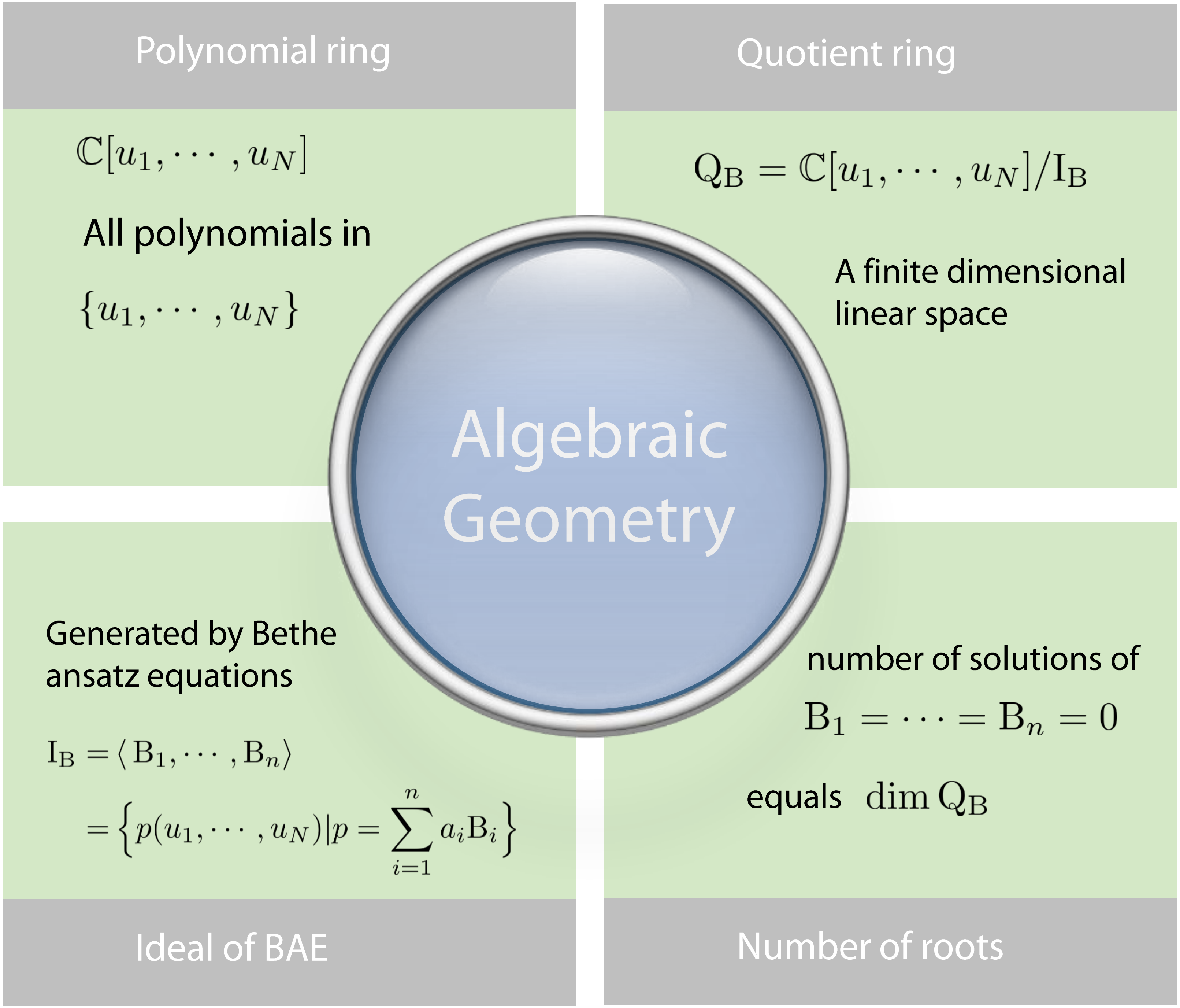
Quotient ring
$\mathrm{Q_B} = \mathbb{C}[u_1, \cdots, u_N]/\mathrm{I_B}$
A finite dimensional linear space
Algebraic Geometry
Generated by Bethe ansatz equations
$\mathrm{I_B} = \langle \mathrm{B}_1, \cdots, \mathrm{B}_n \rangle$
$= \left\{ p(u_1, \cdots, u_N) | p = \sum_{i=1}^{n} a_i \mathrm{B}_i \right\}$
Ideal of BAE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A finite dimensional linear space
Algebraic Geometry
Generated by Bethe ansatz equations
$\mathrm{I_B} = \langle \mathrm{B}_1, \cdots, \mathrm{B}_n \rangle$
$= \left\{ p(u_1, \cdots, u_N) | p = \sum_{i=1}^{n} a_i \mathrm{B}_i \right\}$
number of solutions of
$\mathrm{B}_1 = \cdots = \mathrm{B}_n = 0$
equals $\dim \mathrm{Q_B}$
Ideal of BAE
Number of roots

Differences

More (variables) is different !

Differences

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Single variable

BAE $= q(x) = x^3 - 2x^2 + 7 = 0$

"Remainder" of polynomials "divided" by BAE is well-defined

All remainders in the linear space $\mathrm{Span}_{\mathbb{C}}(x^2, x, 1)$

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$$f_1 = y^2 - 1 \qquad f_2 = xy - 1 \qquad F(x,y) = x^2y + xy^2 + y^2$$

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$$F(x,y) = f_1 + (x+y)\, f_2 + (x+y+1)$$

The remainder is **not unique** !

Groebner Basis

Ideals can be generated by different basis

$$\mathrm{I_B} = \langle\, \mathrm{B}_1, \cdots, \mathrm{B}_n \rangle = \langle\, \mathrm{G}_1, \cdots, \mathrm{G}_s \rangle$$

The Groebner basis : remainders are
well-defined for this basis !

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SINGULAR

For the Groebner basis of BAE, we need more efficient package like SINGULAR

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It is important to have brilliant students !

—— **W. Groebner**

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Quotient ring

Bruno Buchberger

Quotient ring

Standard Basis

All monomials that cannot be divided by $\mathrm{LT}[\mathrm{G}_i]$,
$i = 1, \cdots , s$

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Quotient ring

Standard Basis

All monomials that cannot be divided by $\mathrm{LT}[\mathrm{G}_i]$, $i = 1, \cdots, s$

Bruno Buchberger

Simple Example

$f_1 = y^2 - 1 \quad f_2 = xy - 1$

$$\langle f_1, f_2 \rangle = \langle \mathrm{G}_1, \mathrm{G}_2 \rangle$$

$\mathrm{G}_1 = y^2 - 1 \quad \mathrm{G}_2 = x - y$

Choose the order, $x \succ y$
we have

$$\mathrm{LT}[\mathrm{G}_1] = y^2$$

$$\mathrm{LT}[\mathrm{G}_2] = x$$

The basis of $\mathbb{C}[x, y]/\langle f_1, f_2 \rangle$ is given by $\{y, 1\}$

Indeed, easy to see we have 2 solutions

Bethe Ansatz Equations

$$\mathrm{B}_j = (u_j + i/2)^L Q_{\mathbf{u}}(u_j - i) + (u_j - i/2)^L Q_{\mathbf{u}}(u_j + i)$$

$$Q_{\mathbf{u}}(u) = \prod_{j=1}^{N}(u - u_j) = u^N + \sum_{k=1}^{N-1} s_k\, u^k$$

Additional Constraints

Pairwise distinct

$$\mathrm{A}_{jk} = \frac{\mathrm{B}_j - \mathrm{B}_k}{u_j - u_k}, \qquad j < k$$

Non-singular solution

$$\mathrm{B} = w \prod_{j=1}^{N}(u_j^2 + 1/4) - 1$$

Ideal Combine the two sets of constraints

$$\langle \mathrm{B}_1, \cdots, \mathrm{B}_N, \mathrm{A}_{12}, \cdots, \mathrm{A}_{N-1,N}, \mathrm{B} \rangle$$

“You should try *TQ* relations !”

—— *Grisha Korchemsky*

$$t_{\mathbf{u}}(z)Q_{\mathbf{u}}(z) = \mathrm{a}(z)Q_{\mathbf{u}}(z-i) + \mathrm{d}(z)Q_{\mathbf{u}}(z+i)$$

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"Try *QQ* relation, and twist it..."

—— *Volodya Kazakov*

$$z^L = Q_{\mathbf{u}}(z+i/2)\tilde{Q}_{\mathbf{u}}(z-i/2) - Q_{\mathbf{u}}(z-i/2)\tilde{Q}_{\mathbf{u}}(z+i/2)$$

Baxter's *TQ* relations

$$\boxed{t_{\mathbf{u}}(z)Q_{\mathbf{u}}(z) = \mathrm{a}(z)Q_{\mathbf{u}}(z-i) + \mathrm{d}(z)Q_{\mathbf{u}}(z+i)}$$

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Sum formula

$$t_{\mathbf{u}}(z) = \sum_{k=0}^{L} \mathrm{t}_k \, z^k \qquad \Bigg| \qquad Q_{\mathbf{u}}(z) = z^M + \sum_{k=0}^{M-1} \mathrm{s}_k \, z^k$$

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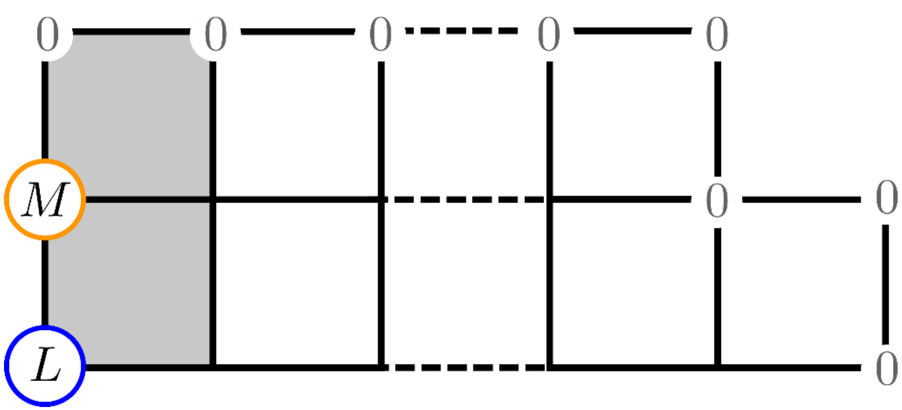
TQ-system gives a system of algebraic equations of t_k and s_k

- The system is **linear** in t_k and s_k
- **More efficient** than original BAE
- Automatically **eliminates coinciding roots**

Rational Q-system



$$Q_{a,s}(z) = z^N + \sum_{k=0}^{N-1} c_{a,s}^{(k)}\, z^k$$

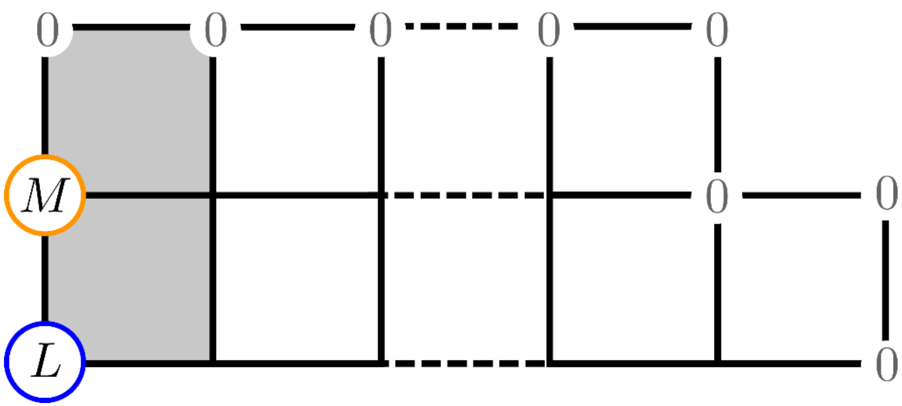
Q-function at each node

QQ-relation + boundary condition

Require Q to be polynomial

[Marboe and Volin, 2016]

Rational Q-system



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[Marboe and Volin, 2016]

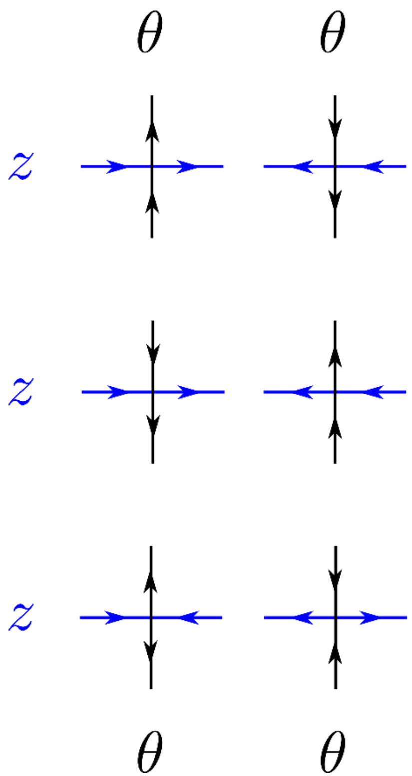
Rational Q-system gives a system of algebraic equations of $c_{a,s}^{(k)}$

- The rational Q-system needs to be generated
- **Even more efficient** than BAE and TQ-relation
- Automatically select **physical solutions**

II. Sum over solutions

Torus partition function, companion matrix

6-vertex model



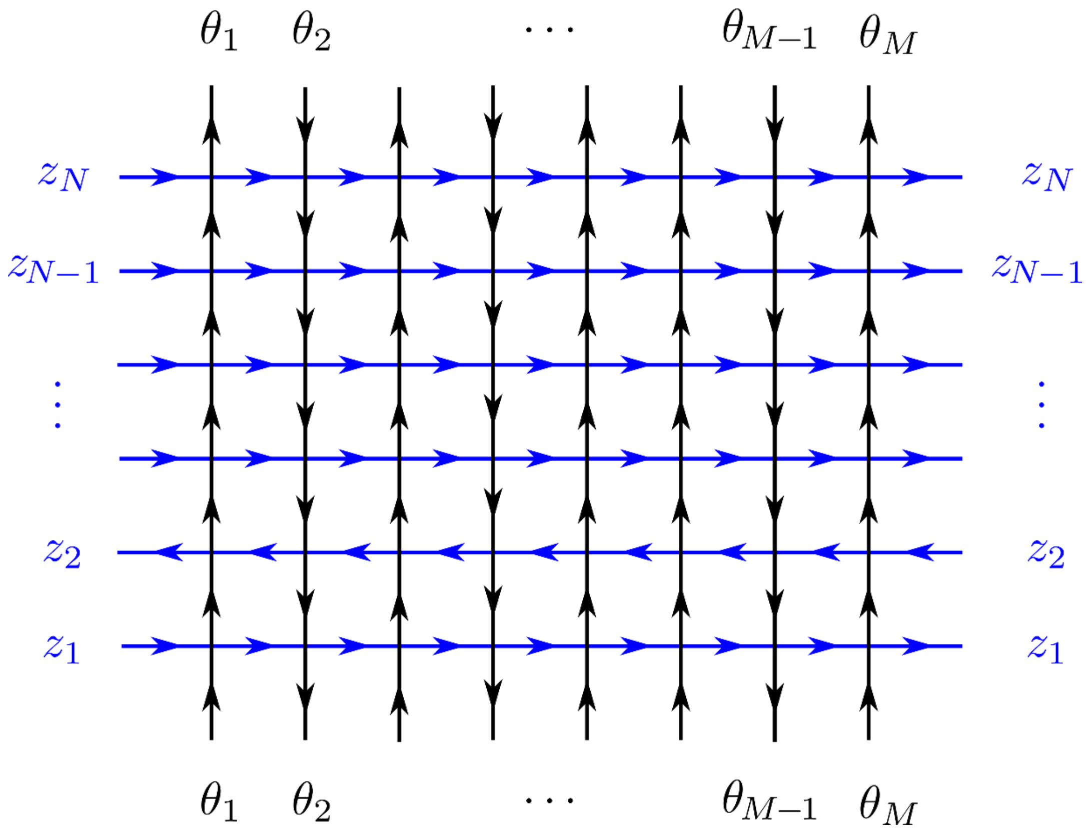
$$a(z,\theta) = z - \theta + \tfrac{i}{2}$$

$$b(z,\theta) = z - \theta - \tfrac{i}{2}$$

$$c(z,\theta) = i$$

Ice-rule
Number of incoming and outgoing arrows are equal

Integrability
The model is integrable and can be solved by Bethe ansatz


θ_1 θ_2 $\cdots$ θ_{M-1} θ_M
z_N
z_{N-1}
$\vdots$
z_2
z_1
z_N
z_{N-1}
$\vdots$
z_2
z_1
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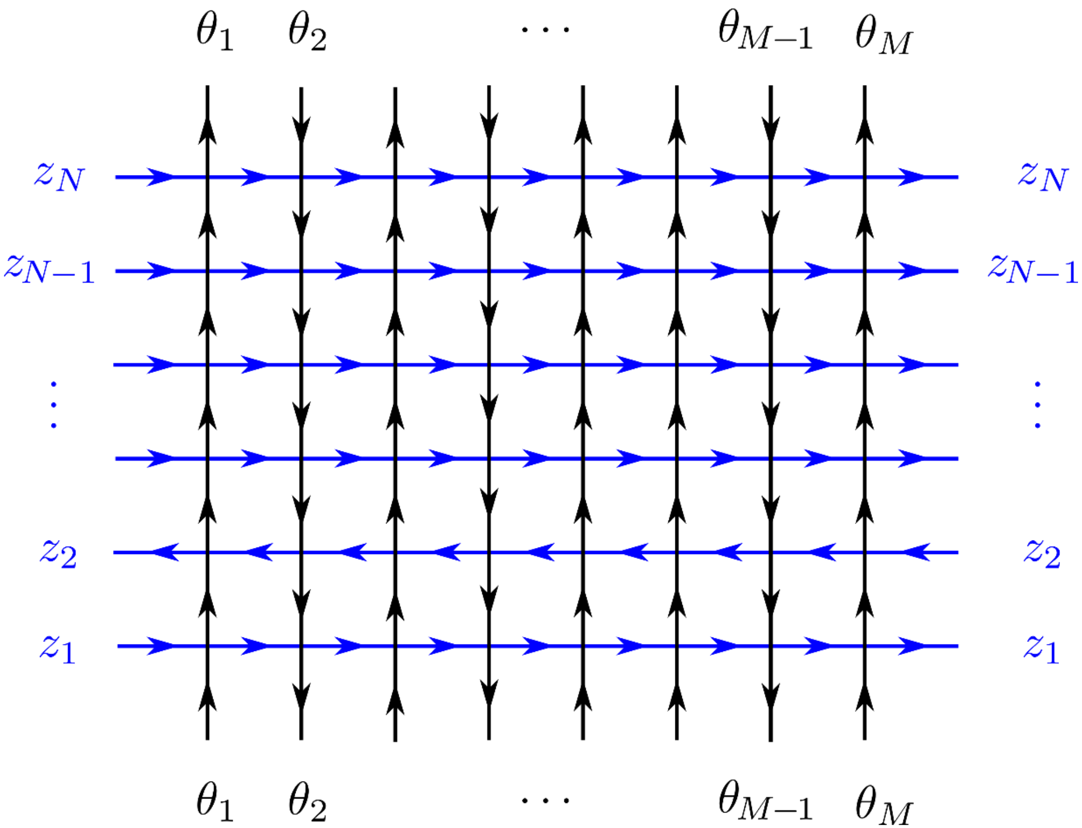
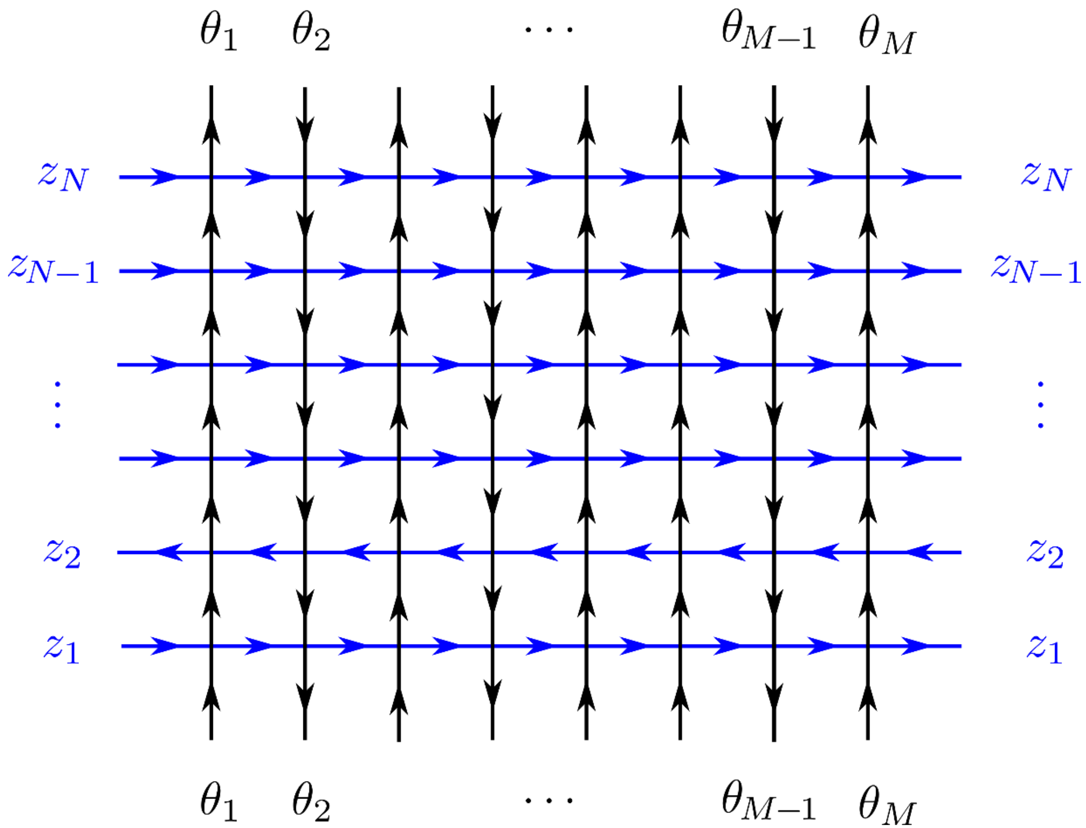


Transfer matrix method

$$T_M(z) = \mathrm{Tr}_a \prod_{n=1}^{M} R_{an}(z-\theta_n)$$

$$Z_{M,N} = \mathrm{Tr}\, T_M(z_1) \cdots T_M(z_N)$$



Homogenous limit

$$\theta_1 = \cdots = \theta_M = 0$$

$$z_1 = \cdots = z_M = z$$

$$\boxed{Z_{M,N} = \mathrm{Tr}\, T_M(z)^N}$$

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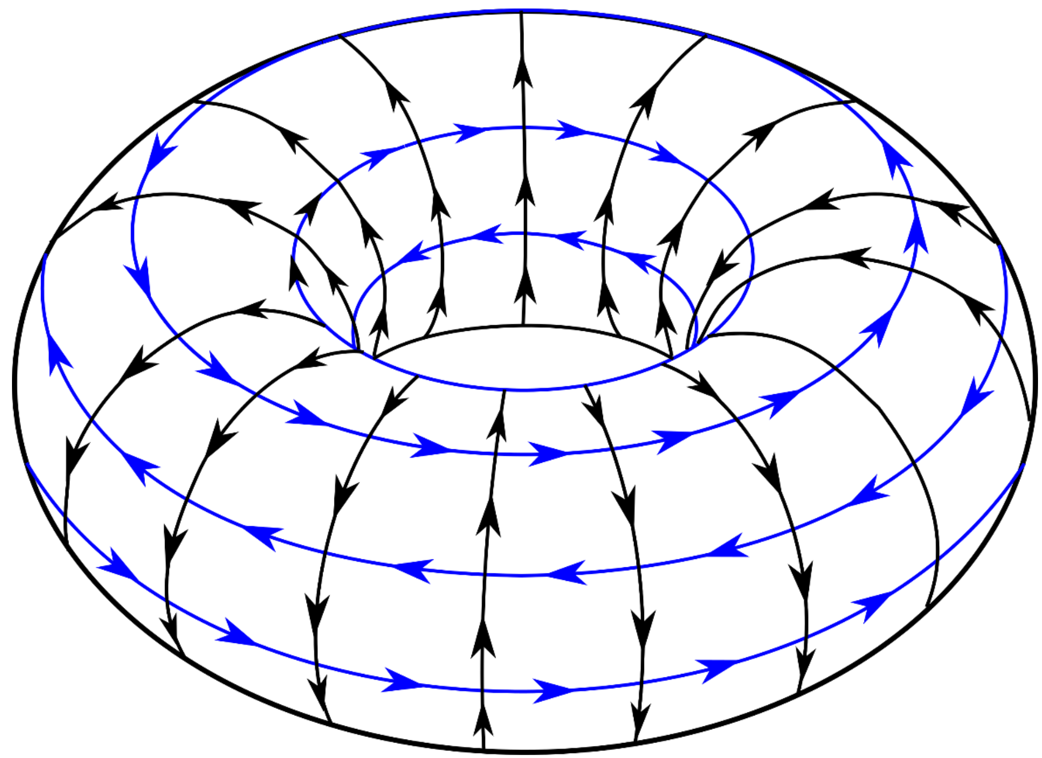
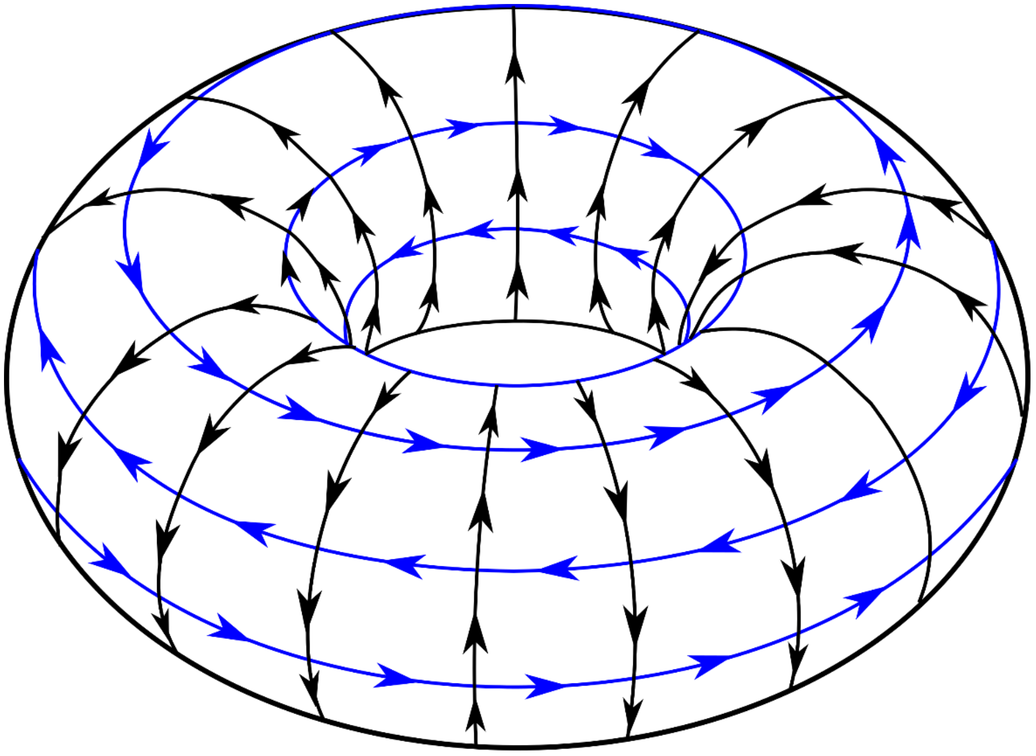
Torus partition function



Brute-force approach

Construct transfer matrix, take matrix power and then take trace

Torus partition function



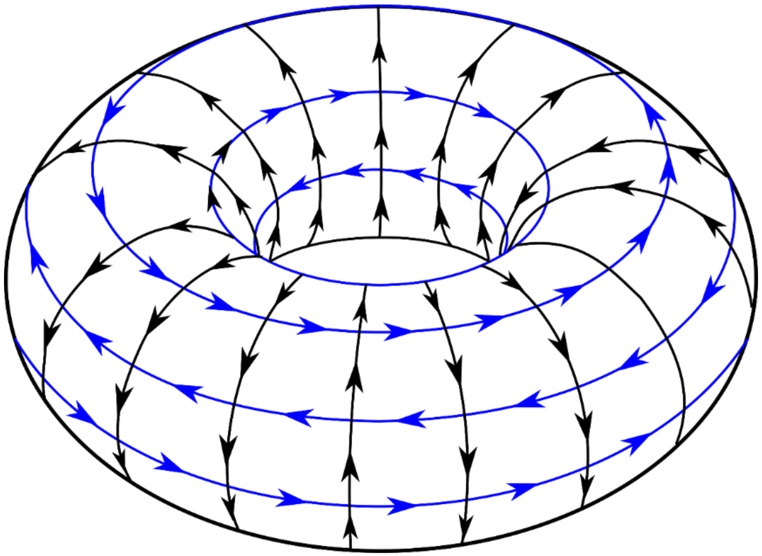
Bethe ansatz

Diagonalize the transfer matrix by Bethe ansatz, find eigenvalues.

Difficulties of Bethe ansatz

$$T(z)|\mathbf{u}\rangle = t_{\mathbf{u}}(z)|\mathbf{u}\rangle$$

Diagonalize transfer matrix by Bethe ansatz. Eigenvalues parameterized by rapidities



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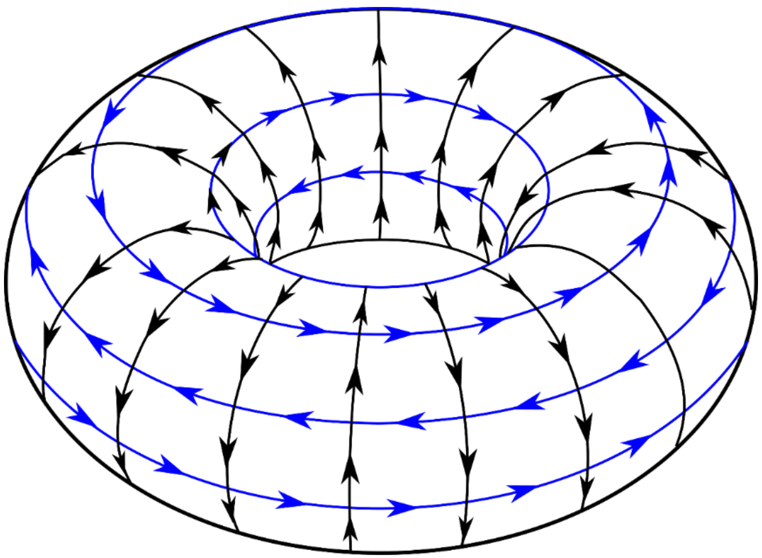

- Need to find **all physical** solutions for fixed quantum numbers

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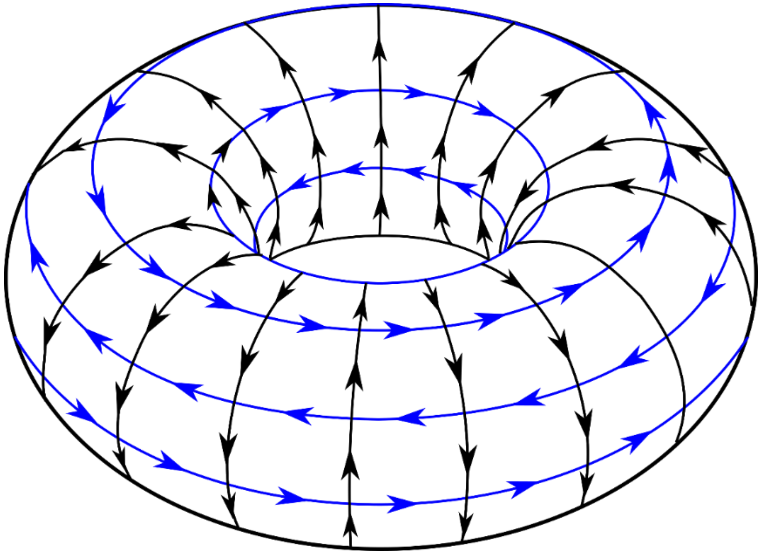

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- Solutions of BAE can only be found **numerically**

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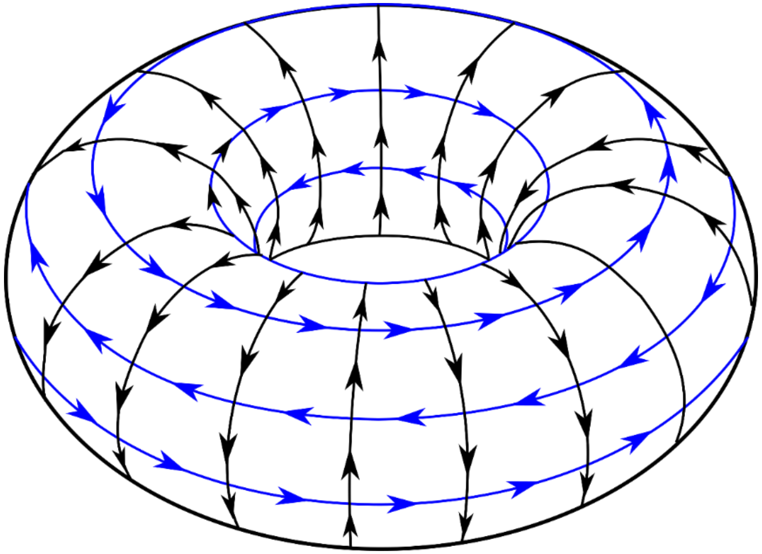

Diagonalize transfer matrix by Bethe ansatz. Eigenvalues parameterized by rapidities

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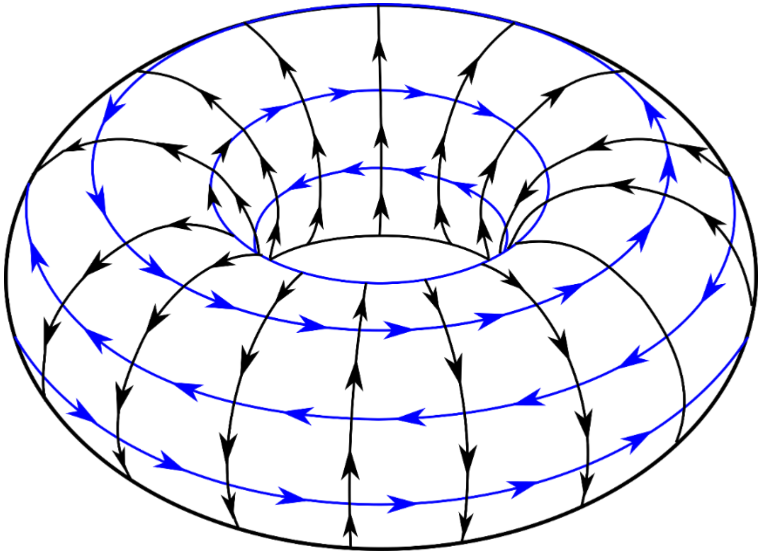

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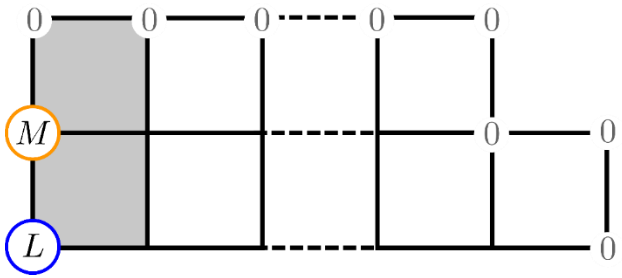
Use algebraic geometry Companion Matrix

Ideal of Q-system



Generate the system,
Compute Groebner basis

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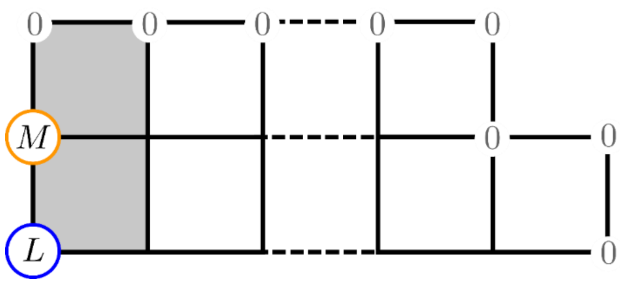

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$$\mathbb{Q}_K = \mathbb{C}[s_1, \cdots, s_K]/\mathrm{I}_K$$

with basis $\{e_1, e_2, \cdots, e_S\}$

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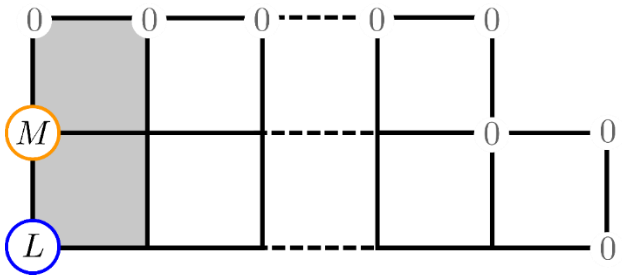

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A polynomial

$$P(s_1, \cdots, s_K)$$

Can be mapped to a finite dimensional matrix

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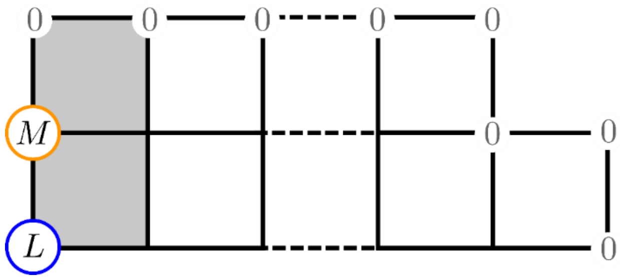

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Companion Matrix

For any e_j , find $P(\mathbf{s})e_j = \sum_{k=1}^{S} a_k G_k + r_j(\mathbf{s})$

Expand in terms of basis $r_j(\mathbf{s}) = \sum_{k=1}^{S} M_{jk}\, e_k$

The matrix $(M_P)_{ij} = M_{ij}$ is called the companion matrix of $P(s_1, \cdots, s_K)$

Properties

$$M_{P_1 \pm P_2} = M_{P_1} \pm M_{P_2}$$

$$M_{P_1 \cdot P_2} = M_{P_1} \cdot M_{P_2}$$

$$M_{P_1 / P_2} = M_{P_1} \cdot M_{P_2}^{-1}$$

Important result

$$\sum_{\text{sol}} P(\mathbf{s}) = \mathrm{Tr}\, M_P$$

Companion Matrix

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Torus partition function from algebraic geometry

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1. Generate **ideal** from the rational Q-system

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2. Compute Groebner basis, construct **quotient ring**

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3. Compute the **companion matrix** of Baxter polynomial

Torus partition function from algebraic geometry

1. Generate **ideal** from the rational Q-system

2. Compute Groebner basis, construct **quotient ring**

3. Compute the **companion matrix** of Baxter polynomial

4. Compute the **companion matrix** of transfer matrix.
   Take matrix power and take trace.

Results I. M=6, any N (20 terms)

$$\begin{aligned}
Z_{6,N} =\, & 7\left(2z^6 - \frac{15}{2}z^4 + \frac{15}{8}z^2 - \frac{1}{32}\right)^N + \cdots \\
& + 5\left(2z^6 - \frac{3}{2}z^4 + \sqrt{3}z^3 - \frac{21}{8}z^2 - \frac{3\sqrt{3}}{4}z - \frac{1}{32}\right)^N \\
& + 5\left(2z^6 - \frac{3}{2}z^4 - \sqrt{3}z^3 - \frac{21}{8}z^2 + \frac{3\sqrt{3}}{4}z - \frac{1}{32}\right)^N \\
& + \cdots \\
& + 3\left(2z^6 + \frac{5}{2}z^4 - \frac{\sqrt{54-6\sqrt{17}}}{2}z^3 + \frac{(2\sqrt{17}-3)}{8}z^2 - \frac{\sqrt{54-6\sqrt{17}}(3+\sqrt{17})}{16}z + \frac{(9+2\sqrt{17})}{32}\right)^N \\
& + 3\left(2z^6 + \frac{5}{2}z^4 + \frac{\sqrt{54-6\sqrt{17}}}{2}z^3 + \frac{(2\sqrt{17}-3)}{8}z^2 + \frac{\sqrt{54-6\sqrt{17}}(3+\sqrt{17})}{16}z + \frac{(9+2\sqrt{17})}{32}\right)^N \\
& + 3\left(2z^6 + \frac{5}{2}z^4 - \frac{\sqrt{54+6\sqrt{17}}}{2}z^3 - \frac{(2\sqrt{17}+3)}{8}z^2 - \frac{\sqrt{54+6\sqrt{17}}(\sqrt{17}-3)}{16}z + \frac{(9-2\sqrt{17})}{32}\right)^N \\
& + 3\left(2z^6 + \frac{5}{2}z^4 + \frac{\sqrt{54+6\sqrt{17}}}{2}z^3 - \frac{(2\sqrt{17}+3)}{8}z^2 - \frac{\sqrt{54+6\sqrt{17}}(\sqrt{17}-3)}{16}z + \frac{(9-2\sqrt{17})}{32}\right)^N \\
& + \cdots \\
& + \left(2z^6 + \frac{9}{2}z^4 + \frac{7-8\sqrt{13}}{8} + \frac{31-8\sqrt{13}}{32}\right)^N \\
& + \left(2z^6 + \frac{9}{2}z^4 + \frac{7+8\sqrt{13}}{8} + \frac{31+8\sqrt{13}}{32}\right)^N
\end{aligned}$$

Closed form expression exist
For all M<7

Results II. M=14, N=100

There are 700 terms, we show one of them here...

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There are 700 terms, we show one of them here...

50592813586572692363859583962575484789973486730786026466
21054523312888328993573004196361515348925761497065214815
61668092120356175488433622495795129294863502389232802959
67617297112705522132714764649758232042045030379951801223
51042684005989806761393756475970714310900496867482000864
34625328933888231460224879617925433535298369895981189289
95286471561175496835388649666891535006503397711044157222
94545408958952041903237427496268723346920302047717672415
6558296575**z^14**/16652691511463038531080384523782536861784
41959562672768851057976064577289547737647681958310677171
86957478745823268496124662137059973175438530238505690837
13220678607215022752553237093359458533680756735072876303
69793870988343607756618714000221510121646677175821649785
73256614665360452089674659597310254700829553091555250248
33642717496781972732846204881391577899041143553136889125
0699665408

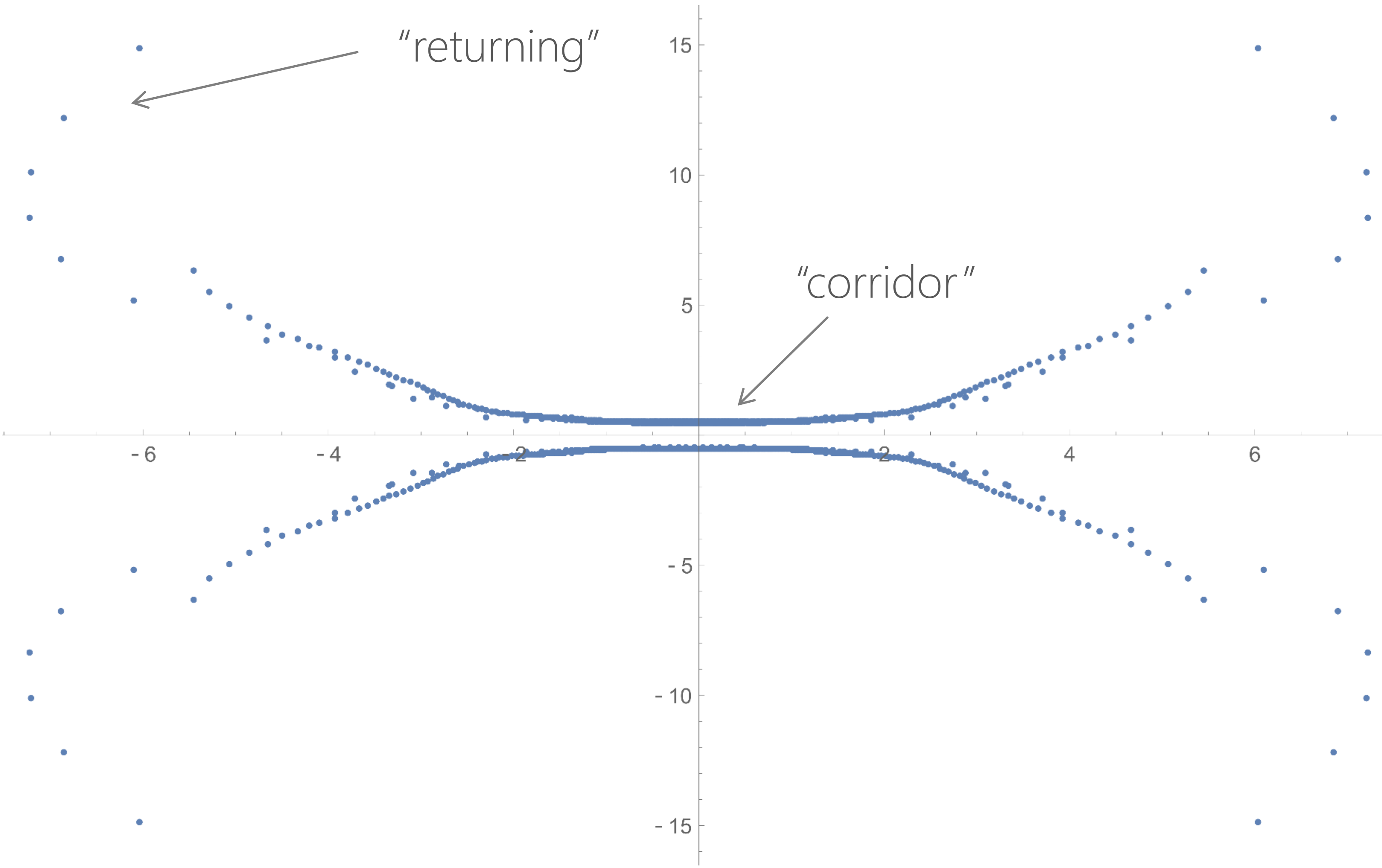
Results II. M=14, N=100

There are 700 terms, we show one of them here…

5059281358657269236385958396257548478997348673078602646 6
2105452331288832899357300419636151534892576149706521481 5
6166809212035617548843362249579512929486350238923280295 9
6761729711270552213271476464975823204204503037995180122 3
5104268400598980676139375647597071431090049686748200086 4
3462532893388823146022487961792543353529836989598118928 9
9528647156117549683538864966689153500650339771104415722 2
9454540895895204190323742749626872334692030204771767241 5
6558296575**z^14**/1665269151146303853108038452378253686178 4
4195956267276885105797606457728954773764768195831067717 1
8695747874582326849612466213705997317543853023850569083 7
1322067860721502275255323709335945853368075673507287630 3
6979387098834360775661871400022151012164667717582164978 5
7325661466536045208967465959731025470082955309155525024 8
3364271749678197273284620488139157789904114355313688912 5
0699665408

Dedicated to the 14th birthday of IGST

Zeros M=14, N=100



Universal feature

Existence of a corridor, condensation curve might be closed

III. Decompose solution space

Lattice momentum, primary decomposition

Recall that ...

$$
\begin{aligned}
Z_{6,N} = & 7\left(2z^6 - \frac{15}{2}z^4 + \frac{15}{8}z^2 - \frac{1}{32}\right)^N + \cdots \\
& + 5\left(2z^6 - \frac{3}{2}z^4 + \sqrt{3}z^3 - \frac{21}{8}z^2 - \frac{3\sqrt{3}}{4}z - \frac{1}{32}\right)^N \\
& + 5\left(2z^6 - \frac{3}{2}z^4 - \sqrt{3}z^3 - \frac{21}{8}z^2 + \frac{3\sqrt{3}}{4}z - \frac{1}{32}\right)^N \\
& + \cdots \\
& + 3\left(2z^6 + \frac{5}{2}z^4 - \frac{\sqrt{54-6\sqrt{17}}}{2}z^3 + \frac{(2\sqrt{17}-3)}{8}z^2 - \frac{\sqrt{54-6\sqrt{17}}(3+\sqrt{17})}{16}z + \frac{(9+2\sqrt{17})}{32}\right)^N \\
& + 3\left(2z^6 + \frac{5}{2}z^4 + \frac{\sqrt{54-6\sqrt{17}}}{2}z^3 + \frac{(2\sqrt{17}-3)}{8}z^2 + \frac{\sqrt{54-6\sqrt{17}}(3+\sqrt{17})}{16}z + \frac{(9+2\sqrt{17})}{32}\right)^N \\
& + 3\left(2z^6 + \frac{5}{2}z^4 - \frac{\sqrt{54+6\sqrt{17}}}{2}z^3 - \frac{(2\sqrt{17}+3)}{8}z^2 - \frac{\sqrt{54+6\sqrt{17}}(\sqrt{17}-3)}{16}z + \frac{(9-2\sqrt{17})}{32}\right)^N \\
& + 3\left(2z^6 + \frac{5}{2}z^4 + \frac{\sqrt{54+6\sqrt{17}}}{2}z^3 - \frac{(2\sqrt{17}+3)}{8}z^2 - \frac{\sqrt{54+6\sqrt{17}}(\sqrt{17}-3)}{16}z + \frac{(9-2\sqrt{17})}{32}\right)^N \\
& + \cdots \\
& + \left(2z^6 + \frac{9}{2}z^4 + \frac{7-8\sqrt{13}}{8} + \frac{31-8\sqrt{13}}{32}\right)^N \\
& + \left(2z^6 + \frac{9}{2}z^4 + \frac{7+8\sqrt{13}}{8} + \frac{31+8\sqrt{13}}{32}\right)^N
\end{aligned}
$$

These four eigenvalues look similar, with some signs flipped

$$
\begin{aligned}
&+\cdots \\
&+3\left(2z^6+\frac{5}{2}z^4-\frac{\sqrt{54-6\sqrt{17}}}{2}z^3+\frac{(2\sqrt{17}-3)}{8}z^2-\frac{\sqrt{54-6\sqrt{17}}(3+\sqrt{17})}{16}z+\frac{(9+2\sqrt{17})}{32}\right)^N \\
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&+\cdots
\end{aligned}
$$

Comments

$$+\cdots$$

$$+3\left(2z^6+\frac{5}{2}z^4-\frac{\sqrt{54-6\sqrt{17}}}{2}z^3+\frac{(2\sqrt{17}-3)}{8}z^2-\frac{\sqrt{54-6\sqrt{17}}(3+\sqrt{17})}{16}z+\frac{(9+2\sqrt{17})}{32}\right)^N$$

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$$+\cdots$$

Comments

- Sum for any *N* gives **rational coefficients**

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&+\cdots\\
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\end{aligned}
$$

Comments

- Sum for any N gives **rational coefficients**
- Each term corresponds to one **physical solution** of BAE.

$$+\cdots$$

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$$+\cdots$$

Comments

- Sum for any *N* gives **rational coefficients**
- Each term corresponds to one **physical solution** of BAE.
- It seems natural to **group this four solutions** together

Primary decomposition

Recall the baby problem

$$q(x) = (x^3 - 2x^2 + 7)(x^2 - 3x + 1)$$

3 solutions

2 solutions

5=3+2

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Multi-variable generalization

Example M=6

The multi-variable generalization can be done systematically. This is called **primary decomposition** of ideals.

K=1	5=1+2+2
K=2	9=1+2+2+4
K=3	5=1+2+2

Conceptually

We find **new internal structure** of the solution space of Bethe ansatz equations

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Example

- CHY formulation of scattering amplitudes, one need to solve the **scattering equations**
- **Primary decomposition** of the solution space, correspond to MHV, NMHV, NNMHV,.... Amplitudes (different helicities)

Physical interpretation

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Shift operator $U = \exp(i\mathrm{P}) = (-i)^L T(i/2)$

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Example M=6

The possible eigenvalues are

$$\boxed{\{1, e^{\frac{i\pi}{3}}, e^{\frac{2i\pi}{3}}, -1, e^{\frac{4i\pi}{3}}, e^{\frac{5i\pi}{3}}\}}$$

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K=1	5=1+2+2
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Eigenvalues of U

Extra conditions for the first two rows

$$\prod_{j=1}^{K} \frac{u_j + i/2}{u_j - i/2} = \pm 1$$

1	$\{+1\}$
2	$\{-1, -1\}$
2	$\{e^{\frac{2i\pi}{3}}, e^{\frac{4i\pi}{3}}\}$
4	$\{e^{\frac{i\pi}{3}}, e^{\frac{i\pi}{3}}, e^{\frac{5i\pi}{3}}, e^{\frac{5i\pi}{3}}\}$

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Eigenvalues of U^3

1	$\{+1\}$
2	$\{-1,-1\}$
2	$\{+1,+1\}$
4	$\{-1,-1,-1,-1\}$

Extra conditions for the last two rows

$$\prod_{j=1}^{K}\left(\frac{u_j+i/2}{u_j-i/2}\right)^3=\pm 1$$

Conclusions

Computational Algebraic Geometry provides **natural language** and **powerful tools** to study BAE

They have **important applications** in many physical problems

It is **more efficient** than solving BAE numerically. It can be used to further study the structure of the solution space.

Purely **algebraic and analytical**, opens the door to new analytical results in integrability.

Outlook

- More on completeness

  Quantum deformation,
  higher rank spin chains

- Partition functions

  Understand distribution of zeros,
  quantum deformation

- Primary decomposition

  Quantum deformation, representations of
  Temperley-Lieb algebra, complete classification
  by Galois theory

  - A website for AG-BA


Thank
You