The Hagedorn temperature of $\text{AdS}_5/\text{CFT}_4$ at finite coupling via the Quantum Spectral Curve

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[1706.03074], [1803.04416] and work in progress with T. Harmark
AdS/CFT correspondence

- Type IIB string theory on $AdS_5 \times S^5$
- $\mathcal{N} = 4$ SYM theory on $\mathbb{R} \times S^3$

Should in particular relate phase transitions, critical behaviour and thermal physics e.g. Hagedorn temperature

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[Hagedorn (1965)]

[Sugishita (2010)]
AdS/CFT correspondence

Type IIB string theory on $AdS_5 \times S^5$

$\mathcal{N} = 4$ SYM theory on $\mathbb{R} \times S^3$

[Maldacena (1997)]

Should in particular relate phase transitions, critical behaviour and thermal physics e.g. Hagedorn temperature [Witten (1998)],

**Hagedorn temperature**

Limiting temperature due to an exponential rise in the density of states $\rho(E)$, where $Z(T) = \sum_E \rho(E) e^{-E/T}$

→ Signals a phase transition [Hagedorn (1965)]
Planar gauge theories on $\mathbb{R} \times S^3$

- Confinement of colour degrees of freedom on $S^3$
  \[ \rightarrow \text{Confinement/deconfinement phase transition similar to QCD or pure YM} \]
Planar gauge theories on $\mathbb{R} \times S^3$

Confinement of colour degrees of freedom on $S^3$

$\rightarrow$ Confinement/deconfinement phase transition similar to QCD or pure YM

$R_{S^3}$ acts as an effective IR cut-off

$\Rightarrow$ Allows for a perturbative comparison between conformal theories, such as $\mathcal{N} = 4$ SYM theory, and non-conformal theories such as pure Yang-Mills theory

[Aharony, Marsano, Minwalla, Papadodimas, Van Raamsdonk (2003)]
State/operator correspondence

- States on $\mathbb{R} \times S^3$ ↔ Gauge-invariant operators on $\mathbb{R}^{1,3}$
- Hamilton operator $H$ on $\mathbb{R} \times S^3$ ↔ Dilatation operator $D$ on $\mathbb{R}^{1,3}$
- Energies $E$ on $\mathbb{R} \times S^3$ ↔ Scaling dimensions $\Delta$ on $\mathbb{R}^{1,3}$

Partition function ($S^3 = 1$):

$$Z(T) = \text{tr}_{\mathbb{R} \times S^3}[e^{-H/T}] = \text{tr}_{\mathbb{R}^{1,3}}[e^{-D/T}]$$

Planar limit: $\Delta \text{tr}() = \Delta \text{tr}() + \Delta \text{tr}() + \ldots$

⇒ Sufficient to look at single-trace operators

Partition function via Pólya theory:

- Tree-level [Sundborg (1999)]
- One-loop [Spradlin, Volovich (2004)]
State/operator correspondence

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Hagedorn temperature of $\mathcal{N} = 4$ SYM theory

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Hagedorn temperature in string theory

### Free (tree-level) string theory

Exponential growth of string states with the energy
### Hagedorn temperature in string theory

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‘Phases’ of type IIB string theory on $AdS_5 \times S^5$

[Aharony, Marsano, Minwalla, Papadodimas, Van Raamsdonk (2003)]

[Inaccessible at large $N$]

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Hagedorn temperature in string theory

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‘Phases’ of type IIB string theory on $AdS_5 \times S^5$
[Aharony, Marsano, Minwalla, Papadodimas, Van Raamsdonk (2003)]
Hagedorn temperature in pp-wave limits of string theory

[Grignani, Orselli, Semenoff, and Trancanelli (2003)]

Special limit

- **string theory** pp-wave limit
- **gauge theory** one-loop SU(2) sector

→ Quantitative match via free energy of Heisenberg spin chain

[Harmark, Orselli (2006)]
Integrability: Scaling dimensions $\Delta(\lambda)$ known for all $\lambda$

Long history: One-loop Bethe equation
→ asymptotic all-loop Bethe equations
→ thermodynamic Bethe ansatz (TBA) equations
→ Y-system equations, T-system equations
→ Quantum spectral curve (QSC)

[Minahan, Zarembo (2002)],…
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[Minahan, Zarembo (2002)],...

Naive idea: Sum over $\Delta$’s yields $Z = \sum e^{-\Delta/T} \rightarrow$ Prohibitive
Integrability for the Hagedorn temperature

**Spectral problem**: Solution on $\mathbb{R} \times S^1_L$ to account for finite-size (wrapping) effects via TBA
Integrability for the Hagedorn temperature

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Hagedorn temperature: Singularity driven by very long spin chains, for which finite-size effects play no role → Solution on $S^1_{1/T} \times \mathbb{R}$ via TBA
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1 Introduction

2 The Hagedorn temperature from the free energy of the spin chain

3 Quantum Spectral Curve for the Hagedorn temperature

4 Solving the Hagedorn QSC

5 Chemical potentials and deformations

6 Conclusion and outlook
Single-trace partition function with $D = D_0 + \delta D$:

$$Z(T) = \text{tr}_{\mathbb{R}^{1,3},\text{single-trace}}[e^{-D/T}] = \sum_{m=2}^{\infty} e^{-\frac{m}{2} \frac{1}{T}} Z_{\text{spin-chain}}^{D_0=\frac{m}{2}}(T)$$

Spin-chain partition function at fixed $D_0 = \frac{m}{2}$:

$$Z_{D_0=\frac{m}{2}}(T) = \text{tr}_{\text{spin-chain},D_0=\frac{m}{2}}[e^{-\delta D/T}]$$
Field-theory and spin-chain partition function

Single-trace partition function with $D = D_0 + \delta D$:

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Spin-chain free energy per unit classical scaling dimension at $D_0 = \frac{m}{2}$:

$$F_m(T) = -T \frac{2}{m} \log Z_{D_0 = \frac{m}{2}}^{\text{spin-chain}}$$
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Spin-chain free energy per unit classical scaling dimension at $D_0 = \frac{m}{2}$:

$$F_m(T) = -T \frac{2}{m} \log Z^{\text{spin-chain}}_{D_0 = \frac{m}{2}}$$

Multi-trace partition function:

$$Z(T) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=2}^{\infty} (-1)^{m(n+1)} e^{-\frac{mn}{2T} (1+F_m(T/n))}$$
Free energy and Hagedorn temperature

Hagedorn singularity = first singularity encountered in \( Z \) when raising the temperature from zero \( \Rightarrow \) stems from \( n = 1 \) term

\[
\sum_{m=2}^{\infty} e^{-\frac{m}{2T} (1+F_m(T))}
\]
Hagedorn singularity = first singularity encountered in $Z$ when raising the temperature from zero $\Rightarrow$ stems from $n = 1$ term

$$\sum_{m=2}^{\infty} e^{-\frac{m}{2T}(1+F_m(T))}$$

**Cauchy root test** Consider $\sum_{n=1}^{\infty} a_n$. Let $r = \lim_{n \to \infty} \sqrt[n]{a_n}$. The series converges for $r < 1$ and diverges for $r > 1$. 

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The Hagedorn temperature of AdS$_5$/CFT$_4$ at finite coupling via the QSC
Free energy and Hagedorn temperature

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$$\sum_{m=2}^{\infty} e^{-\frac{m}{2T}(1+F_m(T))}$$

**Cauchy root test** Consider $\sum_{n=1}^{\infty} a_n$. Let $r = \lim_{n \to \infty} \sqrt[n]{a_n}$. The series converges for $r < 1$ and diverges for $r > 1$.

The Hagedorn temperature is determined by $r \equiv e^{-\frac{1}{2T}(1+F(T))} = 1$ or, equivalently,

$$F(T_H) = -1$$

where

$$F = \lim_{m \to \infty} F_m = -\lim_{m \to \infty} T \frac{2}{m} \log Z_{D_0=D_0}^{\text{spin-chain}}$$

is the thermodynamic limit of the spin-chain free energy per unit scaling dimension!
1. Introduction

2. The Hagedorn temperature from the free energy of the spin chain

3. Quantum Spectral Curve for the Hagedorn temperature

4. Solving the Hagedorn QSC

5. Chemical potentials and deformations

6. Conclusion and outlook

- Starting point: all-loop asymptotic Bethe equations [Beisert, Dippel, Staudacher (2004)], [Beisert, Eden, Staudacher (2006)]
- Replace $L$ by $D_0$
- String hypothesis
- Continuum limit $D_0 \to \infty$

$\Rightarrow$ TBA equations determining in particular $F(T)$ and thus $T_H$

Main difference compared to the TBA equations of the spectral problem:

No double Wick rotation $\rightarrow$ Zhukovsky variable $x(u)$ with a short cut

$$x(u) = \frac{u}{2} \left( 1 + \sqrt{1 - \frac{4g^2}{u^2}} \right) \quad g^2 = \frac{\lambda}{16\pi^2}$$

Note: direct theory with different continuum limit studied in [Cavaglia, Fioravanti, Tateo (2010)]
From the TBA equations to the QSC:

TBA equations: Infinite system of integral equations

Y-system/T-system: Infinite system of finite difference equations

Quantum Spectral Courve: Finite system of finite difference equations

[Gromov, Kazakov, Leurent, Volin (2013)]

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Part of an analytic Q-system with fundamental functions

\[ P_a(u) \quad Q_i(u) \quad \text{and} \quad Q_{a|i}(u) \quad a, i = 1, 2, 3, 4 \]

on a Riemann surface

Finite difference equations with \( f^\pm(u) = f(u \pm \frac{i}{2}) \):

\[ Q^+_{a|i} - Q^-_{a|i} = P_a Q_i \quad P_a = -Q^i Q^+_a \]

with \( Q^i = \chi^{ij} Q_j \) and \(-\chi^{14} = -\chi^{32} = \chi^{23} = \chi^{41} = 1\)

Orthonormality:

\[ Q_{a|i} Q^{b|i} = \delta^b_a \quad Q_{a|i} Q^a{j} = \delta^j_i \]

[Gromov, Kazakov, Leurent, Volin (2013)]
Adjusting the QSC

Applications to spectral problem:

- $\mathcal{N} = 4$ SYM theory [Gromov, Kazakov, Leurent, Volin (2013)]
- Twisted $\mathcal{N} = 4$ SYM theory [Kazakov, Leurent, Volin (2015)]
- $\eta$-deformed $\mathcal{N} = 4$ SYM theory [Klabbers, van Tongeren (2017)]

Applications to structure constants:

- Cusped Wilson lines [Cavaglià, Gromov, Levkovich-Maslyuk (2018)]
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Adjust the QSC to Hagedorn temperature:
- Asymptotics (large $u$)
- Branch cuts
- Gluing conditions
Asymptotics (large $u$)

Asymptotic T-system \[ \text{[Harmark, MW (2017)]} \Rightarrow \text{Asymptotics of } P_a \text{ and } Q_i \]
Asymptotics (large $u$)

Asymptotic T-system [Harmark, MW (2017)] $\Rightarrow$ Asymptotics of $P_a$ and $Q_i$

\[
\begin{align*}
P_1(u) &= A_1 (-e^{-\frac{1}{2T_H}})^{-iu} (1 + \mathcal{O}(u^{-1})) & Q_1(u) &= B_1 (1 + \mathcal{O}(u^{-1})) \\
P_2(u) &= A_2 (-e^{-\frac{1}{2T_H}})^{-iu} (u + \mathcal{O}(u^0)) & Q_2(u) &= B_2 (u + \mathcal{O}(u^0)) \\
P_3(u) &= A_3 (-e^{-\frac{1}{2T_H}})^{+iu} (1 + \mathcal{O}(u^{-1})) & Q_3(u) &= B_3 (u^2 + \mathcal{O}(u^1)) \\
P_4(u) &= A_4 (-e^{-\frac{1}{2T_H}})^{+iu} (u + \mathcal{O}(u^0)) & Q_4(u) &= B_4 (u^3 + \mathcal{O}(u^2))
\end{align*}
\]

with $A_1 A_4 = A_2 A_3 = \frac{i}{\tanh^2 \frac{1}{4T_H}}$ and $B_1 B_4 = \frac{1}{3} B_2 B_3 = -\frac{8i}{3} \cosh^4 \frac{1}{4T_H}$
Asymptotics (large $u$)

Asymptotic T-system [Harmark, MW (2017)] ⇒ Asymptotics of $P_a$ and $Q_i$

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Remarks:

- $T_H$ enters the asymptotics of $P_a$ and the prefactors!
- Gauge freedom: Fix $B_1 = B_2 = 1$. Fix $A_1 = iA_2 = -A_3 = -iA_4 = (\tanh \frac{1}{4T_H})^{-1}$.
- Asymptotics formally agree with those in the twisted spectral problem for $\Delta = S_1 = S_2 = J_1 = J_2 = J_3 = 0$. 
Opposite branch cut structure compared to spectral problem:
\[ \exists \text{ sheet on which } Q_i \text{ has a single cut from } -2g \text{ to } +2g \]
Opposite branch cut structure compared to spectral problem:

∃ sheet on which $Q_i$ has a single cut from $-2g$ to $+2g$

Ansatz: $Q_i(u) = B_i(gx(u))^{i-1} \left( 1 + \sum_{n=1}^{\infty} \frac{c_{i,n}(g)}{x(u)^{2n}} \right)$
Branch cuts and gluing conditions

Opposite branch cut structure compared to spectral problem:
\exists\text{ sheet on which } Q_i \text{ has a single cut from } -2g \text{ to } +2g

\[
Q_i \quad 2g+2i
\]

\[
Q_i \quad 2g+i
\]

\[
2g
\]

\[
Q_i \quad 2g-i
\]

\[
2g-2i
\]

Ansatz: 
\[Q_i(u) = B_i(gx(u))^{i-1}\left(1 + \sum_{n=1}^{\infty} \frac{c_{i,n}(g)}{x(u)^{2n}}\right)\]

Gluing conditions (analytic continuation through branch cut): 
\[\tilde{P}_a(u) = (-1)^{a+1}P_a(u)\]

Note: Formulation of [Gromov, Levkovitch-Maslyuk, Sizov (2015)], no \(\mu_{ab}\).
General strategy

Ansatz for $Q_i$: $Q_i(u) = B_i(gx(u))^{i-1} \left( 1 + \sum_{n=1}^{\infty} \frac{c_{i,n}(g)}{x(u)^{2n}} \right)$

Solve for $Q_a \mid_i$ via $Q_a \mid_i - Q_a \mid_j = -Q_i Q_j \mid_a$

Solve for $P_a = -Q_i Q_j \mid_a$ and $\tilde{P}_a = -\tilde{Q}_i Q_j \mid_a$

Impose gluing conditions $\tilde{P}_a(u) = (-1)^{a+1} P_a(u)$ and asymptotics $P_2(u) P_1(u) = -iu + O(u^0)$
General strategy

1. Ansatz for $Q_i$: $Q_i(u) = B_i(gx(u))^{i-1} \left( 1 + \sum_{n=1}^{\infty} \frac{c_{i,n}(g)}{x(u)^{2n}} \right)$

2. Solve for $Q_a^i$ via $Q_a^+ - Q_a^- = -Q_i Q^j Q_a^+$

3. Solve for $P_a$ via $P_a = -Q_i Q^j Q_a^+$ and $\tilde{P}_a = -\tilde{Q}_i Q^j Q_a^+$

4. Impose gluing conditions $\tilde{P}_a(u) = (-1)^{a+1} P_a(u)$ and asymptotics via $P_2(u) P_1(u) = -iu + O(u^0)$
General strategy

1. Ansatz for $Q_i$: $Q_i(u) = B_i(gx(u))^{i-1} \left( 1 + \sum_{n=1}^{\infty} \frac{c_{i,n}(g)}{x(u)^{2n}} \right)$

2. Solve for $Q^+_{a|i}$ via $Q^+_{a|i} - Q^-_{a|i} = -Q_i Q^i Q^+_a$

3. Solve for $P_a = -Q^i Q^+_{a|i}$ and $\tilde{P}_a = -\tilde{Q}^i Q^+_{a|i}$
General strategy

1. Ansatz for $Q_i$: $Q_i(u) = B_i(g x(u))^{i-1} \left(1 + \sum_{n=1}^{\infty} \frac{c_{i,n}(g)}{x(u)^{2n}}\right)$

2. Solve for $Q_{a|i}$ via $Q^+_{a|i} - Q^-_{a|i} = -Q_i Q^+_{a|j}$

3. Solve for $P_a = -Q^i Q^+_{a|i}$ and $\tilde{P}_a = -\tilde{Q}^i Q^+_{a|i}$.

4. Impose gluing conditions $\tilde{P}_a(u) = (-1)^{a+1} \overline{P_0(u)}$ and asymptotics via $\frac{P_2(u)}{P_1(u)} = -iu + O(u^0)$

$\Rightarrow T_H$ and $c_{i,n}!$
Strategy following \cite{Gromov, Levkovich-Maslyuk, Sizov (2015)}

Workhorse: Solve \( f(u + i) - f(u) = g(u) \)
Strategy following [Gromov, Levkovich-Maslyuk, Sizov (2015)]

Workhorse: Solve $f(u + i) - f(u) = g(u)$

Example: $g(u) = \frac{z^{-iu}}{u^2} \Rightarrow f(u) = -z^{-iu} \sum_{n=0}^{\infty} \frac{z^n}{(u + in)^2}$
Perturbative solution

Strategy following [Gromov, Levkovich-Maslyuk, Sizov (2015)]

Workhorse: Solve \( f(u + i) - f(u) = g(u) \)

Example: \( g(u) = \frac{-iu}{u^2} \Rightarrow f(u) = -z^{-iu} \sum_{n=0}^{\infty} \frac{z^n}{(u+in)^2} \)

Solution given by generalised \( \eta \) functions:

\[
\eta_{s_1, \ldots, s_k}^{z_1, \ldots, z_k}(u) \equiv \sum_{n_1 > n_2 > \cdots > n_k \geq 0} \frac{z_1^{n_1} \cdots z_k^{n_k}}{(u + in_1)^{s_1} \cdots (u + in_k)^{s_k}}
\]

[Cromov, Levkovich-Maslyuk, Sizov (2015)]
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\]

\[\text{[Gromov, Levkovich-Maslyuk, Sizov (2015)]}\]

\( \eta_{s_1, \ldots, s_k}^{z_1, \ldots, z_k}(i) \) is proportional to a multiple polylogarithm:

\[
\text{Li}_{s_1, \ldots, s_k}(z_1, \ldots, z_k) \equiv \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}
\]
Strategy following [Gromov, Levkovich-Maslyuk, Sizov (2015)]

Workhorse: Solve $f(u + i) - f(u) = g(u)$

Example: $g(u) = \frac{z^{-iu}}{u^2} \Rightarrow f(u) = -z^{-iu} \sum_{n=0}^{\infty} \frac{z^n}{(u + in)^2}$

Solution given by generalised $\eta$ functions:

$$\eta_{s_1, \ldots, s_k}^{z_1, \ldots, z_k}(u) \equiv \sum_{n_1 > n_2 > \cdots > n_k \geq 0} \frac{z_1^{n_1} \cdots z_k^{n_k}}{(u + in_1)^{s_1} \cdots (u + in_k)^{s_k}}$$

[Gromov, Levkovich-Maslyuk, Sizov (2015)]

$\eta_{s_1, \ldots, s_k}^{z_1, \ldots, z_k}(i)$ is proportional to a multiple polylogarithm:

$$\text{Li}_{s_1, \ldots, s_k}(z_1, \ldots, z_k) \equiv \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \cdots n_k^{s_k}}$$

Cave: $z_i \in \{1, (2 + \sqrt{3})^{\pm 2}\}$ and e.g. $\text{Li}_s(z)$ has a cut for $z \geq 1$  

\[ \exp(\mp T_0^{(0)} H) \]  

$\Rightarrow i\epsilon$ prescription!
Perturbative results

Perturbative solution at weak coupling:

\[ T_H = \frac{1}{2 \log(2 + \sqrt{3})} + g^2 \frac{1}{\log(2 + \sqrt{3})} \]

\[ \approx 0.3797 \quad [\text{Sundborg (1999)}] + 0.7593 \quad [\text{Spradlin, Volovich (2004)}] \]

\[ + g^4 \left( 48 - \frac{86}{\sqrt{3}} + \frac{48 \text{Li}_1 \left( \frac{1}{(2 + \sqrt{3})^2} \right)}{\log(2 + \sqrt{3})} \right) \]

\[ \approx -4.3676 \quad [\text{Harmark, MW (2017)}] \]

\[ + g^6 \left( 624 \text{Li}_2 \left( \frac{1}{(2 + \sqrt{3})^2} \right) + \frac{432 \text{Li}_1 \left( \frac{1}{(2 + \sqrt{3})^2} \right)^2}{\log(2 + \sqrt{3})} + \frac{312 \text{Li}_3 \left( \frac{1}{(2 + \sqrt{3})^2} \right)}{\log(2 + \sqrt{3})} \right) + \mathcal{O}(g^8) \]

\[ + \left( 384 \sqrt{3} - 864 + 416 \log(2 + \sqrt{3}) \right) \text{Li}_1 \left( \frac{1}{(2 + \sqrt{3})^2} \right) \]

\[ - \frac{20}{\sqrt{3}} + \left( \frac{1900}{3} - 384 \sqrt{3} \right) \log(2 + \sqrt{3}) \]

\[ \approx 37.2253 \]
Perturbative results

\[ T_H^{(4)} = -288 \text{Li}_2,1 \left( \frac{1}{(2+\sqrt{3})^2}, (2+\sqrt{3})^2 \right) - 288 \text{Li}_1 \left( \frac{1}{(2+\sqrt{3})^2} \right) \zeta(2) - \frac{144 \text{Li}_1 \left( \frac{1}{(2+\sqrt{3})^2} \right) \zeta(3)}{\log(2+\sqrt{3})} \\
- 8928 \text{Li}_2 \left( \frac{1}{(2+\sqrt{3})^2} \right) \text{Li}_1 \left( \frac{1}{(2+\sqrt{3})^2} \right) - 5400 \text{Li}_4 \left( \frac{1}{(2+\sqrt{3})^2} \right) + \text{Li}_1 \left( \frac{1}{(2+\sqrt{3})^2} \right) \left( 704 \sqrt{3} + 5952 \log^2(2+\sqrt{3}) - 2560 \sqrt{3} \log^2(2+\sqrt{3}) - 18816 \log(2+\sqrt{3}) + 11904 \sqrt{3} \log(2+\sqrt{3}) \right) \\
+ \text{Li}_2 \left( \frac{1}{(2+\sqrt{3})^2} \right) \left( -1440 \log^2(2+\sqrt{3}) + 8928 \log(2+\sqrt{3}) - 3840 \sqrt{3} \log(2+\sqrt{3}) \right) \\
- \frac{5184 \text{Li}_1 \left( \frac{1}{(2+\sqrt{3})^2} \right)^3}{\log(2+\sqrt{3})} + \text{Li}_1 \left( \frac{1}{(2+\sqrt{3})^2} \right)^2 \left( 15552 - 5952 \sqrt{3} - 6048 \log(2+\sqrt{3}) \right) \\
- \frac{4608 \text{Li}_3 \left( \frac{1}{(2+\sqrt{3})^2} \right) \text{Li}_1 \left( \frac{1}{(2+\sqrt{3})^2} \right)}{\log(2+\sqrt{3})} + \text{Li}_3 \left( \frac{1}{(2+\sqrt{3})^2} \right) \left( 5040 - 1920 \sqrt{3} - 4320 \log(2+\sqrt{3}) \right) \\
- \frac{2700 \text{Li}_5 \left( \frac{1}{(2+\sqrt{3})^2} \right)}{\log(2+\sqrt{3})} + \frac{40}{\sqrt{3}} - \frac{43906 \log^2(2+\sqrt{3})}{3 \sqrt{3}} + 8448 \log^2(2+\sqrt{3}) \\
- 704 \sqrt{3} \log(2+\sqrt{3}) + 1272 \log(2+\sqrt{3}) \\
\approx -372.0410892 \]
Perturbative results

\[ T_H^{(5)} = \left( -\frac{286200}{7} + 14160\sqrt{3} + 46368 \log(2 + \sqrt{3}) \right) \text{Li}_5 \left( \frac{1}{(2 + \sqrt{3})^2} \right) + 54096 \text{Li}_6 \left( \frac{1}{(2 + \sqrt{3})^2} \right) + \frac{27048 \text{Li}_7 \left( \frac{1}{(2 + \sqrt{3})^2} \right)}{\log(2 + \sqrt{3})} + 41 \text{ further terms} \approx 4132.973342 \]

\[ T^{(6)} = -592704 \text{Li}_8 \left( \frac{1}{(2 + \sqrt{3})^2} \right) - \frac{337680 \text{Li}_1 \left( \frac{1}{(2 + \sqrt{3})^2} \right) \text{Li}_7 \left( \frac{1}{(2 + \sqrt{3})^2} \right)}{\log(2 + \sqrt{3})} - \frac{296352 \text{Li}_9 \left( \frac{1}{(2 + \sqrt{3})^2} \right)}{\log(2 + \sqrt{3})} + 97 \text{ terms} \approx -49510.01767 \]

\[ T^{(7)} = + \left( -3423168 + 1282176\sqrt{3} + 6272640 \log(2 + \sqrt{3}) \right) \text{Li}_9 \left( \frac{1}{(2 + \sqrt{3})^2} \right) + 6899904 \text{Li}_{10} \left( \frac{1}{(2 + \sqrt{3})^2} \right) + \frac{3449952 \text{Li}_11 \left( \frac{1}{(2 + \sqrt{3})^2} \right)}{\log(2 + \sqrt{3})} + 261 \text{ terms} \approx 625284.5652 \]
Strategy following [Gromov, Levkovich-Maslyuk, Sizov (2015)]: Numeric solution of QSC

\[ g^2 \approx 0.38042046 \]

\[ T^\text{numeric} \approx 0.6 \text{ loop} \]

Minimalization problem (solve via Levenberg Marquart algorithm)
Strong-coupling behaviour:

\[ T_H(g) \simeq \sqrt{\frac{g}{2\pi}} \approx (0.3989422804 \ldots) \sqrt{g} \]
Numeric results at large coupling

Strong-coupling behaviour:

\[ T_H (g) \approx \sqrt{\frac{g}{2\pi}} \approx (0.3989422804 \ldots) \sqrt{g} \]

= Hagedorn temperature of tree-level type IIB string theory on ten-dimensional Minkowski space [Sundborg 1984]
(Naive explanation: \( \lambda \to \infty \leftrightarrow \text{Curvature} \to 0 \))
Introduce chemical potentials for the R-charges $J_1, J_2, J_3$ and angular momenta $S_1, S_2$:

$$
Z(T, \Omega_i) = \text{tr} \left( e^{-\beta D + \beta \sum_{i=1}^{3} \Omega_i J_i + \beta \sum_{a=1}^{2} \Omega_{a+3} S_a} \right), \quad \beta = 1/T
$$

- Allow to single out contributions from specific fields
- Previously studied at tree level and one-loop [Yamada, Yaffe (2006)], [Harmark, Orselli (2006)], [Suzuki (2017)]

⇒ Include chemical potentials in the QSC approach
Asymptotics (similar to twisted spectral problem [Kazakov, Leurent, Volin (2015)]):

\[
P_a \simeq A_a x_a^{+iu} u + \sum_{b < a} \delta_{x_a x_b} - \sum_{i < a} \delta_{x_a y_i}
\]

\[
P^a \simeq A^a x_a^{-iu} u + \sum_{b > a} \delta_{x_a x_b} - \sum_{i > ca} \delta_{x_a y_i}
\]

\[
Q_i \simeq B_i y_i^{-iu} u - \sum_{a < i} \delta_{x_a y_i} + \sum_{j < i} \delta_{y_i y_j}
\]

\[
Q^i \simeq B^i y_i^{+iu} u - \sum_{a > i} \delta_{x_a y_i} + \sum_{j > i} \delta_{y_i y_j}
\]

with

\[
x_1 = -e^{-\frac{1+\Omega_4+\Omega_5}{2T_H}}
\]

\[
x_2 = -e^{-\frac{1-\Omega_4-\Omega_5}{2T_H}}
\]

\[
x_3 = -e^{\frac{1+\Omega_4-\Omega_5}{2T_H}}
\]

\[
x_4 = -e^{\frac{1-\Omega_4+\Omega_5}{2T_H}}
\]

\[
y_1 = e^{\frac{\Omega_1+\Omega_2-\Omega_3}{2T_H}}
\]

\[
y_2 = e^{\frac{\Omega_1-\Omega_2+\Omega_3}{2T_H}}
\]

\[
y_3 = e^{-\frac{\Omega_1+\Omega_2+\Omega_3}{2T_H}}
\]

\[
y_4 = e^{-\frac{-\Omega_1-\Omega_2-\Omega_3}{2T_H}}
\]
\( \gamma_i \)-deformation [Frolov (2005)] \supset \beta\)-deformation:

- Tree-level partition function trivially the same as in \( \mathcal{N} = 4 \) SYM theory.
- One-loop partition function depends on \( \gamma_i \) but \( T_H^{(1)} \) same as in \( \mathcal{N} = 4 \) SYM theory [Fokken, MW (2014)].
- Integrability approach shows that \( T_H \) is the same as in \( \mathcal{N} = 4 \) SYM theory at any \( \lambda \).
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Conclusions and outlook

Conclusions

- Derived integrability-based QSC equations that determine the Hagedorn temperature of planar $\mathcal{N} = 4$ SYM theory / type IIB string theory on $\text{AdS}_5 \times S^5$ at any value of the ’t Hooft coupling
  $\rightarrow$ Non-perturbative understanding of thermal physics

- Perturbative solution at weak coupling
  $\rightarrow$ Previously unknown $\ell = 2, 3, 4, 5, 6, 7$-loop Hagedorn temperature

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  $\rightarrow T_H(\lambda)$ asymptotes to the Hagedorn temperature of type IIB string theory on 10D flat space

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- Strong-coupling expansion?
- Further observables: critical exponents?
- Flat space holography?
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