The Hagedorn temperature of AdS_5/CFT_4 at finite coupling via the Quantum Spectral Curve

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[1706.03074], [1803.04416] and work in progress with T. Harmark

AdS/CFT correspondence

type IIB string theory
on $AdS_5 \times S^5$

$\mathcal{N} = 4$ SYM theory
on $\mathbb{R} \times S^3$

[Maldacena (1997)]

Should in particular relate phase transitions, critical behaviour and thermal physics e.g. Hagedorn temperature [Witten (1998)], . . .

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Hagedorn temperature

Limiting temperature due to an exponential rise in the density of states $\rho(E)$, where $\mathcal{Z}(T) = \sum_E \rho(E)\, e^{-\frac{E}{T}}$

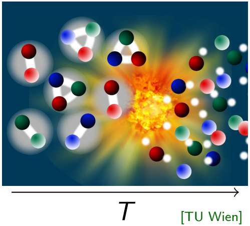
$\rightarrow$ Signals a phase transition [Hagedorn (1965)]

Hagedorn temperature in planar gauge theories

Planar gauge theories on $\mathbb{R} \times S^3$

Confinement of colour degrees of freedom on S^3

$\rightarrow$ Confinement/deconfinement phase transition similar to QCD or pure YM



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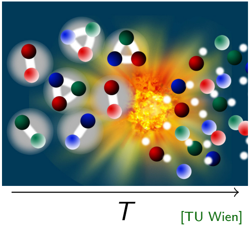

[TU Wien]

R_{S^3} acts as an effective IR cut-off

⇒ Allows for a perturbative comparison between conformal theories, such as $\mathcal{N} = 4$ SYM theory, and non-conformal theories such as pure Yang-Mills theory

[Aharony, Marsano, Minwalla, Papadodimas, Van Raamsdonk (2003)]

Hagedorn temperature of $\mathcal{N} = 4$ SYM theory

State/operator correspondence

States on $\mathbb{R} \times S^3$	$\leftrightarrow$	Gauge-invariant operators on $\mathbb{R}^{1,3}$
Hamilton operator H on $\mathbb{R} \times S^3$	$\leftrightarrow$	Dilatation operator D on $\mathbb{R}^{1,3}$
Energies E on $\mathbb{R} \times S^3$	$\leftrightarrow$	Scaling dimensions Δ on $\mathbb{R}^{1,3}$

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Partition function ($R_{S^3} = 1$)

$$\mathcal{Z}(T) = \mathrm{tr}_{\mathbb{R}\times S^3}[\mathrm{e}^{-H/T}] = \mathrm{tr}_{\mathbb{R}^{1,3}}[\mathrm{e}^{-D/T}]$$

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Planar limit: $\Delta_{\mathrm{tr()\,tr()}\ldots} = \Delta_{\mathrm{tr()}} + \Delta_{\mathrm{tr()}} + \ldots$
$\Rightarrow$ Sufficient to look at single-trace operators

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Partition function via Pólya theory:

- Tree-level [Sundborg (1999)]
- One-loop [Spradlin, Volovich (2004)]

Hagedorn temperature in string theory

Free (tree-level) string theory

Exponential growth of string states with the energy

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Interacting string theory

Connected to Hawking-Page transition [Witten (1998)]

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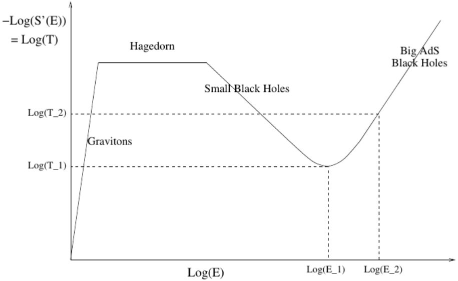

'Phases' of type IIB string theory on $AdS_5 \times S^5$

[Aharony, Marsano, Minwalla, Papadodimas, Van Raamsdonk (2003)]

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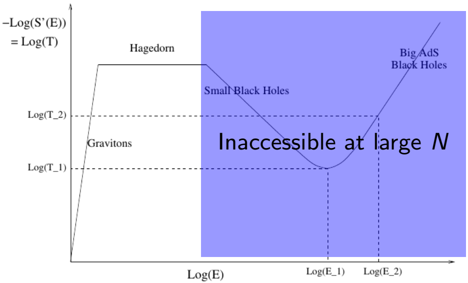

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Hagedorn temperature in pp-wave limits

Hagedorn temperature in pp-wave limits of string theory

[Pando Zayas, Vaman (2002)], [Greene, Schalm, Shiu (2002)], [Brower, Lowe, and Tan (2003)], [Grignani, Orselli, Semenoff, and Trancanelli (2003)]

Special limit

string theory pp-wave limit

gauge theory one-loop SU(2) sector

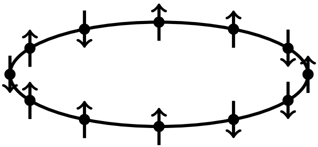
$\rightarrow$ Quantitative match via free energy of Heisenberg spin chain

[Harmark, Orselli (2006)]

Integrability

Integrability: Scaling dimensions $\Delta(\lambda)$ known for all λ

Long history: One-loop Bethe equation
$\rightarrow$ asymptotic all-loop Bethe equations
$\rightarrow$ thermodynamic Bethe ansatz (TBA) equations
$\rightarrow$ Y-system equations, T-system equations
$\rightarrow$ Quantum spectral curve (QSC)
[Minahan, Zarembo (2002)],...



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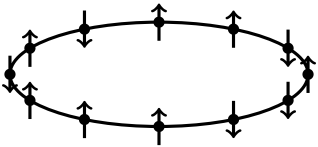
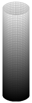


Naive idea: Sum over Δ's yields $\mathcal{Z} = \sum e^{-\Delta/T}$ $\rightarrow$ Prohibitive

Integrability for the Hagedorn temperature

Spectral problem: Solution on $\mathbb{R} \times S^1_L$ to account for finite-size (wrapping) effects via TBA



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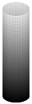
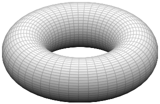


Partition function: Solution on $S^1_{1/T} \times S^1_L$



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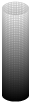

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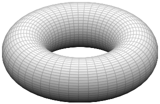

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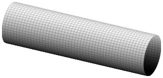
Hagedorn temperature: Singularity driven by *very* long spin chains, for which finite-size effects play no role $\rightarrow$ Solution on $S^1_{1/T} \times \mathbb{R}$ via TBA



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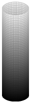

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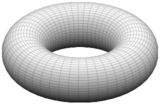

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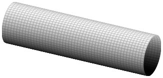
high-D_0



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2. The Hagedorn temperature from the free energy of the spin chain
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Field-theory and spin-chain partition function

Single-trace partition function with $D = D_0 + \delta D$:

$$Z(T) = \mathrm{tr}_{\mathbb{R}^{1,3},\text{single-trace}}[\mathrm{e}^{-D/T}] = \sum_{m=2}^{\infty} \mathrm{e}^{-\frac{m}{2}\frac{1}{T}}\, Z^{\text{spin-chain}}_{D_0=\frac{m}{2}}(T)$$

Spin-chain partition function at fixed $D_0 = \frac{m}{2}$:

$$Z^{\text{spin-chain}}_{D_0=\frac{m}{2}}(T) = \mathrm{tr}_{\text{spin-chain},D_0=\frac{m}{2}}[\mathrm{e}^{-\delta D/T}]$$

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Spin-chain free energy per unit classical scaling dimension at $D_0 = \frac{m}{2}$:

$$F_m(T) = -T\frac{2}{m}\log Z^{\text{spin-chain}}_{D_0=\frac{m}{2}}$$

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Multi-trace partition function:

$$\mathcal{Z}(T) = \exp\sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=2}^{\infty}(-1)^{m(n+1)}\,\mathrm{e}^{-\frac{mn}{2T}(1+F_m(T/n))}$$

Free energy and Hagedorn temperature

Hagedorn singularity = first singularity encountered in $\mathcal{Z}$ when raising the temperature from zero ⇒ stems from $n = 1$ term

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Cachy root test Consider $\sum_{n=1}^{\infty} a_n$. Let $r = \lim_{n\to\infty} \sqrt[n]{a_n}$. The series converges for $r < 1$ and diverges for $r > 1$.

Free energy and Hagedorn temperature

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The Hagedorn temperature is determined by $r \equiv e^{-\frac{1}{2T}(1+F(T))} = 1$ or, equivalently,

$$F(T_\mathrm{H}) = -1$$

where

$$F = \lim_{m\to\infty} F_m = -\lim_{m\to\infty} T\frac{2}{m}\log Z^{\text{spin-chain}}_{D_0=\frac{m}{2}}$$

is the thermodynamic limit of the spin-chain free energy per unit scaling dimension!

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Thermodynamic Bethe Ansatz (TBA) equations

Derivation paralleling the one in the spectral problem: [Arutyunov, Frolov (2009)], [Gromov, Kazakov, Kozak, Vieira (2009)], [Bombardelli, Fioravanti, Tateo (2009)], [Gromov, Kazakov, Vieira (2009)]

- Starting point: all-loop asymptotic Bethe equations [Beisert, Dippel, Staudacher (2004)], [Beisert, Eden, Staudacher (2006)]
- Replace L by D_0
- String hypothesis
- Continuum limit $D_0 \to \infty$

$\Rightarrow$ TBA equations determining in particular $F(T)$ and thus T_{H}

Main difference compared to the TBA equations of the spectral problem:

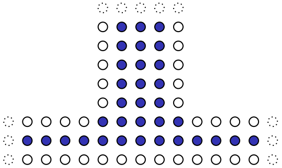
No double Wick rotation $\to$ Zhukovsky variable $x(u)$ with a short cut

$$x(u) = \frac{u}{2}\left(1 + \sqrt{1 - \frac{4g^2}{u^2}}\right) \qquad g^2 = \frac{\lambda}{16\pi^2}$$

Note: direct theory with different continuum limit studied in [Cavaglia, Fioravanti, Tateo (2010)]

From the TBA equations to the QSC

TBA equations:
Infinite system of
integral equations



Y-system/T-system:
Infinite system of
finite difference equations

Quantum Spectral Courve:
Finite system of
finite difference equations
[Gromov, Kazakov, Leurent, Volin (2013)]

The Quantum Spectral Curve

Part of an analytic Q-system with fundamental functions

$$\mathbf{P}_a(u) \qquad \mathbf{Q}_i(u) \quad \text{and} \quad Q_{a|i}(u) \qquad a, i = 1, 2, 3, 4$$

on a Riemann surface

Finite difference equations with $f^{\pm}(u) = f(u \pm \frac{i}{2})$:

$$Q^+_{a|i} - Q^-_{a|i} = \mathbf{P}_a \mathbf{Q}_i \qquad \qquad \mathbf{P}_a = -\mathbf{Q}^i Q^+_{a|i}$$

with $\mathbf{Q}^i = \chi^{ij} \mathbf{Q}_j$ and $-\chi^{14} = -\chi^{32} = \chi^{23} = \chi^{41} = 1$

Orthonormality:

$$Q_{a|i} Q^{b|i} = \delta_a^b \qquad Q_{a|i} Q^{a|j} = \delta_i^j$$

[Gromov, Kazakov, Leurent, Volin (2013)]

Adjusting the QSC

Applications to spectral problem:

- $\mathcal{N} = 4$ SYM theory [Gromov, Kazakov, Leurent, Volin (2013)]
- Twisted $\mathcal{N} = 4$ SYM theory [Kazakov, Leurent, Volin (2015)]
- Pomeron [Alfimov, Gromov, Kazakov (2014)],[Gromov, Levkovich-Maslyuk, Sizov (2015)]
- Cusped Wilson line [Gromov, Levkovich-Maslyuk, Sizov (2015)] and quark-antiquark potential [Gromov, Levkovich-Maslyuk (2015)]
- η-deformed $\mathcal{N} = 4$ SYM theory [Klabbers, van Tongeren (2017)]

Applications to structure constants:

- Cusped Wilson lines [Cavaglià, Gromov, Levkovich-Maslyuk (2018)]

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Adjust the QSC to Hagedorn temperature:

- Asymptotics (large u)
- Branch cuts
- Gluing conditions

Asymptotics (large u)

Asymptotic T-system [Harmark, MW (2017)] $\Rightarrow$ Asymptotics of $\mathbf{P}_a$ and $\mathbf{Q}_i$

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$$\mathbf{P}_1(u) = A_1(-\mathrm{e}^{-\frac{1}{2T_\mathrm{H}}})^{-iu}\left(1+\mathcal{O}(u^{-1})\right) \qquad \mathbf{Q}_1(u) = B_1\left(1+\mathcal{O}(u^{-1})\right)$$

$$\mathbf{P}_2(u) = A_2(-\mathrm{e}^{-\frac{1}{2T_\mathrm{H}}})^{-iu}\left(u+\mathcal{O}(u^{0})\right) \qquad \mathbf{Q}_2(u) = B_2\left(u+\mathcal{O}(u^{0})\right)$$

$$\mathbf{P}_3(u) = A_3(-\mathrm{e}^{-\frac{1}{2T_\mathrm{H}}})^{+iu}\left(1+\mathcal{O}(u^{-1})\right) \qquad \mathbf{Q}_3(u) = B_3\left(u^2+\mathcal{O}(u^{1})\right)$$

$$\mathbf{P}_4(u) = A_4(-\mathrm{e}^{-\frac{1}{2T_\mathrm{H}}})^{+iu}\left(u+\mathcal{O}(u^{0})\right) \qquad \mathbf{Q}_4(u) = B_4\left(u^3+\mathcal{O}(u^{2})\right)$$

with $A_1A_4 = A_2A_3 = \frac{i}{\tanh^2\frac{1}{4T_\mathrm{H}}}$ and $B_1B_4 = \frac{1}{3}B_2B_3 = -\frac{8i}{3}\cosh^4\frac{1}{4T_\mathrm{H}}$

Asymptotics (large u)

Asymptotic T-system [Harmark, MW (2017)] $\Rightarrow$ Asymptotics of $\mathbf{P}_a$ and $\mathbf{Q}_i$

$$\mathbf{P}_1(u) = A_1\left(-e^{-\frac{1}{2T_{\mathrm{H}}}}\right)^{-iu}\left(1+\mathcal{O}(u^{-1})\right) \qquad \mathbf{Q}_1(u) = B_1\left(1+\mathcal{O}(u^{-1})\right)$$

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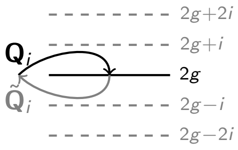
Remarks:

- T_{H} enters the asymptotics of $\mathbf{P}_a$ and the prefactors!
- Gauge freedom: Fix $B_1 = B_2 = 1$. Fix $A_1 = iA_2 = -A_3 = -iA_4 = (\tanh \frac{1}{4T_{\mathrm{H}}})^{-1}$.
- Asymptotics formally agree with those in the twisted spectral problem for $\Delta = S_1 = S_2 = J_1 = J_2 = J_3 = 0$

Branch cuts and gluing conditions

Opposite branch cut structure compared to spectral problem:
$\exists$ sheet on which $\mathbf{Q}_i$ has a single cut from $-2g$ to $+2g$



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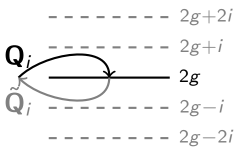
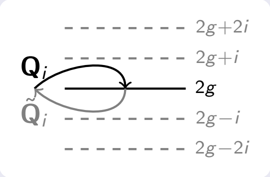


Ansatz: $\mathbf{Q}_i(u) = B_i (g x(u))^{i-1} \left(1 + \sum_{n=1}^{\infty} \frac{c_{i,n}(g)}{x(u)^{2n}}\right)$

Branch cuts and gluing conditions

Opposite branch cut structure compared to spectral problem:
$\exists$ sheet on which $\mathbf{Q}_i$ has a single cut from $-2g$ to $+2g$



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Gluing conditions (analytic continuation through branch cut):

$$\tilde{\mathbf{P}}_a(u) = (-1)^{a+1}\overline{\mathbf{P}_a(u)}$$

Note: Formulation of [Gromov, Levkovich-Maslyuk, Sizov (2015)], no μ_{ab}.

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General strategy

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$\Rightarrow$ T_{H} and $c_{i,n}$!

Perturbative solution

Strategy following [Gromov, Levkovich-Maslyuk, Sizov (2015)]

Workhorse: Solve $f(u+i) - f(u) = g(u)$

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Solution given by generalised η functions:

$$\eta_{s_1,\ldots,s_k}^{z_1,\ldots,z_k}(u) \equiv \sum_{n_1>n_2>\cdots>n_k\geq 0} \frac{z_1^{n_1} \ldots z_k^{n_k}}{(u+in_1)^{s_1} \ldots (u+in_k)^{s_k}}$$

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$\eta_{s_1,\dots,s_k}^{z_1,\dots,z_k}(i)$ is proportional to a multiple polylogarithm:

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Cave: $z_i \in \{1, \underbrace{(2+\sqrt{3})^{\pm 2}}_{\exp(\mp T_{\mathrm{H}}^{(0)})}\}$ and e.g. $\mathrm{Li}_s(z)$ has a cut for $z \geq 1$

$\Rightarrow$ $i\epsilon$ prescription!

Perturbative results

Perturbative solution at weak coupling:

$$T_{\mathrm{H}} = \underbrace{\frac{1}{2\log(2+\sqrt{3})}}_{\approx 0.3797} \text{ [Sundborg (1999)]} + g^2 \underbrace{\frac{1}{\log(2+\sqrt{3})}}_{\approx 0.7593} \text{ [Spradlin, Volovich (2004)]}$$

$$+g^4 \underbrace{\left(48 - \frac{86}{\sqrt{3}} + \frac{48\mathrm{Li}_1\left(\frac{1}{(2+\sqrt{3})^2}\right)}{\log(2+\sqrt{3})}\right)}_{\approx -4.3676} \text{ [Harmark, MW (2017)]}$$

$$+g^6 \underbrace{\left(624\mathrm{Li}_2\left(\frac{1}{(2+\sqrt{3})^2}\right) + \frac{432\mathrm{Li}_1\left(\frac{1}{(2+\sqrt{3})^2}\right)^2}{\log\left(2+\sqrt{3}\right)} + \frac{312\mathrm{Li}_3\left(\frac{1}{(2+\sqrt{3})^2}\right)}{\log\left(2+\sqrt{3}\right)} + \left(384\sqrt{3} - 864 + 416\log\left(2+\sqrt{3}\right)\right)\mathrm{Li}_1\left(\frac{1}{(2+\sqrt{3})^2}\right) - \frac{20}{\sqrt{3}} + \left(\frac{1900}{3} - 384\sqrt{3}\right)\log\left(2+\sqrt{3}\right)\right)}_{\approx 37.2253} + \mathcal{O}(g^8)$$

Perturbative results

$$
\begin{aligned}
T_H^{(4)} =& -288\,\mathrm{Li}_{2,1}\left(\frac{1}{(2+\sqrt{3})^2},(2+\sqrt{3})^2\right) - 288\,\mathrm{Li}_1\left(\frac{1}{(2+\sqrt{3})^2}\right)\zeta(2) - \frac{144\,\mathrm{Li}_1\left(\frac{1}{(2+\sqrt{3})^2}\right)\zeta(3)}{\log(2+\sqrt{3})} \\
& -8928\,\mathrm{Li}_2\left(\frac{1}{(2+\sqrt{3})^2}\right)\mathrm{Li}_1\left(\frac{1}{(2+\sqrt{3})^2}\right) - 5400\,\mathrm{Li}_4\left(\frac{1}{(2+\sqrt{3})^2}\right) + \mathrm{Li}_1\left(\frac{1}{(2+\sqrt{3})^2}\right)\Big(704\sqrt{3} \\
& \quad +5952\log^2(2+\sqrt{3}) - 2560\sqrt{3}\log^2(2+\sqrt{3}) - 18816\log(2+\sqrt{3}) + 11904\sqrt{3}\log(2+\sqrt{3})\Big) \\
& +\mathrm{Li}_2\left(\frac{1}{(2+\sqrt{3})^2}\right)\Big(-1440\log^2(2+\sqrt{3}) + 8928\log(2+\sqrt{3}) - 3840\sqrt{3}\log(2+\sqrt{3})\Big) \\
& -\frac{5184\,\mathrm{Li}_1\left(\frac{1}{(2+\sqrt{3})^2}\right)^3}{\log(2+\sqrt{3})} + \mathrm{Li}_1\left(\frac{1}{(2+\sqrt{3})^2}\right)^2\Big(15552 - 5952\sqrt{3} - 6048\log(2+\sqrt{3})\Big) \\
& -\frac{4608\,\mathrm{Li}_3\left(\frac{1}{(2+\sqrt{3})^2}\right)\mathrm{Li}_1\left(\frac{1}{(2+\sqrt{3})^2}\right)}{\log(2+\sqrt{3})} + \mathrm{Li}_3\left(\frac{1}{(2+\sqrt{3})^2}\right)\Big(5040 - 1920\sqrt{3} - 4320\log(2+\sqrt{3})\Big) \\
& -\frac{2700\,\mathrm{Li}_5\left(\frac{1}{(2+\sqrt{3})^2}\right)}{\log(2+\sqrt{3})} + \frac{40}{\sqrt{3}} - \frac{43906\log^2(2+\sqrt{3})}{3\sqrt{3}} + 8448\log^2(2+\sqrt{3}) \\
& -704\sqrt{3}\log(2+\sqrt{3}) + 1272\log(2+\sqrt{3}) \\
\approx& -372.0410892
\end{aligned}
$$

Perturbative results

$$T_H^{(5)} = \left(-\frac{286200}{7} + 14160\sqrt{3} + 46368\log(2+\sqrt{3})\right)\mathrm{Li}_5\left(\frac{1}{(2+\sqrt{3})^2}\right)$$
$$+54096\,\mathrm{Li}_6\left(\frac{1}{(2+\sqrt{3})^2}\right) + \frac{27048\,\mathrm{Li}_7\left(\frac{1}{(2+\sqrt{3})^2}\right)}{\log(2+\sqrt{3})} + 41 \text{ further terms}$$
$$\approx 4132.973342$$

$$T^{(6)} = -592704\,\mathrm{Li}_8\left(\frac{1}{(2+\sqrt{3})^2}\right) - \frac{337680\,\mathrm{Li}_1\left(\frac{1}{(2+\sqrt{3})^2}\right)\mathrm{Li}_7\left(\frac{1}{(2+\sqrt{3})^2}\right)}{\log(2+\sqrt{3})}$$
$$-\frac{296352\,\mathrm{Li}_9\left(\frac{1}{(2+\sqrt{3})^2}\right)}{\log(2+\sqrt{3})} + 97 \text{ terms}$$
$$\approx -49510.01767$$

$$T^{(7)} = +\left(-3423168 + 1282176\sqrt{3} + 6272640\log(2+\sqrt{3})\right)\mathrm{Li}_9\left(\frac{1}{(2+\sqrt{3})^2}\right)$$
$$+6899904\,\mathrm{Li}_{10}\left(\frac{1}{(2+\sqrt{3})^2}\right) + \frac{3449952\,\mathrm{Li}_{11}\left(\frac{1}{(2+\sqrt{3})^2}\right)}{\log(2+\sqrt{3})} + 261 \text{ terms}$$
$$\approx 625284.5652$$

Numeric results at finite coupling

Strategy following [Gromov, Levkovich-Maslyuk, Sizov (2015)]: Numeric solution of QSC
$\rightsquigarrow$ Minimalization problem (solve via Levenberg Marquart algorithm)

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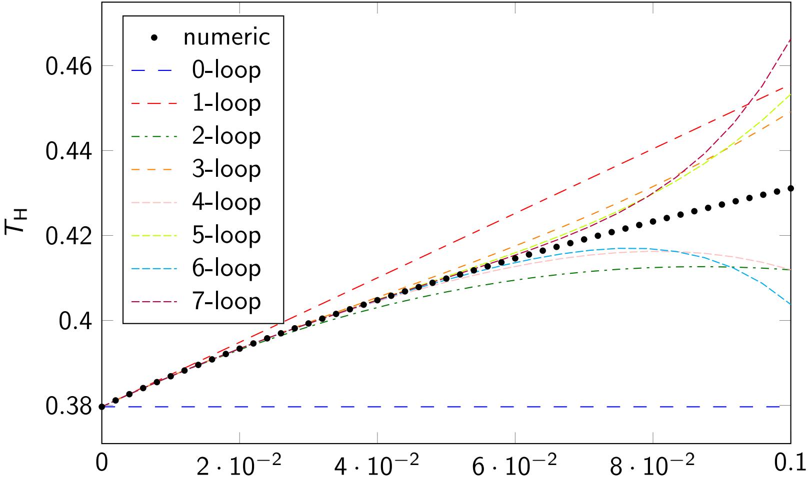
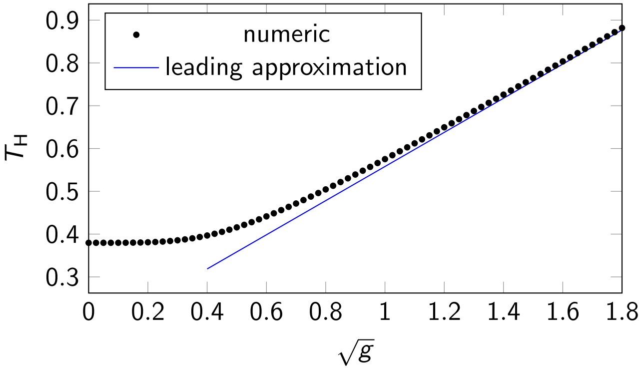


Numeric results at large coupling



Strong-coupling behaviour:

$$T_{\mathrm{H}}(g) \simeq \sqrt{\frac{g}{2\pi}} \approx (0.3989422804\ldots)\sqrt{g}$$

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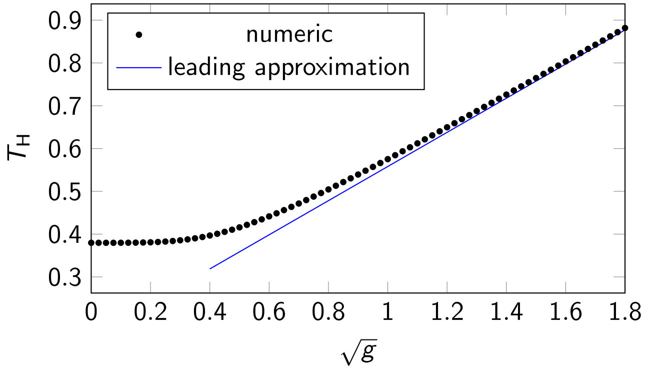

Strong-coupling behaviour:

$$T_{\mathrm{H}}(g) \simeq \sqrt{\frac{g}{2\pi}} \approx (0.3989422804\dots)\sqrt{g}$$

$=$ Hagedorn temperature of tree-level type IIB string theory on ten-dimensional Minkowski space [Sundborg 1984]
(Naive explanation: $\lambda \to \infty \leftrightarrow$ Curvature $\to 0$)

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Chemical potentials

Introduce chemical potentials for the R-charges J_1, J_2, J_3 and angular momenta S_1, S_2:

$$\mathcal{Z}(T,\Omega_i) = \operatorname{tr}\left(e^{-\beta D+\beta\sum_{i=1}^{3}\Omega_i J_i+\beta\sum_{a=1}^{2}\Omega_{a+3}S_a}\right), \qquad \beta = 1/T$$

- Allow to single out contributions from specific fields
- Previously studied at tree level and one-loop [Yamada, Yaffe (2006)], [Harmark, Orselli (2006)], [Suzuki (2017)]

$\Rightarrow$ Include chemical potentials in the QSC approach

Chemical potentials in the QSC

Asymptotics (similar to twisted spectral problem [Kazakov, Leurent, Volin (2015)]):

$$\mathbf{P}_a \simeq \mathrm{A}_a \mathrm{x}_a^{+iu} u^{+\sum_{b<a}\delta_{\mathrm{x}_a\mathrm{x}_b} - \sum_{i<a}\delta_{\mathrm{x}_a\mathrm{y}_i}}$$
$$\mathbf{P}^a \simeq \mathrm{A}^a \mathrm{x}_a^{-iu} u^{+\sum_{b>a}\delta_{\mathrm{x}_a\mathrm{x}_b} - \sum_{i>ca}\delta_{\mathrm{x}_a\mathrm{y}_i}}$$
$$\mathbf{Q}_i \simeq \mathrm{B}_i \mathrm{y}_i^{-iu} u^{-\sum_{a<i}\delta_{\mathrm{x}_a\mathrm{y}_i} + \sum_{j<i}\delta_{\mathrm{y}_i\mathrm{y}_j}}$$
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with

$$\mathrm{x}_1 = -\,\mathrm{e}^{\frac{-1+\Omega_4+\Omega_5}{2T_\mathrm{H}}} \quad \mathrm{x}_2 = -\,\mathrm{e}^{\frac{-1-\Omega_4-\Omega_5}{2T_\mathrm{H}}} \quad \mathrm{x}_3 = -\,\mathrm{e}^{\frac{1+\Omega_4-\Omega_5}{2T_\mathrm{H}}} \quad \mathrm{x}_4 = -\,\mathrm{e}^{\frac{1-\Omega_4+\Omega_5}{2T_\mathrm{H}}}$$
$$\mathrm{y}_1 = \mathrm{e}^{\frac{\Omega_1+\Omega_2-\Omega_3}{2T_\mathrm{H}}} \quad \mathrm{y}_2 = \mathrm{e}^{\frac{\Omega_1-\Omega_2+\Omega_3}{2T_\mathrm{H}}} \quad \mathrm{y}_3 = \mathrm{e}^{\frac{-\Omega_1+\Omega_2+\Omega_3}{2T_\mathrm{H}}} \quad \mathrm{y}_4 = \mathrm{e}^{\frac{-\Omega_1-\Omega_2-\Omega_3}{2T_\mathrm{H}}}$$

Deformations

γ_i-deformation [Frolov (2005)] $\supset$ β-deformation:

- Tree-level partition function trivially the same as in $\mathcal{N} = 4$ SYM theory
- One-loop partition function depends on γ_i but $T_H^{(1)}$ same as in $\mathcal{N} = 4$ SYM theory [Fokken, MW (2014)]
- Integrability approach shows that T_H is the same as in $\mathcal{N} = 4$ SYM theory at any λ

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- Derived integrability-based QSC equations that determine the Hagedorn temperature of planar $\mathcal{N} = 4$ SYM theory / type IIB string theory on $\mathrm{AdS}_5 \times S^5$ at any value of the 't Hooft coupling
  $\rightarrow$ Non-perturbative understanding of thermal physics
- Perturbative solution at weak coupling
  $\rightarrow$ Previously unknown $\ell = 2, 3, 4, 5, 6, 7$-loop Hagedorn temperature
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- Strong-coupling expansion?
- Further observables: critical exponents?
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