Counting instantons in N=1 theories of class S_k

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Copenhagen

[1512.06079 Coman,EP,Taki,Yagi] [1703.00736 Mitev,EP] [1712.01288 Bourton,EP]

Motivation: N=2 exact results

- ✱ *Seiberg-Witten theory: effective theory in the IR* Classical Integrability
- ✱ *Nekrasov: instanton partition function* Quantum Integrability
- ✱ *Pestun: observables in the UV (path integral on the sphere localizes)*

▷ String/M-/F-theory realizations

- ✱ *Gaiotto: 4D N=2 **class S**: 6D (2,0) on Riemann surface* $C_{g,n}$
- ✱ *AGT: 4D partition functions = 2D CFT correlators*
- ✱ *4D SC Index = 2D correlation function of a TFT*

2D/4D relations

What can we do for N=1 theories?

- ☑ *Superconformal Index* [Romelsberge 2005] [Kinney,Maldacena,Minwalla,Raju 2005]
- ☑ *Intriligator and Seiberg: generalized SW technology*
- ☑ *Witten: IIA/M-theory approach to curves*

- ☐ *No Localization: No Nekrasov! No Pestun!*

What can we do for N=1 theories?

- ☑ *Can construct **conformal** N=1 theories.* [Leigh,Strassler 1995]
- ☑ *AdS/CFT natural route to several examples.* [Kachru,Silverstein 1998] [Lawrence,Nekrasov,Vafa1998]
- ☑ *6D (1,0) on a Riemann Surface.* [Gaiotto,Razamat 2015] [Heckman,Vafa….]

Class S_k (S_Γ):

[Gaiotto,Razamat 2015]

2D/4D
relation

- *Conformal*
- *Obtained by orbifolding N=2 (inheritance)*
- *Labeled by punctured Riemann Surface*
- *Index = 2D correlation function of a TFT*

Plan

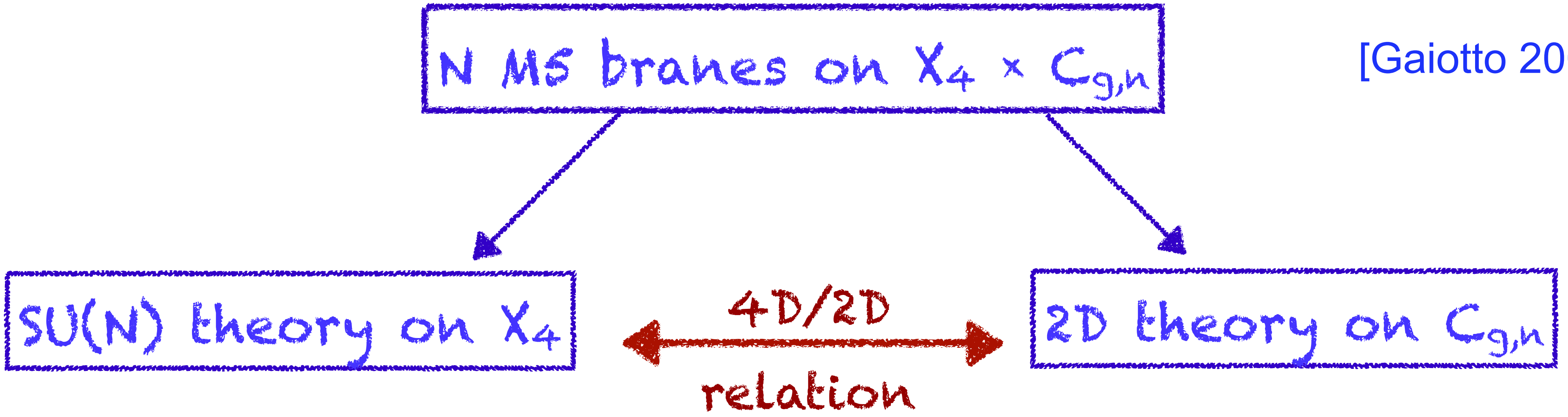
Is there AGT_k?
4D partition functions = 2D CFT correlators

- Introduce N=1 theories in class S_k
- Spectral curves for N=1 theories in class S_k
- From the curves: 2D symmetry algebra and representations
- Conformal Blocks $\longrightarrow$ Instanton partition function
- Instanton partition function from Dp/D(p-4) branes on orbifold

Class S_k

Class S and S_k

[Gaiotto 2009]



6D **(2,0)** SCFT on Riemann surface: 4D N=2 theories of **class S**

Transverse C^2/Z_k Orbifold the 6D (2,0) SCFT to 6D (1,0) SCFT

6D **(1,0)** SCFT on Riemann surface: 4D N=1 theories of **class S_k**

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}
N M5-branes	—	—	—	—	.	.	—	.	.	.	—
A_{k-1} orbifold	.	.	.	.	—	—	.	—	—	.	.

[Gaiotto,Razamat 2015]

Class S_k

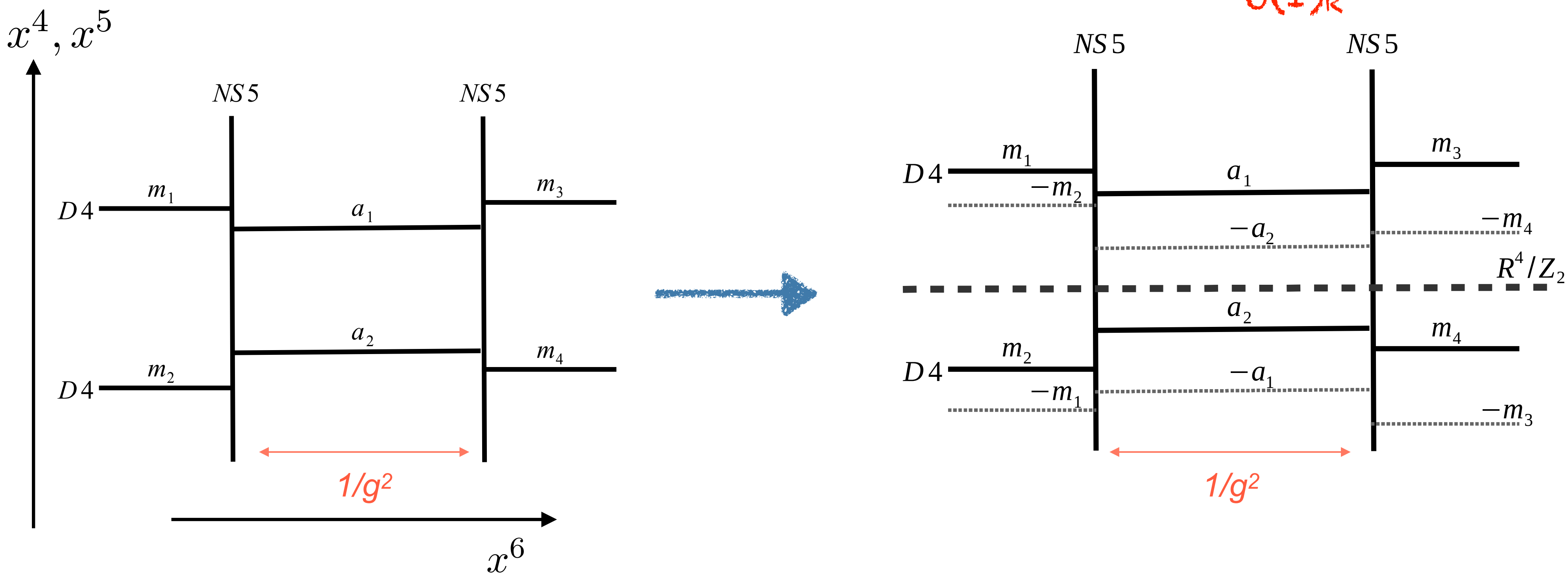
[Gaiotto,Razamat 2015]

Type IIA

$U(1)_r$ ~~$SU(2)_R$~~ of $N=2$

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	(x^{10})
M NS5 branes	—	—	—	—	—	—	.	.	.	.	.
N D4-branes	—	—	—	—	.	.	—	.	.	.	—
A_{k-1} orbifold	.	.	.	.	—	—	.	—	—	.	.

$U(1)_R$



YM coupling: $1/g^2$

$$\epsilon = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\epsilon = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^6\epsilon = \underline{\Gamma^4\Gamma^5\Gamma^7\Gamma^8}\epsilon$$

Class S_k

Type IIB

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
A_{M-1} orbifold	.	.	.	.	.	.	—	—	—	—
N D3-branes	—	—	—	—	.	.	.	.	.	.
A_{k-1} orbifold	.	.	.	.	—	—	.	—	—	.

- *N=1 orbifold daughter of N=4 SYM* $\Gamma = \mathbb{Z}_k \times \mathbb{Z}_M$
- *Useful for AdS/CFT (**orbifold inheritance**)* $AdS_5 \times \mathbb{S}^5 / \left(\mathbb{Z}_k \times \mathbb{Z}_M\right)$

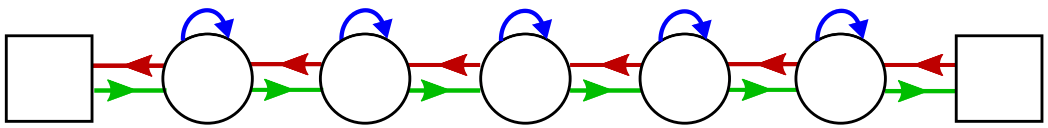
  [Bershadsky, Kakushadze,Vafa 1998]
- *String theory technics to calculate instantons*

  [Dorey, Hollowood, Khoze, Mattis,…] [Lerda,…]

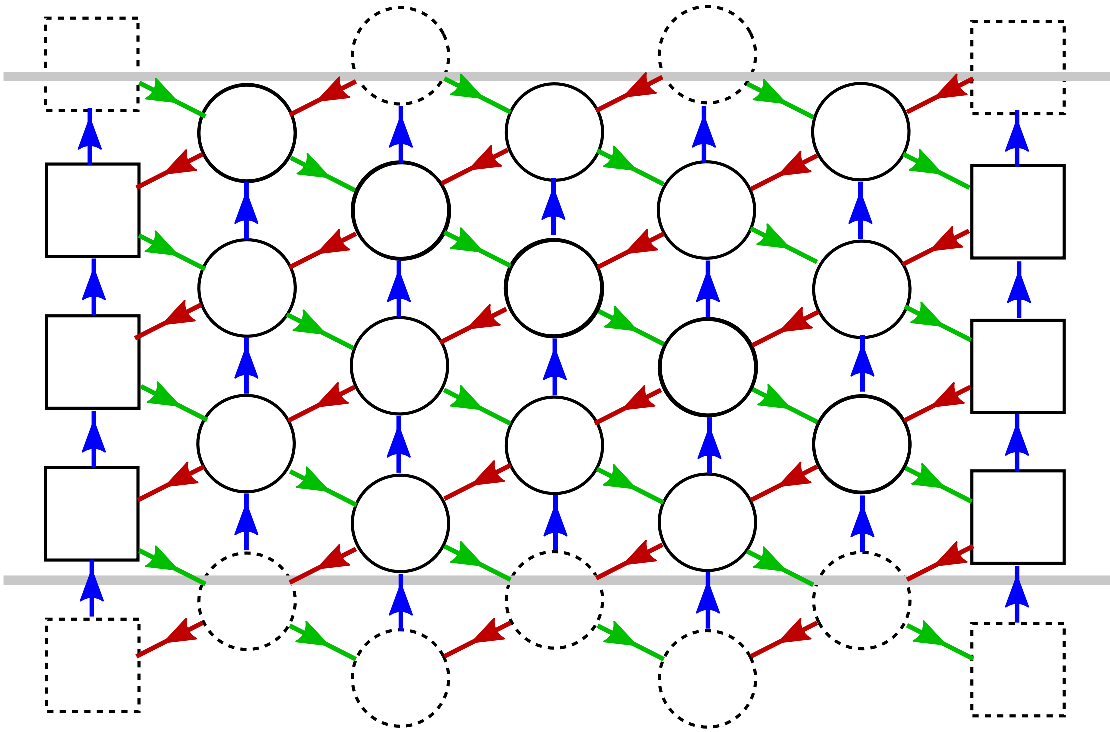
Class S_k

[Gaiotto,Razamat 2015]

4D field theory point of view



N=2 class S mother theory



N=1 class S_k daughter theory

	$U(1)_t$	$U(1)_{\alpha_c}$	$U(1)_{\beta_{i+1-c}}$	$U(1)_{\gamma_i}$
$V_{(i,c)}$	0	0	0	0
$\Phi_{(i,c)}$	-1	0	-1	$+1$
$Q_{(i,c-1)}$	$+1/2$	-1	$+1$	0
$\widetilde{Q}_{(i,c-1)}$	$+1/2$	$+1$	0	-1

Large global symmetry group

Class S_k

Begin with N=2 class S with SU(kN) factors:

$$W_{\mathcal{S}} = \sum_{c=1}^{M-1} \left(Q_{(c-1)} \Phi_{(c)} \tilde{Q}_{(c-1)} - \tilde{Q}_{(c)} \Phi_{(c)} Q_{(c)} \right)$$

N=2 class S mother theory

<u>*Orbifold projection:*</u> [Douglas,Moore 1996]

$$\underset{kN\times kN}{\Phi_{(c)}} = \begin{pmatrix} & \Phi_{(1,c)} & & & \\ & & \Phi_{(2,c)} & & \\ & & & \ddots & \\ & & & & \Phi_{(k-1,c)} \\ \underset{N\times N}{\Phi_{(k,c)}} & & & & \end{pmatrix}$$

$$Q_{(c)} = \begin{pmatrix} Q_{(1,c)} & & & \\ & Q_{(2,c)} & & \\ & & \ddots & \\ & & & Q_{(k,c)} \end{pmatrix}$$

$$\tilde{Q}_{(c)} = \begin{pmatrix} & & & \tilde{Q}_{(k,c)} \\ \tilde{Q}_{(1,c)} & & & \\ & \ddots & & \\ & & \tilde{Q}_{(k-1,c)} & \end{pmatrix}$$

$$W_{\mathcal{S}_k} = \sum_{i=1}^{k} \sum_{c=1}^{M-1} \left(Q_{(i,c-1)} \Phi_{(i,c)} \tilde{Q}_{(i,c-1)} - \tilde{Q}_{(i,c)} \Phi_{(i,c)} Q_{(i+1,c)} \right)$$

N=1 class S_k daughter theory

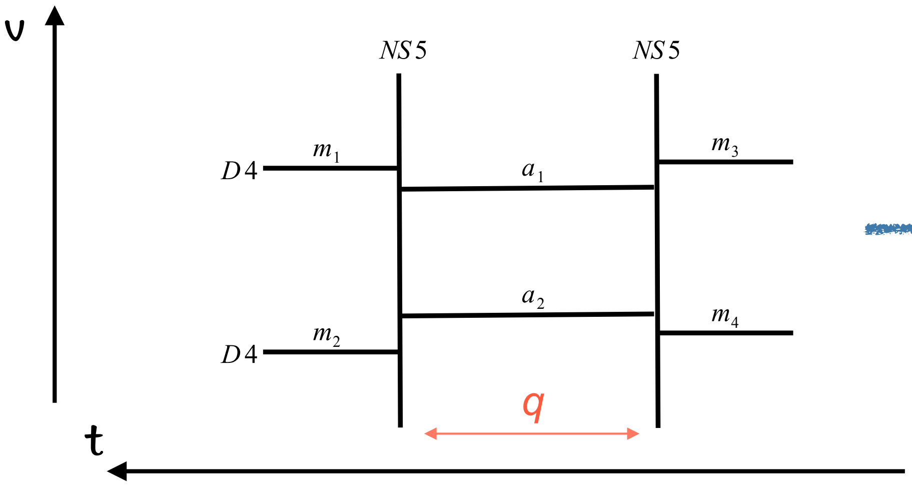
Curves

[1512.06079 Coman,EP,Taki,Yagi]

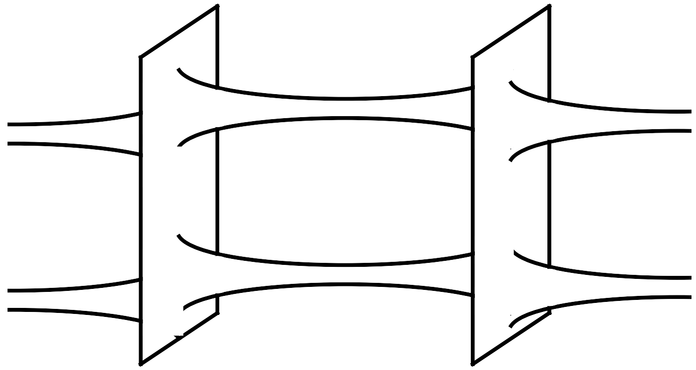
Curves from M-theory

[Witten 1997]

$v = x^4 + ix^5$



$t = e^{-\frac{x^6+ix^{10}}{R_{10}}}$



2D surface F(t,v)=0 in the 4D space $\{x^4, x^5, x^6, x^{10}\}=\{v,t\}$.

The NS5/D4 is the classical configuration.

Take in account tension of the branes: include quantum effects.

M-theory: a single M5 brane with non trivial topology

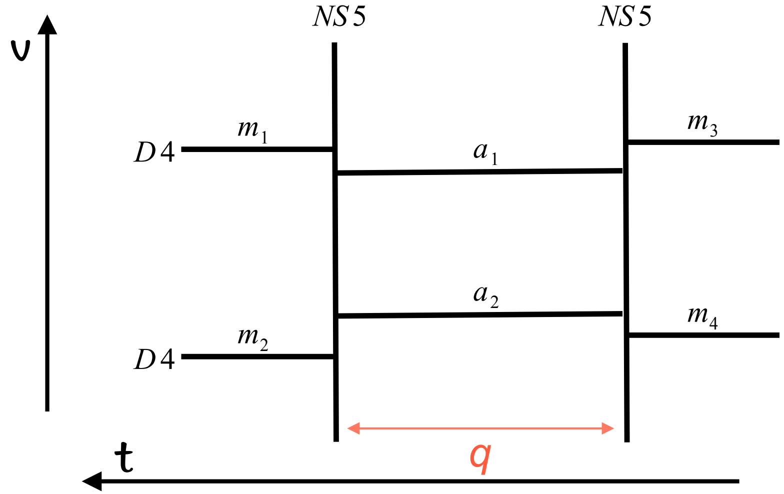
$$(t-1)(t-q)v^2 - P_1(t)v + P_2(t) = 0$$

$$(v-m_1)(v-m_2)t^2 + \left(-(1+\underline{q})v^2 + q\underline{M}v + \underline{u}\right)t + q(v-m_3)(v-m_4) = 0$$

coupling constant $q = e^{2\pi i \tau_{UV}}$ $u = \mathrm{tr}\,\phi^2$ $M = m_1 + m_2 + m_3 + m_4$

Class S Curve

SU(2) with 4 flavors



$$(t-1)(t-q)v^2 - P_1(t)v + P_2(t) = 0$$

$$(v-m_1)(v-m_2)t^2 + \left(-(1+q)v^2 + qMv + u\right)t + q(v-m_3)(v-m_4) = 0$$

$q = e^{2\pi i \tau_{UV}}$ $\quad$ $u = \mathrm{tr}\,\phi^2$ $\quad$ $M = m_1+m_2+m_3+m_4$

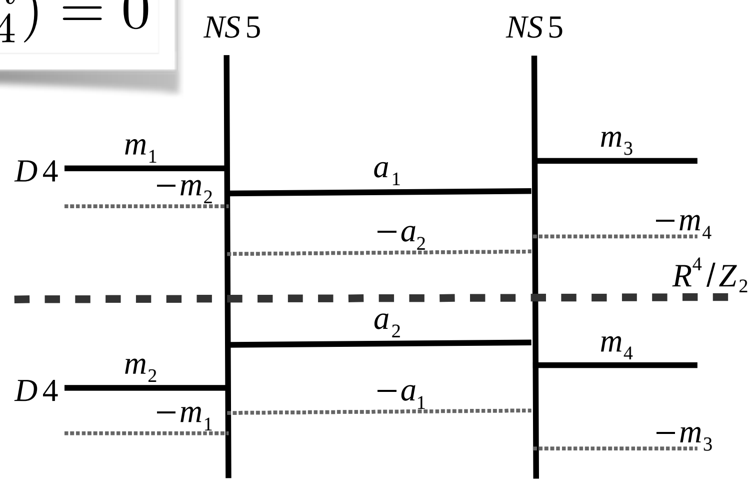
Class S_k Curve

$$v \sim e^{\frac{2\pi i}{k}} v$$

$$\boxed{(v^k - m_1^k)(v^k - m_2^k)t^2 + P(v)t + q(v^k - m_3^k)(v^k - m_4^k) = 0}$$

$$P(v) = -(1+q)v^{2k} + u_k v^k + u_{2k}$$

vevs of gauge invariant operators:
parameterize the Coulomb branch



$$\boxed{\langle \mathrm{tr}\left(\Phi_{(1)} \cdots \Phi_{(k)}\right)\rangle \sim u_k}$$

$$\boxed{\langle \mathrm{tr}\left(\Phi_{(1)} \cdots \Phi_{(k)}\right)^2\rangle \sim u_{2k}}$$

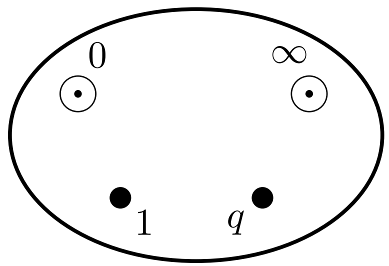
S_k curves

SW or IR curve Σ of $g=kN-1$ → Gaiotto or UV curve $C_{0,n}$ a sphere with n punctures

$$x^{kN} = -\sum_{\ell=1}^{N} \phi^{(4)}_{k\ell}(t)\, x^{k(N-\ell)}$$

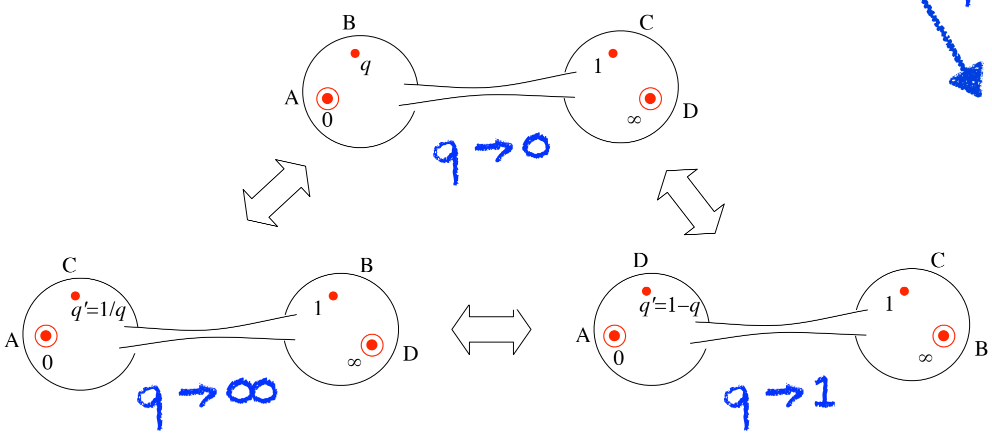
$$\mathfrak{c}^{(s,k)} = \sum_{i_1<\cdots<i_s=1}^{N} m^k_{i_1}\cdots m^k_{i_s}$$

$$\phi^{(4)}_{k\ell}(t) = \frac{(-1)^\ell\, \mathfrak{c}^{(\ell,k)}_L\, t^2 + u_{k\ell}\, t + (-1)^\ell\, \mathfrak{c}^{(\ell,k)}_R\, q}{t^{k\ell}(t-1)(t-q)}$$

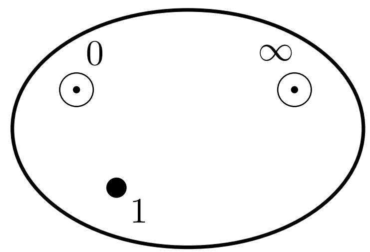


$\mathcal{N}=1$ SU(N) SCQCD$_k$

q → 0



$$\phi^{(3)}_{k\ell}(t) = (-1)^\ell\, \frac{\mathfrak{c}^{(\ell,k)}_L\, t - \mathfrak{c}^{(\ell,k)}_R}{t^{k\ell}(t-1)}$$



Free Trinion $\mathcal{C}^{(k)}_{0,3}$

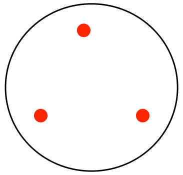
Instantons from the 2D Blocks

$$\boxed{\mathcal{Z}_{\text{inst}} = \mathcal{B}_{\mathbf{w}}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4 | q)}$$

[1703.00736 Mitev,EP]

The AGT relation from the curve

Example:
N=2 SU(2) Free trinion



Close to the punctures:

$$\boxed{\phi_2^{(3)}(z) \sim \frac{m_j^2}{(z-z_j)^2}}$$

Recall 2D Ward Identities:

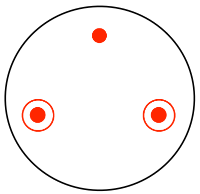
$$\langle T(z) \mathsf{V}_1(z_1)\mathsf{V}_2(z_2)\mathsf{V}_3(z_3)\rangle = \sum_{j=1}^{3}\left[\frac{h_j}{(z-z_j)^2}+\frac{\partial_j}{z-z_j}\right]\langle \mathsf{V}_1(z_1)\mathsf{V}_2(z_2)\mathsf{V}_3(z_3)\rangle$$

$$\boxed{h_i = -m_i^2}$$

$$\phi_2^{(3)}(z) = \frac{\langle\, T(z)\mathsf{V}_1(z_1)\mathsf{V}_2(z_2)\mathsf{V}_3(z_3)\,\rangle}{\langle\, \mathsf{V}_1(z_1)\mathsf{V}_2(z_2)\mathsf{V}_3(z_3)\,\rangle}$$

Free trinions curves are equivalent to Ward identities!

N=2 SU(N)
Free trinion



$$\boxed{\phi_\ell^{(3)}(z) = \frac{\langle\, W_\ell(z)\mathsf{V}_\odot(z_1)\mathsf{V}_\bullet(z_2)\mathsf{V}_\odot(z_3)\,\rangle}{\langle\, \mathsf{V}_\odot(z_1)\mathsf{V}_\bullet(z_2)\mathsf{V}_\odot(z_3)\,\rangle}}$$

From the curves to the 2D CFT

$$\lim_{\epsilon_{1,2}\to 0} \langle\langle\, J_\ell(t)\,\rangle\rangle_n = \phi_\ell^{(n)}(t)$$

$$\langle\langle\, J(t)\,\rangle\rangle_n \stackrel{\text{def}}{=} \frac{n\text{-point W-block with insertion of } J(t)}{n\text{-point W-block}}$$

- *The symmetry algebra that underlies the 2D CFT = W_{kN} algebra*
- *The reps are **standard** reps of the W_{kN} algebra*
- *Obtain them from the N=2 SU(kN) after replacing:*

$$m_{j+Ns}^{\mathrm{SU}(Nk)} \longmapsto m_j\, \mathrm{e}^{\frac{2\pi i}{k}s} \qquad a_{j+Ns}^{\mathrm{SU}(Nk)} \longmapsto a_j\, \mathrm{e}^{\frac{2\pi i}{k}s}$$

[1703.00736 Mitev,EP]

2D Conformal Blocks = Instanton P.F.

- *We have the reps of the W_{kN} algebra for $\varepsilon_{1,2} = 0$ (from the curve)*
- *Demand: the structure of the multiplet (null states) not change $\varepsilon_{1,2} \neq 0$*
- *The blocks for $\varepsilon_{1,2} \neq 0$: proposal for the instanton partition functions:*

$$\boxed{\mathcal{Z}_{\text{inst}} = \mathcal{B}_{\mathbf{w}}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4 | q)}$$

- *If w and c turn on $Q \neq 0$ as in Liouville/Toda,*

  then we obtain them from the N=2 SU(kN) after replacing:

$$\boxed{m_{j+Ns}^{\mathrm{SU}(Nk)} \longmapsto m_j \, \mathrm{e}^{\frac{2\pi i}{k} s} \qquad a_{j+Ns}^{\mathrm{SU}(Nk)} \longmapsto a_j \, \mathrm{e}^{\frac{2\pi i}{k} s}}$$

[1703.00736 Mitev,EP]

Instantons from D(p-4) branes

[1712.01288 Bourton, EP]

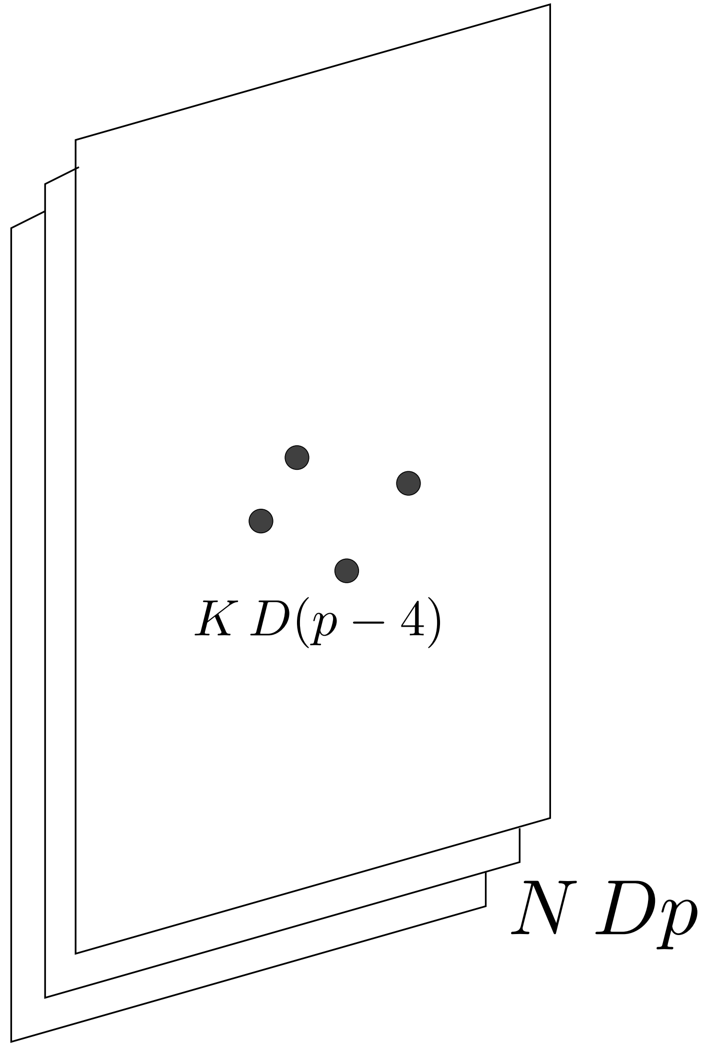
Instantons from D branes

$$\mathrm{Tr} \int_{Dp} d^{p+1}x \;\; C_{p-3} \wedge F \wedge F$$

Instanton on Dp brane = D(p-4) brane

$$\mu^{\mathbf{C}} := [B_1, B_2] + IJ = 0 \ ,$$

$$\mu^{\mathbf{R}} := [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0$$



The Dp-D(p-4) strings give the NxK I, J† and the D(p-4)-D(p-4) the KxK B₁, B₂ auxiliary matrices of the ADHM construction

$$\boxed{\mathcal{M}^{Dp}_{K\text{-inst}} \simeq \mathcal{M}^{D(p-4)}_{\text{Higgs}}}$$

$$= \left\{B_1, B_2, I, J \,|\, \mu^{\mathbf{C}} = 0, \mu^{\mathbf{R}} = 0\right\} / U(K)$$

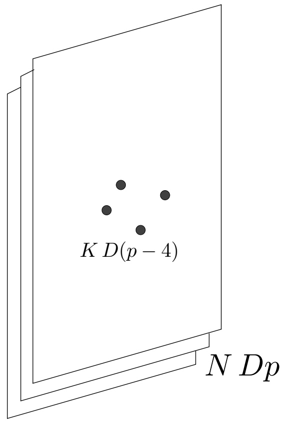
$$= \left\{X^{\perp}_{Dp} = 0, \mathcal{V}_{D(p-4)} = 0\right\} / U(K)$$

[Witten 1995, Douglas 1995, Dorey 1999, ...]

Instanton Partition Function from 2D SCI

$$\mathcal{Z}_{K\text{-inst}}(a,m,\epsilon_1,\epsilon_2) = \mathrm{Tr}_{\mathcal{M}_{K\text{-inst}}} \underline{(-1)^F e^{\epsilon_1 J_L} e^{\epsilon_2 J_R} e^{\xi R} e^{a J_G} e^{m J_F}}$$

Nekrasov's Instantons: deformed Witten index



$$\mathcal{M}^{Dp}_{K\text{-inst}} \simeq \mathcal{M}^{D(p-4)}_{\text{Higgs}}$$

For p=5 and when the D1s wrapping a T^2:

$$\boxed{\begin{array}{c}\text{Instanton's index} = \\ \text{2D SCI} = \text{flavoured elliptic genius}\end{array}}$$

of the 2D gauge theory living on the D1s

2D R-symmetry $J_{69}-J_{78}$

J_{01} , J_{23}

$$\mathcal{Z}^{\text{D5}}_{K\text{-inst}}(a,m,\epsilon_1,\epsilon_2) = \mathcal{Z}^{K\ \text{D1}}_{\text{Higgs}}(a,m,\epsilon_1,\epsilon_2) = \mathcal{I}_{2D} = \mathrm{Tr}_{\mathcal{M}^{K\ D1}_{\text{Higgs}}}(-1)^F e^{\epsilon_1 J_L} e^{\epsilon_2 J_R} e^{\xi R} e^{a J_G} e^{m J_F}$$

$J_{69}+J_{78}$ 2D Flavor

2D SCI is very well studied, very easy to compute!

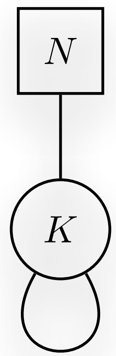
[Gadde, Gukov, Putrov]
[Benini,Eager,Hori,Tachikawa]

Mass deformed N=4 SYM

First we practice without the orbifolds:

- *Obtain the* $\mathcal{Z}_{\rm inst}$ *for mass deformed N=4 SYM (N=2*)*

  from a ***2D SCI computation:*** 2D theory with (4,4) susy



$$\boxed{D3/D(-1) \xleftrightarrow{\text{T-duality}} D5/D1}$$

- *The 2D SCI of the gauge theory on the D1 branes*

  = *instantons of the 6D (2,0) on* $\mathbb{R}^4 \times T^2$ = *M-strings.*

- *KK reducing: instantons of mass deformed 5D N=2 MSYM on* $\mathbb{S}^1$

  and further KK reduce to mass deformed N=4 SYM in 4D.

Instantons with an orbifold

- *"**Orbifold**" the 2D SCI of mass deformed N=4 SYM with one, say the Z_M orbifold: we get M-strings on a transverse orbifold*

  = instantons of an $SU(N)^M$ quiver when reduce down to 5D/4D

  2D theory with (0,4) susy

- *Further "**Orbifold**" the 2D SCI with the Z_k orbifold we get something new that should correspond to instantons of class Sk, the rational, trigonometric and elliptic uplift.*

  2D theory with (0,2) susy

S_k Instantons

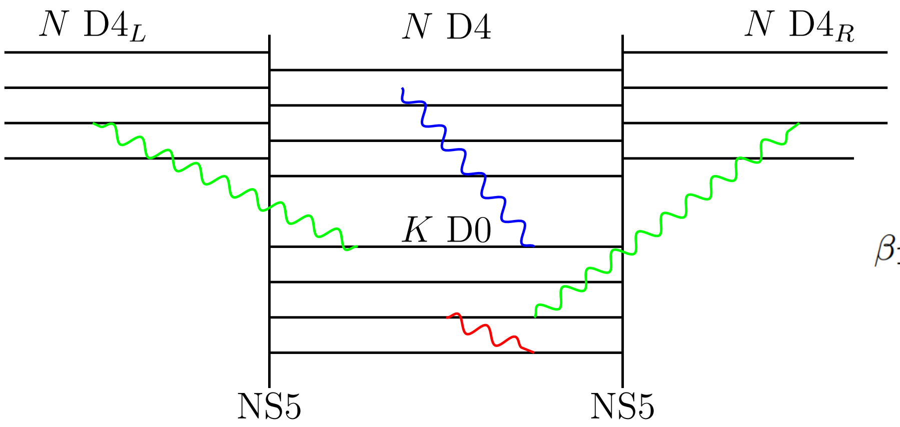
$$\mathcal{Z}^{\mathrm{D5}}_{K\text{-inst}}(a,m,\epsilon_1,\epsilon_2) \;=\; \mathcal{Z}^{K\ \mathrm{D1}}_{\mathrm{Higgs}}(a,m,\epsilon_1,\epsilon_2) = \mathcal{I}_{2D} \;=\; \mathrm{Tr}_{\mathcal{M}^{K\ D1}_{\mathrm{Higgs}}}(-1)^F e^{\epsilon_1 J_L} e^{\epsilon_2 J_R} e^{\xi R} e^{a J_G} e^{m J_F}$$

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
N D5	$-$	$-$	$-$	$-$	$-$	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\mathbb{Z}_M$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\times$	$\times$	$\times$	$\times$
$\mathbb{Z}_k$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\times$	$\times$	$\cdot$	$\times$	$\times$	$\cdot$
K D1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-$	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

$$\begin{array}{ccccccc}
 & & \mathbb{R}^4\times T^2 & & \mathbb{R}^4\times\mathbb{S}^1 & & \mathbb{R}^4 \\
\mathcal{I}_{D1} & = & \mathcal{Z}^{(2,0)}_{\mathrm{inst}} & \xrightarrow{\mathrm{KK}} & \mathcal{Z}^{\mathcal{N}=2}_{\mathrm{inst}} & \xrightarrow{\mathrm{KK}} & \mathcal{Z}^{\mathcal{N}=4}_{\mathrm{inst}} \\
\mathcal{I}^{\mathbb{Z}_M}_{D1} & = & \mathcal{Z}^{(1,0)}_{\mathrm{inst}} & \xrightarrow{\mathrm{KK}} & \mathcal{Z}^{\mathcal{N}=1}_{\mathrm{inst}} & \xrightarrow{\mathrm{KK}} & \mathcal{Z}^{\mathcal{N}=2}_{\mathrm{inst}} \\
\mathcal{I}^{\mathbb{Z}_M\times\mathbb{Z}_k}_{D1} & = & \mathcal{Z}^{(1,0)\mathrm{w/defect}}_{\mathrm{inst}} & \xrightarrow{\mathrm{KK}} & \mathcal{Z}^{\mathcal{N}=1\mathrm{w/defect}}_{\mathrm{inst}} & \xrightarrow{\mathrm{KK}} & \boxed{\mathcal{Z}^{\mathcal{N}=1}_{\mathrm{inst}}}
\end{array}$$

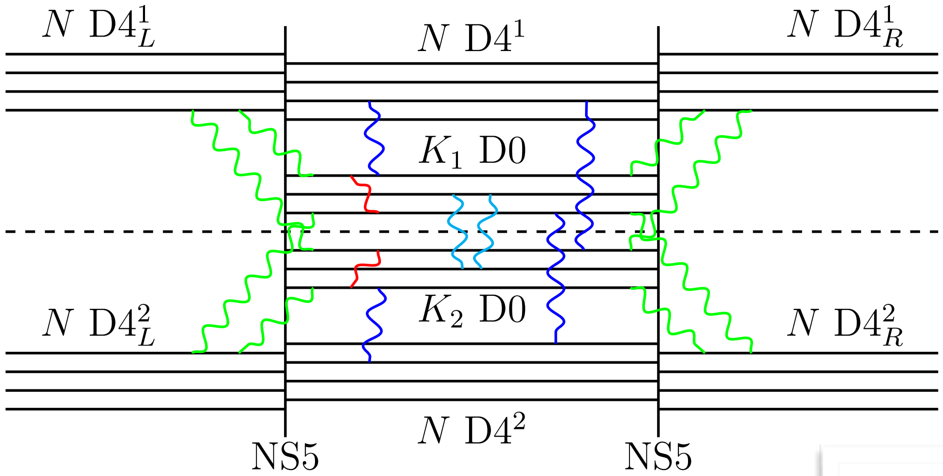
✱ *The same answer as in* [1703.00736 Mitev,EP] *from conformal blocks!*

Instantons from D0 branes



$$\lim_{\beta_1,\beta_2\to 0} I_{\mathrm{SCI}} = \frac{1}{K!}\int \prod_{I=1}^{K} du_I \prod_{I,J=1}^{K} \frac{u'_{IJ}\left(u_{IJ}-2\epsilon_+\right)}{\left(u_{IJ}+\epsilon_1\right)\left(u_{IJ}+\epsilon_2\right)}$$

$$\times \prod_{I=1}^{K}\prod_{A=1}^{N} \frac{\left(u_I-m_{L,A}\right)\left(u_I-m_{R,A}\right)}{\left(u_I-a_A-\epsilon_+\right)\left(u_I-a_A+\epsilon_+\right)}$$



$$\lim_{\beta_1,\beta_2\to 0} I^{\mathrm{orb}}_{\mathrm{SCI}} \propto \prod_{i=1}^{k}\int \prod_{I=1}^{K_i} du_{i,I} \prod_{I=1}^{K_i} \frac{u'_{ii,IJ}\prod_{j\neq i}\prod_{J=1}^{K_j}\left(u_{ij,IJ}-2\epsilon_+\right)}{\prod_{j=1}^{k}\prod_{J=1}^{K_j}\left(u_{ij,IJ}+\epsilon_1\right)\left(u_{ij,IJ}+\epsilon_2\right)}$$

$$\times \prod_{j=1}^{k}\prod_{I=1}^{K_i}\prod_{A=1}^{N} \frac{\left(u_{i,I}-\widetilde{m}_{L,j,A}\right)\left(u_{i,I}-\widetilde{m}_{R,j,A}\right)}{\left(u_{i,I}-\widetilde{a}_{j,A}-\epsilon_+\right)\left(u_{i,I}-\widetilde{a}_{j,A}+\epsilon_+\right)}$$

Summary

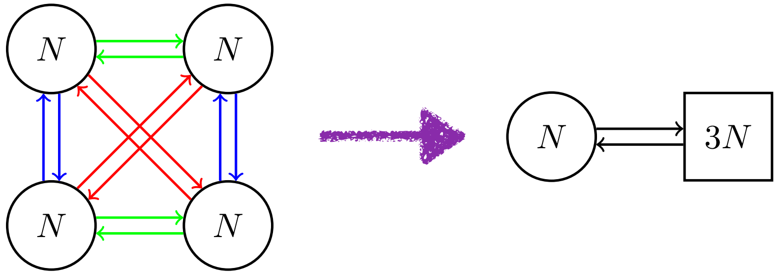
- ☑ *We constructed spectral **curves** for N=1 theories in class S_k*
- ☑ *The curves: 2D symmetry algebra (W_{kN}) and representations*
- ☑ *Conformal Blocks $\longrightarrow$ Instanton partition function*
- ☑ *Instanton partition function from Dp/D(p-4) on orbifold*

Future

- *Compute one, two instantons with standard QFT techniques*

- *Go away from the orbifold point*

  [to appear Bourton, EP]



- *Other N=1 theories*

- *The perturbative part? N=1 partition function on S^4!*

- *Get the AGT$_k$ from (1,0) 6D à la Cordova and Jafferis*

Thank you!