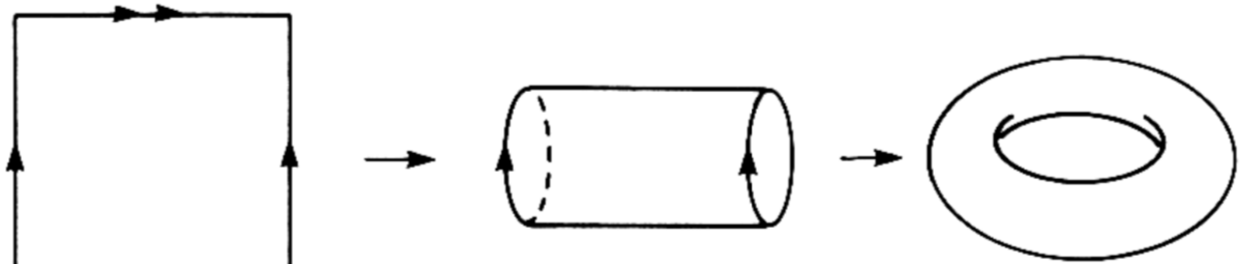
Non-planar Scattering Amplitudes

Amit Sever, Tel-Aviv University & CERN

With Roy Ben Israel & Alexander Tumanov 1802.09395

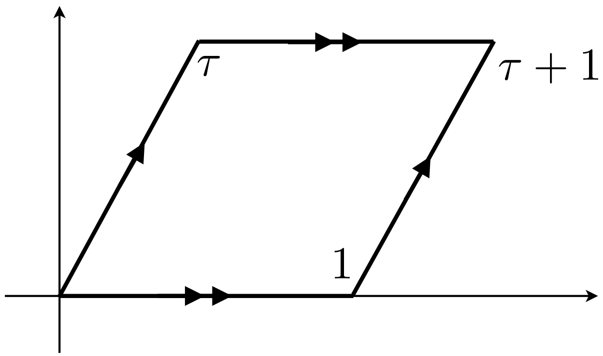
Overview and plan

Main idea — To go from planar to non-planar, we **glue planar objects** together to form closed cycles



Sum over the string Hilbert space

Higher genus surfaces have moduli space that we integrate/sum over



$\Rightarrow$ Gluing has to be done separately at any point on the moduli space

We will do this for **scattering amplitudes** in **momentum space**

Overview and plan

Application — Non planar duality with Wilson loops

- In a toy model (fishnet)
- In N=4 SYM theory
- Checks
  - Dual string in AdS
  - One loop

What it is good for?

- Restore dual conformal symmetry
- Integrability based pentagon OPE
- Non planar loop integrand

Extensions — OPE for form factors

Local twist operators (see Kolya's talk)

Review — 't Hooft expansion of scattering amplitudes

Scattering amplitudes of particles in the adjoint representation of SU(N_c)

$$A_n(\{k_i, T_i, \epsilon_i\}) = \sum_{\sigma_n} \mathrm{Tr}\left(T_{\sigma(1)} \cdot \ldots \cdot T_{\sigma(n)}\right) \mathcal{A}_n(\sigma(1), \ldots, \sigma(n))$$

k_i: momenta; T_i: color matrix; ϵ_i: polarization

$\mathcal{A}_n$: single trace partial amplitude

$$+ \sum_{c,a} \mathrm{Tr}\left(T_{a_1} \cdot \ldots \cdot T_{a_c}\right) \mathrm{Tr}\left(T_{a_{c+1}} \cdot \ldots \cdot T_{a_n}\right) \mathcal{A}_{c,n-c}(a_1, \ldots, a_c; a_{c+1}, \ldots, a_n)$$

$\mathcal{A}_{c,n-c}$: double trace partial amplitude

$$+ \ldots$$

't Hooft large N_c limit of N=4 SYM theory $\quad N_c \to \infty\,, \quad \lambda = g_{YM}^2 N_c$ fixed

$$\mathcal{A}_n = g_{YM}^{n-2}\left(f_0(\lambda) + \frac{\lambda}{N^2} f_1(\lambda) + \ldots\right)$$

at tree level all amplitudes are planar

$f_0(\lambda)$: planar partial amplitude

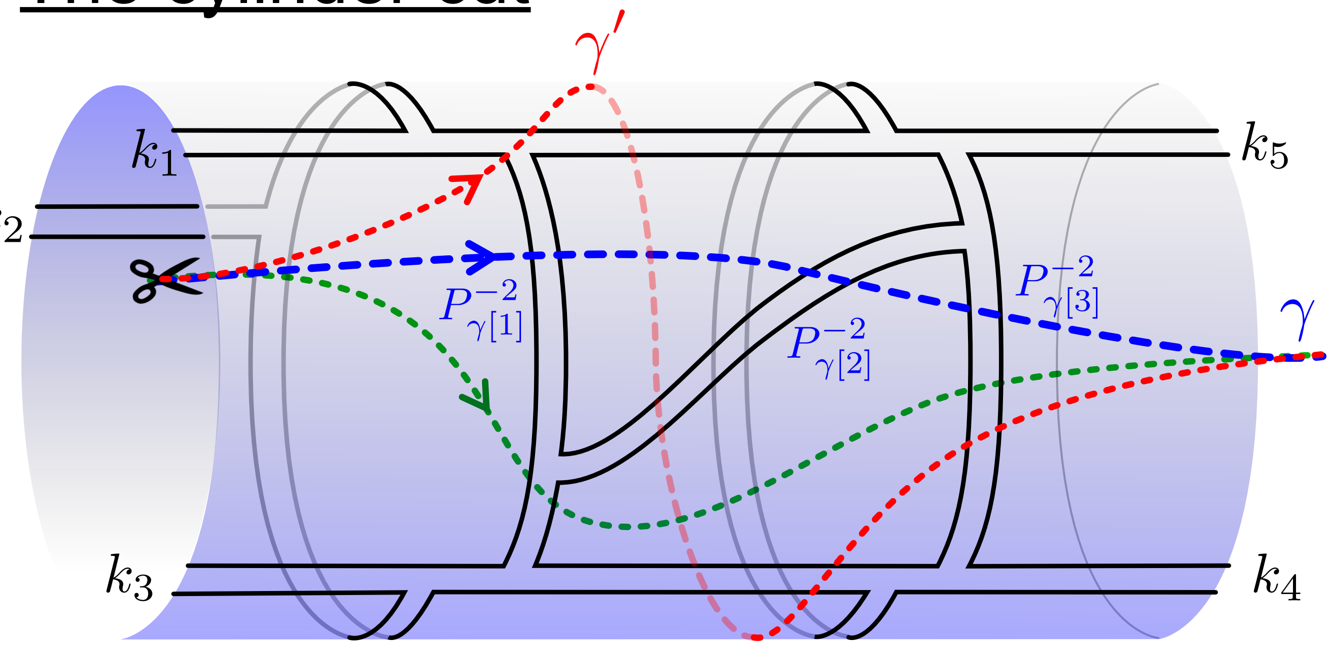
$$\mathcal{A}_{n,m} = g_{YM}^{n-2}\left(\boxed{\frac{\lambda}{N} h_0(\lambda)} + \frac{\lambda^2}{N^3} h_1(\lambda) + \ldots\right)$$

starts at 1-loop

$\frac{\lambda}{N} h_0(\lambda)$: first non-planar correction

$$\vdots$$

The cylinder cut



$q = k_1 + k_2 + k_3$ →

momentum flow through the cylinder

momentum flow around the cylinder

$$\prod_{i=1}^{L} \int d^4 l_i \, G(l_1, \dots, l_L) = \int d^4 l \underbrace{\left[\prod_{i=1}^{L} \int d^4 l_i \, G(l_1, \dots, l_L) \times \delta^4 (l - \sum_j P_{\gamma(j)}) \right]}_{\mathcal{A}^{\gamma}_{n;m}(l)}$$

$$\mathcal{A}^{\gamma'}_{n;m}(l) = \mathcal{A}^{\gamma}_{n;m}(l+q) \quad \Rightarrow \quad \mathbb{A}_{n,m}(l) = \sum_a \mathcal{A}^{\gamma}_{n,m}(l + aq)$$

physical (independent of γ)

$\mathbb{A}_{n,m}(l+q) = \mathbb{A}_{n,m}(l)$

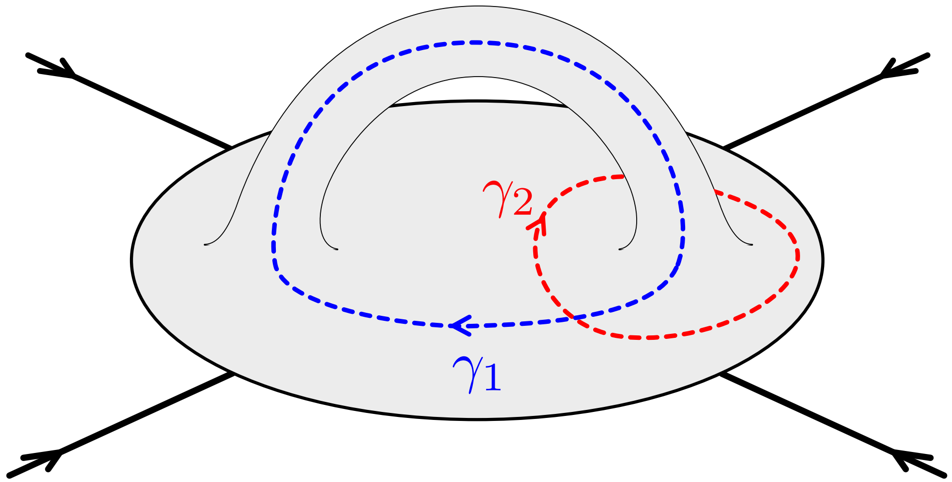
$l \to l + q$ cylinder modular transformation

$$A_{n,m} = \frac{\lambda}{N} \int_{l \simeq l+q} d^4 l \, \mathbb{A}_{n,m}(l)$$

full amplitude

Higher orders in 1/N

$1/N^2$ correction to single trace



$$\delta^4(l_1 - \sum_j P_{\gamma_1(j)}) \qquad \delta^4(l_2 - \sum_j P_{\gamma_2(j)})$$

Now we have two cycles

$\Rightarrow$ two cuts

$$l_1 \simeq l_1 + l_2$$
$$l_2 \simeq l_2 + l_1$$

Dual torus modular parameter $\tau = l_2/l_1$

In 2d complex plane spreed by l_1 and l_2

$$\mathbb{A}(l_1, l_2) = \sum_{g \in SL(2,\mathbb{Z})} \mathcal{A}^{\gamma_1\,\gamma_1}((l_1, l_2) \cdot g)$$

$$\int d^4 l_1\, d^4 l_2 = \int_{orientation} d^5 M \times \int_{size} d^1 \lambda \times \int_{modular} d^2 \tau$$

The cycles cut amplitude is the "*string loop integrand*"

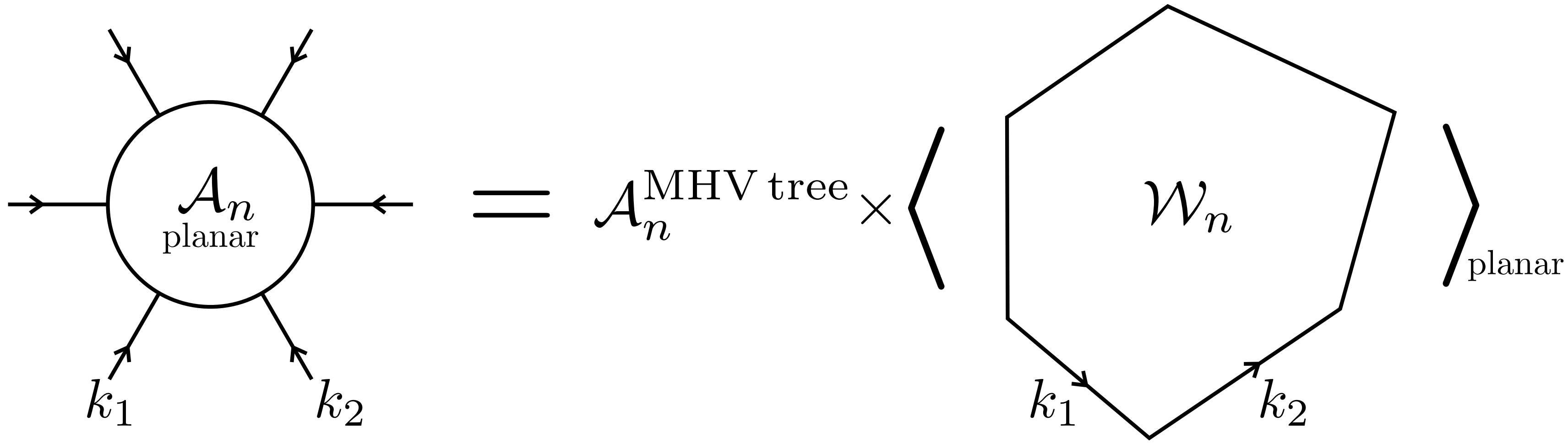
Planar amplitude - Wilson loop duality

In the past 10y — huge progress in the study of planar amplitudes

Important tools

- Dual conformal symmetry ($\Rightarrow$ Yangian)
- All loops integrand
- Non-perturbative operator product expansion

Generated by a strange looking duality



[Alday, Maldacena], [Brandhuber, Heslop, Travaglini], [Drummond, Henn, Korchemsky, Sokatchev], [Beisert, Ricci, Tseytlin, Wolf]

- How does this strange duality come about?

  T-duality in string theory

  In perturbation theory - toy example

  } $\xrightarrow{\text{cylinder cut}}$ Non-planar

Toy example - the fishnet model

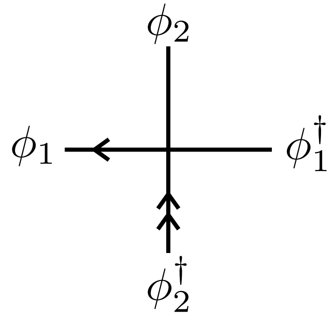
Two complex $N \times N$ scalars

$$\mathcal{L} = \frac{1}{g^2} \mathrm{Tr} \left(-\partial_\mu \phi_1^\dagger \partial^\mu \phi_1 - \partial_\mu \phi_2^\dagger \partial^\mu \phi_2 + \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2 \right)$$

no c.c.

[Gürdoan, Kazakov]

- Planar limit $N \to \infty \, , \quad \lambda = g^2 N$ fixed
- One interaction vertex



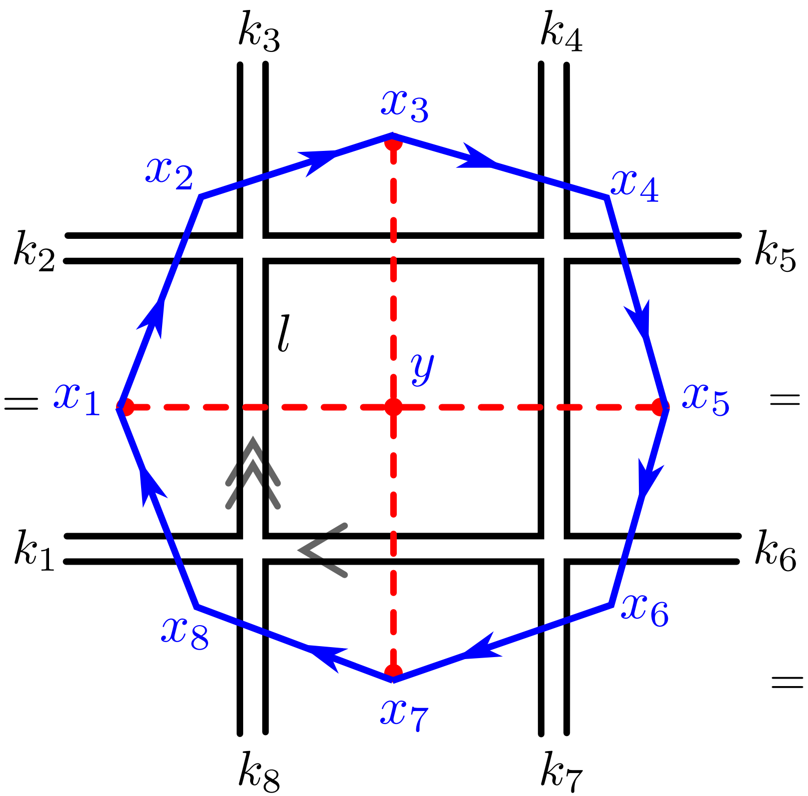
- Non unitary
- Integrable and conformal in the planar limit (add double trace)

[Sieg, Wilhelm], [Grabner, Gromov, Kazakov, Korchemsky]

- Planar amplitudes are fully accounted for by a single diagram

Planar amplitude - Wilson loop duality in the fishnet model

$\mathcal{A}_8^{\text{fishnet}} =$



$$= \int \frac{g^6 \lambda\, \delta^4\left(\sum k_i\right)\, d^4 l}{l^2 (l+k_2+k_3)^2 (l+k_2+k_3+k_4+k_5)^2 (l-k_1-k_8)^2}$$

$$= \int \underbrace{\frac{g^6 \lambda\, \delta^4\left(\sum k_i\right)\, d^4 y}{(y-x_1)^2 (y-x_3)^2 (y-x_5)^2 (y-x_7)^2}}_{\text{Integrand}}$$

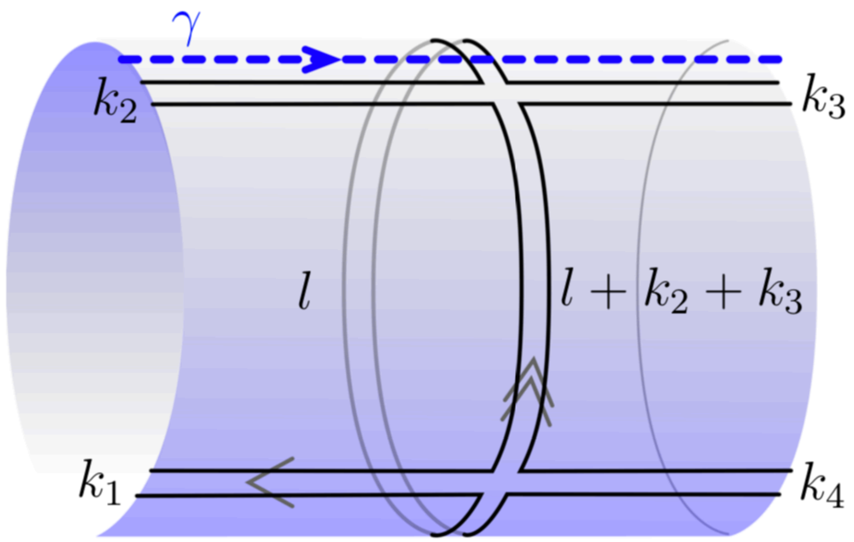
Dual coordinates $\quad x_i - x_{i-1} = k_i \quad$ **and** $\quad y - x_1 = l$

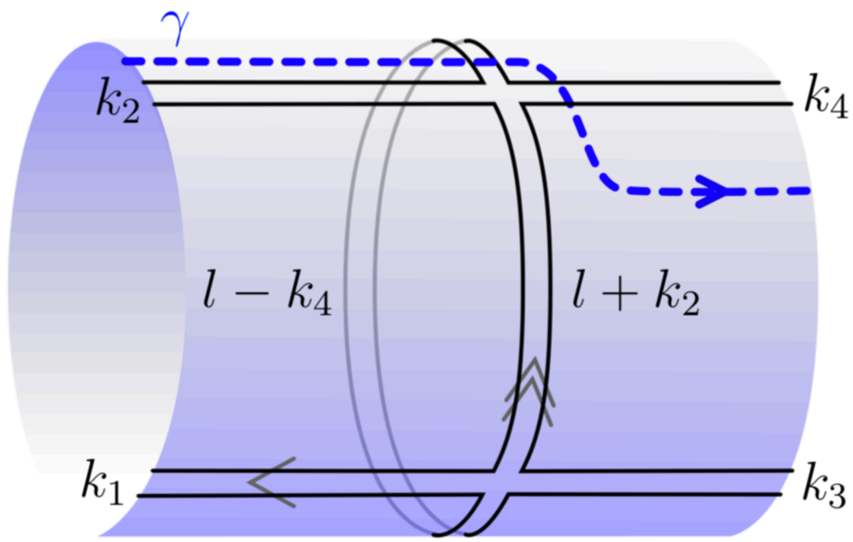
$$\Rightarrow \quad A_8^{\text{fishnet}} \propto \delta^4\left(\sum k_i\right) \times \langle \mathbb{W}_8^{\text{fishnet}} \rangle$$

$$\mathbb{W}_8^{\text{fishnet}} = \frac{1}{N} \mathrm{Tr}\left(\phi_2^\dagger(x_7)\, \phi_1^\dagger(x_5)\, \phi_2(x_3)\, \phi_1(x_1)\right)$$

Double trace amplitude in the fishnet model

Cylinder cut of the 1-loop amplitude



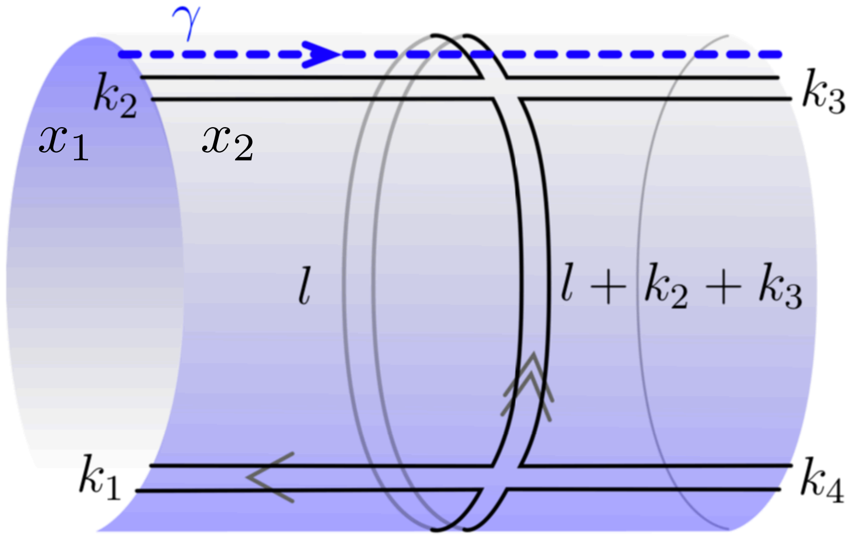


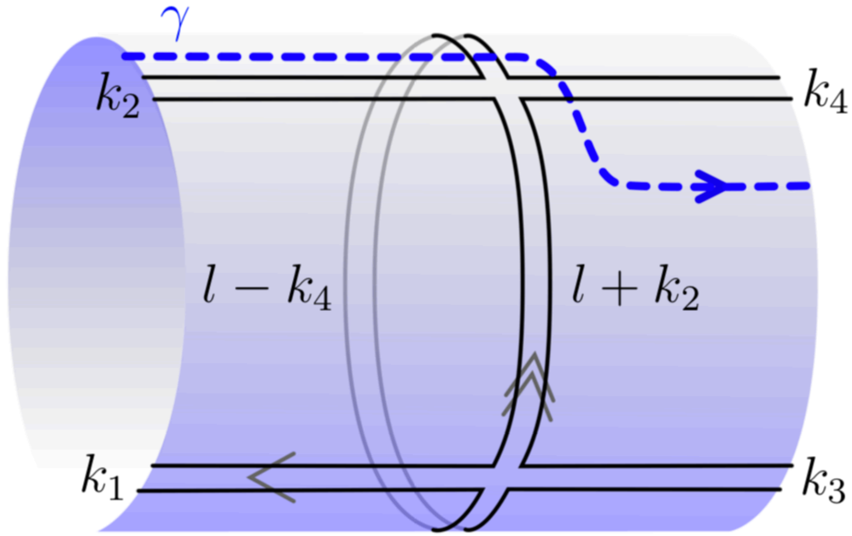
$$\mathcal{A}^{\text{fishnet}}_{2,2\,\text{tree}}(l) = g^2\,\delta^4\Big(\sum k_i\Big)\left(\frac{1}{l^2(l+k_2+k_3)^2} + \frac{1}{(l+k_2)^2(l-k_4)^2}\right),$$

$$\mathbb{A}^{\text{fishnet}}_{2,2}(l) = \sum_{a=-\infty}^{\infty} \mathcal{A}^{\text{fishnet}}_{2,2}(l + a\,q)\,, \qquad A^{\text{fishnet}}_{2,2} = \frac{\lambda}{N}\int_{l\simeq l+q} d^4 l\,\mathbb{A}^{\text{fishnet}}_{2,2}(l)$$

Double trace amplitude in the fishnet model

Cylinder cut of the 1-loop amplitude





We now try to do as as in the planar case $x_2 - x_1 = k_2$

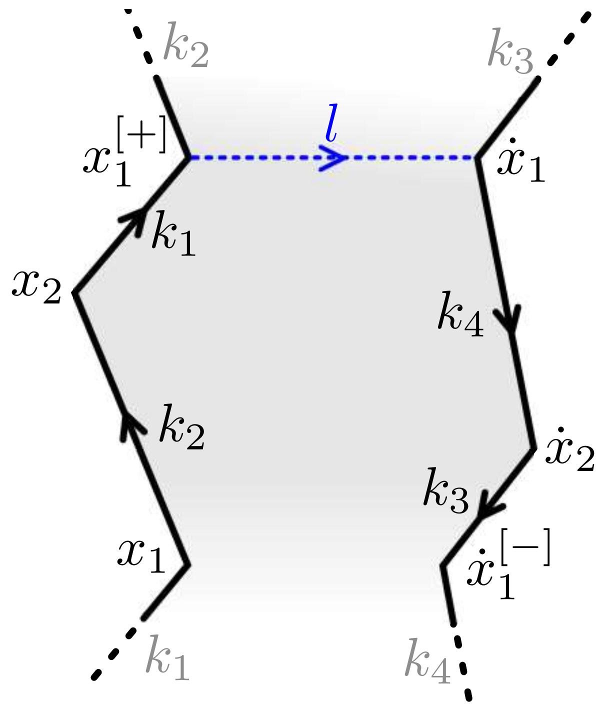
but now $x_1 - x_2 = -k_2 = k_1 - q \neq k_1$

total momentum through the cylinder $q = k_1 + k_2$

Universal cover $x_1^{[+]} = x_1 + q$

$x_1^{[+]} - x_2 = k_1$

Similarly $\dot{x}_2 - \dot{x}_1 = k_4 \qquad \dot{x}_1^{[-]} - \dot{x}_2 = k_3$

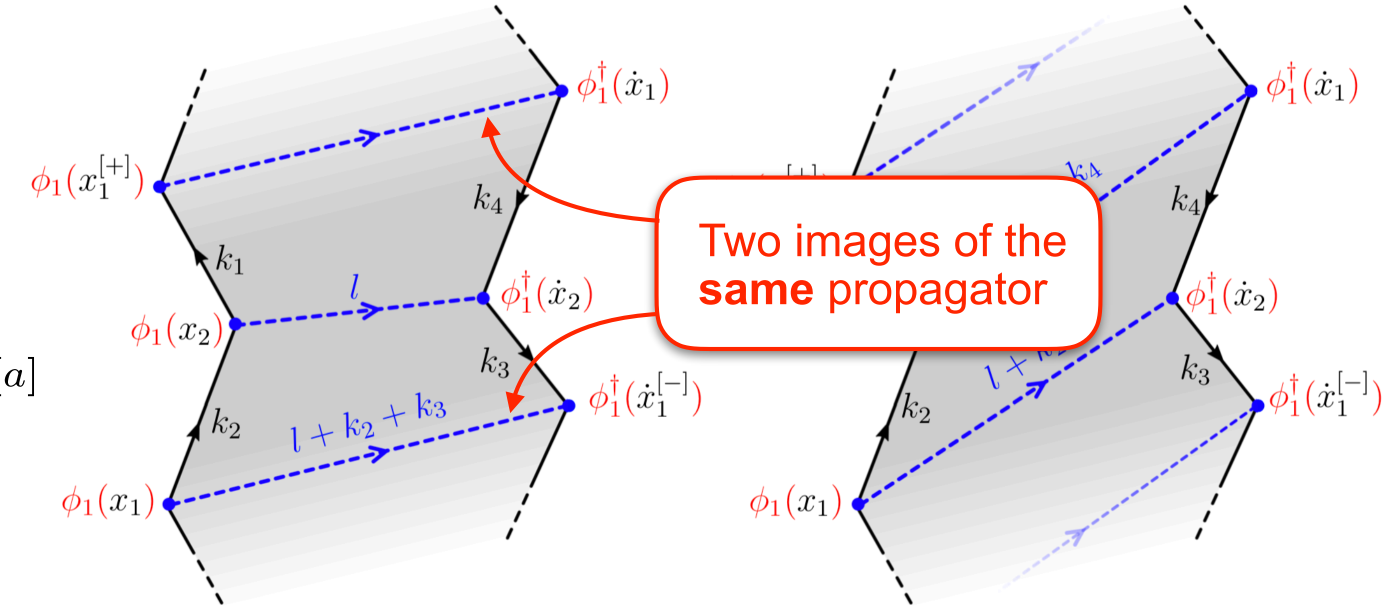


Double trace in the fishnet model - **Wilson loop dual**

$$\mathbb{A}_{2,2}^{\text{fishnet}}(l) \propto \delta^4\left(\sum k_i\right)\,\mathbb{W}_{2,2}^{\text{fishnet}}(l)$$

$$\mathbb{W}_{2,2}^{\text{fishnet}}(l) = \left\langle \text{Tr}\left(\ldots\phi_1(x_1)\phi_1(x_2)\ldots\right)\text{Tr}\left(\ldots\phi_1^\dagger(\dot{x}_1)\phi_1^\dagger(\dot{x}_2)\ldots\right)\right\rangle_{\underline{\text{cylinder}}}$$

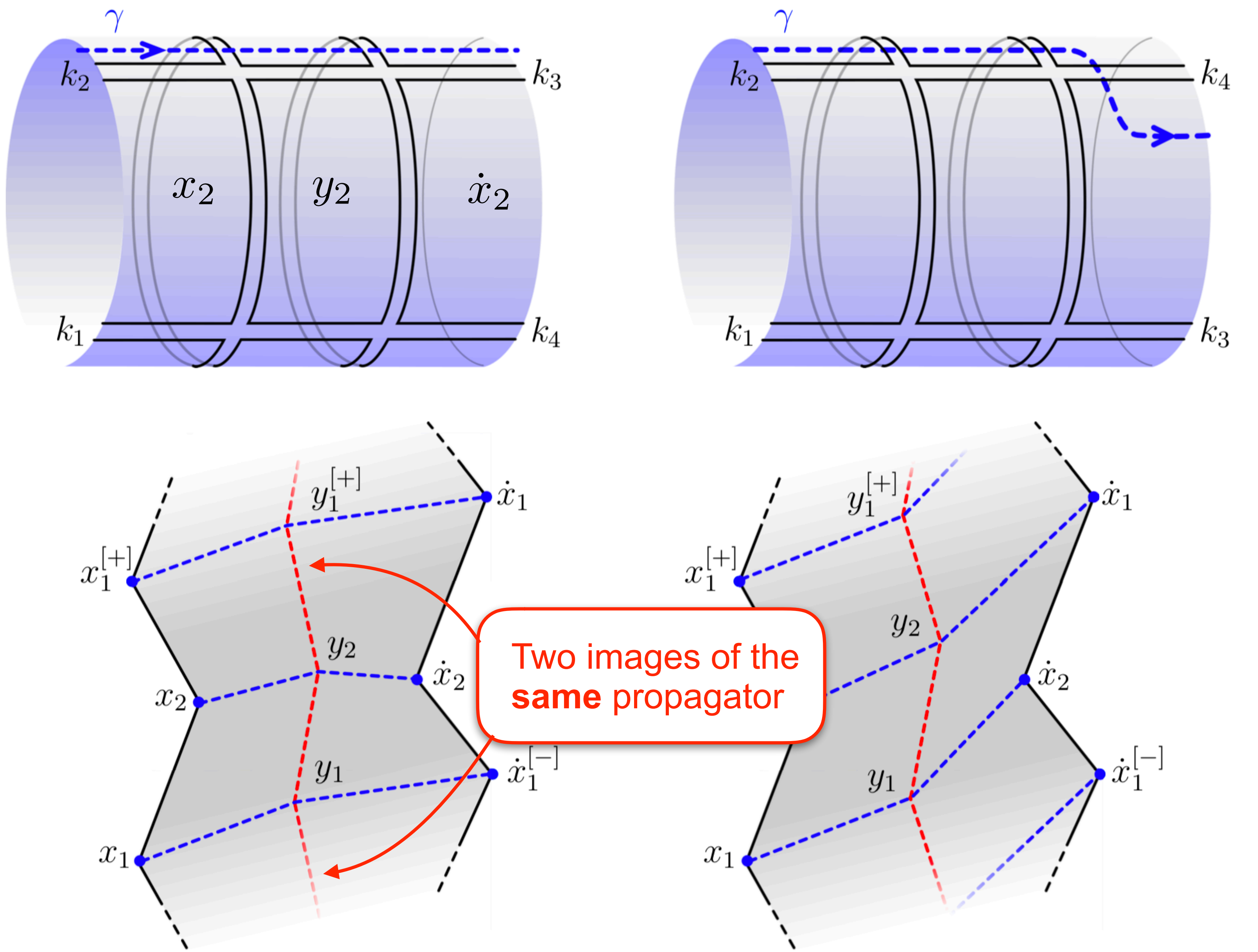
$$= \sum_{\text{shift }[a]}$$



$$= \sum_{a=-\infty}^{\infty}\left(\frac{1}{(\dot{x}_1^{[a-1]}-x_1)^2(\dot{x}_2^{[a]}-x_2)^2}+\frac{1}{(\dot{x}_2^{[a]}-x_1)^2(\dot{x}_1^{[a]}-x_2)^2}\right)$$

Double trace in the fishnet model - **higher loops**

Cylinder cut of 2 loop amplitude



Full N=4 SYM cylindrical duality

N=4 SYM multiplet $|i\rangle = |g_i^+\rangle + \tilde{\eta}_i^A|\psi_{iA}\rangle + \frac{1}{2}\tilde{\eta}_i^A\tilde{\eta}_i^B|\phi_{iAB}\rangle + \frac{1}{3!}\epsilon_{ABCD}\tilde{\eta}_i^A\tilde{\eta}_i^B\tilde{\eta}_i^C|\tilde{\psi}_i^D\rangle + \frac{1}{4!}\tilde{\eta}_i^4|g_i^-\rangle$

$$A_{n;m} = \frac{g_{\rm YM}^{n+m-2}\,\delta^4\left(\sum k_i\right)\delta^8\left(\sum \lambda_i\tilde{\eta}_i\right)}{\left(\langle 1\,2\rangle\ldots\langle n\,1\rangle\right)\left(\langle n+1\,m+2\rangle\ldots\langle n+m\,n+1\rangle\right)} \times \frac{\lambda}{N} \times \int_{l\simeq l+q} d^4l \underbrace{\int d^8\theta\,\widehat{\mathcal{W}}_{n;m}(l,\theta)}_{\mathbb{W}_{n;m}(l)}$$

Cusp supercoordinate

$$\theta_i - \theta_{i-1} = \lambda_i\tilde{\eta}_i \qquad \theta_{i+n} - \theta_i = Q = \sum_{i=1}^{n}\lambda_i\tilde{\eta}_i \qquad \theta = \dot{\theta}_m - \theta_n$$

$$\dot{\theta}_j - \dot{\theta}_{j-1} = \dot{\lambda}_j\dot{\tilde{\eta}}_j \qquad \theta_i^{[a]} = \theta_i + a\,Q \qquad (l,\theta) \simeq (l+q, \theta+Q)$$

SUSY charge going through the cylinder

$$\widehat{\mathcal{W}}_{n;m}(l,\theta) = \Big\langle \mathrm{Tr}\left(\ldots\mathcal{P}e^{\int_0^1 dt\,\mathcal{E}_1(t)}\mathcal{V}_{12}e^{\int_0^1 dt\,\mathcal{E}_2(t)}\mathcal{V}_{23}\ldots e^{\int_0^1 dt\,\mathcal{E}_n(t)}\mathcal{V}_{n1^{[+]}}\ldots\right)$$

$$\times\mathrm{Tr}\left(\ldots\mathcal{P}e^{\int_0^1 dt\,\mathcal{E}_{\dot{1}}(t)}\mathcal{V}_{\dot{1}\dot{2}}e^{\int_0^1 dt\,\mathcal{E}_{\dot{2}}(t)}\mathcal{V}_{\dot{2}\dot{3}}\ldots e^{\int_0^1 dt\,\mathcal{E}_{\dot{m}}(t)}\mathcal{V}_{\dot{m}\dot{1}^{[-]}}\ldots\right)\Big\rangle_{\text{cylinder}}$$

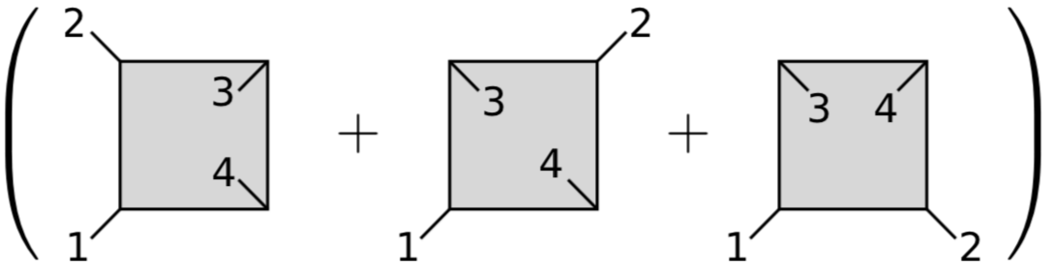
[Caron-Huot]

$$\mathcal{E}_i(t) = -k_i\cdot A + \ldots \qquad \text{and} \qquad \mathcal{V}_{i\,i+1} = 1 + \ldots$$

Explicate perturbative check

$$A^{\text{1-loop}}_{2;2} = 2\,A_4^{\text{1-loop}}(1,2,3,4) + 2\,A_4^{\text{1-loop}}(1,3,2,4) + 2\,A_4^{\text{1-loop}}(1,3,4,2)$$

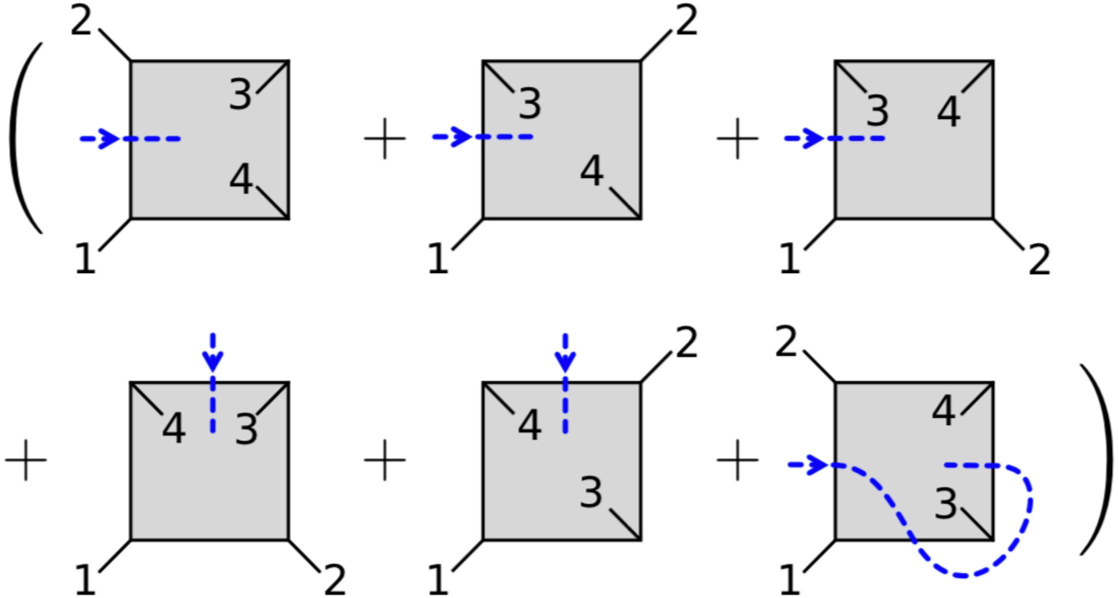
$$= 2i\, g_{YM}^2\, s_{12}s_{23}\, A_4^{tree}(1,2,3,4)$$



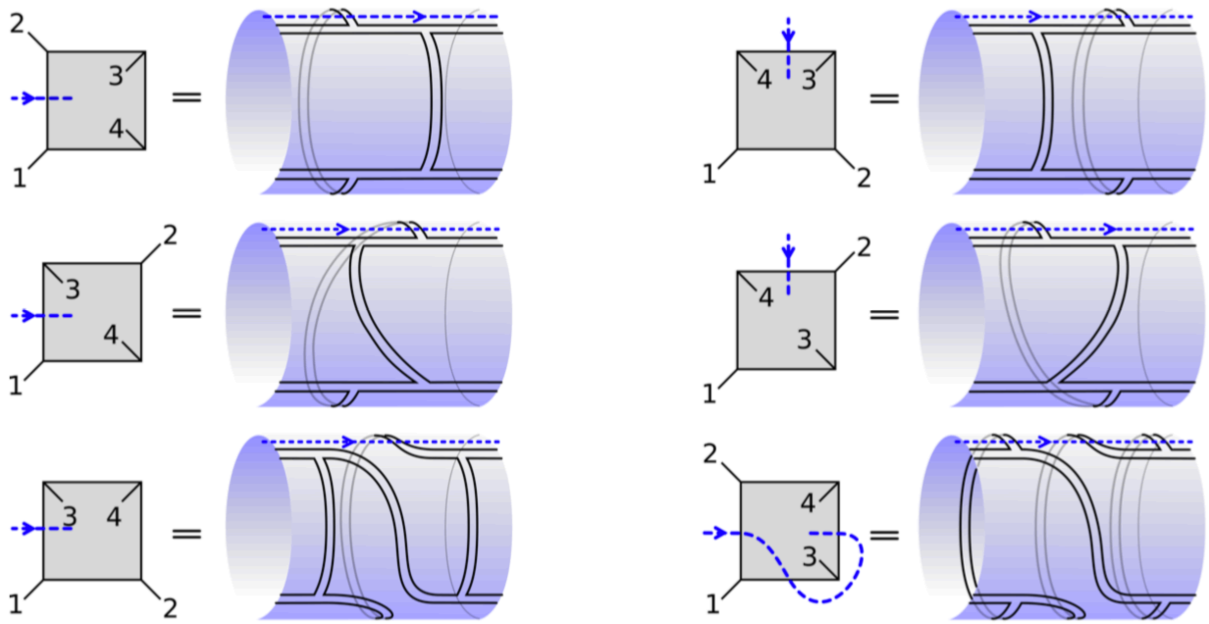
[Bern,Dixon,Dunbar,Kosower], [Feng,Jia,Huang]

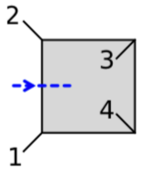
$$\mathbb{A}_{n;m}(l) = \sum_a \mathcal{A}_{n;m}(l + a\,q)$$

$$\mathcal{A}^{\text{1-loop}}_{2;2}(l) = g_{YM}^2 s_{12}s_{23}\, A_4^{tree} \times$$

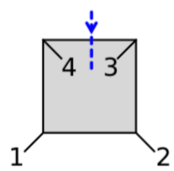


Explicate perturbative check

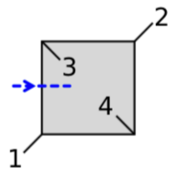




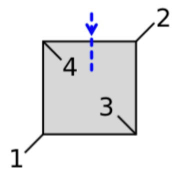
$$= \frac{1}{l^2\,(l-k_1)^2\,(l+k_2)^2\,(l+k_2+k_3)^2},$$



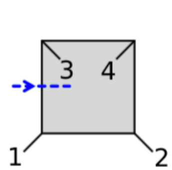
$$= \frac{1}{l^2\,(l+k_3)^2\,(l-k_4)^2\,(l+k_2+k_3)^2},$$



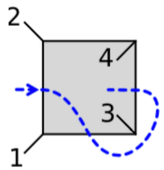
$$= \frac{1}{l^2\,(l-k_1)^2\,(l+k_3)^2\,(l+k_2+k_3)^2},$$



$$= \frac{1}{l^2\,(l+k_2)^2\,(l-k_4)^2\,(l+k_2+k_3)^2},$$



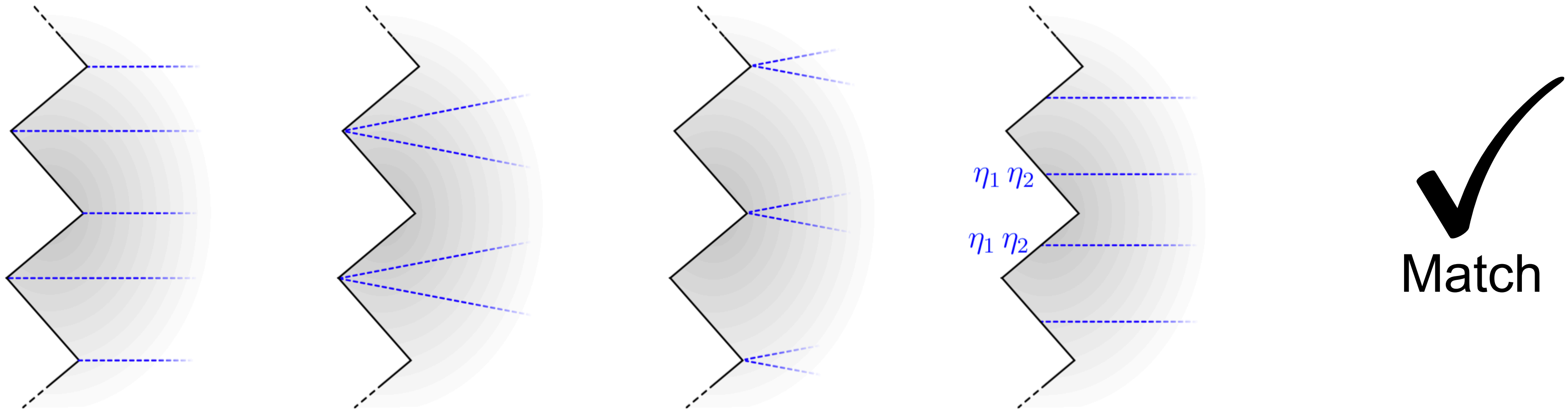
$$= \frac{1}{l^2\,(l-k_1)^2\,(l+k_3)^2\,(l-k_1-k_2)^2},$$



$$= \frac{(l-k_1+k_3)^{-2}}{(l-k_1)^2\,(l+k_3)^2\,(l+k_2+k_3)^2}.$$

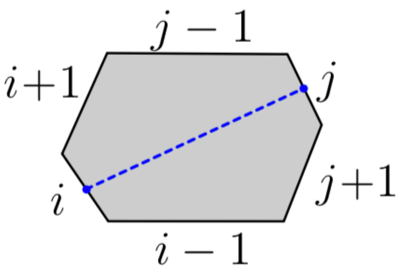
Explicate perturbative check

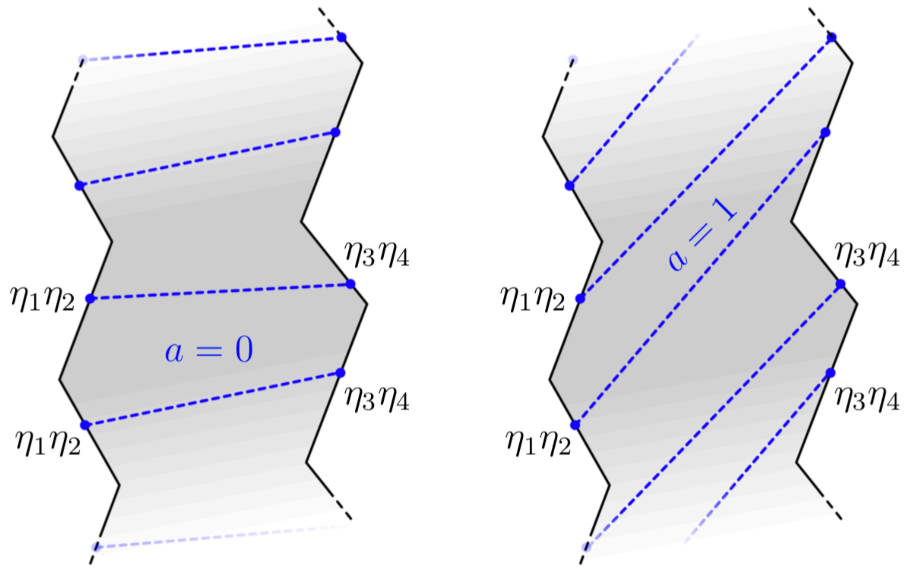
$$\mathbb{A}^{\mathrm{MHV}}_{2;2}(l) = g^4_{YM}\,\delta^4\left(\sum k_i\right)\delta^8\left(\sum \lambda_i\tilde{\eta}_i\right) \times \int d\eta^1_1\, d\eta^2_1\, d\eta^1_2\, d\eta^2_2\, d\eta^3_3\, d\eta^4_3\, d\eta^3_4\, d\eta^4_4\, \mathcal{W}_{2;2}(\eta_1,\eta_2;\eta_3,\eta_4|l)$$



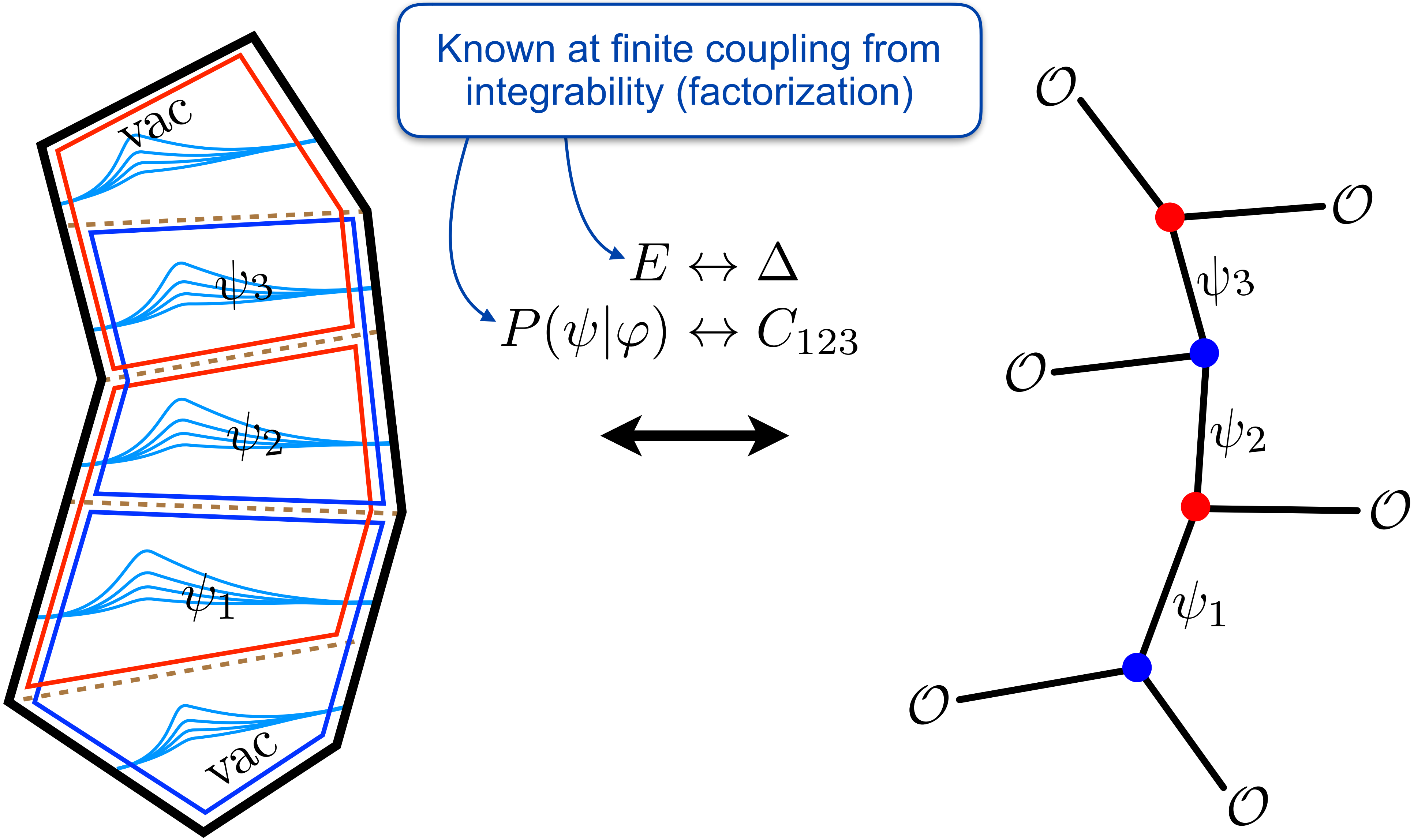
$$\mathbb{W}^{\mathrm{N^2MHV}}_{2;2\ \mathrm{tree}}(l) = \langle 1\,2\rangle^2\langle 3\,4\rangle^2 \sum_{a=-\infty}^{\infty} \mathcal{B}(\{Z_i\},\{\dot{Z}^{[a]}_j\})$$

$$EE(i,j) = R^{(i,i,j,j)}_{\mathrm{tree}} =$$





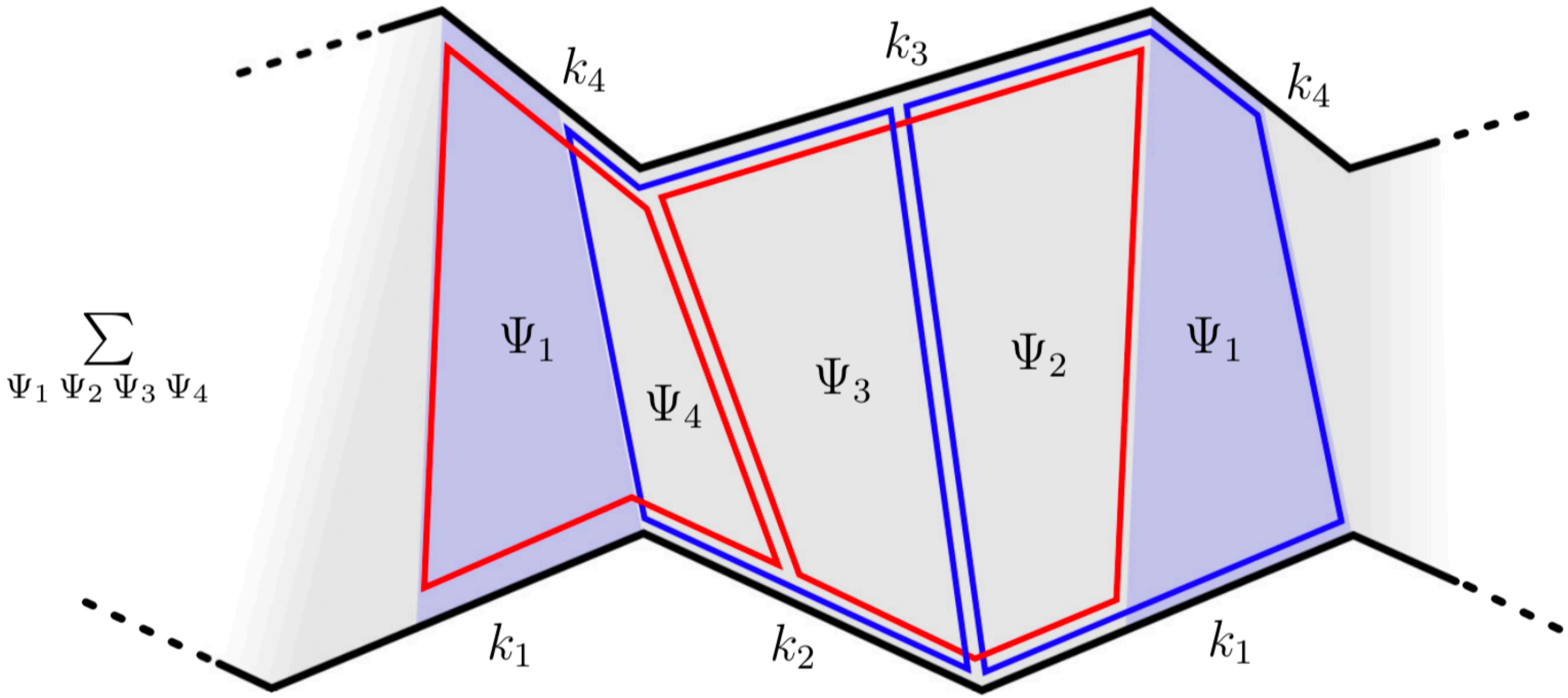
The pentagon operator product expansion



OPE for Wilson loops

OPE for correlation functions

Pentagon operator product expansion



Periodicity is natural in the POPE —

glue periodically by summing over a complete basis of GKP states

Gauged dual super conformal symmetry

Twistors

$$\mathcal{Z}_i^{[a]} = \begin{pmatrix} \lambda_i \\ x_i^{[a]}\lambda_i \\ \theta_i^{[a]}\lambda_i \end{pmatrix} \qquad \dot{\mathcal{Z}}_i^{[a]} = \begin{pmatrix} \dot\lambda_j \\ \dot x_j^{[a]}\dot\lambda_j \\ \dot\theta_j^{[a]}\dot\lambda_j \end{pmatrix}$$

left line   right line

- Trivialize momentum conservation and supersymmetry
- Transform linearly under the dual superconformal transformations

$$\mathcal{Z} \;\to\; \widetilde{\mathcal{Z}} = \mathcal{K}\cdot\mathcal{Z}$$

conformal transformation

Variables

$$\mathcal{W}_{n;m} = \mathcal{W}_{n;m}(\{\mathcal{Z}_i\}_{i=1}^n, \{\dot{\mathcal{Z}}_j\}_{j=1}^m; \mathcal{P}) \qquad \mathcal{P} = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ q & \mathbb{1} & 0 \\ Q & 0 & \mathbb{1} \end{pmatrix}, \qquad \mathcal{Z}^{[a]} = \mathcal{P}^a\cdot\mathcal{Z}$$

shift by (q,Q)

Transformation of variables

$$\mathcal{P} \;\to\; \widetilde{\mathcal{P}} = \mathcal{K}\cdot\mathcal{P}\cdot\mathcal{K}^{-1} \qquad \Rightarrow \qquad \mathcal{Z}^{[a]} \;\to\; \widetilde{\mathcal{Z}}^{[a]} = \mathcal{K}\cdot\mathcal{Z}^{[a]} = \widetilde{\mathcal{P}}^a\cdot\widetilde{\mathcal{Z}}$$

twisted periodicity

The anomaly is the same

$$\beta_\mu\mathcal{K}^\mu \circ \mathcal{W}_{n,m}^{\text{finite}}(\{\mathcal{Z}_i\}_{i=1}^n, \{\dot{\mathcal{Z}}_j\}_{j=1}^m; \mathcal{P}) \;=\; \frac{1}{2}\Gamma_{\text{cusp}}(g)\,\beta_\mu \left(\sum_{i=1}^n x_{i,i+1}^\mu \log\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} + \sum_{j=1}^m \dot x_{j,j+1}^\mu \log\frac{\dot x_{j,j+2}^2}{\dot x_{j-1,j+1}^2}\right)$$

[Drummond,Henn,Korchemsky,Sokatchev]

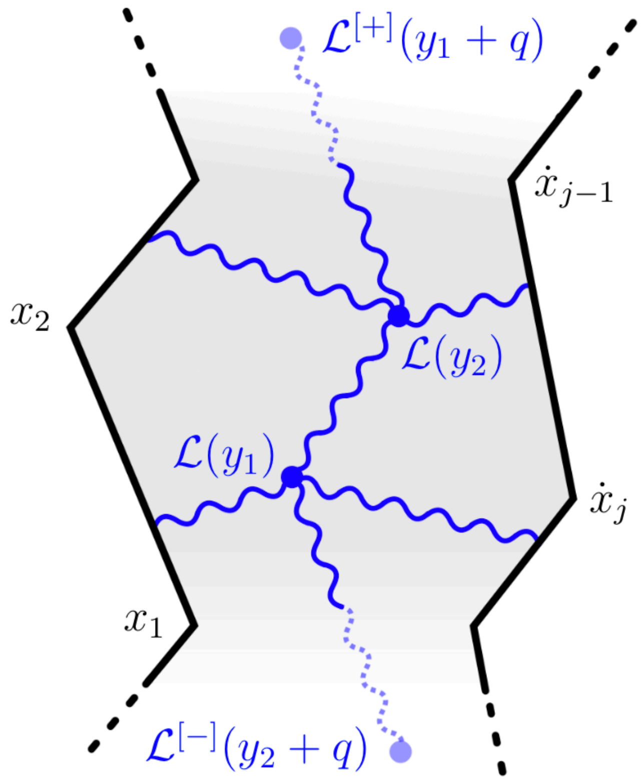
Cylinder integrand

$$\mathcal{W}_{n,n;L}(\{k\}_n;\{\dot{k}\}_m;l;\{y\}_L) =$$

$$\frac{1}{l!}\left\langle W_{n,m}\frac{i\mathcal{L}_{\text{on-shell}}(y_1)}{g^2}\cdots\frac{i\mathcal{L}_{\text{on-shell}}(y_L)}{g^2}\right\rangle_{\text{cylinder, tree}} =$$

$$= \left(\prod_{i=1}^{L}\langle A_i,B_i\rangle^4\right)\mathcal{W}_{n,n;L}(\{\mathcal{Z}\}_n;\{\dot{\mathcal{Z}}\}_m;\mathbb{P};\{A,B\}_L)$$

$y \propto Z_A \wedge Z_B$



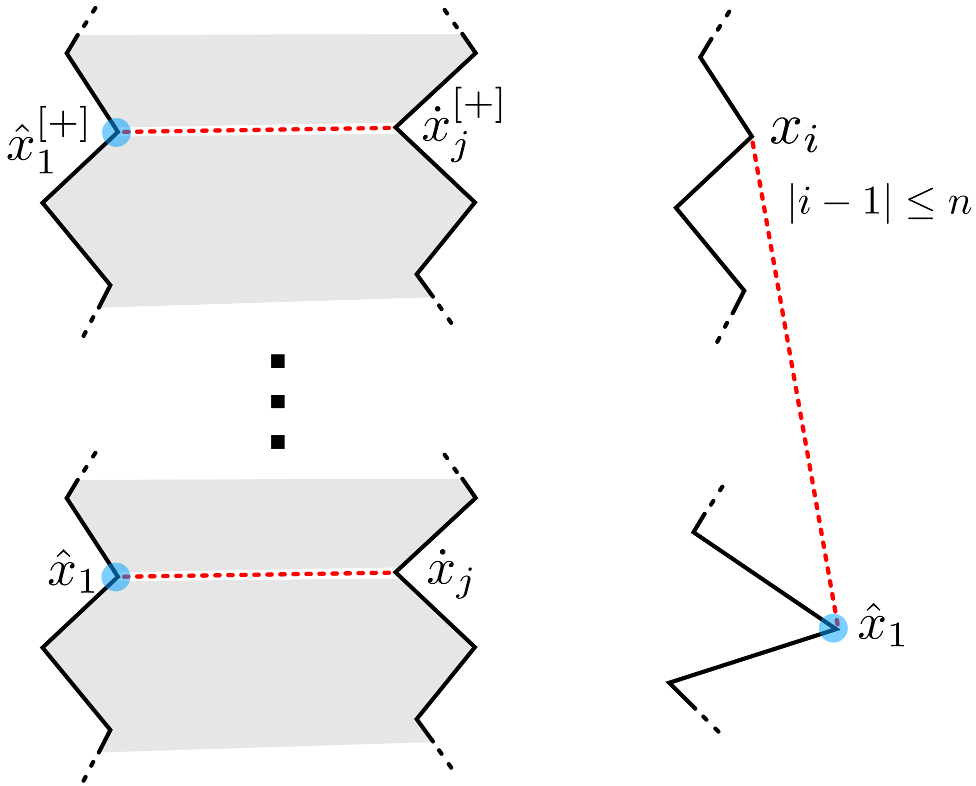
Cylinder integrand is dual conformal invariant (no anomaly)

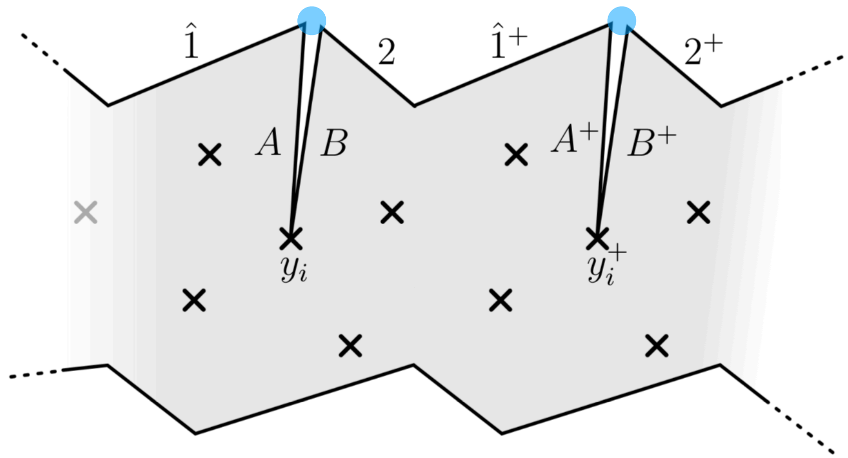
$$\mathcal{W}_{n;m;L}(\{\mathcal{Z}_i\}_{i=1}^n,\{\dot{\mathcal{Z}}_j\}_{j=1}^m;\{A,B\}_L;\mathcal{P}) = \mathcal{W}_{n;m;L}(\{\widetilde{\mathcal{Z}}_i\}_{i=1}^n,\{\dot{\widetilde{\mathcal{Z}}}_j\}_{j=1}^m;\{\widetilde{A},\widetilde{B}\}_L;\widetilde{\mathcal{P}})$$

All loop recursion relation

$$x_1^{[a]} \to \hat{x}_1^{[a]}(z) \qquad \mathcal{Z}_1 \to \mathcal{Z}_1 + z\mathcal{Z}_{n[-]} \qquad \Rightarrow \qquad \mathcal{Z}_{1[i]} \to \mathcal{Z}_{1[i]} + z\mathcal{Z}_{n[i-1]}$$

$$\mathcal{W}_{n,m;L}(l) = \oint \frac{dz}{2\pi i\, z} \widehat{\mathcal{W}}_{n,m;L}(l;z)$$



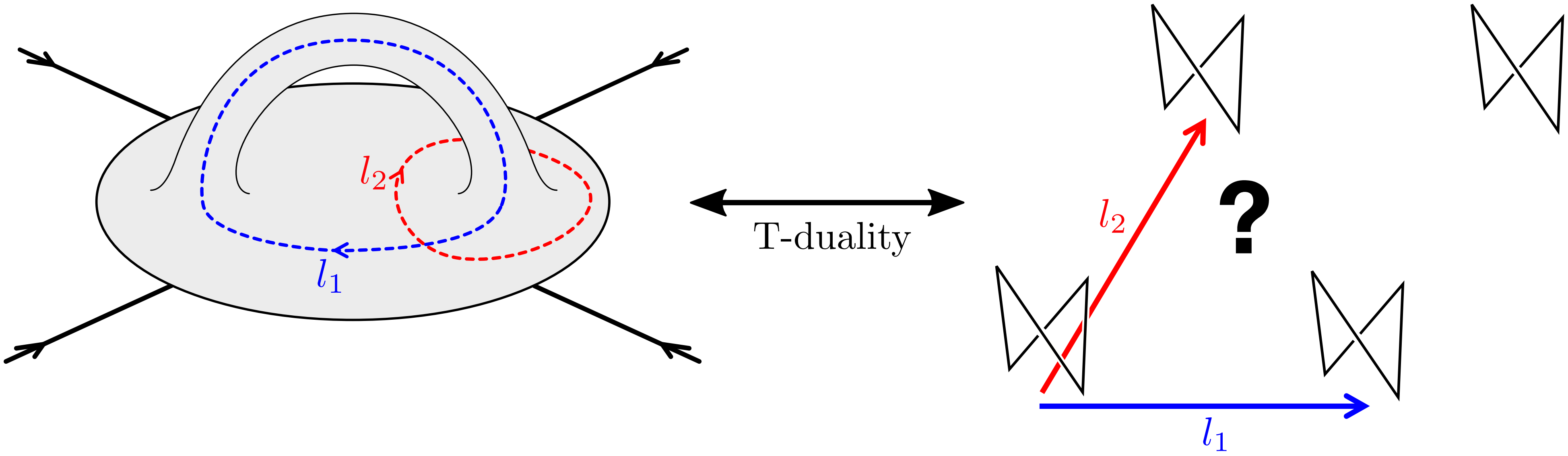


Factorization poles

Forward limits

[Britto, Cachazo,Feng,Witten], [Arkani-Hamed,Bourjaily,Cachazo,Caron-Huot,Trnka], [Caron-Huot]

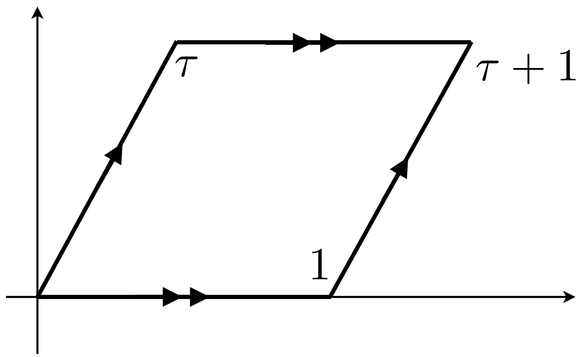
Higher orders in 1/N



$$l_1 \simeq l_1 + l_2$$
$$l_2 \simeq l_2 + l_1$$

$$\tau = l_2/l_1$$

torus modular parameter
in (l_1, l_2) plane

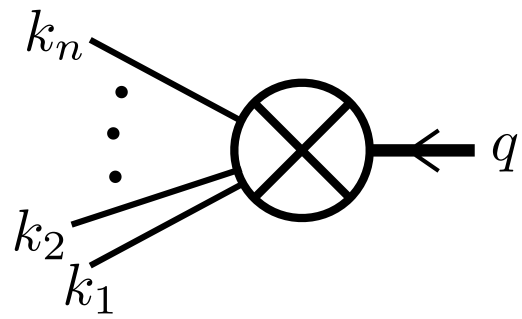


two periodicity constraints

Extension - Form factors

Amplitude of a local operator to create an n-gluon asymptotic state

$$F_{\mathcal{O}}(k_1, \ldots, k_n) = \langle k_1, \ldots, k_n | \mathcal{O}(q) | 0 \rangle =$$

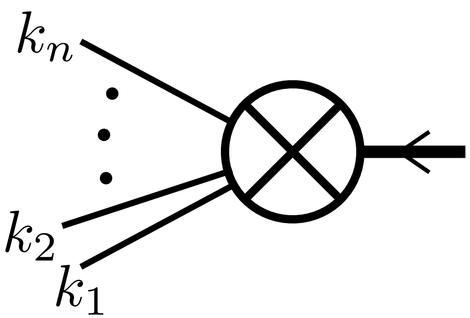


Leaves in momentum space and has the topology of a cylinder

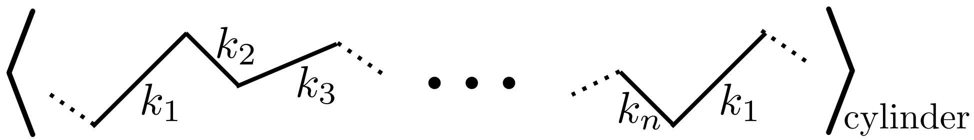
<u>Claims</u> [Tumanov, Wilhelm, AS] in progress

- Dual to quantum periodic Wilson loop

see also [Brandhuber, Spence, Travaglini, Yang]



$$\mathcal{L}(q) =$$



- Can compute at finite coupling by a generalization of the OPE

Thank you!