Planar and non-planar correlation functions in AdS/CFT

Alessandro Sfondrini

based on

1710.10212, 1806.06051

with M. de Leeuw, B. Eden, Y. Jiang, T. Meier, D. le Plat

1804.01998, 1806.00422

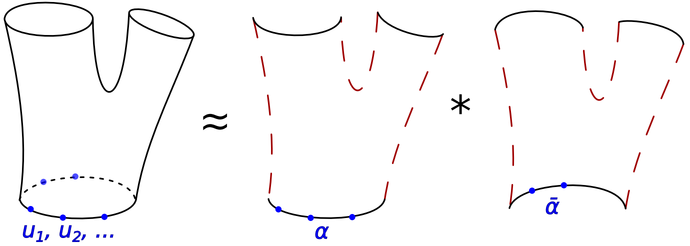
with M. Baggio, A. Dei.

Plan

Plan

Three-point functions and hexagon tessellations

[Basso, Komatsu, Vieira 15]



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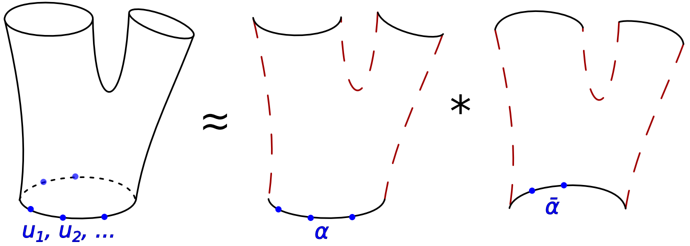

$$C_{ijk} \sim \sum_{v_1, v_2, \ldots} \int \mathrm{d}\mu_{\{v\}} \sum_{u=\alpha\cup\bar{\alpha}} \omega(\alpha, \bar{\alpha})\, H(\alpha, v)\, H(\bar{v}, \bar{\alpha})$$

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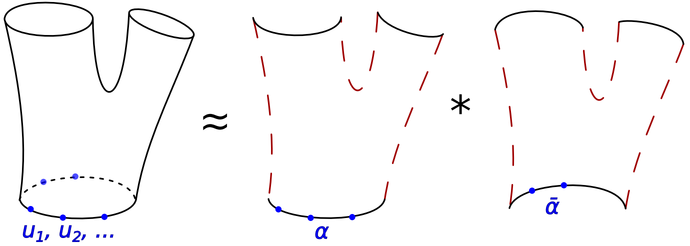

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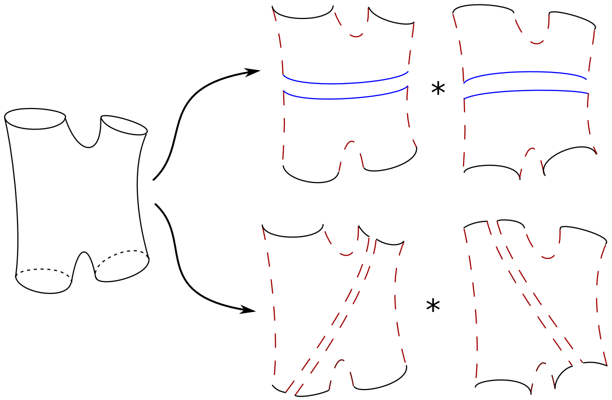
There are closed formulae for ω and H. Integration comes from "cutting"

$$\mathbf{1} = |0\rangle\langle 0| + \sum_i \int \mathrm{d}\mu_p\, |p,i\rangle\langle p,i| + \ldots$$

Evaluating mirror corrections is hard in practice. [Basso, Goncalves, Komatsu 17]

Tessellating four-point functions

Two natural ways to tessellate the 4-point functions with hexagons:



OPE decomposition requires sum over intermediate physical states.

Alternative: sum over mirror magnons. [Eden, AS 16] [Fleury, Komatsu 16]

from 3 to 4 points: cross-ratios

4-pt functions depend on **conformal cross-ratios**:

$$z\bar{z} = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2}, \qquad (1-z)(1-\bar{z}) = \frac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2}.$$

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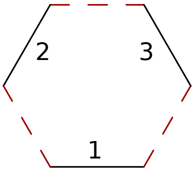

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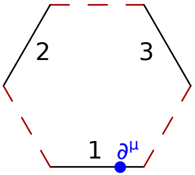
Consider a derivative ∂^μ at position 1.

We get a conformal vector:



$$(V_{1;23})^\mu = \frac{(x_{12})^\mu}{(x_{12})^\nu (x_{12})_\nu} - \frac{(x_{13})^\mu}{(x_{13})^\nu (x_{13})_\nu}$$

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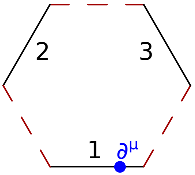

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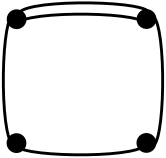
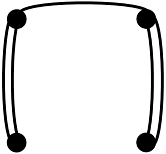
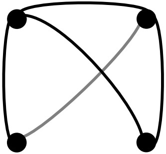
Recipe: weight of $(V_{i;jk})^\pm$ for $\partial^\pm$ at i, etc. [Eden, AS 16] [Fleury, Komatsu 16]

Sketch of a 4pt function computation

Several tree-level diagrams appear for 4pt functions:

 $+$  $+$  $+ \cdots$

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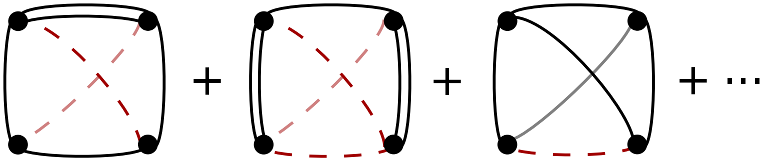

Cut along the propagators (and add extra cuts where needed) to tessellate.

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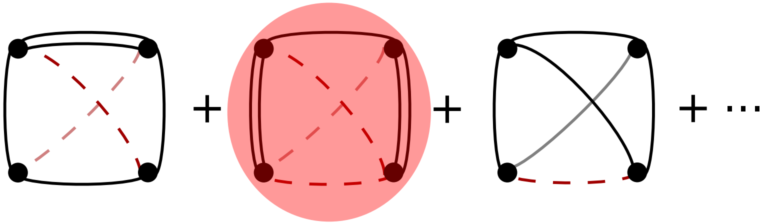

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Prescription: discard wrapping (only!) of one-edge reducible diagrams.

Checked on BPS^4 and BMN-BPS^3 diagrams @ 1 loop [Fleury, Komatsu 16].

More restrictions on allowed diagrams

More systematic investigation: consider BMN^2-BPS^2 diagrams.

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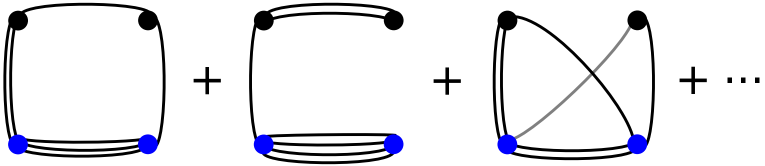

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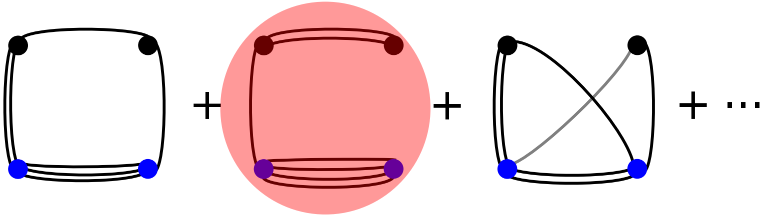

Surprise: discard certain diagrams to reproduce $SU(N_c)$ SYM at tree level!

Better prescription needed!

Longer operators and mixing

For longer (non-BPS) operators, non-planar corrections appear:

$$\mathbb{X}_7 = \mathcal{X}_7 - \frac{1}{N_c}\left(\frac{3}{2\sqrt{2}}\,\mathcal{X}_4 \cdot \mathcal{O}_3 + 2\sqrt{2}\,\mathcal{X}_5 \cdot \mathcal{O}_2\right) + \dots$$

and at leading order in $1/Nc$

$$\langle \mathbb{X}_7\, \mathcal{X}_5\, \mathcal{O}_2\, \mathcal{O}_2 \rangle \neq \langle \mathcal{X}_7\, \mathcal{X}_5\, \mathcal{O}_2\, \mathcal{O}_2 \rangle$$

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Tessellations compute single-trace part $\langle \mathcal{X}_7\, \mathcal{X}_5\, \mathcal{O}_2\, \mathcal{O}_2 \rangle$.

Lessons learned

Must discard certain diagrams to reproduce $SU(N_c)$, or $U(N_c)$, SYM.

$\longrightarrow$ prescription: explicitly account for **colour factors**.

We reproduce ~ 350 "data points" @ tree level for $SU(N_c)$ or $U(N_c)$.

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Tessellation computes single-trace diagrams.

$\longrightarrow$ need to account for non-planar terms separately.

[Eden, le Plat, Jiang, AS 17]

Plan

Two-point functions at order $1/N_c^2$

Consider a (non-BPS) operator of the form

$$\mathbb{X}_n = \mathcal{X}_n + \frac{\kappa}{N_c}\mathcal{X}_m \cdot \mathcal{O}_{n-m} + \dots$$

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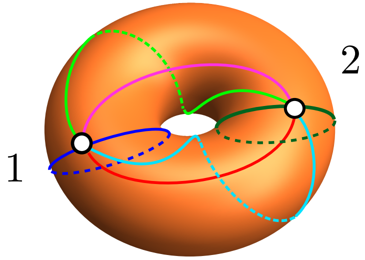
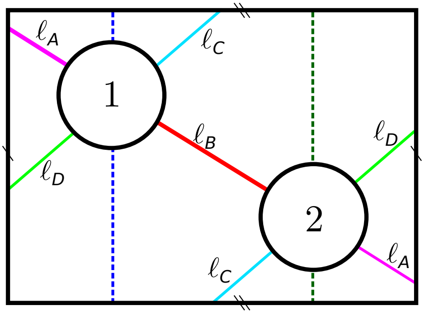
Tessellating the torus





$\ell_A,\ \ell_B, \dots$ are the number of propagators (“bridge length”).

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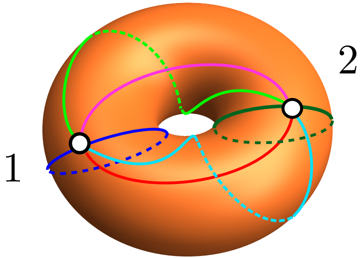

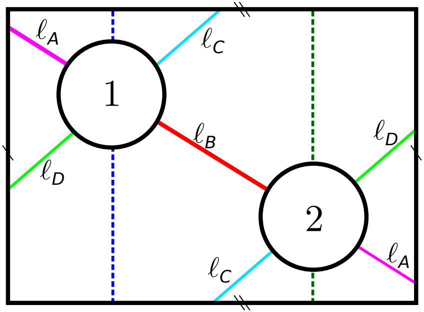

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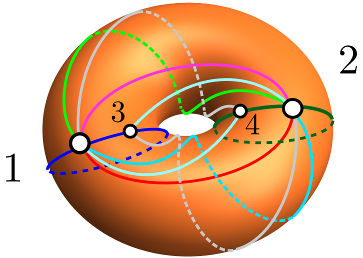

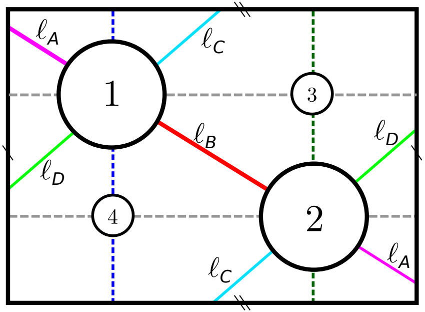

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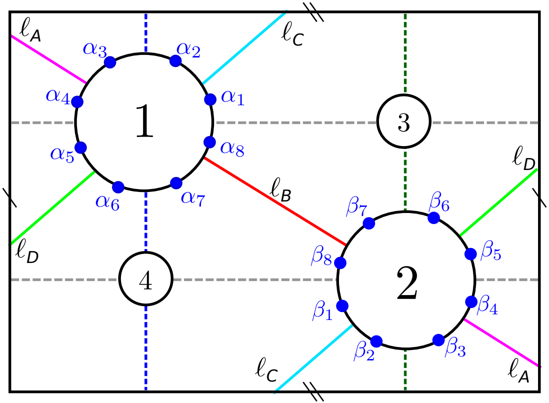
Solution: introduce regularisation by identity operators $\mathbf{1}(x_3)$, $\mathbf{1}(x_4)$.

The tessellation in detail



- Solid ℓs give different diagrams; weight them by **colour factors**.
- Sum over partitions α_i, β_i, final result is x_3, x_4-independent.
- Note: the dashed ℓs are zero!

Single-trace / double-trace contribution

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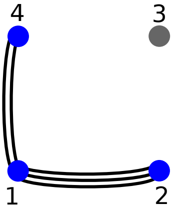

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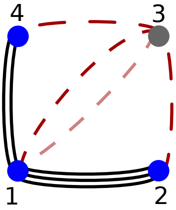

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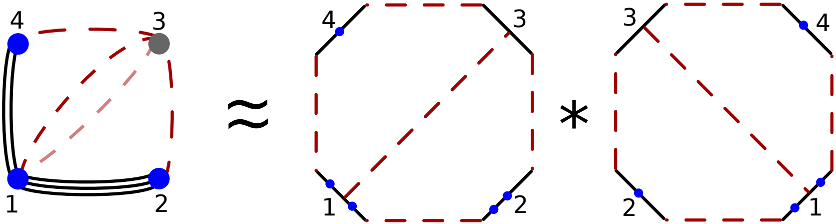

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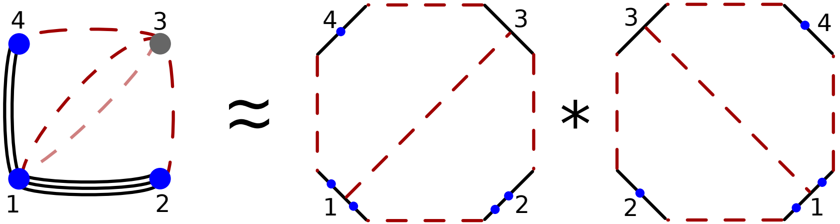

Limit $x_4 \to x_2$ manifestly regular. **Perfect match** with field theory.

Plan

Wrapping corrections

When cutting the worldsheet we introduce corrections

$$\mathbf{1} = |0\rangle\langle 0| + \sum_i \int \mathrm{d}\mu_p \, |p,i\rangle\langle p,i| + \dots$$

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Problem 2: mirror-anti-mirror interactions can go like g^{-2}.

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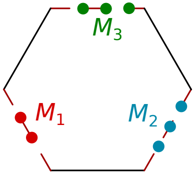

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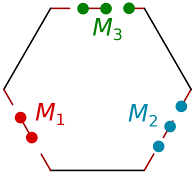

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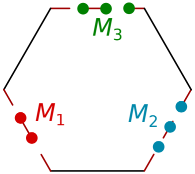

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- $\sum_{j=1}^{3} M_i(M_i - 1)/2$ for mirror-mirror interactions.

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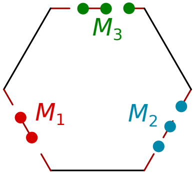

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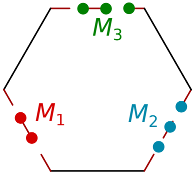
- $-(M_1\, M_2 + M_1\, M_3 + M_2\, M_3)$ from mirror-anti-mirror interactions.

Naïve estimate of wrapping order



Consider an hexagon with M_1, M_2, M_3 mirror particles.

Order in g^{2p}?

- $\sum_{j=1}^{3} M_j(\ell_j + 1)$ for each magnon.
- $\sum_{j=1}^{3} M_i(M_i - 1)/2$ for mirror-mirror interactions.
- $-(M_1\, M_2 + M_1\, M_3 + M_2\, M_3)$ from mirror-anti-mirror interactions.

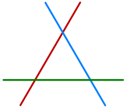
Hence naïvely

$$p = \sum_{j=1}^{3} M_j\,\ell_j + (M_1 - M_2)^2 + M_3^2 - 2M_3(M_1 + M_2)$$

Not positive definite!

Refining the estimate

Crucial observation: chain of mirror-anti-mirror hexagon excitations

$$H_{12}(u_1, u_2)\, H_{13}(u_1, u_3)\, H_{23}(u_2, u_3) =$$  $$\sim g^{-2}$$

(really a property of the mirror-anti-mirror $su(2|2)$ S matrix)

Much better than expected g^{-6}!

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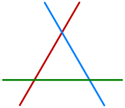
$$H_{12}({\color{red}u_1},{\color{blue}u_2})\,H_{13}({\color{red}u_1},{\color{green}u_3})\,H_{23}({\color{blue}u_2},{\color{green}u_3}) =$$  $$\sim g^{-2}$$

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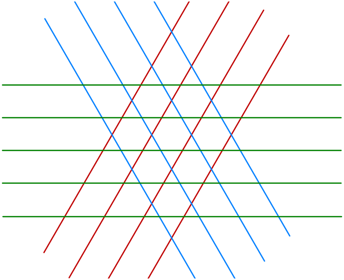
Count maximal number of triangles for M_1, M_2, M_3 and adjust estimate.



Refined estimate and open issues

Improved estimate for g^{2p}

$$p = \sum_{j=1}^{3} M_j\, \ell_j + (M_3 - M_1 - M_2)^2\,.$$

[de Leeuw, Eden, Jiang, le Plat, Meier, Sfondrini 18]

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- Field theory gives constraints: can be used to simplify the computation, avoid infinite sets of magnons.
- Still, mirror corrections are outstanding challenge.

Plan

Integrable AdS_3/CFT_2: lightning review

Much of the dynamics is illustrated by the dispersion relation:

$$E(p) = \sqrt{\left(\mu + \frac{k}{2\pi} p\right)^2 + 4h^2 \sin^2\left(\frac{p}{2}\right)}$$

[Borsato, O.Sax, AS 12] [Hoare, Tseytlin 13] [Borsato, O.Sax, AS, Stefanski, Torrielli 13-15]

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- k, h generic: typical point in moduli space
  - S-matrix known from symmetry, analytic structure not understood
- $h = 0$ corresponds to level-k WZW model
  - Very simple spectrum: closed formulae!
  - Nothing known from integrability (until recently!)

Pure-NSNS $AdS_3 \times S^3 \times T^4$ action

In uniform l.c. gauge, we have a free parameter a

$$x^+ = (1-a)\,t + a\,\phi = \tau, \qquad x^- = \phi - t, \quad p_- = 1.$$

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At the same time

$$E = \int_0^{R(a)} \mathrm{d}\sigma\,\mathcal{H}_{\text{lc}}, \qquad R(a) = J + a\,E\,.$$

$$\frac{\mathrm{d}}{\mathrm{d}a}E = 0 \;\Rightarrow\; \text{equivalently,} \begin{cases} \textbf{non-linear} \text{ theory in volume } R = J, \\ \textbf{free theory} \text{ in volume } R = J + E/2. \end{cases}$$

CDD factor from $T\bar{T}$ deformation

[Arutyunov, Frolov 05] [Cavaglià et al. 16] [Smirnov, Zamolodchikov 16]

Above, classical observation for flat T^4. How about quantum level? (non-trivial!)

$$e^{ip_i(J+E/2)} = 1, \qquad p_i = \frac{2\pi\, n_i}{R+E/2}, \qquad E = \sum_i \frac{k}{2\pi}|p_i| = \frac{2k\,\mathcal{N}}{J+E/2}.$$

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Matches with WZW result for T^4 modes. Natural CDD interpretation

$$e^{ip_i(J+E/2)} = 1 \quad \Leftrightarrow \quad e^{ip_i J}\prod_j e^{\frac{i}{2}\Phi_{\text{CDD}}} = 1, \qquad \Phi_{\text{CDD}} = p_i E_j - p_j E_i$$

Arbitrary excitations

Without restricting to T^4, we have

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$$\Phi_{\text{CDD}} = p_i(E_j - \hat{\mu}_j) - p_j(E_i - \hat{\mu}_i)$$

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$$\begin{aligned} E &= k\frac{\sum_i n_i\,\text{sgn}_i}{J + (E - \hat{\mu})/2} + \sum_i \hat{\mu}_i \\ &= \frac{2\,k\,\mathcal{N}}{J + (E - \hat{\mu})/2} + \hat{\mu} \end{aligned}$$

And solving for E

$$E = \sqrt{J^2 + 4k\mathcal{N}} - J + \hat{\mu}$$

Spectrum and S-matrix

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Wait, what?!

Mirror TBA and vanishing of wrapping

The mirror TBA equations can be easily written

$$\epsilon_a(u) = \psi_a + R\,E^*_a(u) - [\Lambda_b * K_{ba}](u), \qquad K_{ij}(u,v) \approx \partial_u \Phi_{ij}(u,v)$$

$$\psi_a = \begin{cases} 0 & \text{bosons} \\ i\pi & \text{fermions} \end{cases} \qquad \Lambda_a = \begin{cases} -\log(1-e^{-\epsilon_a}) & \text{bosons} \\ +\log(1+e^{-\epsilon_a}) & \text{fermions} \end{cases}$$

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Wrapping vanishes! (also for excited states)

$$\sum_b \int \Lambda_b = \sum_b \int [-\log(1-e^{\epsilon_b}) + \log(1+e^{\epsilon_b + i\pi})] = \mathbf{0}$$

Plan

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- Huge proliferation of wrapping corrections.
  Perturbation theory is positive **"semi-definite" only**.
- Study a simpler model: **pure-NSNS integrable AdS$_3$**.
  - A "true spin chain" (no wrapping).
  - Plenty of data WZW approach.
  - Wrapping manageable for 3pt? Simple result expected from WZW.
  - Simple, diagonal S matrix $\to$ simple hexagon form factor?
- *Maybe it's time to try a simple problem!*