Handling Handles: Non-Planar AdS/CFT Integrability

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+ further work in progress

Integrability in Gauge and String Theory

Copenhagen, August 2018

Planar Limit & Genus Expansion

Gauge theory with adjoint matter in the large N_c limit: ['t Hooft 1974]

- Feynman diagrams are double-line diagrams.
- Operator traces induce a definite ordering of connecting propagators.
- All color lines (propagators and traces) form closed oriented loops.
- Assign an oriented disk (face) to each loop.
- Obtain a compact oriented surface (operators form boundaries).
- The genus of the surface defines the genus of the diagram.

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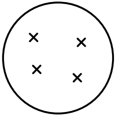
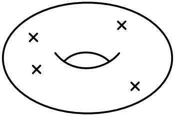
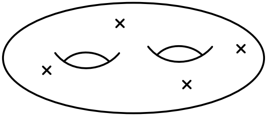
Correlators of single-trace operators:

Count powers of N_c and g_YM^2 for propagators ($\sim g_\mathrm{YM}^2$), vertices ($\sim 1/g_\mathrm{YM}^2$), and faces ($\sim N_\mathrm{c}$), define $\lambda = g_\mathrm{YM}^2 N_\mathrm{c}$, use Euler formula:

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \frac{1}{N_\mathrm{c}^{n-2}} \sum_{g=0}^{\infty} \frac{1}{N_\mathrm{c}^{2g}} G_n^{(g)}(\lambda) \qquad \mathcal{O}_i = \mathrm{Tr}(\Phi_1 \Phi_2 \dots)$$

$\sim \frac{1}{N_\mathrm{c}^2}$  $+ \frac{1}{N_\mathrm{c}^4}$  $+ \frac{1}{N_\mathrm{c}^6}$  $+ \dots$

Proposal

Concrete and explicit realization of the general genus expansion:

$$\langle Q_1 \dots Q_n \rangle = \frac{\prod_{i=1}^{n} \sqrt{k_i}}{N_{\mathrm{c}}^{n-2}}\, \mathcal{S} \circ \sum_{\Gamma \in \boldsymbol{\Gamma}} \frac{1}{N_{\mathrm{c}}^{2g(\Gamma)}} \times$$
$$\times \left[\prod_{b \in \boldsymbol{b}(\Gamma_\triangle)} d_b^{\ell_b} \int_{\boldsymbol{M}_b} \mathrm{d}\psi_b\, \mathcal{W}(\psi_b) \right] \prod_{a=1}^{2n+4g(\Gamma)-4} \mathcal{H}_a \,.$$

Remarkable fact:

For $\mathcal{N} = 4$ SYM, all ingredients of the formula are well-defined and explicitly known as functions of the 't Hooft coupling λ.

Goal of this talk:

- Explain all ingredients of the formula.
- Demonstrate match with known data.
- Show some predictions.

Proposal: Ingredients I

$$\langle Q_1 \dots Q_n \rangle = \frac{\prod_{i=1}^{n} \sqrt{k_i}}{N_{\mathrm{c}}^{n-2}} \, \mathcal{S} \circ \sum_{\Gamma \in \boldsymbol{\Gamma}} \frac{1}{N_{\mathrm{c}}^{2g(\Gamma)}} \times \\ \times \left[\prod_{b \in \boldsymbol{b}(\Gamma_\triangle)} d_b^{\ell_b} \int_{\boldsymbol{M}_b} \mathrm{d}\psi_b \, \mathcal{W}(\psi_b) \right] \prod_{a=1}^{2n+4g(\Gamma)-4} \mathcal{H}_a \, .$$

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Half-BPS operators: $Q_i = Q(\alpha_i, x_i, k_i) = \mathrm{Tr}\big[(\alpha_i \cdot \Phi(x_i))^{k_i}\big] \,, \quad \alpha_i^2 = 0 \,.$
Internal polarizations α_i, positions x_i, weights (charges) k_i.

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Set of all Wick contractions $\Gamma \in \boldsymbol{\Gamma}$ of the free theory, genus $g(\Gamma)$.

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- Collect homotopically equivalent lines in "bridges" $\rightarrow$ skeleton graph. The number of lines in a bridge b is the bridge length (width) ℓ_b.

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- Subdivide all faces into triangles by inserting zero-length bridges.

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Set of all bridges: $\boldsymbol{b}(\Gamma_\triangle)$.

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On each bridge $b \in \boldsymbol{b}(\Gamma_\triangle)$:

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Insert a kinematical weight factor $\mathcal{W}(\psi_b)$, it depends on the cross ratios.

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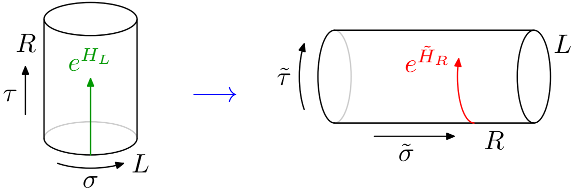
Finally, $\mathcal{S}$: Stratification. Sum over graphs quantizes the integration over the string moduli space. $\mathcal{S}$ accounts for contributions at the boundaries.

The Sum over Mirror States

On each bridge lives a mirror theory, which is obtained from the physical worldsheet theory by an analytic continuation, a double-Wick (90 degree) rotation $(\sigma, \tau) \to (i\tilde{\tau}, i\tilde{\sigma})$ that exchanges space and time:



In all computations, the volume R can be treated as infinite.

$\Rightarrow$ Mirror states are free multi-magnon Bethe states, characterized by rapidities u_i, bound state indices a_i, and flavor indices $(A_i, \dot{A}_i)$.

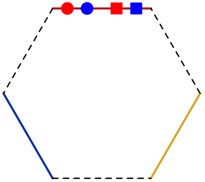
The mirror integration therefore expands to

$$\int_{\boldsymbol{M}_b} d\psi_b = \sum_{m=0}^{\infty} \prod_{i=1}^{m} \sum_{a_i=1}^{\infty} \sum_{A_i, \dot{A}_i} \int_{u_i=-\infty}^{\infty} du_i \, \mu_{a_i}(u_i) \, e^{-\tilde{E}_{a_i}(u_i) \, \ell_b} \, .$$

μ_{a_i}: measure factor, $\tilde{E}$: mirror energy, ℓ_b: length of bridge b.

The Hexagon Form Factors

Hexagon = Amplitude that measures the overlap between three mirror and three physical off-shell Bethe states. Worldsheet branching operator that creates an excess angle of π. [Basso,Komatsu Vieira '15]



Explicitly: $\mathcal{H}(\chi^{A_1}\chi^{\dot{A}_1}\chi^{A_2}\chi^{\dot{A}_2}\ldots\chi^{A_n}\chi^{\dot{A}_n})$

$$= (-1)^{\mathfrak{F}}\left(\prod_{i<j} h_{ij}\right)\langle\chi^{A_1}\chi^{A_2}\ldots\chi^{A_n}|S|\chi^{\dot{A}_n}\ldots\chi^{\dot{A}_2}\chi^{\dot{A}_1}\rangle$$

- $\chi^A, \chi^{\dot{A}}$: Left/Right $\mathfrak{su}(2|2)$ fundamental magnons
- $\mathfrak{F}$: Fermion number operator
- S: Beisert S-matrix
- $h_{ij} = \dfrac{x_i^- - x_j^-}{x_i^- - x_j^+}\dfrac{x_j^+ - 1/x_i^-}{x_2^+ - 1/x_1^+}\dfrac{1}{\sigma_{ij}}\,,\qquad x^\pm(u) = x(u\pm\tfrac{i}{2})\,,\quad \tfrac{u}{g} = x + \tfrac{1}{x}$
  σ_{ij}: BES dressing phase

Some Remarks

$$\langle Q_1 \dots Q_n \rangle = \frac{\prod_{i=1}^n \sqrt{k_i}}{N_{\mathrm{c}}^{n-2}} \mathcal{S} \circ \sum_{\Gamma \in \boldsymbol{\Gamma}} \frac{1}{N_{\mathrm{c}}^{2g(\Gamma)}} \times$$
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Nicely separates sum over graphs and topologies from λ dependence.

Should in principle hold at any value of the coupling λ.

Still a sum over infinitely many mirror contributions that cannot be evaluated in general. But may admit high-loop or even exact expansions in specific limits.

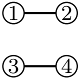
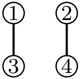
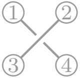
Stratification operator $\mathcal{S}$ looks innocent, but is in fact non-trivial.

Known Non-Planar Data

Half-BPS operators: First non-trivial correlator: Four-point function.

$$\mathcal{Q}_i^k \equiv \operatorname{Tr}\big[(\alpha_i \cdot \Phi(x_i))^k\big]\,, \quad \Phi = (\phi_1, \dots, \phi_6)\,, \quad \alpha_i^2 = 0\,.$$

Specialize to equal weights $k_1 \dots k_4 = k$,
and to specific polarizations α_i with $\alpha_1 \cdot \alpha_4 = \alpha_2 \cdot \alpha_3 = 0$.
Possible propagator structures:

$X \equiv \dfrac{\alpha_1 \cdot \alpha_2\, \alpha_3 \cdot \alpha_4}{x_{12}^2 x_{34}^2} =$  , $\quad Y \equiv$  , $\quad \big(Z \equiv$  $\big)$.

Correlators for general N_c: [Arutyunov, Sokatchev '03] [Arutyunov, Penati '03, Santambrogio, Sokatchev]

$$G_k \equiv \langle \mathcal{Q}_1^k \mathcal{Q}_2^k \mathcal{Q}_3^k \mathcal{Q}_4^k \rangle^{\text{loops}} = R \sum_{m=0}^{k-2} \mathcal{F}_{k,m}\, X^m Y^{k-2-m}$$

Supersymmetry factor: $R = z\bar{z}X^2 - (z+\bar{z})XY + Y^2$

Main data: Coefficients $\mathcal{F}_{k,m} = \mathcal{F}_{k,m}(\lambda, N_\mathrm{c}; z, \bar{z})$

First Test: Genus One, Large Charges

Focus on leading order in large k $\to$ several simplifications:
Data, sum over graphs, and loop expansion (mirror states) all simplify.

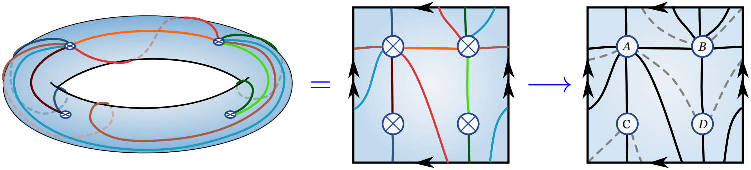
Data: $\mathcal{F}^{(1),\mathrm{U}}_{k,m}(z,\bar{z}) = -\frac{2k^2}{N_c^2}\Big\{1+\frac{1}{N_c^2}\Big[\big[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\big]k^4+\mathcal{O}(k^3)\Big]\Big\}t\,F^{(1)}\,,$

$$\mathcal{F}^{(2),\mathrm{U}}_{k,m}(z,\bar{z}) = \frac{4k^2}{N_c^2}\Bigg[\Big\{1+\frac{1}{N_c^2}\Big[\big[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\big]k^4+\mathcal{O}(k^3)\Big]\Big\}t\,F^{(2)} + \Big\{1+\frac{1}{N_c^2}\Big[\big[\tfrac{29}{6}r^4-\tfrac{11}{4}r^2+\tfrac{15}{32}\big]k^4+\mathcal{O}(k^3)\Big]\Big\}\frac{t^2}{4}\big(F^{(1)}\big)^2\Bigg]\,,$$

where $r=(m+1)/k-1/2$. $\mathcal{F}_{k,m}$: Coefficient of X^mY^{k-2-m}.

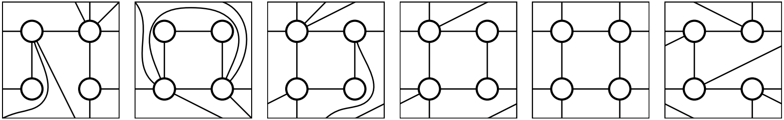
Step 1: Sum over propagator graphs: Split in two steps:

- Sum over torus "skeleton graphs" with non-parallel edges ($\equiv$ bridges).
- Sum over distributions of parallel propagators on bridges.



First Test: Large Charges: Graphs

Genus-one four-point graphs with the maximal number of bridges:



B G L M P Q

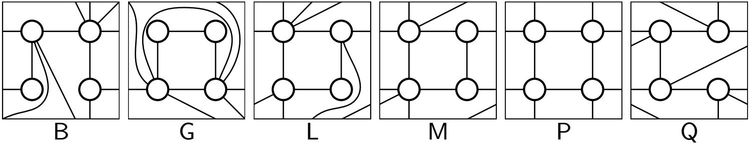
Large k: Combinatorics of distributing propagators on bridges:
Sum over distributions of m propagators on $j+1$ bridges $\rightarrow m^j/j!$
$\Rightarrow$ Only graphs with a maximum number of bridges contribute.

Sum over labelings:

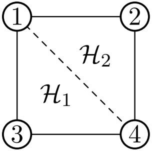
Case	Inequivalent Labelings (clockwise)	Combinatorial Factor
B	$(1,2,4,3),(2,1,3,4),(3,4,2,1),(4,3,1,2)$	$m^3(k-m)/6$
B	$(1,3,4,2),(3,1,2,4),(2,4,3,1),(4,2,1,3)$	$m(k-m)^3/6$
G	$(1,2,4,3),(3,4,2,1)$	$m^4/24$
G	$(1,3,4,2),(2,4,3,1)$	$(k-m)^4/24$
L	$(1,2,4,3),(3,4,2,1),(2,1,3,4),(4,3,1,2)$	$m^2/2\cdot(k-m)^2/2$
M	$(1,2,4,3),(2,1,3,4),(1,3,4,2),(3,1,2,4)$	$m^2(k-m)^2/2$
P	$(1,2,4,3)$	$m^2(k-m)^2/2$
Q	$(1,2,4,3)$	$m^2(k-m)^2$

First Test: Large Charges: Hexagons

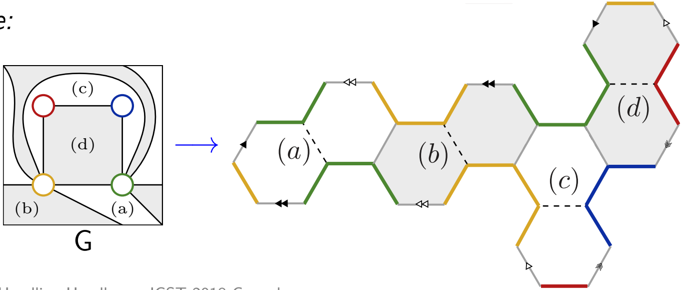
Graphs:



All graphs consist of only octagons!
Split each octagon into two hexagons with a zero-length bridge.



Example:



First Test: Large Charges: Mirror Particles

Loop Counting:

Expand mirror propagation $\mu(u)\, e^{-\ell \tilde{E}(u)}$ and hexagons $\mathcal{H}$ in coupling g.

$\rightarrow$ n particles on bridge of size ℓ: $\mathcal{O}(g^{2(n\ell+n^2)})$

All graphs consist of octagons framed by parametrically large bridges.

$\rightarrow$ Only excitations on zero-length bridges inside octagons survive.

Excited Octagons:

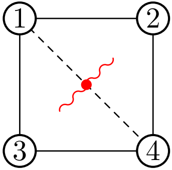
n particles on a zero-length bridge $\rightarrow$ $\mathcal{O}(g^{2n^2})$

$\rightarrow$ Octagons with $1/2/3/4$ particles start at $1/4/9/16$ loops.

Octagon 1–2–4–3 with 1 particle:

[Fleury '16, Komatsu] [TB, Caetano, Fleury, Komatsu, Vieira '18]

$$\mathcal{M}(z,\alpha) = \left[z + \bar{z} - (\alpha + \bar{\alpha})\frac{\alpha\bar{\alpha} + z\bar{z}}{2\alpha\bar{\alpha}}\right] \cdot \left(g^2 F^{(1)}(z) - 2g^4 F^{(2)}(z) + 3g^6 F^{(3)}(z) + \dots\right)$$



For $Z = 0$: R-charge cross ratios $\alpha = z\bar{z}\, X/Y$ and $\bar{\alpha} = 1$.

First Test: Large Charges: Match & Prediction

We are Done:

Sum over graph topologies and labelings (with bridge sum factors),
Sum over one-particle excitations of all octagons.
$\Rightarrow$ Result matches data and produces prediction for higher loops!

Summing all octagons gives:

$$\mathcal{F}^{\mathrm{U}}_{k,m}(z,\bar{z})\big|_{\text{torus}} = -\frac{2k^6}{N_c^4}\Big\{g^2\left[\tfrac{17}{6}r^4 - \tfrac{7}{4}r^2 + \tfrac{11}{32}\right]t\,F^{(1)} \qquad \checkmark \text{ match}$$

$$-2g^4\left[\left[\tfrac{17}{6}r^4 - \tfrac{7}{4}r^2 + \tfrac{11}{32}\right]t\,F^{(2)} + \left[\tfrac{29}{6}r^4 - \tfrac{11}{4}r^2 + \tfrac{15}{32}\right]\frac{t^2}{4}\left(F^{(1)}\right)^2\right] \qquad \checkmark \text{ match}$$

$$+3g^6\left[\left[\tfrac{17r^4}{6} - \tfrac{7r^2}{4} + \tfrac{11}{32}\right]t\,F^{(3)} + \left[\tfrac{29r^4}{18} - \tfrac{11r^2}{12} + \tfrac{5}{32}\right]t^2\,F^{(2)}F^{(1)}\right.$$

$$\left. + \left[\tfrac{(1-4r^2)^2}{96}\right]\left(F^{(1)}\right)^3\right] + \mathcal{O}(g^8) + \mathcal{O}(1/k)\Big\}. \qquad \text{prediction!}$$

In fact, the octagon can be evaluated to much higher loop orders, is a polynomial in ladder integrals. [Coronado to appear]
$\Rightarrow$ Immediate high-loop prediction for the four-point function.

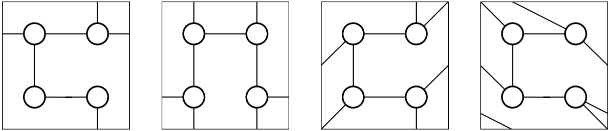
More Tests: $k = 2, 3, 4, 5, \ldots$

Small and finite k:

Few propagators $\rightarrow$ Fewer bridges $\rightarrow$ Graphs with fewer edges

$\Rightarrow$ Graphs composed of not only octagons, but bigger polygons

Example: Graphs for $k = 3$:



Hexagonalization:

Each $2n$-gon: Split into $n-2$ hexagons by $n-3$ zero-length briges.

Loop Expansion: Much more complicated!

All kinds of excitation patterns already at low loop orders

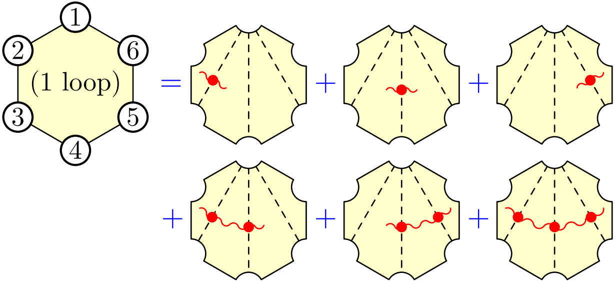
- Single particles on several adjacent zero-length (or $\ell = 1$) bridges
- Strings of excitations wrapping around operators

Finite k: One Loop: Sum over ZLB-Strings

Restrict to one loop: Only single particles on one or more adjacent zero-length bridges contribute.

$\Rightarrow$ Excitations confined to single polygons bounded by propagators.

For each polygon: Sum over all possible one-loop strings:



One-strings: [Fleury '16 Komatsu], two-strings: [Fleury '17 Komatsu].

Can in fact evaluate all one-loop polygons in terms of

$$m(z,\alpha) = g^2 \frac{(z+\bar{z})-(\alpha+\bar{\alpha})}{2} F^{(1)}(z,\bar{z})$$

Finite k, One Loop: Result

Done! Sum over all graphs, expand all polygons to their one-loop values.

Numbers of labeled graphs with assigned bridge sizes:

k:	2	3	4	5
$g=0$:	3	8	15	24
$g=1$:	0	32	441	2760

Data:

$$\mathcal{F}^{(1),\mathrm{U}}_{k,m}(z,\bar{z}) = \\ -\frac{2k^2}{N_c^2}\left\{1+\frac{1}{N_c^2}\left[\left[\tfrac{17}{6}r^4-\tfrac{7}{4}r^2+\tfrac{11}{32}\right]k^4+\left[\tfrac{9}{2}r^2-\tfrac{13}{8}\right]k^3+\left[\tfrac{1}{6}r^2+\tfrac{15}{8}\right]k^2-\tfrac{1}{2}k\right]\right\}F^{(1)}\,,$$

where $r=(m+1)/k-1/2$. $\mathcal{F}_{k,m}$: Coefficient of X^mY^{k-2-m}.

Result: For $k=2,3,4,5,\ldots$:
Matches the $\mathrm{U}(N_c)$ data $\mathcal{F}_{k,m}$, up to a copy of the planar term!

$$\mathcal{F}_{k,m}:\quad \underbrace{\texttt{Result}=(\texttt{torus data})}_{\checkmark\checkmark\checkmark}+\frac{1}{N_c^2}\underbrace{(\texttt{planar data})}_{???}$$

What does this mean?? ⇒ Puzzle.

Resolution of Mismatch: Stratification

We have based the computation on a sum over genus-one graphs of the free theory that cover all cycles of the torus.

We therefore miss contributions from purely virtual handles. In the language of hexagons, these come from graphs where a handle of the torus is traversed only by zero-length bridges (no propagators).

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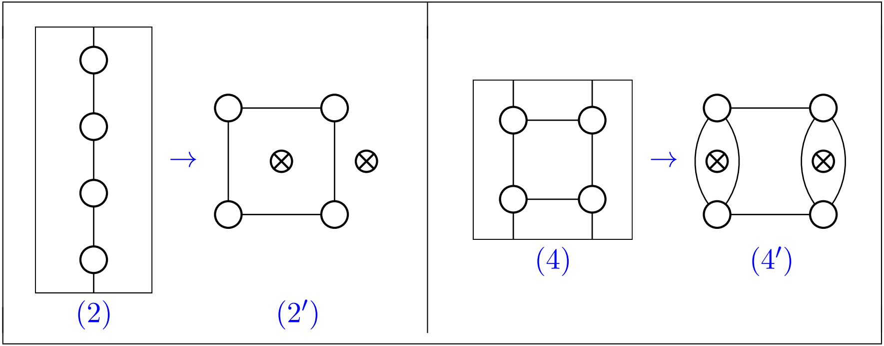
Resolution: Include graphs that are by themselves planar, but drawn on the torus, and fully tessellate the torus by zero-length bridges (as before).

This adds the missing contributions (mirror states traversing a handle that contains no propagators). But it also adds many genuinely planar (and therefore unwanted) contributions.

Get rid of these unwanted contributions by subtracting the same graphs, but now drawn on a degenerate torus where the empty handle has been pinched, such that the torus becomes a sphere with two marked points.

This goes under the name of stratification ($\mathcal{S}$ in our formula).

Stratification: Examples



Stratification & Moduli Space

Stratification is also natural from the string theory point of view: Interpret the sum over graphs as a quantization of the integration over the moduli space of Riemann surfaces.

Strebel theory: Can assign a unique quadratic differential to each point in moduli space (i.e. to each complex structure on a given surface). The differential in turn defines a graph whose cubic vertices are located at the zeros of the differential. Each edge of the graph has a natural (real) geometric length.

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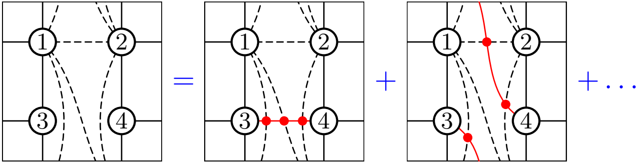
Our graphs are dual to the Strebel graphs, the cubic vertices now being our hexagons, and the edge lengths becoming our (integer) bridge lengths.

Moduli space quantization has been considered before, and one has to carefully account for contributions at the boundaries. [Chekhov 1995]

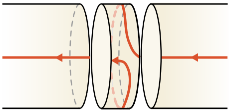
The right treatment in the known cases (stratification) is in line with the prescription explained above.

Stratification: Evaluation

Stratification contributions have many zero-length bridges.
Cannot honestly evaluate all mirror contributions. Example:



However, have simple and consistent rules for which contributions should be taken into account. In particular, mod out by Dehn twists (part of the modular group, leave complex structure invariant).



All relevant contributions can be honestly computed.

Including stratification indeed accounts for the $(\text{planar})/N_c^2$ term.

$\Rightarrow$ Now have a perfect match for $k = 2, 3, 4, 5$!

Summary & Outlook

Summary: Method to compute higher-genus terms in $1/N_\mathrm{c}$ expansion.

- Sum over free graphs, decompose into planar hexagons.
- Infinite sum over mirror excitations.
- Quantizes string moduli space integration.
- Non-trivial match with various one/two-loop correlators.

Outlook: There are many things to do!

- Study more examples: Higher loops / genus, more general operators.
- Understand details/implications of stratification beyond one loop
- Better understand summation/integration of mirror particles!
- Find a limit that can be resummed (λ and/or $1/N_\mathrm{c}$).
- Most promising: Large-charge limit. No stratification.

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Thank you!

Graph Statistics

Numbers of maximal graphs for various g and numbers of insertions:

genus :	0	1	2	3	4
$n = 2$:	1	1	4	82	7325
$n = 3$:	1	3	38	661	
$n = 4$:	2	16	760	122307	
$n = 5$:	4	132	18993		
$n = 6$:	14	1571	487293		
$n = 7$:	66	20465			
$n = 8$:	409	278905			
$n = 9$:	3078				
$n = 10$:	26044				