Quantum complexity in AdS/CFT and quantum field theory

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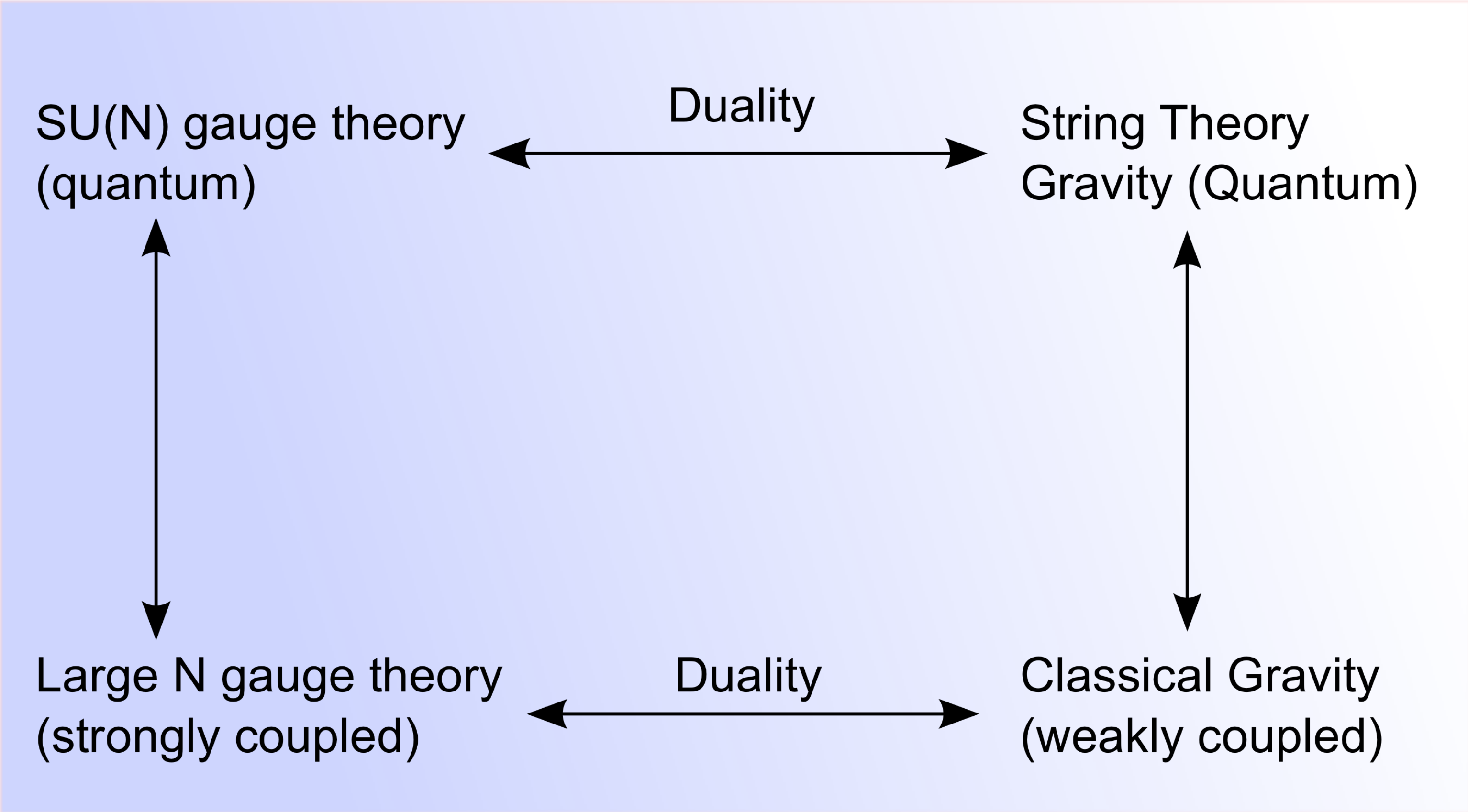
Motivation: The big picture



Motivation: Generalizations of AdS/CFT

How general is the relation between geometry and quantum field theory realized in AdS/CFT?

Motivation: AdS/CFT and quantum information

Starting point: Holographic realization of entanglement entropy

(Ryu, Takayanagi 2006)

Further information theoretic quantities discussed in AdS/CFT:

Fisher information metric, complexity, quantum error correcting codes

Simpler realizations of Hilbert spaces using Tensor Networks

Quantum information

Entangled $|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$ vs. product states $|\uparrow\rangle|\downarrow\rangle$

Density matrix $\rho = \sum_n p_n |\Psi_n\rangle\langle\Psi_n|$

Von Neumann entropy $S_{\mathrm{v}N} = -\mathrm{Tr}(\rho \ln \rho)$

Maximised when ρ diagonal with equal entries,
vanishes for pure states where $\rho^2 = \rho$

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Consider product Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Reduced density matrix

$$\rho_A = \mathrm{Tr}_B \rho_{\mathrm{tot}}$$

Entanglement entropy

$$S_A = -\mathrm{Tr}_A \rho_A \ln \rho_A$$

Analogy to black hole entropy
('Lost information' hidden in B)

Complexity:

Consider set of predefined unitary transformations in a Hilbert space

How many of these need to be applied to reach any given state
from a reference state?

Complexity

Consider a reference state $|r\rangle$ and a set of unitary operators U_1, U_2, ... (gates)

The complexity $\mathcal{C}(|\psi\rangle)$ of a state $|\psi\rangle$ is given by the minimal number of gates required to map $|r\rangle$ to $|\psi\rangle$ up to a given tolerance

$$C(|\psi\rangle) = \min \{ n \in \mathbb{N} | \, U_{i_1} \dots U_{i_n} |r\rangle = |\psi\rangle, \text{up to tolerance} \}$$

- Well-defined for pure states in finite-dimensional Hilbert spaces
- No standard definition for quantum field theories

  (Recent progress for free field theories (Myers et al, Heller et al)

  also Headrick et al, 1804.01561)

Fisher information metric

Fisher metric in information theory: Metric on space of probability distributions

Fisher information metric

Fisher metric in information theory: Metric on space of probability distributions

Probability distribution $p(x,\vec{\theta})$, x a stochastic variable, $\vec{\theta}$ a set of n external parameters

Spectrum $\gamma(x,\vec{\theta}) \equiv -\ln p(x,\theta)$

Fisher metric

$$g_{\mu\nu}(\vec{\theta}) \;=\; \int dx\, p(x,\vec{\theta})\frac{\partial\gamma(x,\theta)}{\partial\theta^\mu}\frac{\partial\gamma(x,\theta)}{\partial\theta^\nu} \;=\; \langle\partial_\mu\gamma\partial_\nu\gamma\rangle$$

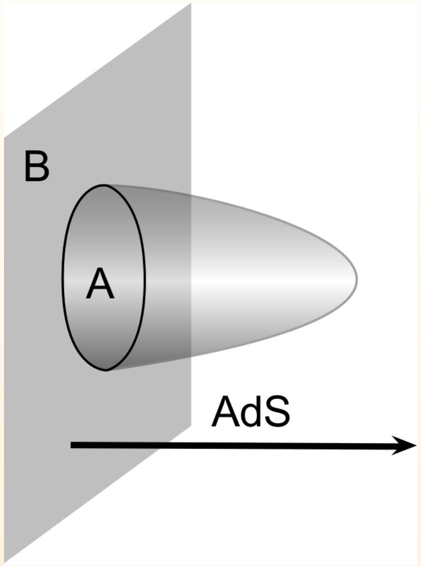
For Gaussian distribution (saddle point approximation)

$$p(x_1,\ldots,x_n) = \frac{1}{(\sqrt{2\pi}\sigma)^n}\exp\left(-\sum_{i=1}^{n}\frac{(x_i-\bar{x}_i)^2}{2\sigma^2}\right)$$

Fisher metric gives Anti-de Sitter space:

$$ds^2 = \frac{1}{\sigma^2}\left(d\bar{x}_i d\bar{x}^i + 2n d\sigma^2\right)$$

Entanglement entropy in gauge/gravity duality



Ryu-Takayanagi 2006:

$$S_A = \frac{\mathrm{Area}\gamma_A}{4G_N}$$

γ_A: Minimal area bulk surface with $\partial A = \partial \gamma_A$

Cut-off regularization near AdS boundary

Satisfies strong subadditivity

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B}$$

Entanglement entropy: Quantum field theory

Conformal field theory in 1+1 dimensions (Cardy, Calabrese):

$$S = \frac{c}{3} \ln(\ell \Lambda)$$

Reproduced by Ryu-Takayanagi result

$\Lambda \propto 1/\epsilon$, ϵ boundary cut-off in radial direction

$c = 3L/(2G_3)$

Finite temperature (at small ℓ):

$$S(\ell) = \frac{c}{3} \ln \left(\frac{1}{\pi \epsilon T} \sinh(2\pi \ell T) \right)$$

Holographic entanglement entropy: Arbitrary dimensions

J.E., Miekley 1709.07016, JHEP 1803 (2018) 034

Analytic expression in closed form for strip region:

$$S_{EE} = \frac{L^{d-1}\left(\tilde{\ell}/\epsilon\right)^{d-2}}{2(d-2)G_N} + \frac{\sqrt{\pi}L^{d-1}}{4(d-1)G_N}\frac{\tilde{\ell}^{d-2}}{z_\star^{d-2}}\sum_{\Delta m=0}^{\frac{2(d-1)}{\chi}-1}\frac{(1/2)_{\Delta m}}{\Delta m!}\frac{\Gamma\left(\frac{d}{\chi}a^{\mathrm{EE}}_{-1/2}\right)}{\Gamma\left(\frac{d}{\chi}a^{\mathrm{EE}}_{0}\right)}\left(\frac{z_\star}{z_h}\right)^{\Delta m d} \tag{5.5b}$$

$$\times_{\frac{3d-2}{\chi}+1}F_{\frac{3d-2}{\chi}}\left(1, a^{\mathrm{EE}}_{-\frac{1}{2}}, \dots, a^{\mathrm{EE}}_{\frac{d}{\chi}-\frac{3}{2}}, b^{\mathrm{EE}}_{\frac{1}{2}}, \dots, b^{\mathrm{EE}}_{\frac{2(d-1)}{\chi}-\frac{1}{2}}; a^{\mathrm{EE}}_{0}, \dots, a^{\mathrm{EE}}_{\frac{d}{\chi}-1}, b^{\mathrm{EE}}_{1}, \dots, b^{\mathrm{EE}}_{\frac{2(d-1)}{\chi}}; \left(\frac{z_\star}{z_h}\right)^{\frac{2(d-1)d}{\chi}}\right)$$

z_*: Turning point of minimal surface

Given implicity in terms of strip width ℓ

Holographic entanglement entropy: Arbitrary dimensions

Entanglement density

$$\sigma = \frac{S(T) - S(T=0)}{vol(A)}$$

Consider strip entangling region $A = a \cdot \ell$

For small ℓ: $S(T) - S(T=0) = \langle T_{tt} \rangle \cdot a \cdot \ell \Rightarrow \sigma \propto \ell$

Blanco, Casini, Hung, Myers; Bhattacharya, Nozaki, Takayanagi, Ugajin

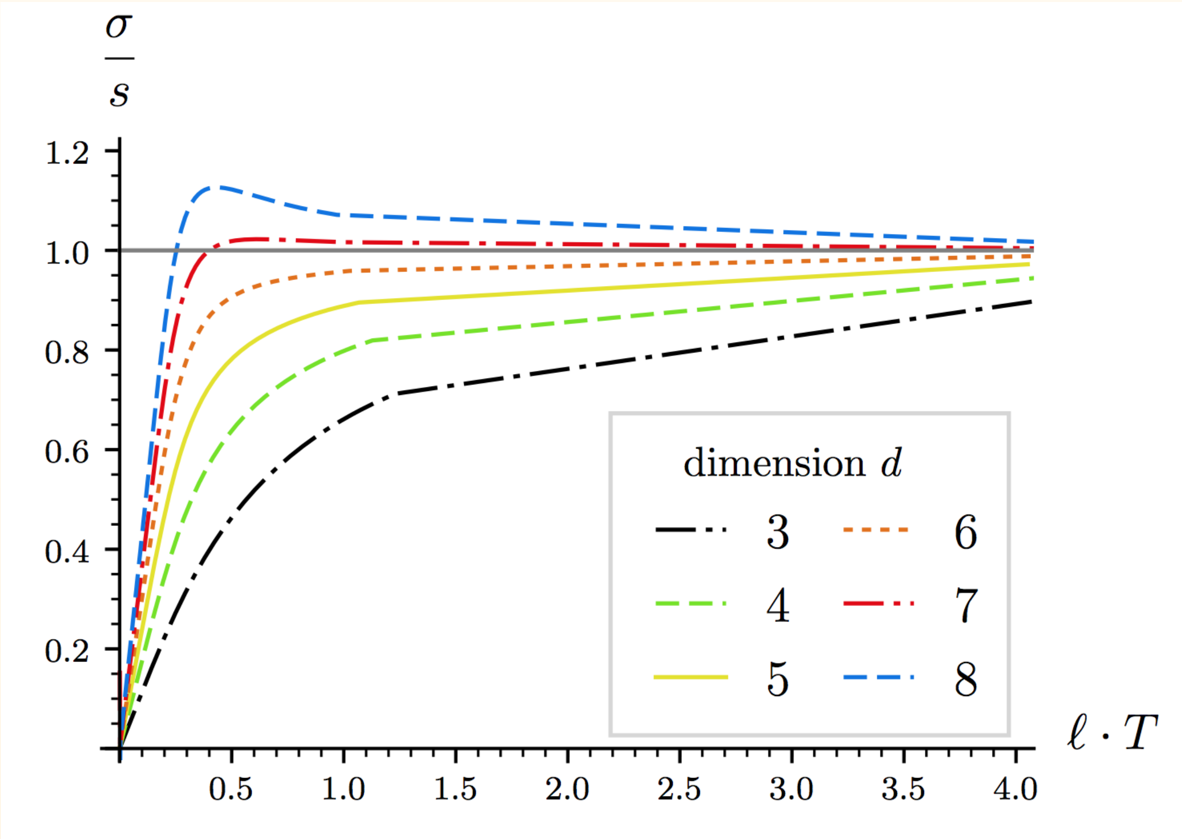
Modular Hamiltonian, positivity of relative entropy

For large ℓ: $S(T) = s \cdot V - \Delta\alpha \cdot a + \ldots$

Area theorem Casini: $\Delta\alpha > 0$

Holographic entanglement entropy: Arbitrary dimensions

J.E., Miekley 1709.07016



Non-monotonic behaviour signals violation of area theorem

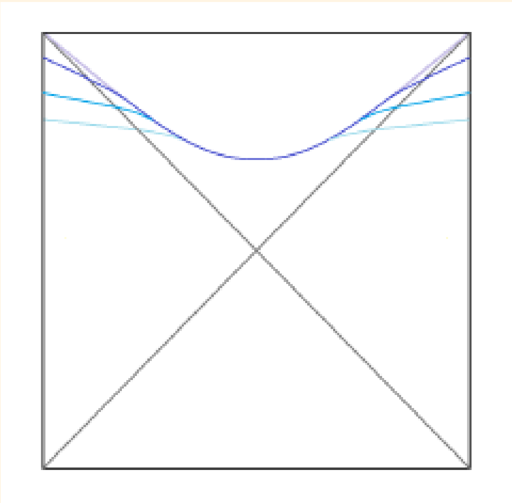
(cf. Gushterov, O'Bannon, Rodgers 1708.09376)

Holographic complexity proposals

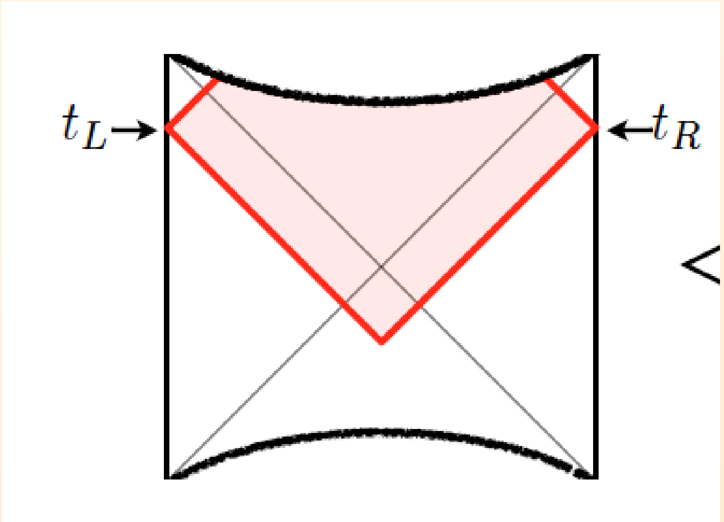
Susskind et al: 1402.5674, 1509.07876

Consider evolution of two copies of a CFT initially entangled in the thermofield double state

‘Complexity = Volume’:
Volume of Einstein-Rosen bridge



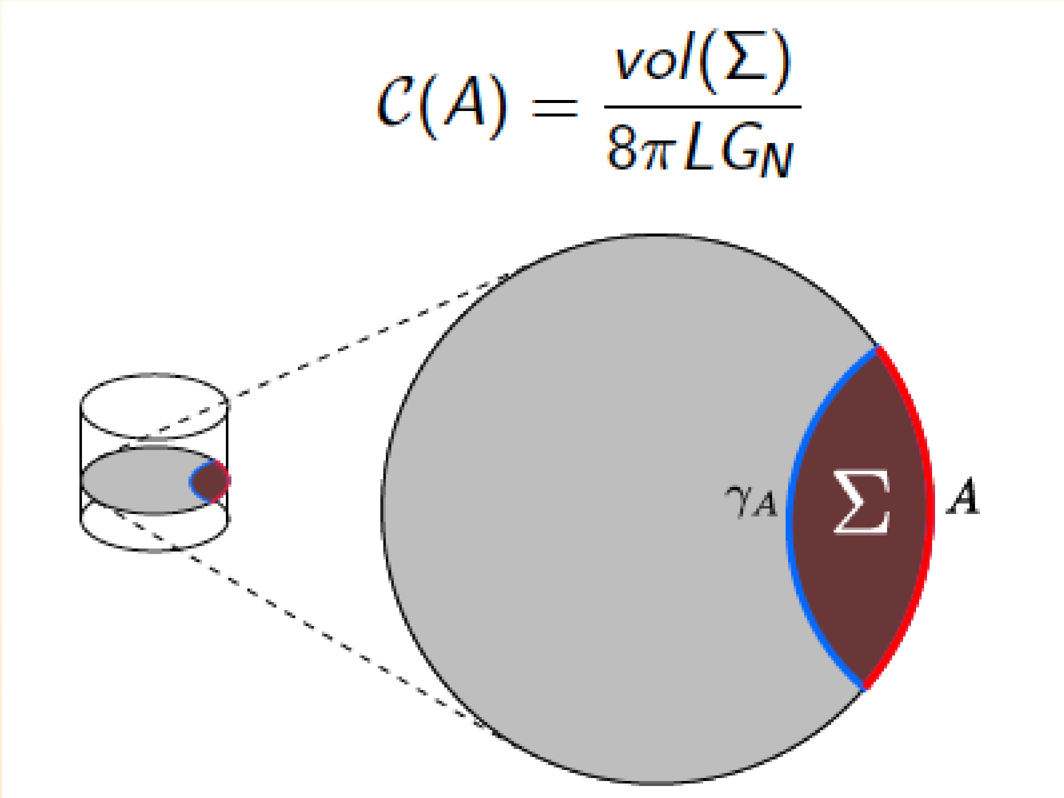
‘Complexity = Action’:
Action on Wheeler-de Witt patch



Both evolve linearly with time

Holographic subregion complexity

Alishahiha PRD 92 (2015):



Kinematic space

Czech, Lamprou, McCandlish, Sully PRD 90 (2014), JHEP (2015), (2016)

- Ryu-Takayanagi proposal offers one-to-one correspondence between

  entangling intervals and boundary-anchored geodesics

- Kinematic space exploits this correspondence

  treats geodesics and entangling intervals on equal footing

- From CFT point of view:

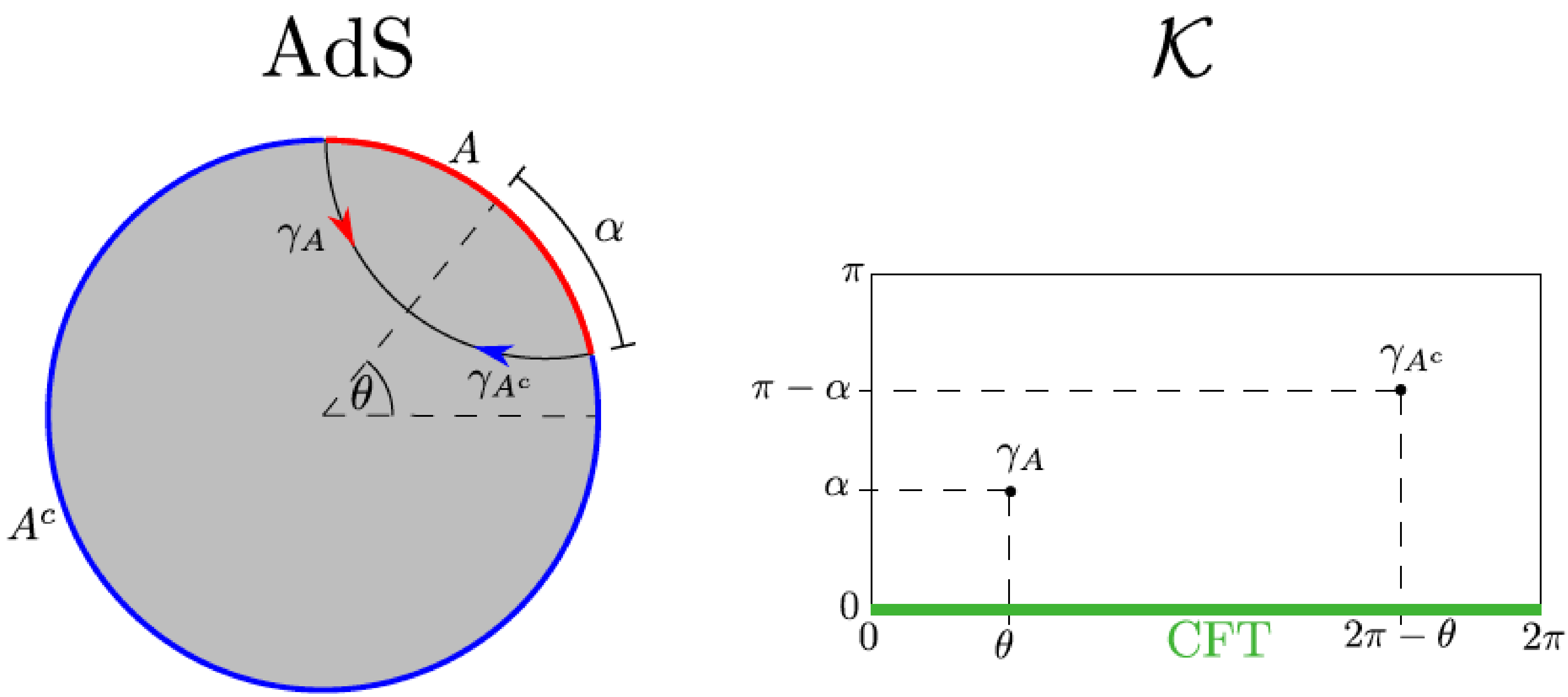
  Auxiliary Lorentzian geometry whose metric is defined from conditional mutual information

  Organizes entanglement pattern of CFT

- From bulk gravity:

  Space of bulk geodesics studied in integral geometry

We choose the opening angle α and the position θ of the center of geodesics/entanglement intervals as coordinates on $\mathcal{K}$.



Kinematic space

Metric:

$$ds^2 = -\frac{1}{2}\partial_\alpha^2 S(-d\alpha^2 + d\theta^2)$$

$S(\alpha)$ entanglement entropy of interval with opening angle α

Crofton form:

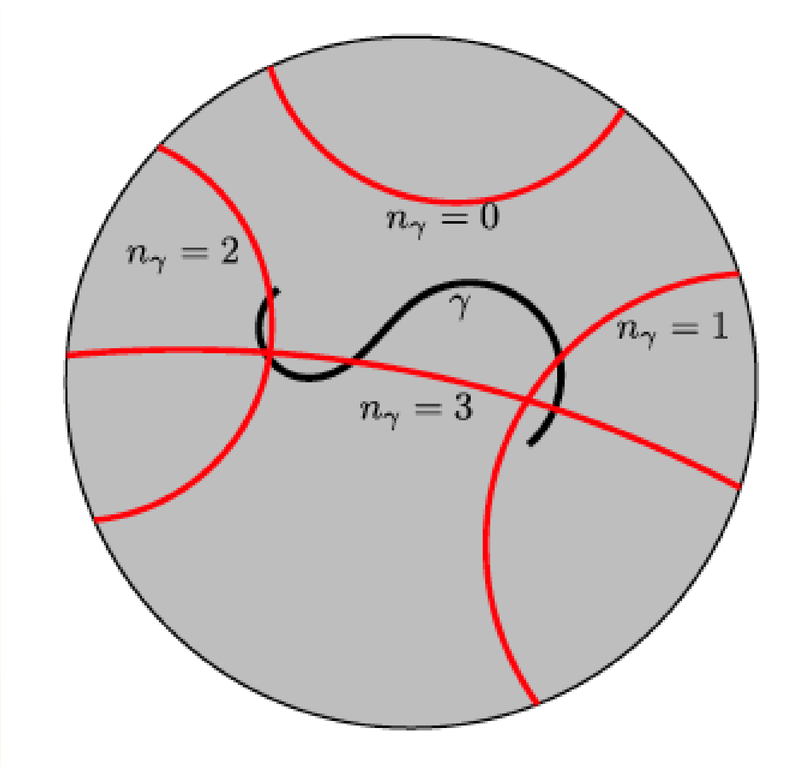
$$\omega = -\frac{1}{2}\partial_\alpha^2 S d\theta \wedge d\alpha$$

Length of any bulk curve:

$$\frac{\ell[\gamma]}{4G_N} = \frac{1}{4}\int_{\mathcal{K}} \omega n_\gamma$$

$n_\gamma(\theta, \alpha)$: number of intersection points of the geodesic defined by (θ, α) with γ

Kinematic space: Length of curve in AdS$_3$



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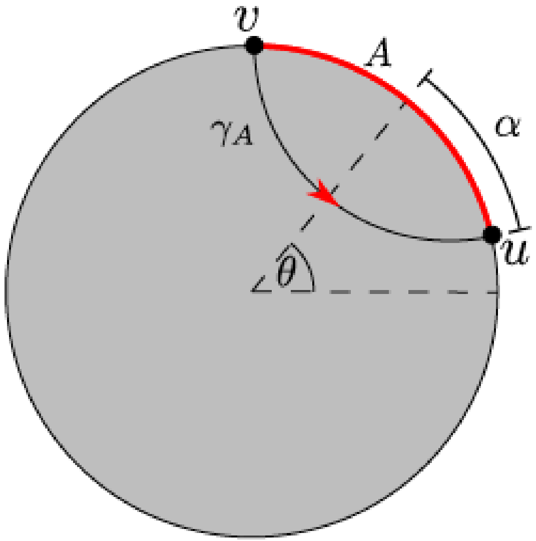
Interpretation of Crofton form

- We can interpret ω in terms of **conditional mutual information**

$$I(B, C|A) = S(AB) + S(AC) - S(ABC) - S(A) \ .$$

- In the coordinates $u = \theta - \alpha$ and $v = \theta + \alpha$ we have

$$ds^2_{\mathcal{K}} = \partial_u \partial_v S du dv \quad \text{and} \quad \omega = \partial_u \partial_v S du \wedge dv \ .$$



So ω can be interpreted as an infinitesimal conditional mutual information of neighboring intervals:

$$I(B, C|A) = \partial_u \partial_v S du dv \sim \omega \ .$$

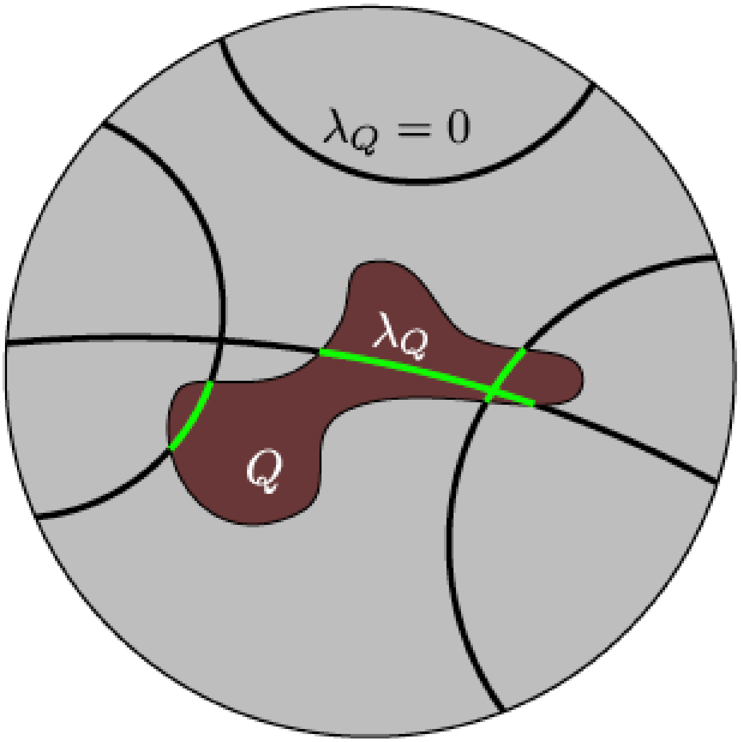
Bulk volumes from kinematic space

R. Abt, J.E., H. Hinrichsen, C. Melby-Thompson, R. Meyer, C. Northe, I. Reyes 1710.01327;
R. Abt, J.E., M. Gerbershagen, C. Melby-Thompson, C. Northe 1805.10298

Calculate volume of bulk region Q

- We probe Q with geodesics.
- Each geodesic (θ, α) is weighted with the the length $\lambda_Q(\theta, \alpha)$ of the segment that intersects Q (**chord length**).
- The volume of Q is then given by the integral over λ_Q:

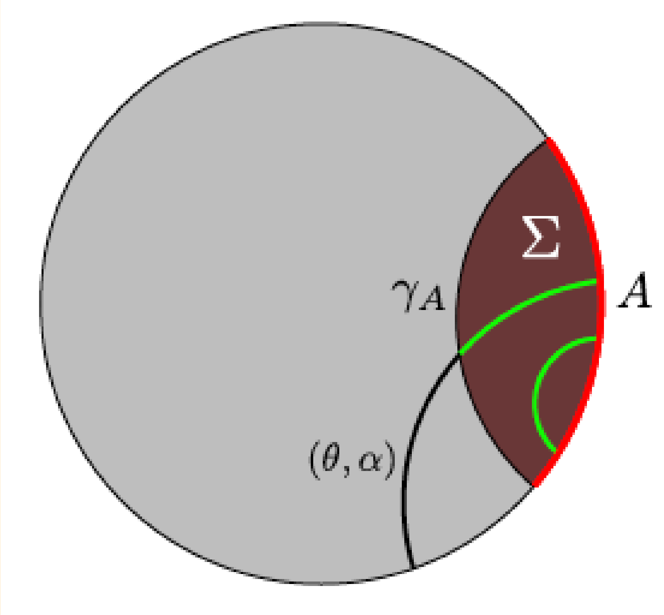
$$\frac{vol(Q)}{4G_N} = \frac{1}{2\pi}\int_{\mathcal{K}} \omega\lambda_Q$$



Use general result to calculate volume bounded by Ryu-Takayanagi surface:

$$\frac{\mathrm{vol}(\Sigma)}{4G_N} = \frac{1}{2\pi}\int_{\mathcal{K}} \omega\lambda_\Sigma$$

$\lambda_\Sigma(\theta, \alpha)$ is the length of the part of the geodesic given by (θ, α) that lies inside Σ



$\Rightarrow$ Use this volume formula to evaluate

holographic subregion complexity proposal

$$\mathcal{C}(A) = \frac{\mathrm{vol}(\Sigma)}{8\pi L G_N}$$

Holographic subregion complexity in kinematic space

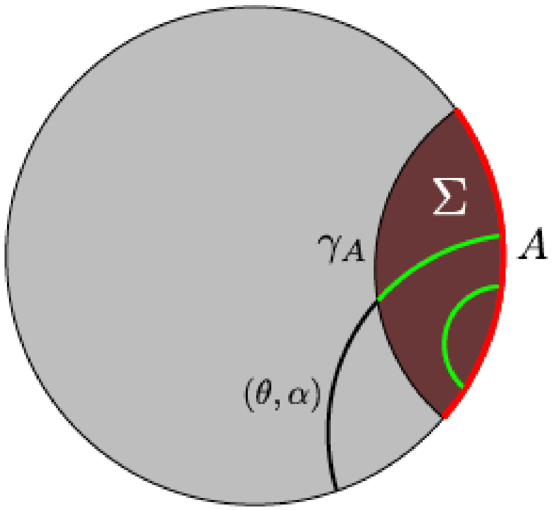
We obtain an expression for holographic subregion complexity in terms of entanglement entropies:

$$\mathcal{C}(A) = \frac{3}{32\pi^2 c} \int_{\mathcal{K}} d\theta d\alpha \int_{\Delta} d\theta' d\alpha' \partial^2_{\alpha'} S(\alpha') \partial^2_{\alpha} S(\alpha) \; .$$

- Where we used $c = \frac{3L}{2G_N}$ and

$$\frac{\lambda_\Sigma(\theta,\alpha)}{4G_N} = \frac{1}{4}\int_{\mathcal{K}} \omega n_\gamma = \frac{1}{4}\int_{\Delta} \omega \; .$$

- $\Delta(\theta,\alpha)$ is the set of all geodesics that intersect the part of (θ,α) that lies inside of Σ.



Holographic subregion complexity proposal: Field theory insights

$$\mathcal{C}(A) = \frac{3}{32\pi^2 c} \int_{\mathcal{K}} d\theta d\alpha \int_{\Delta} d\theta' d\alpha' \partial^2_{\alpha'} S(\alpha') \partial^2_{\alpha} S(\alpha) \ .$$

The proposal relates complexity to entanglement entropy

R.h.s. is defined purely in terms of field-theory quantities

Holographic entanglement entropy: Finite temperature

Hubeny, Rangamani; Takayanagi

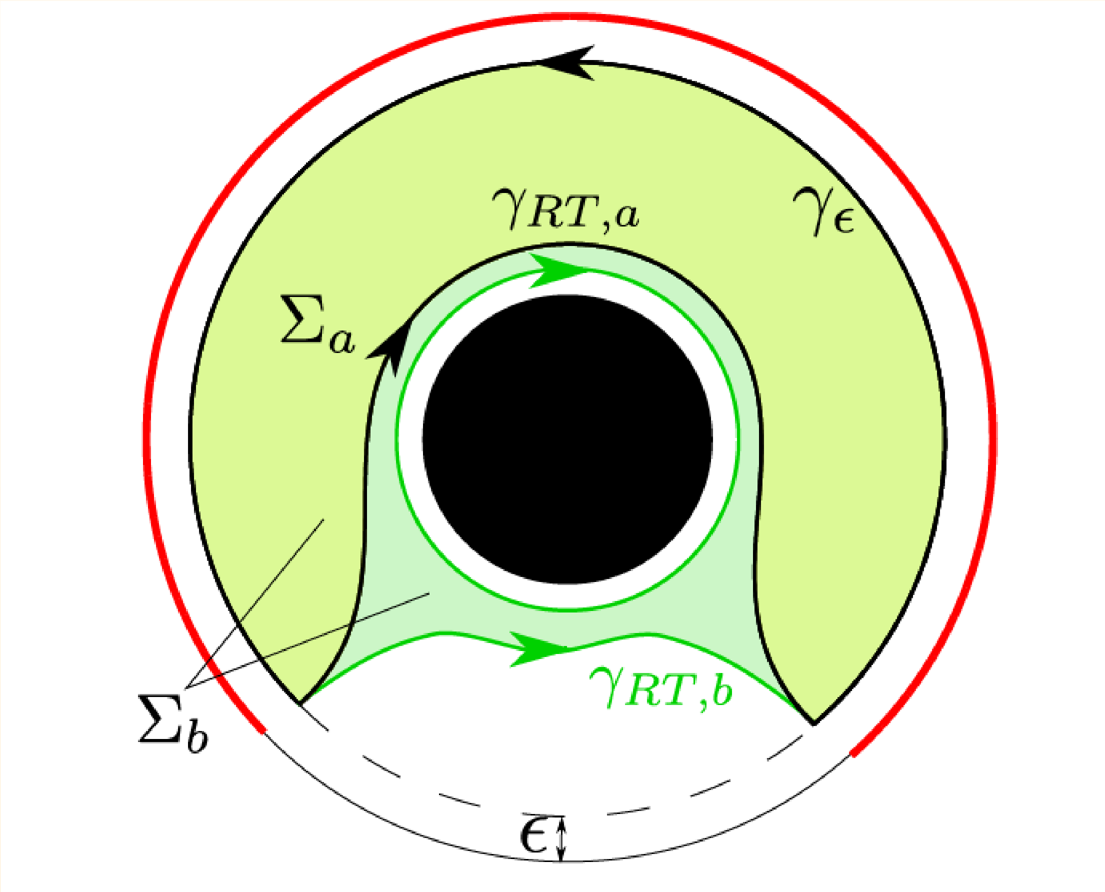


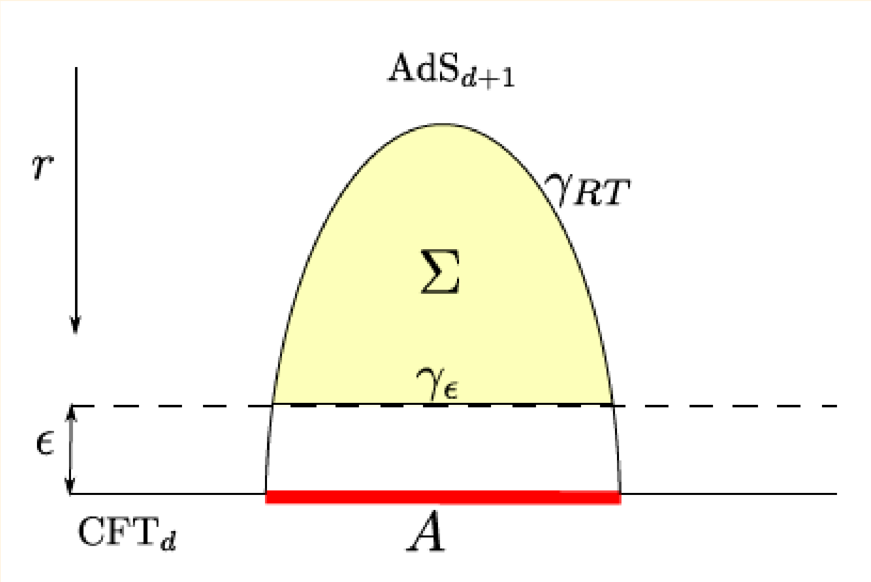
figure by Raimond Abt

Holographic Subregion Complexity:

Abt, J.E., Hinrichsen, Melby-Thompson, Meyer, Northe, Reyes 1710.01327:

Consider volume proposal in the form

$$\mathcal{C}(\mathcal{A}) = -\frac{1}{2}\int_{\Sigma} R d\sigma$$



Holographic Subregion Complexity

For $d = 2$, evaluate complexity using Gauss-Bonnet theorem

For black hole:

$$\mathcal{C} = \frac{x}{\epsilon} - \pi$$

$$\Delta\mathcal{C} = 2\pi$$

Holographic Subregion Complexity

For $d = 2$, evaluate complexity using Gauss-Bonnet theorem

For black hole:

$$\mathcal{C} = \frac{x}{\epsilon} - \pi$$

$$\Delta\mathcal{C} = 2\pi$$

This is reproduced using random tensor networks

Many-body states of many qubits have a vast Hilbert space, $\mathrm{dim}\mathbb{H} \sim 2^N$

Tensor networks: Approximation method for determining states, in particular for ground states of local Hamiltonians

Reduction to small corner of Hilbert space

μ_k labels a complete basis in the Hilbert space $\mathbb{H}_k$; dimension D_k

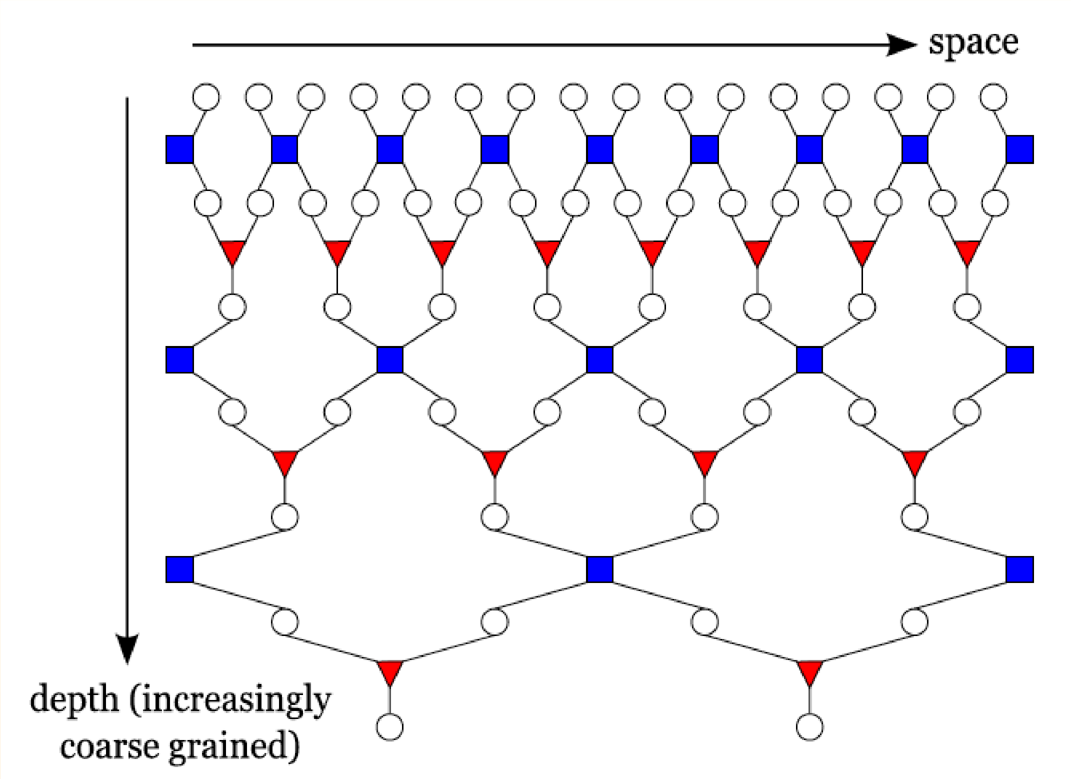
Tensor $T_{\mu_1,\ldots,\mu_n}$ corresponds to wave function of a quantum state in the product Hilbert space $\otimes_{k=1}^n \mathbb{H}_k$

$$|T\rangle = \sum_{\{\mu_k\}} T_{\mu_1\mu_2\ldots\mu_n} |\mu_1\rangle \otimes |\mu_2\rangle \otimes \cdots \otimes |\mu_n\rangle$$

A tensor network is obtained by contracting indices

Contracting two tensors by an internal line corresponds to the projection onto a maximally entangled state

Entanglement entropy: Tensor networks



MERA networks:

Implement RG idea

Networks defined on discretizations of hyperbolic space

cf. AdS/CFT: Extra dimension corresponds to RG scale

MERA Network:
Entanglement entropy bounded from above by Ryu-Takayanagi formula
(Swingle 0905.1317)

Random tensor network:

Observables obtained by averaging over tensor network states
built from random tensors living on a fixed graph

Random tensor networks may be mapped to an associated Ising model
Hayden et al 1601.01694

Average value of second Renyi entropy related to partition function

$$\overline{\mathrm{Tr}(\rho_A^2)} \sim Z_A$$

Numerical simulation of entanglement entropy in black hole background

Map to associated Ising model 1710.01327

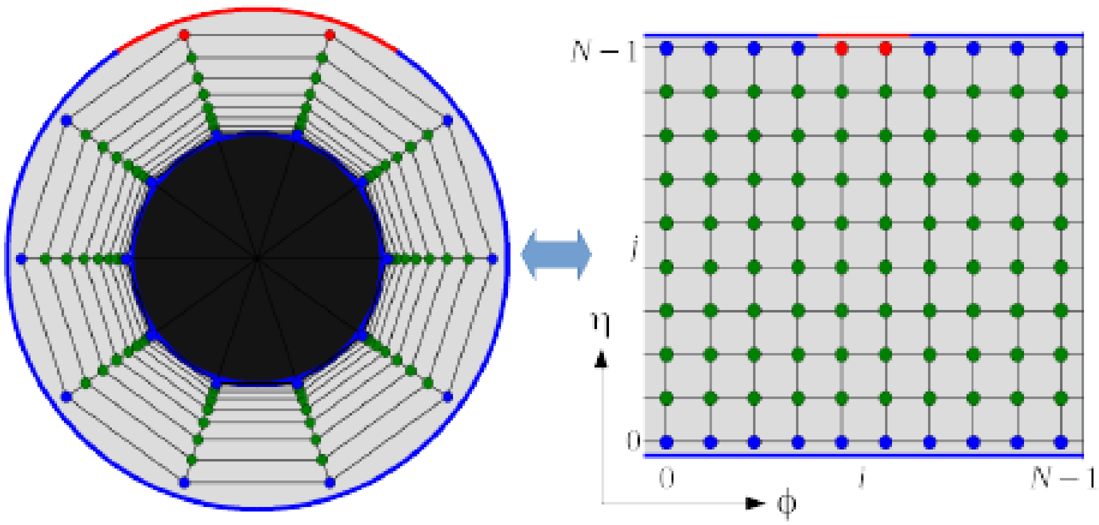


FIG. 6. BTZ black hole (left) with radial coordinate $\arctan(r/L)$ mapped to conformal coordinates ϕ, η (right) where an Ising model on a square lattice is embedded. The top and the bottom row of spins are fixed according to the respective boundary conditions (red=↑, blue=↓) while the green spins are allowed to fluctuate.

Complexity: Random tensor networks

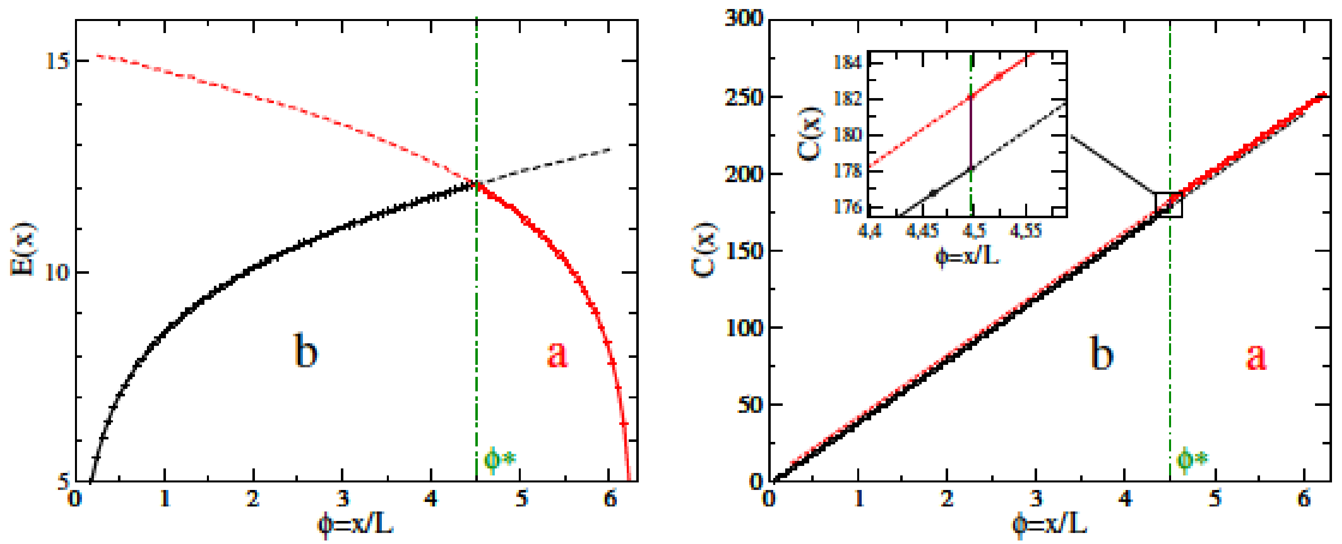


FIG. 8. Numerical results on a lattice with 200×200 sites for a BTZ black hole with mass $M = 0.1$. Left: Numerically measured entanglement of the two solutions as functions of the subregion size. As can be seen, the lines cross precisely at the theoretically expected transition point, marked by the vertical green dashed line. Right: Corresponding complexity, reproducing the linear law. The inset shows a magnification where the discontinuous jump occurs.

Overview of recent results

- Relating Fisher information for mixed states to volume changes induced by metric and operator perturbations

  Banerjee, J.E., Sarkar 1701.02319

- Time evolution   J.E., Fernandez, Flory, Megias, Straub, WItkowski 1705.04696

- Holographic subregion complexity for 1+1-dimensional field theories at finite temperature from gravity and tensor network analysis

  R. Abt, J.E., H. Hinrichsen, C. Melby-Thompson, R. Meyer, C. Northe, I. Reyes 1710.01327

- Holographic subregion complexity from kinematic space

  R. Abt, J.E., M. Gerbershagen, C. Melby-Thompson, C. Northe 1805.10298

- General properties of modular Hamiltonian   Abt, J.E. to appear

Conclusion and outlook

- Relation between holographic subregion complexity and entanglement entropy

  based on kinematic space

- Further example for relation between information and geometry

- Holographic subregion complexity evaluated in tensor network approach

- Outlook:

  Complexity for interacting QFT

  Consider CFT in 1+1 dimensions, use of OPE and integrability?