Self-correction in Wegner’s 3D Ising lattice gauge theory

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Outline

1. Introduction & Motivation
2. Wegner’s Ising lattice gauge model
3. Homology
4. Numerical results
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Passive device that stores information reliably
- At finite temperature.
- Without fine-tuning, i.e. its parameters can fluctuates a little.
- With no imposed symmetry.

We are interested in systems
- Composed of finite state systems (Ising spins).
- Interacting locally on a lattice.

Contrasts with
- Local cellular automata.
- Fault-tolerant computer.

which have time-dependent hamiltonians and/or consume power.
**2D Ferromagnetic Ising model**

\[ H = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z \]

- Degenerate ground state = 1 bit of information.
- At finite temperature, error droplet \( D \) has energy cost \( 2J|\partial D| \).
  - Large droplets suppressed by Boltzmann factor \( e^{-2\beta J|\partial D|} \).
- Magnetization \( \langle M \rangle = \frac{1}{N} \sum_j \langle \sigma_j^z \rangle \) retains information.
- This protection is due to a symmetry \( U = \prod_j \sigma_j^x \), \( UHU = H \).
- Symmetry is spontaneously broken below Curie temperature (Landau-Ginsburg).
Consider a symmetry-breaking perturbation (magnetic field)

\[ H = J \sum_{\langle i,j \rangle} \sigma^z_i \sigma^z_j + B \sum_j \sigma^z_j \]

Unique ground state with extensive energy bias, one meta-stable states.

At finite temperature, in meta stable state, error droplet \( D \) has energy cost \( 2J|\partial D| - 2B|D| \).

- Large droplets favored by Boltzmann factor \( e^{-\beta J|\partial D|+\beta B|D|} \).

All initial states are driven to state with \( \langle M \rangle < 0 \).

Self-correcting behavior requires symmetry, does not survive generic perturbation.

Extends to all systems with local order parameter \( A \), by adding local perturbation \( \propto A \) to the Hamiltonian.
Why care? 1. Classical

- Self-correcting memory is possible under local cellular automaton, even in 1D (Gács’86)
  - Why not time-independent Hamiltonians, e.g. including clock?
- Gibbs’ phase rule suggests it should be impossible for time-independent Hamiltonians
  - Co-existence of $d + 1$ phases only occurs in codimension-$d$ region of phase space.
  - Only on line $B = 0$ of $(B/J, T/J)$ phase diagram.
Stable quantum memory = quantum hard drive

- Proposals in 2 and 3 dimensions (Bacon'06, Hamma et al. ’08, Hutter et al. ’12, Pedrocchi et al. ’12, Michnicki’14, Brell’16)
- ... have been disputed (Haah & Preskill’12, Landon-Cardinal & Poulin ’13, Landon-Cardinal et al. ’15)
- Known to be possible in 4D (Dennis et al. ’02).
- Stable quantum memory is more difficult to achieve:
  - If $E_0$ and $E_1$ differ even by a constant $\epsilon$, then
    $\psi(t) = \alpha \phi_0 + e^{-i\epsilon t/\hbar} \beta \phi_1$ leads to decoherence.
  - Contrasts with extensive energy bias leading to thermal instability.
- Should understand classical case before moving on to quantum.
  - I.e. Ising model with field on single site is stable.
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The model

- Cubic lattice with Ising spins $\sigma^z_\epsilon = \pm 1$ on each edge.
- Hamiltonian $H = -J \sum_P A_P$ where $A_P = \prod_{e \in \partial P} \sigma^z_\epsilon$.
- Local (gauge) symmetry associated to each vertex
  $U_v = \prod_{e : v \in \partial e} \sigma^x_\epsilon$.
  $U_v H U_v = H$

- Thermal state is gauge invariant.
- Extensive degeneracy.
- $U_v \sigma^z_\epsilon U_v = -\sigma^z_\epsilon$ for $v \in \partial e$
- So $\langle \sigma^z_\epsilon \rangle = 0$.
- Elitzur’s theorem.
Gauge invariant observables are Wilson loops, \( W_{\ell} = \prod_{e \in \ell} \sigma^z_e \) such that \( \partial \ell = 0 \).

\[ U_v W_{\ell} U_v = W_{\ell} \]

Consider homologically trivial loop \( \ell = \partial S \) for some surface \( S \).

\[ W_{\ell} = \prod_{P \subset S} A_P \]

\[
\langle W_{\ell} \rangle = \frac{\sum_{\{\sigma\}} \prod_P e^{-\beta \sigma^z_{P,1} \sigma^z_{P,2} \sigma^z_{P,3} \sigma^z_{P,4}} W_{\ell}}{\sum_{\{\sigma\}} \prod_P e^{-\beta \sigma^z_{P,1} \sigma^z_{P,2} \sigma^z_{P,3} \sigma^z_{P,4}}} \tag{1}
\]

\[
= \frac{\sum_{\{\sigma\}} \prod_P (1 + \sigma^z_{P,1} \sigma^z_{P,2} \sigma^z_{P,3} \sigma^z_{P,4} \tanh \beta) W_{\ell}}{\sum_{\{\sigma\}} \prod_P (1 + \sigma^z_{P,1} \sigma^z_{P,2} \sigma^z_{P,3} \sigma^z_{P,4} \tanh \beta)} \tag{2}
\]

Expanding the red product gives the sum over all trivial loops \( \ell = \partial S \), with a weight \( (\tanh \beta)^{|S|} \).

Since \( \sum_{\sigma} \sigma = 0 \) and \( \sum_{\sigma} \sigma^2 = 2 \), the leading contribution to \( \langle W_{\ell} \rangle \) at low \( \beta \) is \( (\tanh \beta)^{|S|} \sim e^{-\alpha |S|} \).
Consider homologically trivial loop $\ell = \partial S$ for some surface $S$.

- $W_\ell = \prod_{P \in S} A_P$

In the ground state, we have $A_P = 1$.

A flipped spin costs $8J$ in energy (4 neighboring plaquettes).

In a diluted spin flip regime, the probability of a flipped spin is thus $p_f \propto e^{-8\beta J}$.

$\langle W_\ell \rangle = (\text{prob. } \ell \text{ has even # flips}) - (\text{prob } \ell \text{ has odd # flips})$

$\propto e^{-\gamma |\ell|}$

High temperature

$\langle W_\ell \rangle \propto e^{-\alpha |S|}$

Area law

Low temperature

$\langle W_\ell \rangle \propto e^{-\gamma |\partial S|}$

Perimeter law

Phase transition at some finite $T$. 
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Homologically non trivial loops

- On some lattices, there exists $\ell \neq \partial S$ with $\partial \ell = 0$.
  - Bulk with a puncture.
  - Special boundary conditions.
  - Hyper-torus.
- $W_\ell$ is not a product of $A_P$, so $\langle W_\ell \rangle$ is not fixed in the ground state: degeneracy beyond gauge symmetry.
  - One encoded bit.

With all spins up, $W_z = +1$.

- Extensive degeneracy.
- Flipping all spins in $xy$ plane sets $W_z = -1$.
- $L_{xy} H L_{xy} = H$.

How does phase transition manifests itself on nontrivial $W_\ell$?
Magnetic field

- A magnetic field $-B \sum_e \sigma^Z_e$ breaks the gauge symmetry.
- Elitzur’s theorem tells us that system doesn’t magnetize, $\chi = 0$.
- Average energy difference between two sectors $W_z = \pm 1$ is at most $L^2$: sub-extensive.
  - Less when considering gauge group average.

With field...

- Unique ground state $\sigma^Z_e = 1$.
- In sector $W_z = 1$.
- Lowest energy state in sector $W_z = -1$ obtained from $L_{xy}$.
- Energy difference $2BL^2$ (sub-ext.).
- Much like 2D ferromagnetic Ising, but entropy fluctuates error membrane out of plane.
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Simulate thermal dynamics using Metropolis-Hastings MCMC.
  - Use single-spin flip.
  - Use gauge flip.

Begin in lowest energy state with $W_z = -1$.

Ramp up temperature to $T_{\text{hold}}$.

Hold temperature for some macroscopic time.
  - $10,000 \times L^3$ MC updates.

Ramp down temperature to $T = 0$, and measure $W_z$. 
Numerical results

Experiment results

Without magnetic field $B/J = 0$

With magnetic field $B/J = 0.2$

Topological
Paramagnetic
Disordered

1.25
T
hold
$h/J$

Glassy

1.32

$T_{\text{hold}}/J$

Disordered

1.32
1.25

Glassy

Topological

Paramagnetic

$h/J$
Self-correction is interesting in the classical and quantum setting.

Self-correction is ruled out in symmetry broken phases with local order parameter.

Elitzur’s theorem rules out a local order parameter in systems with gauge symmetry.

Well known phase transition $\langle W_{\ell} \rangle \propto e^{-\alpha |S|}$ vs $\propto e^{-\gamma |\partial S|}$.

Additional degeneracy from homology.
- No extensive energy splitting from any local field.
- Numerical results consistent with finite $T_{\text{hold}}$ and $h$ phase transition.
- Unconventional order parameter at finite $T$: defined algorithmically.

Open questions
- Can we prove stability?
- Complete phase diagram?
- Reduced to stability of 2D quantum model?