Self-correction in Wegner's 3D Ising lattice gauge theory

David Poulin, Roger Melko, and Matthew Hastings

Institut Quantique & Département de Physique
Université de Sherbrooke
Canadian Institute for Advanced Research

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Outline

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Self correcting memory

- Passive device that stores information reliably
  - At finite temperature.
  - Without fine-tuning, i.e. its parameters can fluctuates a little.
  - With no imposed symmetry.
- We are interested in systems
  - Composed of finite state systems (Ising spins).
  - Interacting locally on a lattice.

  Contrasts with
  - Local cellular automata.
  - Fault-tolerant computer.

  which have time-dependent hamiltonians and/or consume power.

2D Ferromagnetic Ising model

$$H = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z$$

- Degenerate ground state = 1 bit of information.
- At finite temperature, error droplet D has energy cost $2J|\partial D|$.
    - Large droplets suppressed by Boltzmann factor $e^{-2\beta J|\partial D|}$.
- Magnetization $\langle M \rangle = \frac{1}{N} \sum_j \langle \sigma_j^z \rangle$ retains information.
- This protection is due to a symmetry $U = \prod_j \sigma_j^x$, $UHU = H$.
- Symmetry is spontaneously broken below Curie temperature (Landau-Ginsburg).

2D Ferromagnetic Ising model

- Consider a symmetry-breaking perturbation (magnetic field)

$$H = J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z + B \sum_j \sigma_j^z$$

- Unique ground state with extensive energy bias, one meta-stable states.
- At finite temperature, in meta stable state, error droplet D has energy cost $2J|\partial D| - 2B|D|$.
  - Large droplets favored by Boltzmann factor $e^{-\beta J|\partial D| + \beta B|D|}$.
- All initial states are driven to state with $\langle M \rangle < 0$.
- Self-correcting behavior requires symmetry, does not survive generic perturbation.
- Extends to all systems with local order parameter A, by adding local perturbation $\propto A$ to the Hamiltonian.

Why care? 1. Classical

- Self-correcting memory is possible under local cellular automaton, even in 1D (Gács'86)
  - Why not time-independent Hamiltonians, e.g. including clock?
- Gibbs' phase rule suggests it should be impossible for time-independent Hamiltonians
  - Co-existence of $d+1$ phases only occurs in codimension-d region of phase space.
  - Only on line $B=0$ of $(B/J, T/J)$ phase diagram.

Why care? 2. Quantum

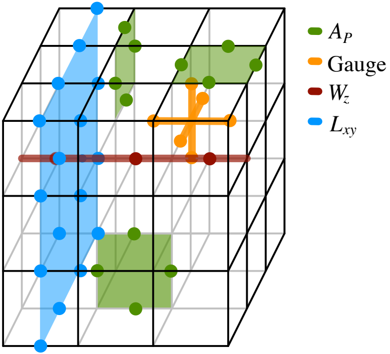
Stable quantum memory = quantum hard drive

- Proposals in 2 and 3 dimensions (Bacon'06, Hamma et al. '08, Hutter et al. '12, Pedrocchi et al. '12, Michnicki'14, Brell'16)
- ... have been disputed (Haah & Preskill'12, Landon-Cardinal & Poulin '13, Landon-Cardinal et al. '15)
- Known to be possible in 4D (Dennis et al. '02).
- Stable quantum memory is more difficult to achieve:
  - If E_0 and E_1 differ even by a constant ϵ, then $|\psi(t)\rangle = \alpha|\phi_0\rangle + e^{-i\epsilon t/\hbar}\beta|\phi_1\rangle$ leads to decoherence.
  - Contrasts with extensive energy bias leading to thermal instability.
- Should understand classical case before moving on to quantum.
  - I.e. Ising model with field on single site is stable.

Outline

The model

- Cubic lattice with Ising spins $\sigma_e^z = \pm 1$ on each edge.
- Hamiltonian $H = -J \sum_P A_P$ where $A_P = \prod_{e \in \partial P} \sigma_e^z$.
- Local (gauge) symmetry associated to each vertex $U_v = \prod_{e: v \in \partial e} \sigma_e^x$.
  - $U_v H U_v = H$



- Thermal state is gauge invariant.
- Extensive degeneracy.
- $U_v \sigma_e^z U_v = -\sigma_e^z$ for $v \in \partial e$
- So $\langle \sigma_e^z \rangle = 0$.
- Elitzur's theorem.

Wilson loops – High temperature

- Gauge invariant observables are Wilson loops, $W_\ell = \prod_{e\in\ell}\sigma_e^z$ such that $\partial\ell = 0$.
  - $U_v W_\ell U_v = W_\ell$
- Consider homologically trivial loop $\ell = \partial S$ for some surface S.
  - $W_\ell = \prod_{P\in S} A_P$

$$\langle W_\ell\rangle = \frac{\sum_{\{\sigma\}}\prod_P e^{-\beta\sigma^z_{P,1}\sigma^z_{P,2}\sigma^z_{P,3}\sigma^z_{P,4}}W_\ell}{\sum_{\{\sigma\}}\prod_P e^{-\beta\sigma^z_{P,1}\sigma^z_{P,2}\sigma^z_{P,3}\sigma^z_{P,4}}} \tag{1}$$

$$= \frac{\sum_{\{\sigma\}}\prod_P(1+\sigma^z_{P,1}\sigma^z_{P,2}\sigma^z_{P,3}\sigma^z_{P,4}\tanh\beta)W_\ell}{\sum_{\{\sigma\}}\prod_P(1+\sigma^z_{P,1}\sigma^z_{P,2}\sigma^z_{P,3}\sigma^z_{P,4}\tanh\beta)} \tag{2}$$

- Expanding the red product gives the sum over all trivial loops $\ell = \partial S$, with a weight $(\tanh\beta)^{|S|}$.
- Since $\sum_\sigma \sigma = 0$ and $\sum_\sigma \sigma^2 = 2$, the leading contribution to $\langle W_\ell\rangle$ at low β is $(\tanh\beta)^{|S|} \sim e^{-\alpha|S|}$.

Wilson loops – Low temperature

- Consider homologically trivial loop $\ell = \partial S$ for some surface S.
  - $W_\ell = \prod_{P\in S} A_P$
- In the ground state, we have $A_P = 1$.
- A flipped spin costs $8J$ in energy (4 neighboring plaquettes).
- In a diluted spin flip regime, the probability of a flipped spin is thus $p_f \propto e^{-8\beta J}$.
- $\langle W_\ell \rangle$ = (prob. ℓ has even # flips) - (prob ℓ has odd # flips)
  $\propto e^{-\gamma|\ell|}$

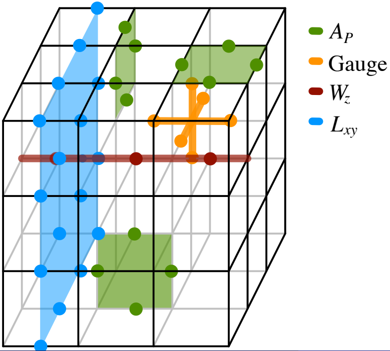
High temperature	Low temperature
$\langle W_\ell \rangle \propto e^{-\alpha|S|}$	$\langle W_\ell \rangle \propto e^{-\gamma|\partial S|}$
Area law	Perimeter law

Phase transition at some finite T.

Outline

Homologically non trivial loops

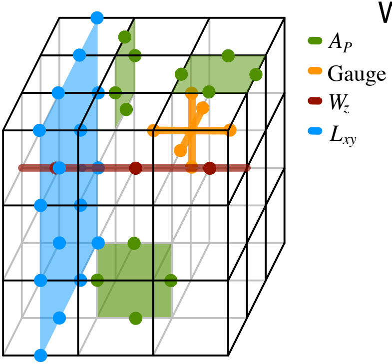
- On some lattices, there exists $\ell \neq \partial S$ with $\partial \ell = 0$.
  - Bulk with a puncture.
  - Special boundary conditions.
  - Hyper-torus.
- W_ℓ is not a product of A_P, so $\langle W_\ell \rangle$ is not fixed in the ground state: degeneracy beyond gauge symmetry.
  - One encoded bit.



- With all spins up, $W_z = +1$.
- Extensive degeneracy.
- Flipping all spins in xy plane sets $W_z = -1$.
- $L_{xy} H L_{xy} = H$.
- How does phase transition manifests itself on nontrivial W_ℓ?

Magnetic field

- A magnetic field $-B\sum_e \sigma_e^z$ breaks the gauge symmetry.
- Elitzur's theorem tells us that system doesn't magnetize, $\chi = 0$.
- Average energy difference between two sectors $W_z = \pm 1$ is at most L^2: sub-extensive.
  - Less when considering gauge group average.



With field...

- Unique ground state $\sigma_e^z = 1$.
- In sector $W_z = 1$.
- Lowest energy state in sector $W_z = -1$ obtained from L_{xy}.
- Energy difference $2BL^2$ (sub-ext.).
- Much like 2D ferromagnetic Ising, but entropy fluctuates error membrane out of plane.

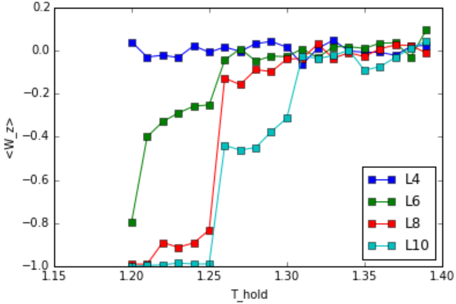
Outline

Experiment description

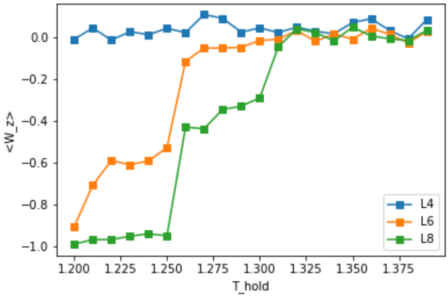
- Simulate thermal dynamics using Metropolis-Hastings MCMC.
  - Use single-spin flip.
  - Use gauge flip.
- Begin in lowest energy state with $W_z = -1$.
- Ramp up temperature to T_{hold}.
- Hold temperature for some macroscopic time.
  - $10{,}000 \times L^3$ MC updates.
- Ramp down temperature to $T = 0$, and measure W_z.

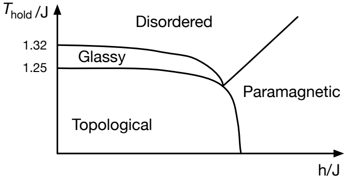
Experiment results

Without magnetic field $B/J = 0$



With magnetic field $B/J = 0.2$





Conclusion

- Self-correction is interesting in the classical and quantum setting.
- Self-correction is ruled out in symmetry broken phases with local order parameter.
- Elitzur's theorem rules out a local order parameter in systems with gauge symmetry.
- Well known phase transition $\langle W_\ell \rangle \propto e^{-\alpha|S|}$ vs $\propto e^{-\gamma|\partial S|}$.
- Additional degeneracy from homology.
  - No extensive energy splitting from any local field.
  - Numerical results consistent with finite $T_{\rm hold}$ and h phase transition.
  - Unconventional order parameter at finite T: defined algorithmically.
- Open questions
  - Can we prove stability?
  - Complete phase diagram?
  - Reduced to stability of $2D$ quantum model?