Stabilizer description of Absolutely Maximally Entangled states and associated quantum error correction codes

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Absolutely entangled state

A state $|\psi\rangle$ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ with $\mathcal{H}_i \cong \mathbb{C}^d$ is absolutely maximally entangled (AME) if it satisfies the following equivalent conditions for arbitrary disjoint bipartition into subsystems A and B, $|B| = m \leq \lfloor n/2 \rfloor$:

1. $Tr_A|\psi\rangle\langle\psi|=\mathbb{I}_B$

2. $S(Tr_A|\psi\rangle\langle\psi|)=S(B) = |B|\log d$

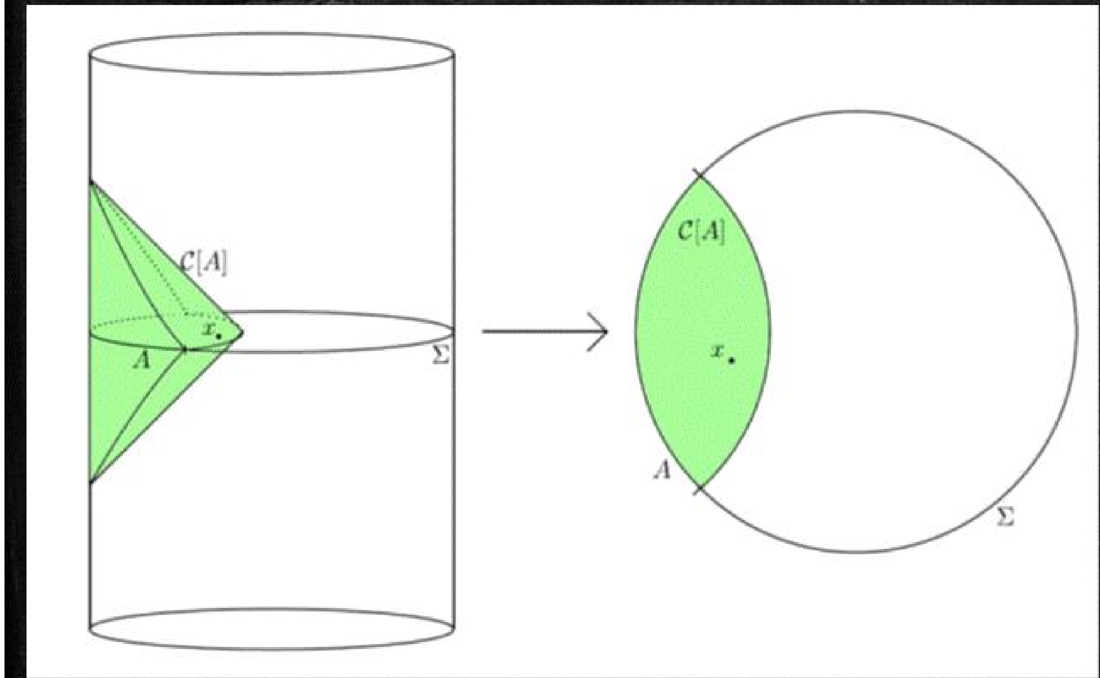
3. $|\psi\rangle = \frac{1}{\sqrt{d^m}}\sum_{k\in\mathbb{Z}_d^m} |k_1\rangle_{B_1} \ldots |k_m\rangle_{B_m}|\phi(k)\rangle_A$

   with $\langle\phi(k)|\phi(k')\rangle = \delta_{kk'}$.

Generalization of k-uniformity: tracing out k subsystems of n party system, $n-k$ party state is maximally mixed

AdS/CFT isomorphism

Holographic mapping between local operators in the bulk and non-local operators on the edge:



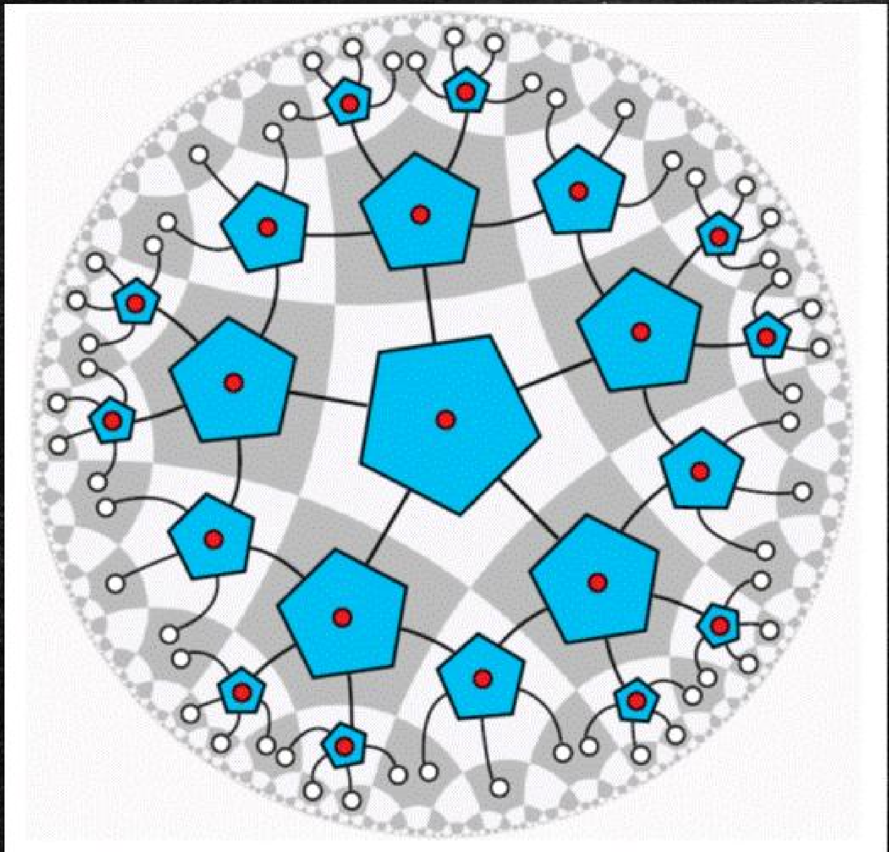
Operator reconstruction:
For every point Y on the edge, an image of O(x) commutes with O(Y).

Unique mapping -> image of O(x) is Identity

Non-unique mapping -> logical subspace, protected against erasure errors

A. Almheiri, X. Dong, D. Harlow, JHEP 1504:163,(2015)
F. Pastawski, B. Yoshida, D. Harlow, J. Preskill, JHEP 06, 149 (2015)

Toy models: AME networks defining ECC codes



- Non-contracted legs on the edge represent qubits
- Red dots represent logical qubits

$$S_A = \frac{Area(\gamma_A)}{4G}$$

- obstruction: monogamy of entanglement
- Classification of AME states for qubits:

a) n=2, d=2

$$|\psi\rangle \sim |00\rangle + |11\rangle$$

b) n=3, d=2

$$|\psi\rangle \sim |000\rangle + |111\rangle$$

c) n=4, d=2 – no AME state [1]

$$|\psi\rangle = \sum_{ijkl} t^{ijkl} |i\rangle|j\rangle|k\rangle|l\rangle$$

[1] A. Higuchi, A. Sudbery, Phys. Lett. A 272, 213 (2000)

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$$|\psi\rangle = \sum_{ijkl} t^{ijkl} |i\rangle|j\rangle|k\rangle|l\rangle$$

E.g

$$\mathrm{tr}_{3,4}\left[|0000+1111\rangle\langle 0000+1111|\right]$$

$$=$$

$$|00\rangle\langle 00| + |11\rangle\langle 11|$$

$$\neq$$

$$\mathbb{1}$$

[1] A. Higuchi, A. Sudbery, Phys. Lett. A 272, 213 (2000)

d) n=5,6 and d=2 – there are AME states found numerically [1]

$$\pi_{ME} = \sum_{|B| = \lfloor \frac{n}{2} \rfloor} \mathrm{Tr}(\rho_B^2)$$

e) No AME for d=2 and n=7 [2] and n≥8 [3]

f) for arbitrary n, there is AME if d high enough!

[1] P. Facchi, G. Florio, G. Parisi, S. Pascazio, Phys. Rev. A 77, 060304 (2008)
[2] F. Huber, O. Guhne, J. Siewert, arxiv 1608.06228
[3] A. J. Scott, Phys. Rev. A 69, 052330 (2004)

Quantum secret shering schemes [1]

A secret

$$|S\rangle = \sum_i a_i |i\rangle \quad \in \mathcal{H} \cong \mathbb{C}^d$$

is encoded as a state

$$|T\rangle = \sum_i a_i |T_i\rangle \in \mathcal{H}^{\otimes 2m-1}$$

shared between the parties. If numer of cooperating parties is at least m, then the secret can be perfectly deduced from their reduced state. If not, no information is provided.

But for AME

$$|\psi\rangle = \frac{1}{\sqrt{d^m}} \sum_{k\in\mathbb{Z}_d^{m-1},i} |i\rangle_{B_1}|k_2\rangle_{B_2} \ldots |k_m\rangle_{B_m}|\phi(k,i)\rangle_A$$

one can take

$$|T_i\rangle = \langle i|T\rangle = \frac{1}{\sqrt{d^m}} \sum_k |k_2\rangle_{B_2} \ldots |k_m\rangle_{B_m} |\phi(k,i)\rangle_A$$

[1] W. Helwig, W. Cui, J. I. Latorre, A. Riera, H. K. Lo Phys. Rev. A 86, 052335 (2012)

$$|B| = m, \quad |A| = m$$

But for AME

$$|\psi\rangle = \frac{1}{\sqrt{d^m}} \sum_{k \in \mathbb{Z}_d^{m-1}, i} |i\rangle_{B_1} |k_2\rangle_{B_2} \dots |k_m\rangle_{B_m} |\phi(k, i)\rangle_A$$

one can take $|t_i\rangle = \langle i | t\rangle = \frac{1}{\sqrt{d^m}} \sum_k |k_2\rangle_{B_2} \dots |k_m\rangle_{B_m} |\phi(k,i)\rangle_A$

A can apply joint unitary:

$$U\, |\phi(k,i)\rangle_A = |i\rangle_{A_1} |k_2\rangle_{A_2} \dots |k_m\rangle_{A_m}$$

That leads to

$$\sum_i a_i |t_i\rangle_{A, B\setminus\{B_1\}} \rightarrow \sum_i a_i |i\rangle_{A_1} \otimes \sum_{k_2} |k_2\rangle_{A_2} |k_2\rangle_{B_2} \otimes \dots \otimes \sum_{k_m} |k_m\rangle |k_m\rangle$$

Equivalence with perfect tensors

linear map $T: \mathcal{H}_B \to \mathcal{H}_A$ preserving the inner product (isometry) $\Leftrightarrow$ $\sum_a T^\dagger_{b'a} T_{ab} = \delta_{b'b}$

$T: |b\rangle \to \sum_a |a\rangle T_{ab}$

But $|\psi\rangle = \sum_{a,b} T_{ab} |a\rangle|b\rangle$

Demand $\mathrm{tr}_A |\psi\rangle\langle\psi| = \sum_{b,b',a} T_{ab} \overline{T}_{ab'} |b\rangle\langle b'| = \sum_b |b\rangle\langle b|$

$\Updownarrow$

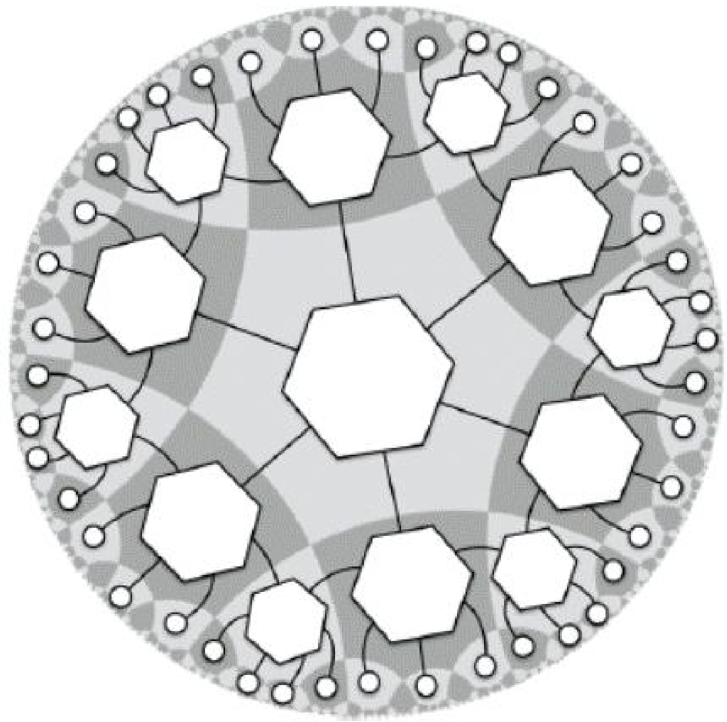
$\sum_a T^\dagger_{b'a} T_{ab} = \delta_{b'b}$

Ryu-Takayanagi formula

For a CFT whose gravitational dual is well-approximated by Einstein gravity at low energies, in any static state with a geometric bulk description the entropy S_A of a boundary subregion A at fixed time obeys

$$S_A = \frac{Area(\gamma_A)}{4G}$$

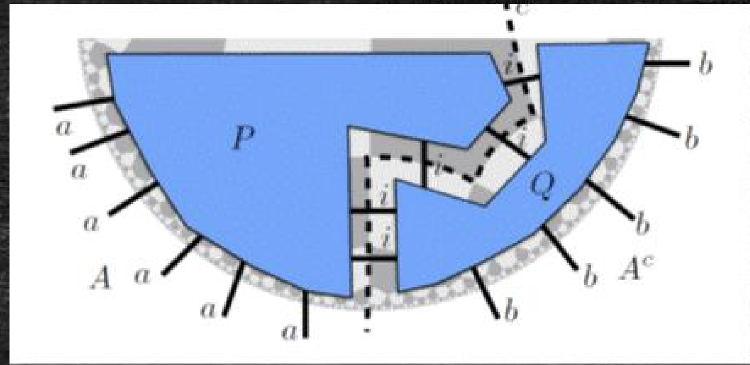
γ_A- minimal-area codimension-two bulk surface with boundary matching ∂A.



F. Pastawski, B. Yoshida, D. Harlow, J. Preskill, JHEP 06, 149 (2015)

$$P : |i\rangle \to \sum_a |a\rangle P_{ai}$$

$$Q : |i\rangle \to \sum_b |b\rangle Q_{bi}$$



State on the boundary:

$$|\Psi\rangle = \sum_{a,b,i} |a\,b\rangle P_{ai} Q_{bi} = \sum_i |P_i\rangle_A \otimes |Q_i\rangle_{A^c}$$

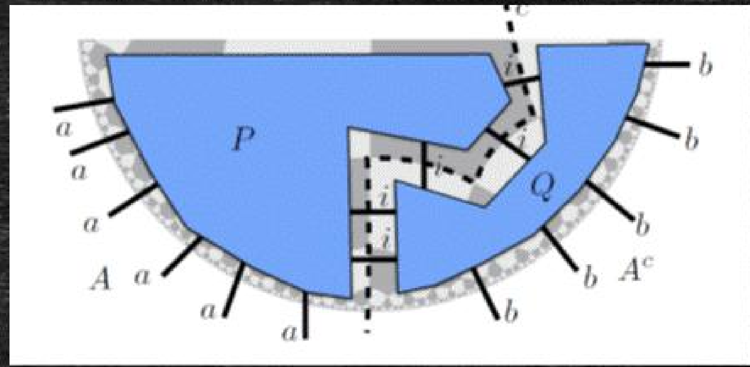
$$\rho_A = \sum_{i,i'} \langle Q_{i'}|Q_i\rangle\, |P_i\rangle\langle P_{i'}| \qquad S(\rho_A) \leq \gamma \log d$$

F. Pastawski, B. Yoshida, D. Harlow, J. Preskill, JHEP 06, 149 (2015)

Isometries:

$$P: |i\rangle \to \sum_a |a\rangle P_{ai}$$

$$Q: |i\rangle \to \sum_b |b\rangle Q_{bi}$$



State on the boundary:

$$|\psi\rangle = \sum_{a,b,i} |ab\rangle P_{ai} Q_{bi} = \sum_i |P_i\rangle_A \otimes |Q_i\rangle_{A^c}$$

$$\rho_A = \sum_{i,i'} \underbrace{\langle Q_{i'}|Q_i\rangle}_{\delta_{i,i'}} |P_i\rangle\langle P_{i'}| \qquad S(\rho_A) = \gamma \log d$$

F. Pastawski, B. Yoshida, D. Harlow, J. Preskill, JHEP 06, 149 (2015)

Quantum error correction properties

AME state defines a map

$$|\psi_i\rangle = \langle i|\psi\rangle = \frac{1}{\sqrt{d^m}} \sum_k |k_2\rangle_{B_2} \cdots |k_m\rangle_{B_m} |\phi(k,i)\rangle_A$$

Encoding 1 qubit into 2m-1 qubits s.t. the logical information is protected against erasure of m-1 qubis. That's the highest possible value due to no-cloning principle.

What's the relations between AME states and their respective codes? How one can produce these codes?

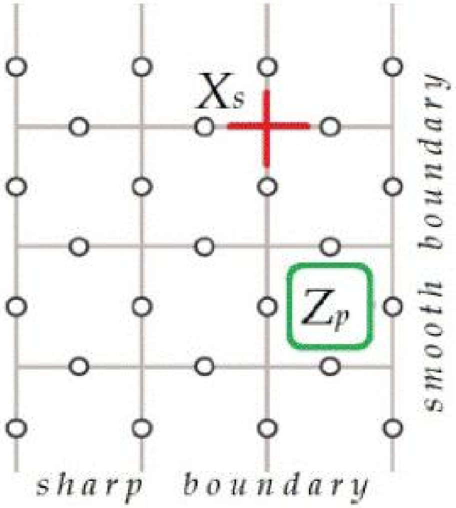
Stabilizer formalism

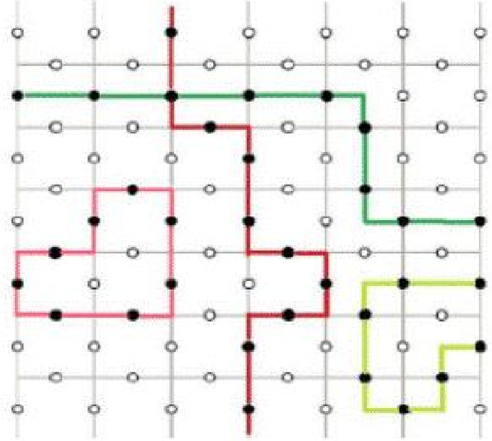
- n -qubit Clifford group C_n is a finite subgroup of $\mathbb{U}(2^n)$ generated by Hadamard gate, C-NOT gate and a phase gate
- Pauli group $\mathcal{P}_n \subset \mathrm{C}_n$ is generated by Pauli operators acting on n qubits
- An abelian subgroup S of $\mathcal{P}_n$ is called a stabilizer group if $-\mathbb{I} \notin S$.
  S defines a subspace: Code $= \{|\psi\rangle : s|\psi\rangle = |\psi\rangle\ \forall s \in S\}$

Error correction:

- If an operator $U(2^n)$ anticommutes with some element of S -> error detection
- If an operator $U(2^n)$ commutes with S, then it belongs to S or is a logical operator.

Paths of logical operators on the topological space are uncontractible to the point. The stabilizer group generated by star and plaquette parity operators responsible for detection of bit- and phase-errors.





Stabilizer group generators of Kitaev's topological code on a plane:
$Z_p = \otimes_{i=1}^{4} Z_i^p$, $X_p = \otimes_{i=1}^{4} X_i^p$

$X \otimes X \otimes \cdots \otimes X$ - logical bit flip operator
$X \otimes X \otimes \cdots \otimes X$ - trivial bit flip operator
$Z \otimes Z \otimes \cdots \otimes Z$ - logical phase flip operator
$Z \otimes Z \otimes \cdots \otimes Z$ - trivial phase flip operator

$L^2 + (L-1)^2$ links, $2L(L-1)$ stabilizer generators $\rightarrow$ 1 logical qubit encoded

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E. Dennis, A. Kitaev, A. Landahl and J. Preskill, J. Math. Phys. 43, 4452 (2002)

Stabilizer lists for qubits: updating rules

Unitary operation U on a state: $|\psi\rangle = s|\psi\rangle \rightarrow U\,|\psi\rangle = UsU^{\dagger}U|\psi\rangle$

Measurement of O on a state:

- If O can be expressed as a product of generators, no changes made

$$S = \langle Z_1, -Z_2 \rangle,\quad O = Z_1 Z_2 \longrightarrow S' = \langle Z_1, -Z_2 \rangle,\quad -Z_1 Z_2 \in S$$

- If O cannot be expressed as a product of generators, and commutes with all of them, then append it to a generator list with an appropriate sign

$$S = \langle Z_1 Z_2, Z_2 Z_3 \rangle \longrightarrow S' = \langle Z_1 Z_2, Z_2 Z_3, \pm Z_2 \rangle$$

$$O = Z_2$$

- If O cannot be expressed as a product of generators, and does not commute with some of them, then replace an anticommuting generator by O (with appropriate sign), and multiply all the other anticommuting generators by the generator removed

$$S = \langle Z_1 Z_2, \; Z_2 Z_3, \; Z_3 Z_4 \rangle \longrightarrow S = \langle Z_1 Z_2, \; \cancel{Z_2 Z_3}\; X_3, \; \cancel{Z_3} Z_4 \rangle$$

$$O = X_3 \qquad\qquad Z_2 Z_3 \cdot Z_3 Z_4 = Z_2 Z_4$$

Codes based on (6,2) AME state

$$S_1 = X_1 Z_2 Z_3 X_4 1_5 1_6$$

$$S_2 = 1\,X\,Z\,Z\,X\,1$$

$$S_3 = X\,1\,X\,Z\,Z\,1$$

$$S_4 = Z\,X\,1\,X\,Z\,1$$

$$S_5 = X\,X\,X\,X\,X\,X$$

$$S_6 = Z\,Z\,Z\,Z\,Z\,Z$$

$$
\begin{aligned}
|\psi\rangle = \frac{1}{\sqrt{32}} \Big\{ \Big[& |00000 - 00011 + 00101 - 00110 + \\
& + 01001 + 01010 - 01100 - 01111 + \\
& - 10001 + 10010 + 10100 - 10111 + \\
& - 11000 - 11011 - 11101 - 11110\rangle \Big] |0\rangle \\
+ \Big[& |-00001 - 00010 - 00100 - 00111 + \\
& - 01000 + 01011 + 01101 - 01110 + \\
& - 10000 - 10001 + 10101 + 10110 + \\
& - 11001 + 11010 - 11100 + 11111\rangle \Big] |1\rangle \Big\}
\end{aligned}
$$

Codes based on (6,2) AME state

$$S_1 = X_1 Z_2 Z_3 X_4 \mathbb{1}_5 \mathbb{1}_6 \quad \mathbb{1}_7$$
$$S_2 = \mathbb{1}\, X\, Z\, Z\, X\, \mathbb{1} \quad \mathbb{1}$$
$$S_3 = X\, \mathbb{1}\, X\, Z\, Z\, \mathbb{1} \quad \mathbb{1}$$
$$S_4 = Z\, X\, \mathbb{1}\, X\, Z\, \mathbb{1} \quad \mathbb{1}$$
$$S_5 = X\, X\, X\, X\, X\, X \quad \mathbb{1}$$
$$S_6 = Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 \quad \mathbb{1}_7$$

append ancilla

$$O^1 = X_6 X_7 \longrightarrow$$

$$S_1 = X_1 Z_2 Z_3 X_4 \mathbb{1}_5 \mathbb{1}_6 \quad \mathbb{1}_7$$
$$S_2 = \mathbb{1}\, X\, Z\, Z\, X\, \mathbb{1} \quad \mathbb{1}$$
$$S_3 = X\, \mathbb{1}\, X\, Z\, Z\, \mathbb{1} \quad \mathbb{1}$$
$$S_4 = Z\, X\, \mathbb{1}\, X\, Z\, \mathbb{1} \quad \mathbb{1}$$
$$S_5 = X\, X\, X\, X\, X\, X \quad \mathbb{1}$$
$$S_6 = \mathbb{1}\, \mathbb{1}\, \mathbb{1}\, \mathbb{1}\, \mathbb{1}\, X_6 \quad X_7$$

removed: $Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 \ \mathbb{1}$

Codes based on (6,2) AME state

$$
\begin{aligned}
S_1 &= X_1 Z_2 Z_3 X_4 \mathbb{1}_5 \mathbb{1}_6 \quad \mathbb{1}_7 \\
S_2 &= \mathbb{1}\, X\, Z\, Z\, X\, \mathbb{1} \quad \mathbb{1} \\
S_3 &= X\, \mathbb{1}\, X\, Z\, Z\, \mathbb{1} \quad \mathbb{1} \\
S_4 &= Z\, X\, \mathbb{1}\, X\, Z\, \mathbb{1} \quad \mathbb{1} \\
S_5 &= X\, X\, X\, X\, X\, X \quad \mathbb{1} \\
S_6 &= Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 \quad \mathbb{1}_7
\end{aligned}
$$

append ancilla

$$O^{(1)} = X_6 X_7$$

$$\longrightarrow$$

$$O^{(2)} = Z_6 Z_7$$

$$
\begin{aligned}
S_1 &= X_1 Z_2 Z_3 X_4 \mathbb{1}_5 \mathbb{1}_6 \quad \mathbb{1}_7 \\
S_2 &= \mathbb{1}\, X\, Z\, Z\, X\, \mathbb{1} \quad \mathbb{1} \\
S_3 &= X\, \mathbb{1}\, X\, Z\, Z\, \mathbb{1} \quad \mathbb{1} \\
S_4 &= Z\, X\, \mathbb{1}\, X\, Z\, \mathbb{1} \quad \mathbb{1} \\
S_5 &= \mathbb{1}\, \mathbb{1}\, \mathbb{1}\, \mathbb{1}\, \mathbb{1}\, Z_6 \quad Z_7 \\
S_6 &= \mathbb{1}\, \mathbb{1}\, \mathbb{1}\, \mathbb{1}\, \mathbb{1}\, X_6 \quad X_7
\end{aligned}
$$

removed: $Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 \mathbb{1}$

$X_1 X_2 X_3 X_4 X_5 X_6 \mathbb{1}$

Codes based on (6,2) AME state

$$S_1 = X_1 Z_2 Z_3 X_4 1_5 1_6 \;\; 1_7$$
$$S_2 = 1\,X\,Z\,Z\,X\,1 \;\; 1$$
$$S_3 = X\,1\,X\,Z\,Z\,1 \;\; 1$$
$$S_4 = Z\,X\,1\,X\,Z\,1 \;\; 1$$
$$S_5 = X\,X\,X\,X\,X\,X \;\; 1$$
$$S_6 = Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 \;\; 1_7$$

append ancilla

$$O^{(1)} = X_6 X_7$$
$$\longrightarrow$$
$$O^{(2)} = Z_6 Z_7$$

stabilizers of 1→5 code

$$S_1 = X_1 Z_2 Z_3 X_4 1_5 \; 1_6 \;\; 1_7$$
$$S_2 = 1\,X\,Z\,Z\,X \; 1 \;\; 1$$
$$S_3 = X\,1\,X\,Z\,Z \; 1 \;\; 1$$
$$S_4 = Z\,X\,1\,X\,Z \; 1 \;\; 1$$
$$S_5 = 1\,1\,1\,1\,1 \; Z_6 \;\; Z_7$$
$$S_6 = 1\,1\,1\,1\,1 \; X_6 \;\; X_7$$

logical operators of 1→5 codes

removed: $Z_1 Z_2 Z_3 Z_4 Z_5 \; Z_6 \;\; 1$

$X_1 X_2 X_3 X_4 X_5 \; X_6 \;\; 1$

$$
|+\rangle = \frac{1}{\sqrt{32}} \Big\{ \underbrace{\big[\, |00000 - 00011 + 00101 - 00110 + 01001 + 01010 - 01100 - 01111 - 10001 + 10010 + 10100 - 10111 - 11000 - 11011 - 11101 - 11110\rangle \big]}_{|\bar{0}\rangle} |0\rangle + \underbrace{\big[\, |-00001 - 00010 - 00100 - 00111 - 01000 + 01011 + 01101 - 01110 - 10000 - 10001 + 10101 + 10110 - 11001 + 11010 - 11100 + 11111\rangle \big]}_{|\bar{1}\rangle} |1\rangle \Big\}
$$

$$
\boxed{\begin{matrix} X_1 & Z_2 & Z_3 & X_4 & 1_5 \\ 1 & X & Z & Z & X \\ X & 1 & X & Z & Z \\ Z & X & 1 & X & Z \end{matrix}} = \langle S \rangle
$$

$$\bar{X}|\bar{0}\rangle = |\bar{1}\rangle$$

$$\bar{X}|\bar{1}\rangle = |\bar{0}\rangle$$

$$\bar{Z}|\bar{0}\rangle = |\bar{0}\rangle$$

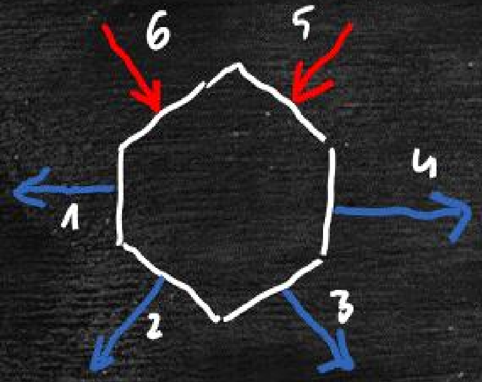
$$\bar{Z}|\bar{1}\rangle = -|\bar{1}\rangle$$

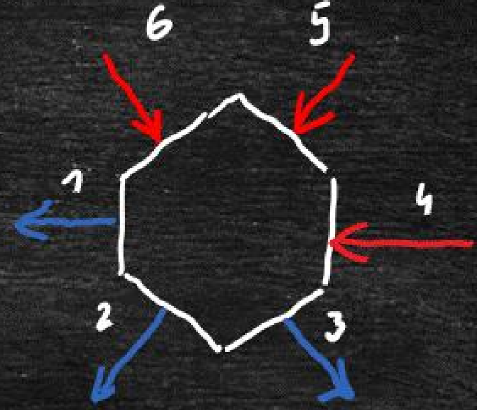
logical operators: $X_1 X_2 X_3 X_4 X_5 = \bar{X}$

$Z_1 Z_2 Z_3 Z_4 Z_5 = \bar{Z}$

$$|+\rangle_{enc} = \alpha|\bar{0}\rangle + \beta|\bar{1}\rangle$$

In this way one obtains 2->4 and 3->3 codes:





$$S = \langle X_1 Z_2 Z_3 X_4, \; Y_1 X_2 X_3 Y_4 \rangle$$

$$\bar{X}_5 = X_1 1_2 Y_3 Y_3$$

$$\bar{Z}_5 = Y Z Y 1$$

$$\bar{X}_6 = 1 X Z Z$$

$$\bar{Z}_6 = X 1 X Z$$

$$S = \emptyset$$

$$\bar{X}_4 = Z_1 X_2 Z_3$$

$$\bar{Z}_4 = Y Z Y$$

$$\bar{X}_5 = X Z Z$$

$$\bar{Z}_5 = Z Y Y$$

$$\bar{X}_6 = Z Z X$$

$$\bar{Z}_6 = Y Y Z$$

Is this method universal for stabilizer AME states?

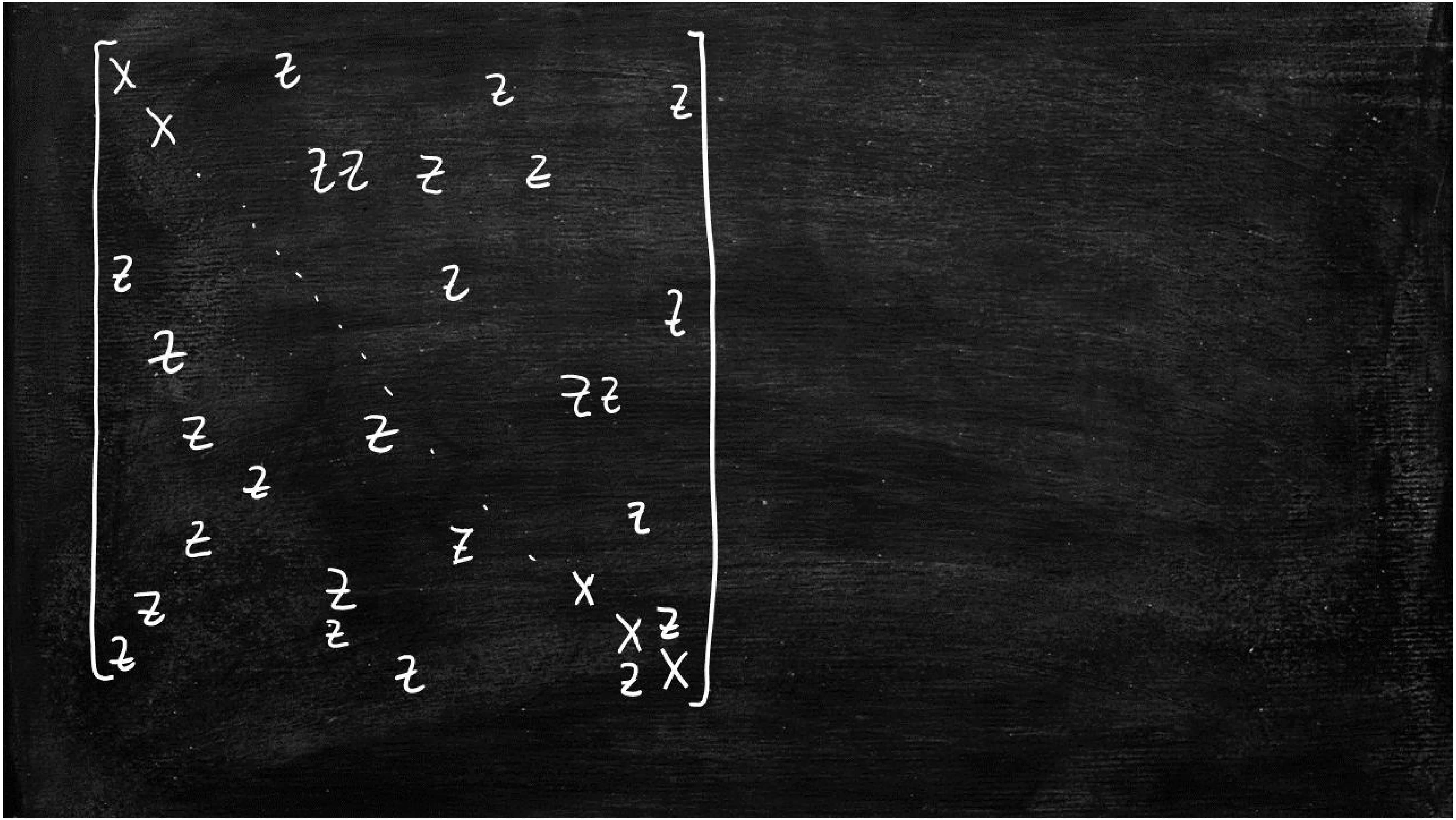
Every stabilizer state can be transformed into a graph state by local Clifford operators [1]:

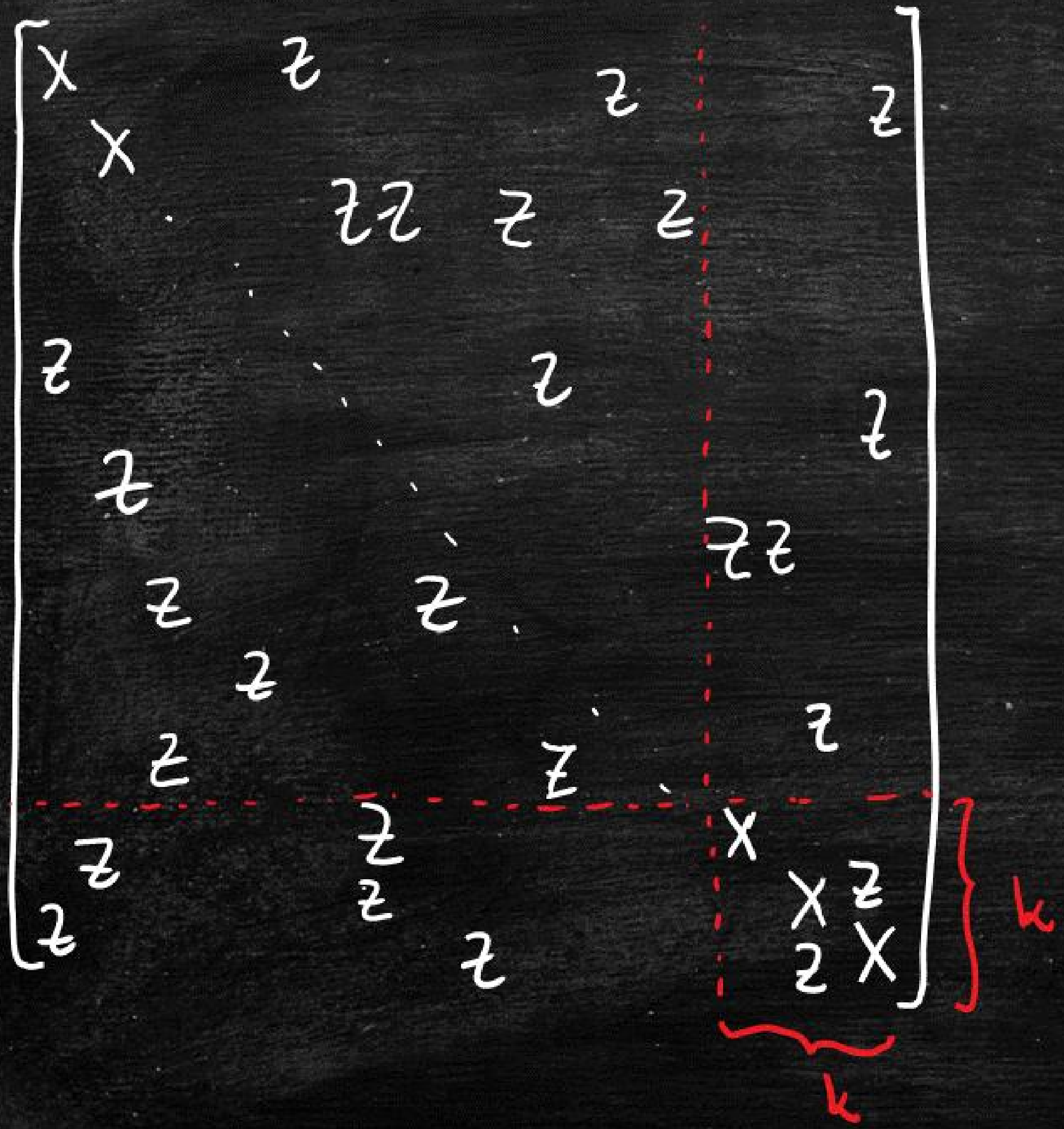
$$\mathcal{U}^{\otimes n} \ni |\psi\rangle \longrightarrow \prod_{i>j} \left(Z_{ij}^{A_{ij}}\right) |+\rangle^{\otimes n}$$

A_{ij} - adjacency matrix $\qquad g_i = X_i \prod_j Z_j^{A_{ij}}$

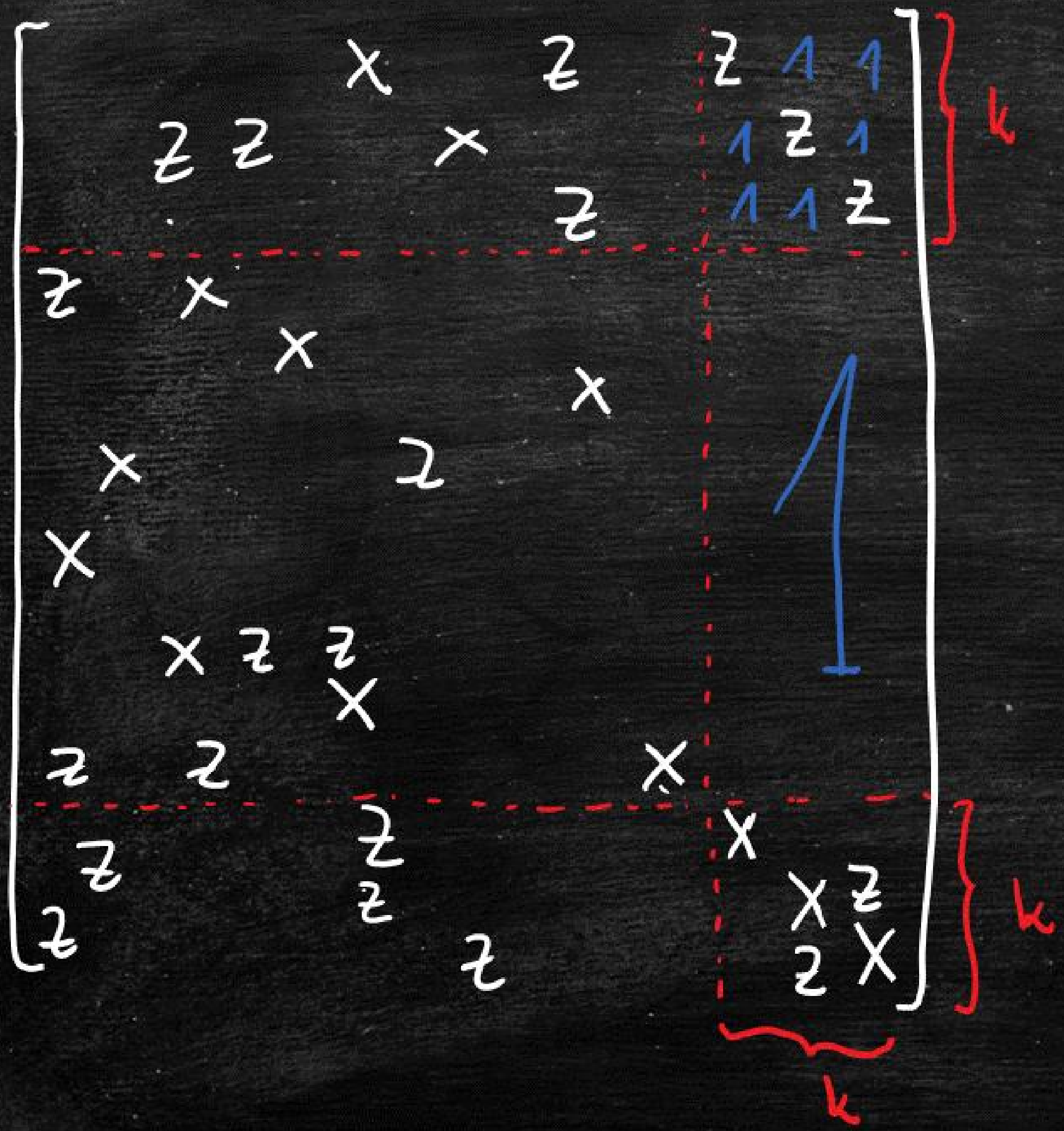
Theorem 7 [2]
A graph state with adjacency matrix A is AME $\Leftrightarrow$ after removing up to $\left\lfloor \frac{n}{2} \right\rfloor$ rows/columns in A, the truncated columns/rows remain linearly independent

[1] D. Schlingemann, Quant. Inf. & Comp. 2, 307 (2002)
[2] W. Helwig, **arXiv:1306.2879**





take $k \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor\}$


k
k
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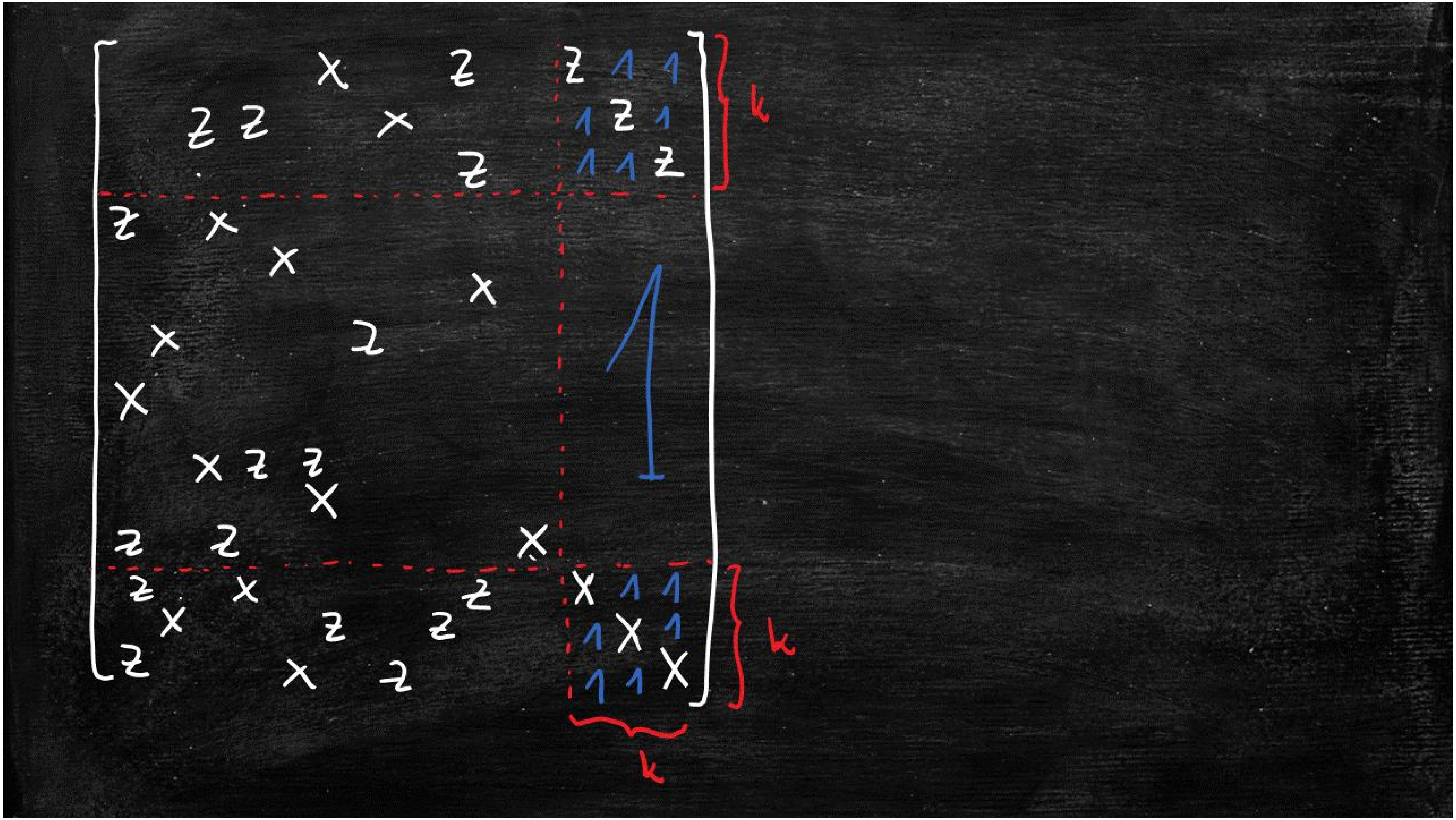
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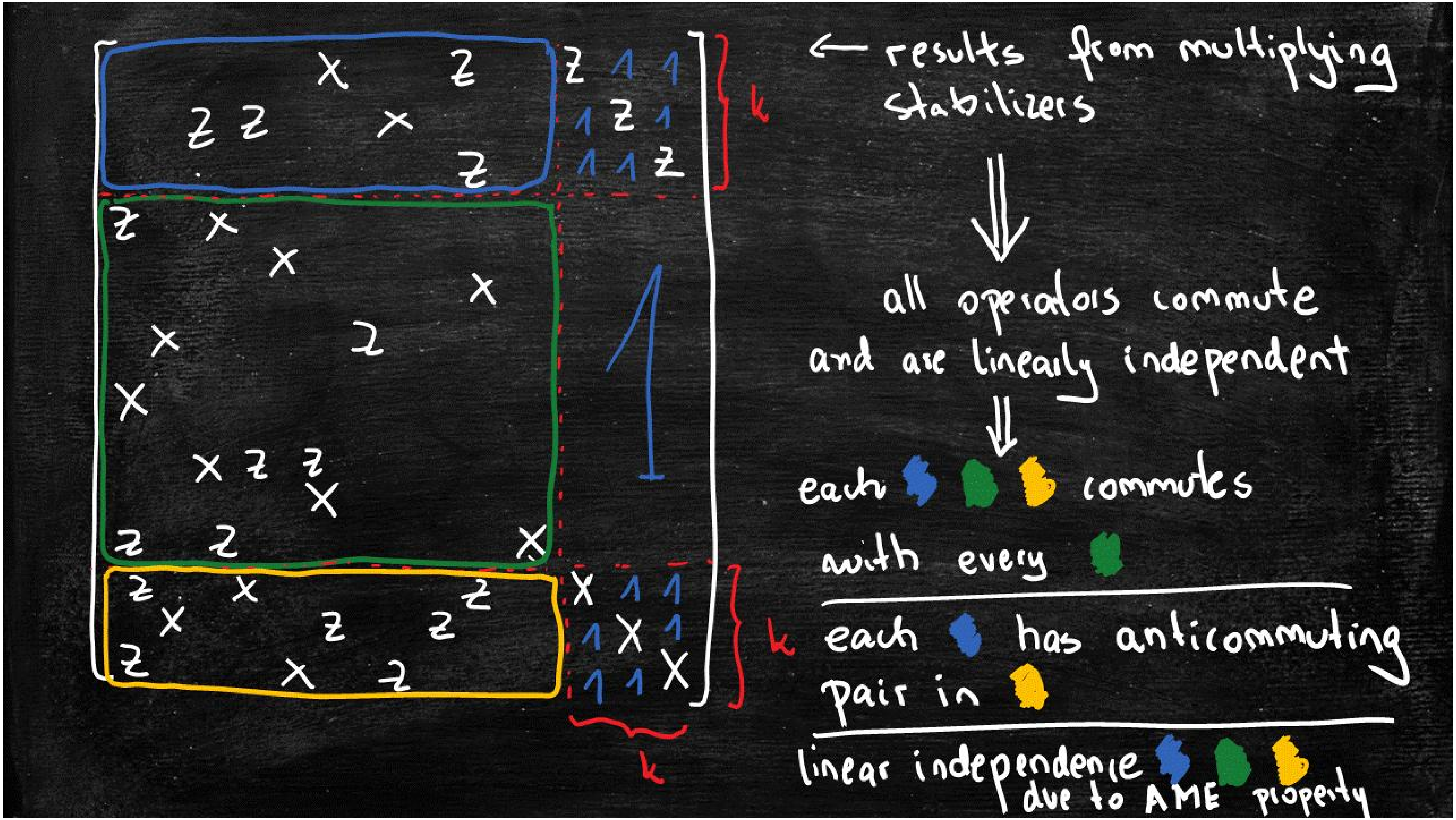
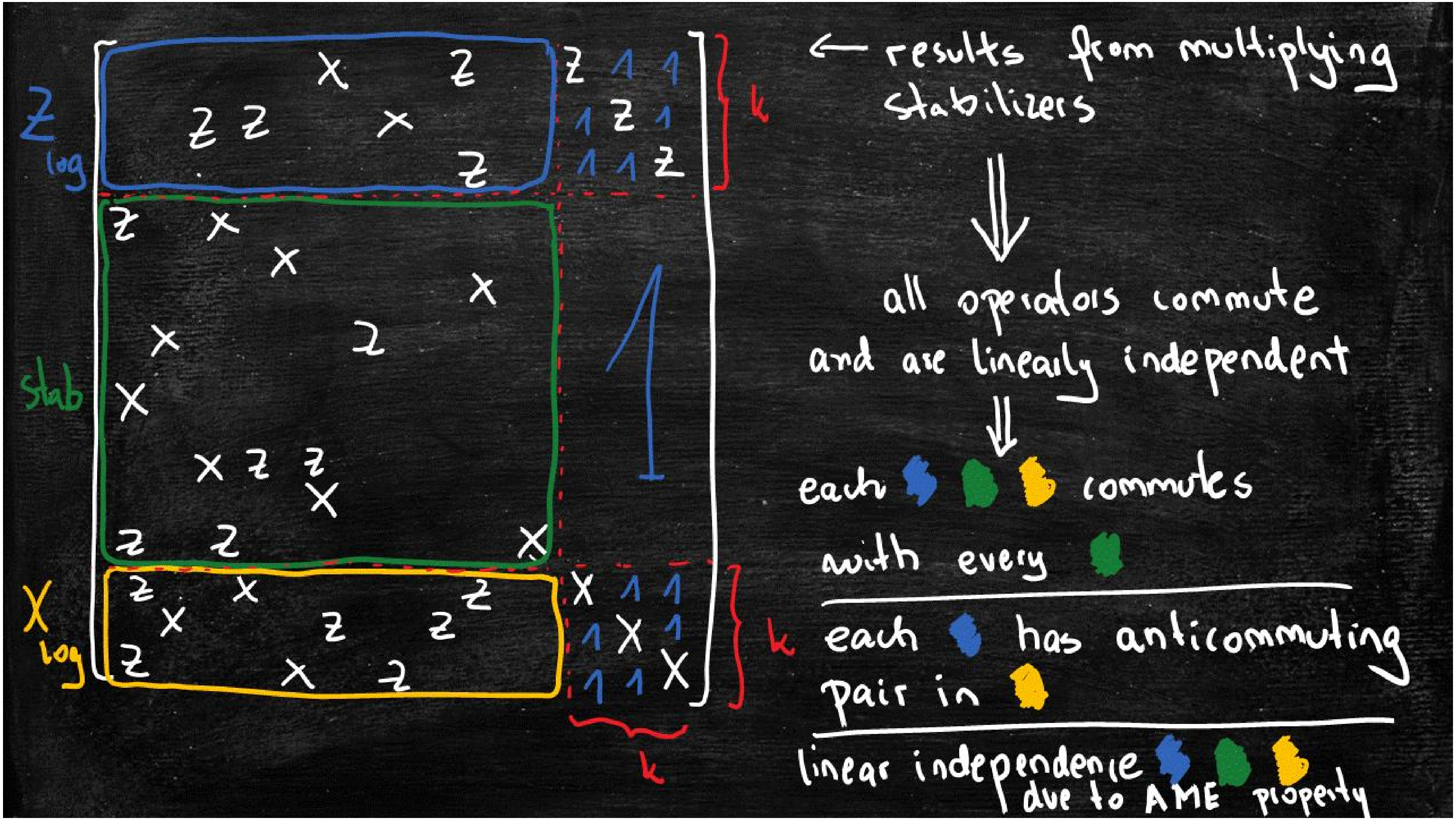

← results from multiplying stabilizers
all operators commute and are linearly independent
each commutes with every
each has anticommuting pair in
linear independence
due to AME property
k
k
k



Z log
stab
X log
k
k
k
← results from multiplying stabilizers
all operators commute and are linearly independent
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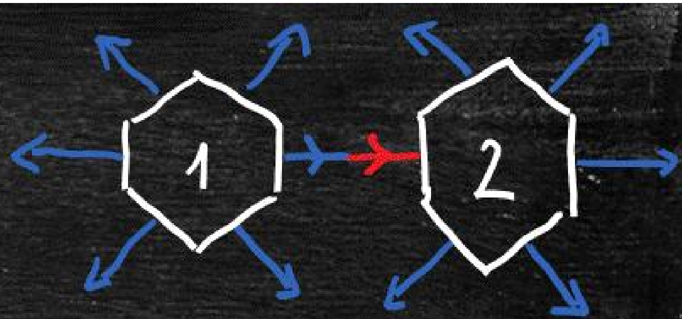
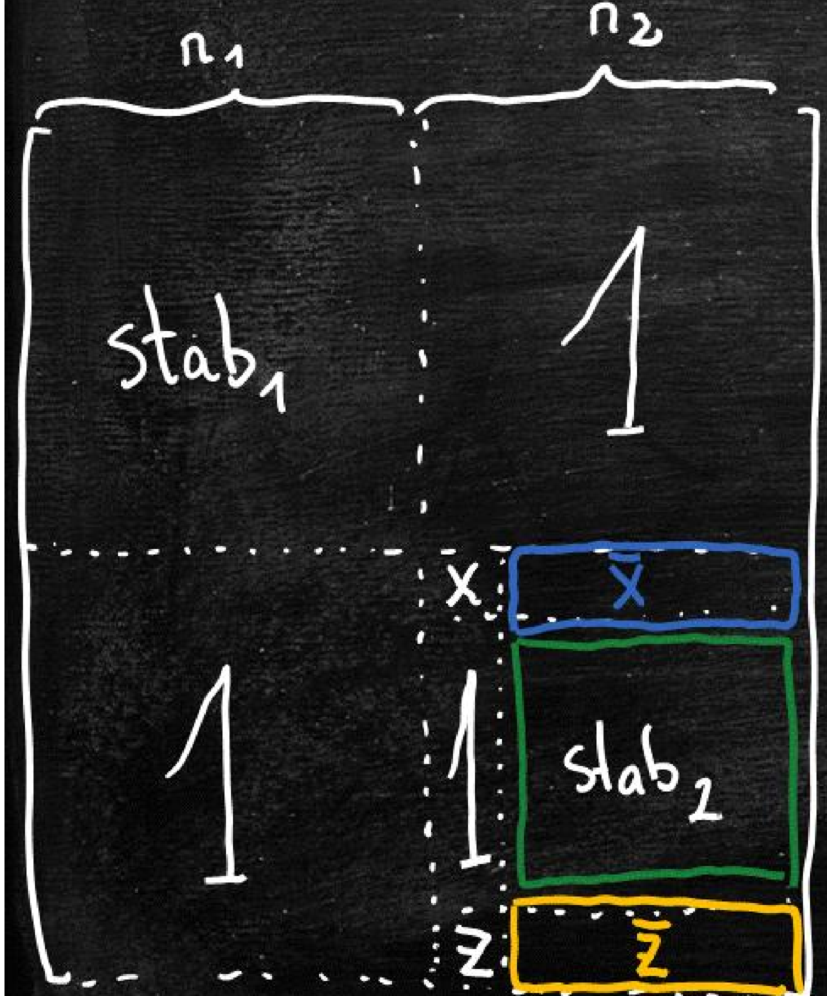
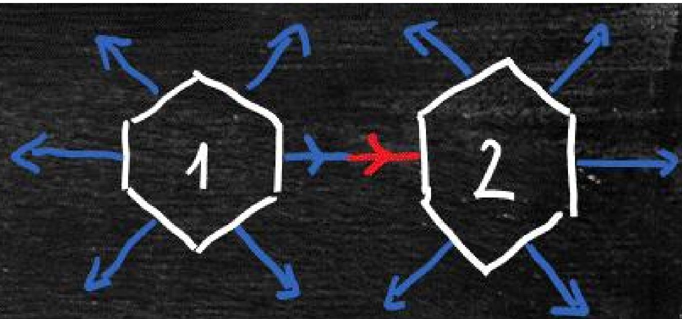
linear independence due to AME property

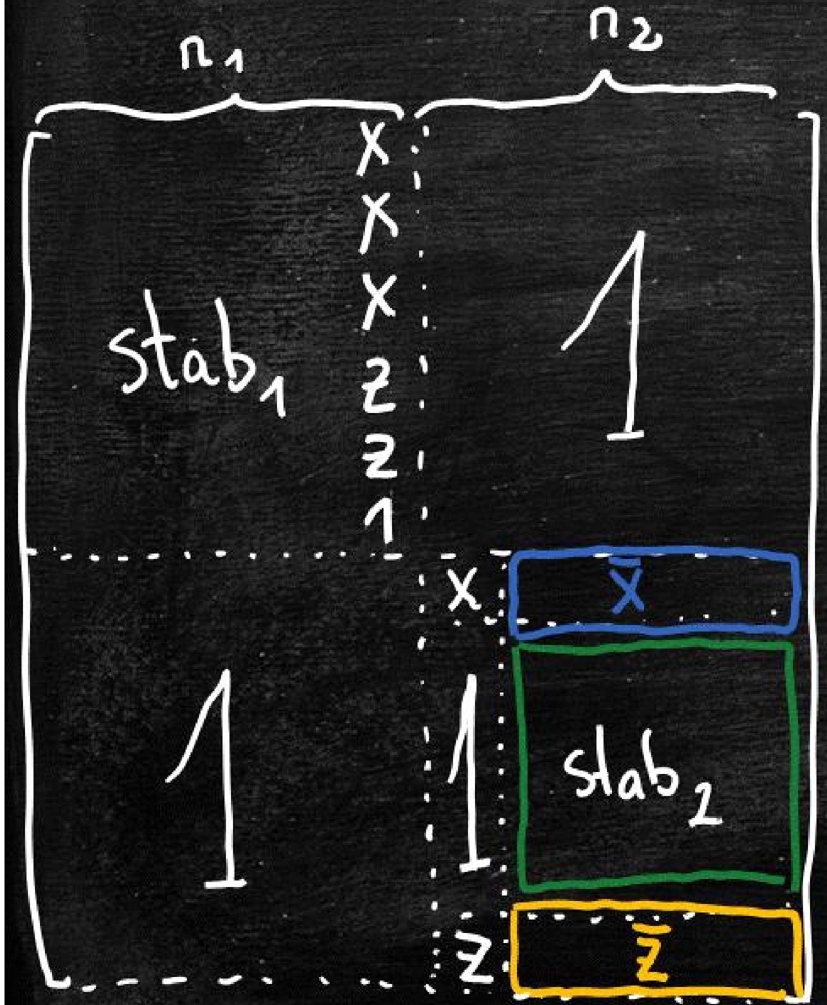
What about contatenation?





What about contatenation?





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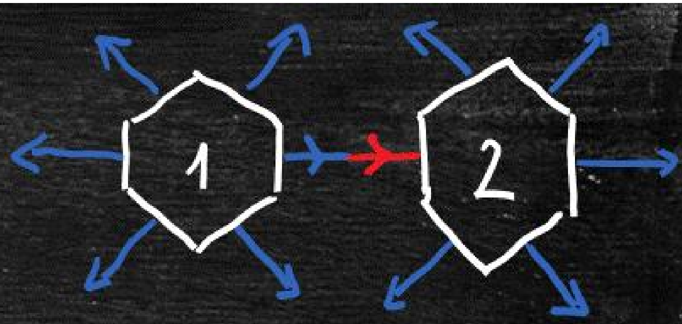

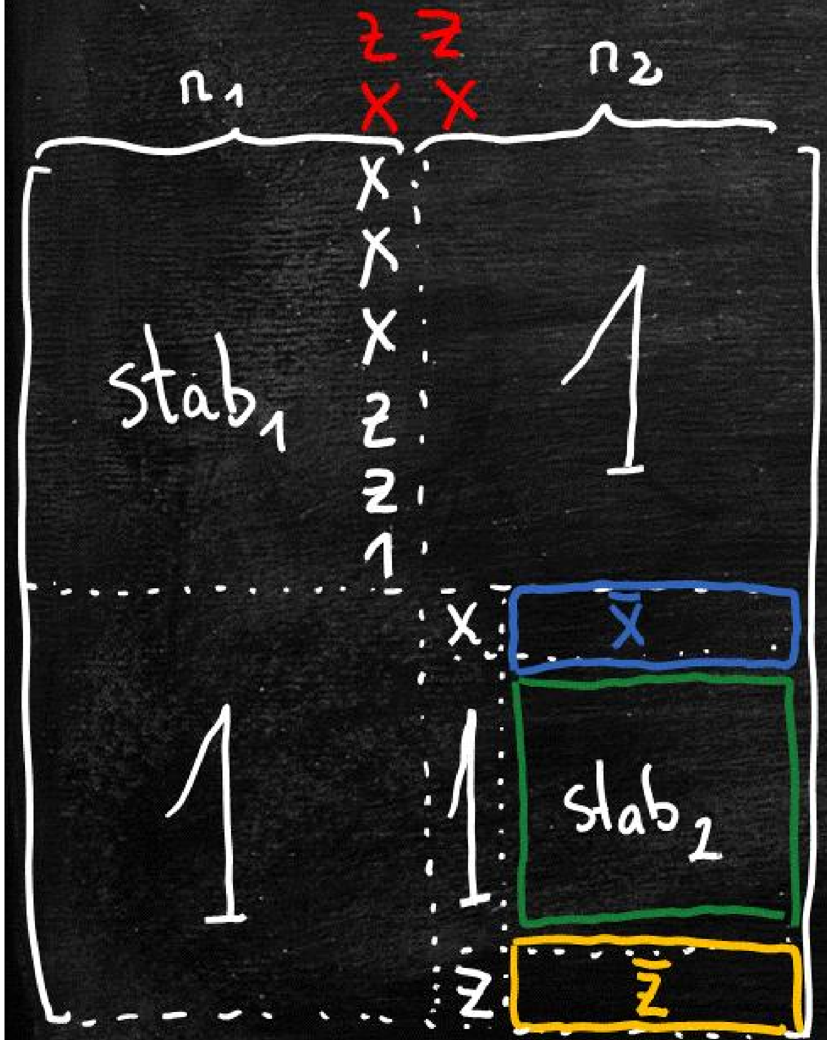

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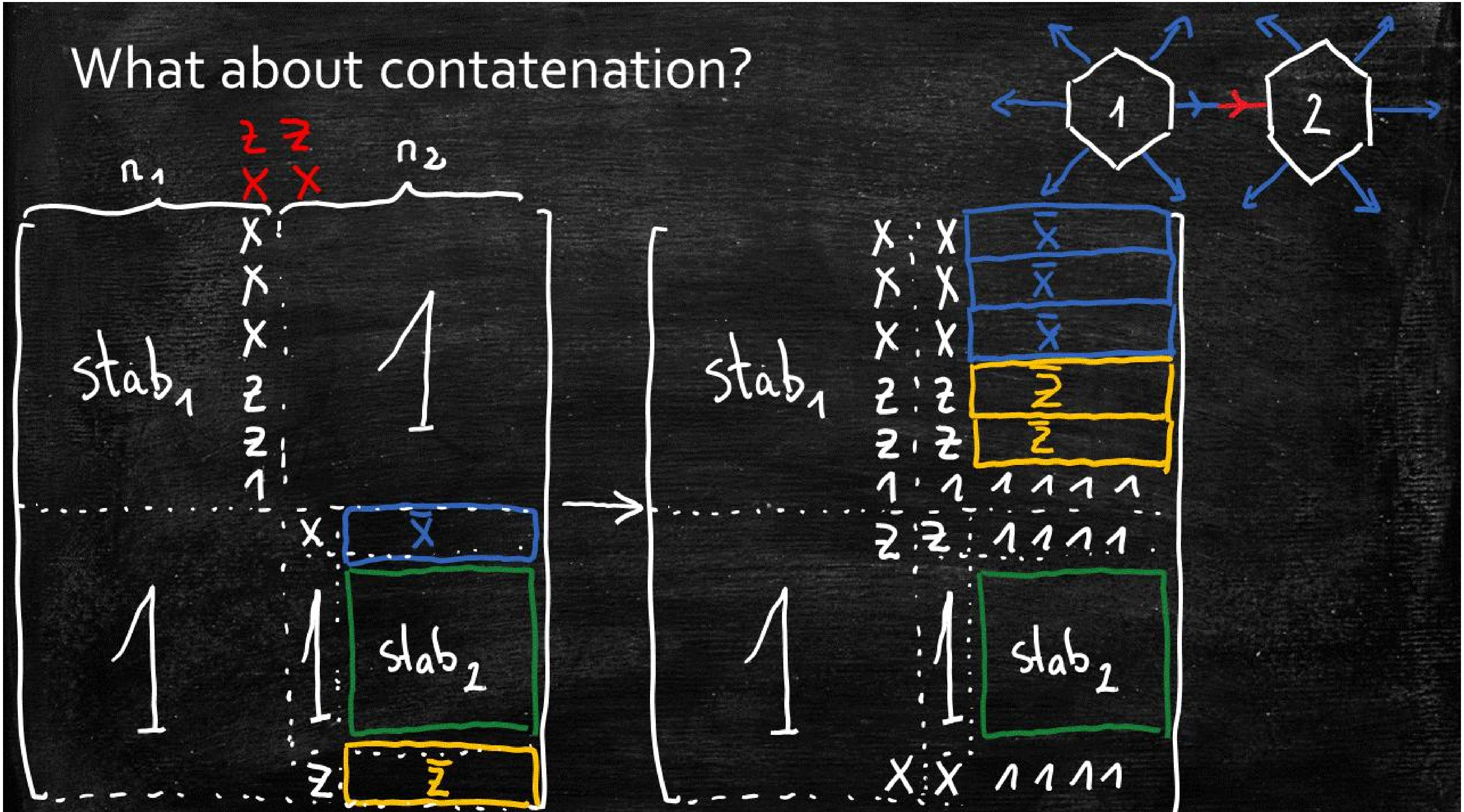

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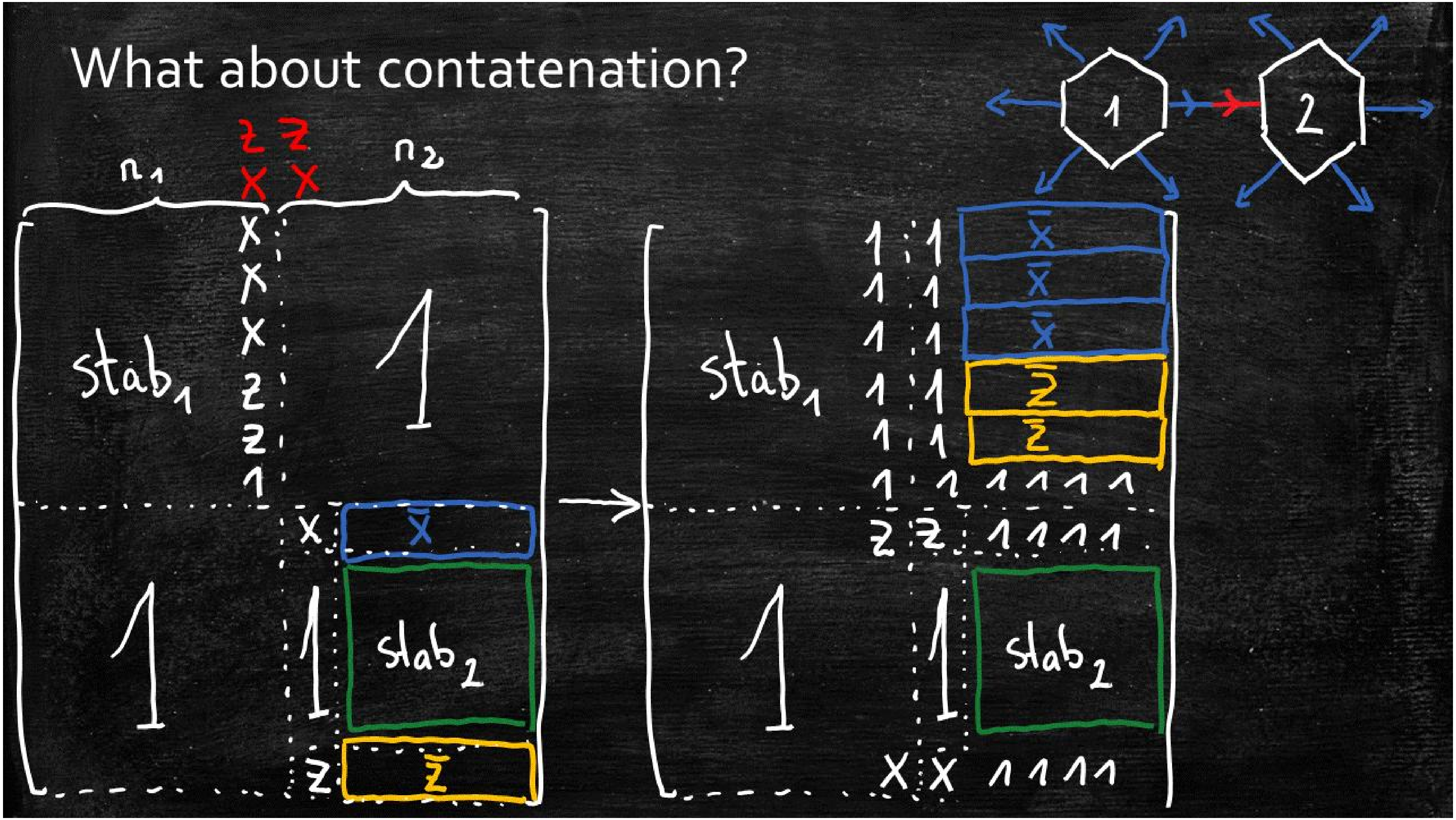
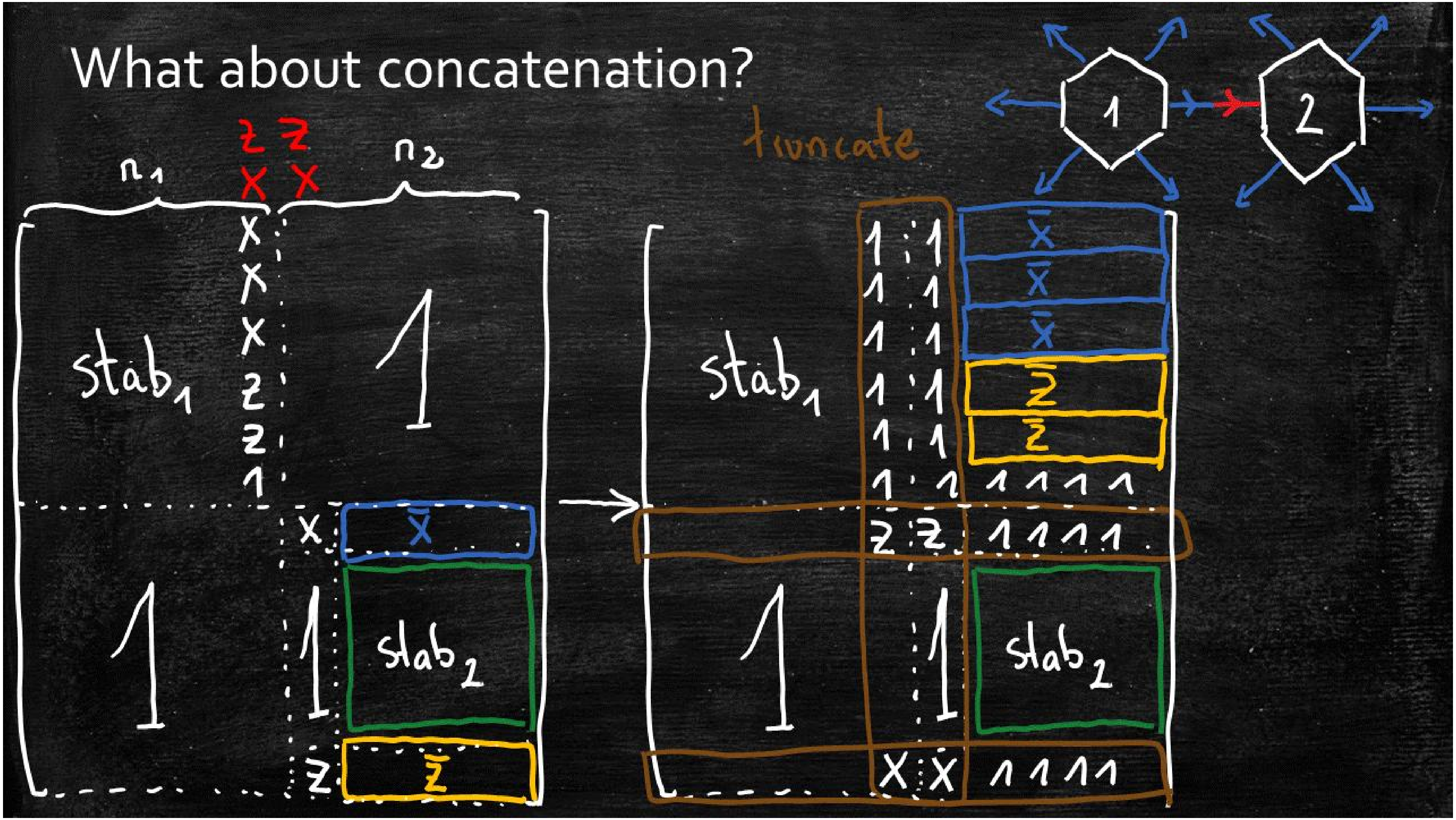


What about concatenation?



Spread of information in the network