Asymptotic restriction and spectral points

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1 The asymptotic restriction problem and quantum entanglement

1.1 Tensor restriction

We work with tensors of some fixed order k over the field $\mathbb{C}$, vector spaces are finite dimensional.

Definition 1.1 (Tensor restriction). *$s \in V_1 \otimes \cdots \otimes V_k$ restricts to $t \in W_1 \otimes \cdots \otimes W_k$ if there are linear maps $A_1, \ldots, A_k$, $A_j : V_j \to W_j$ such that $(A_1 \otimes \cdots \otimes A_k)s = t$. Notation: $s \geq t$.*

Proposition 1.2. *$\leq$ is a preorder (i.e. reflexive and transitive)*

Proof. Let $t \in W_1 \otimes \cdots \otimes W_k$. Then $(\mathrm{id}_{W_1} \otimes \cdots \otimes \mathrm{id}_{W_k})t = t$, so $t \geq t$.

Let $r \in U_1 \otimes \cdots \otimes U_k$, $s \in V_1 \otimes \cdots \otimes V_k$ and $t \in W_1 \otimes \cdots \otimes W_k$. Suppose $r \geq s$ and $s \geq t$. This means there are linear maps $A_1, \ldots, A_k$, $B_1, \ldots, B_k$, $A_j : U_j \to V_j$, $B_j : V_j \to W_j$ with $s = (A_1 \otimes \cdots \otimes A_k)r$ and $t = (B_1 \otimes \cdots \otimes B_k)s$. Therefore

$$\begin{aligned} t &= (B_1 \otimes \cdots \otimes B_k)s \\ &= (B_1 \otimes \cdots \otimes B_k)(A_1 \otimes \cdots \otimes A_k)r \\ &= (B_1 A_1 \otimes \cdots \otimes B_k A_k)r, \end{aligned} \tag{1}$$

which implies $t \geq r$. □

Definition 1.3 (Equivalence of tensors). *We say that s and t are equivalent if $s \geq t$ and $t \leq s$.*

Remark 1.4. *Equivalence does not imply isomorphism since the spaces can be of different dimension. For the purposes of these lectures it is safe to identify equivalent tensors.*

Example 1.5 (Characterisation of rank with restriction). *$\langle r \rangle \in \mathbb{C}^r \otimes \cdots \otimes \mathbb{C}^r$ is the unit tensor*

$$\langle r \rangle = \sum_{i=1}^{r} e_i \otimes e_i \otimes \cdots \otimes e_i \tag{2}$$

where $e_1, e_2, \ldots, e_r$ is the standard basis of $\mathbb{C}^r$.

If $t \in W_1 \otimes \cdots \otimes W_k$ then $\langle r \rangle \geq t$ means there are linear maps $A_1, \ldots, A_k$, $A_j : \mathbb{C}^r \to W_j$ such that

$$\begin{aligned} t &= (A_1 \otimes \cdots \otimes A_k) \sum_{i=1}^{r} e_i \otimes e_i \otimes \cdots \otimes e_i \\ &= \sum_{i=1}^{r} A_1(e_i) \otimes A_2(e_i) \otimes \cdots \otimes A_k(e_i) \end{aligned} \tag{3}$$

which is equivalent to $\mathrm{R}(t) \leq r$.

Definition 1.6 (Kronecker product of tensors). *Let $s \in V_1 \otimes \cdots \otimes V_k$ and $t \in W_1 \otimes \cdots \otimes W_k$. Their product is $s \otimes t \in (V_1 \otimes W_1) \otimes \cdots \otimes (V_k \otimes W_k)$, regarded as a tensor of order k. $t \otimes t \otimes \cdots \otimes t$ (n factors) is abbreviated as $t^{\otimes n}$.*

Exercise 1. *Prove that $s_1 \geq t_1$ and $s_2 \geq t_2$ implies $s_1 \otimes s_2 \geq t_1 \otimes t_2$.*

Definition 1.7 (Asymptotic rank)**.** *Let t be a tensor. Its asymptotic rank is defined as $\underset{\sim}{\mathrm{R}}(t) = \lim_{n\to\infty} \sqrt[n]{\mathrm{R}(t^{\otimes n})}$.*

Example 1.8. *Let $\langle l, m, n\rangle$ denote the tensor corresponding to matrix multiplication as a bilinear map $\mathrm{Mat}_{l\times m} \times \mathrm{Mat}_{m\times n} \to \mathrm{Mat}_{l\times n}$. Then $\langle l_1, m_1, n_1\rangle \otimes \langle l_2, m_2, n_2\rangle$ is equivalent to $\langle l_1 l_2, m_1 m_2, n_1 n_2\rangle$. In particular, $\langle 2, 2, 2\rangle^{\otimes n} = \langle 2^n, 2^n, 2^n\rangle$, which implies $\underset{\sim}{\mathrm{R}}(\langle 2, 2, 2\rangle) = 2^\omega$ where ω is the exponent of matrix multiplication.*

Since $\langle r_1\rangle \otimes \langle r_2\rangle$ is equivalent to $\langle r_1 r_2\rangle$, we may also write

$$\omega = \inf \left\{ \frac{m}{n} \middle| \langle 2\rangle^{\otimes m} \geq \langle 2, 2, 2\rangle^{\otimes n} \right\}. \tag{4}$$

This motivates the following generalisation:

$$\omega(s, t) := \inf \left\{ \frac{m}{n} \middle| s^{\otimes m} \geq t^{\otimes n} \right\} = \lim_{n\to\infty} \frac{1}{n} \min \left\{ m \in \mathbb{N} \middle| s^{\otimes m} \geq t^{\otimes n} \right\}. \tag{5}$$

We also write $s \gtrsim t$ if $\omega(s, t) \leq 1$ and say that s asymptotically restricts to t.

Exercise 2. *Prove that $\omega(\langle 2\rangle, t) = \log \underset{\sim}{\mathrm{R}}(t)$.*

Remark 1.9. *If $s \unrhd t$ then $\omega(s, t) \leq 1$.*

1.2 Basic tools of quantum information theory

Definition 1.10 (State)**.** *Let $\mathcal{H}$ be a Hilbert space. A state is a positive semidefinite linear operator ρ satisfying $\operatorname{Tr}\rho = 1$. The set of states is denoted by $\mathcal{S}(\mathcal{H})$. $\mathcal{B}_1(\mathcal{H})$ is the space of linear operators equipped with the Schatten 1-norm $\|A\|_1 = \operatorname{Tr}|A|$.*

A state ρ is called pure if it is an extremal point in the convex set $\mathcal{S}(\mathcal{H})$.

A state is pure iff $\rho = \rho^2$ iff $\operatorname{rk}\rho = 1$.

The literature on quantum information theory makes extensive use of the so-called bra-ket notation. The bra-ket notation means two things:

1) A vector $\psi \in \mathcal{H}$ is sometimes "wrapped" as $|\psi\rangle$ in a "ket". This does not really mean anything different from ψ. Since $\mathcal{H}$ is a Hilbert space, a vector $\psi \in \mathcal{H}$ also determines a linear form $\varphi \mapsto \langle \psi, \varphi\rangle$. In bra-ket notation, this linear form is denoted as $\langle\psi|$. The idea is that $\langle\psi|\,|\varphi\rangle = \langle\psi, \varphi\rangle$ are visually very similar. We might as well replace the double vertical line with a single one and write $\langle\psi|\varphi\rangle$. Another combination which is frequently encountered is $|\psi\rangle\langle\psi|$, especially if $\|\psi\| = 1$, in which case it is the orthogonal projection onto the subspace spanned by ψ.

2) A slightly different use of bras and kets is to denote elements of a labelled basis using only the labels. Thus if $\mathcal{X}$ is some index set then $|x\rangle$ may denote an element of the orthonormal basis $\{|x\rangle\}_{x\in\mathcal{X}}$. Orthonormality implies that $\langle x|$ is the corresponding element of the dual basis. This means $\langle x|x'\rangle = \delta_{xx'}$ and $\sum_{x\in\mathcal{X}} |x\rangle\langle x| = I = \mathrm{id}_{\mathcal{H}}$.

   Elements of the standard basis of $\mathbb{C}^n$ are often denoted $|0\rangle, |1\rangle, \ldots, |n-1\rangle$.

Definition 1.11 (Quantum channel)**.** *A linear map $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{K})$ is called positive if $\rho \geq 0$ implies $T(\rho) \geq 0$, and completely positive if $T \otimes \mathrm{id}_{\mathcal{B}_1(\mathbb{C}^n)}$ is positive for every $n \in \mathbb{N}$. A channel is a completely positive trace preserving map.*

Channels model physical processes like unitary evolution

$$\rho \mapsto U\rho U^* \tag{6}$$

and measurement

$$\rho \mapsto \sum_{x\in\mathcal{X}} \operatorname{Tr}(E_x\rho)\, |x\rangle\langle x| \tag{7}$$

where $E_x \geq 0$ and $\sum_{x\in\mathcal{X}} E_x = I$.

In general, a channel $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{K})$ can be written in the form

$$T(\rho) = \sum_{i\in I} K_i \rho K_i^* \tag{8}$$

where the linear maps $K_i : \mathcal{H} \to \mathcal{K}$ satisfy

$$\sum_{i\in I} K_i^* K_i = I = \mathrm{id}_{\mathcal{H}} \tag{9}$$

and are called Kraus operators. Conversely, every map of this form is a channel.

Information theoretic viewpoint: any channel we can possibly write down corresponds to an operation that one can in principle (i.e. given advanced technology, unlimited space, time and energy, etc.) be carried out in a lab.

States and channels are best viewed as generalisations of probability distributions and stochastic maps as the following exercise shows.

Exercise 3. *Let $(p_x)_{x\in\mathcal{X}}$ be a probability distribution on the finite set $\mathcal{X}$ and $(S_{xy})_{x\in\mathcal{X},y\in\mathcal{Y}}$ a stochastic map, i.e. a collection of nonnegative numbers such that*

$$\forall x : \sum_{y\in\mathcal{Y}} S_{xy} = 1. \tag{10}$$

If a random quantity X with distribution p is processed via the random operation modeled by S, resulting in a random quantity Y with distribution q, then $q_y = \sum_{x\in\mathcal{X}} S_{xy} p_x$.

Let

$$\rho_p = \sum_{x\in\mathcal{X}} p_x \left|x\right\rangle\!\left\langle x\right| \in \mathcal{B}_1(\mathbb{C}^{\mathcal{X}}) \tag{11}$$

$$\rho_q = \sum_{y\in\mathcal{Y}} q_y \left|y\right\rangle\!\left\langle y\right| \in \mathcal{B}_1(\mathbb{C}^{\mathcal{Y}}) \tag{12}$$

and

$$T_S(\rho) = \sum_{\substack{x\in\mathcal{X}\\ y\in\mathcal{Y}}} S_{xy} \left\langle x\right| \rho \left|x\right\rangle \left|y\right\rangle\!\left\langle y\right|. \tag{13}$$

Show that $T_S(\rho_p) = \rho_q$.

Remark 1.12. *An important difference is that after "applying" a quantum channel, the input is no longer available, whereas in the classical case we may assume that the channel copies the random variable before processing, and thus effectively preserves the input. Any such mechanism is ruled out in the quantum theory by the no-cloning theorem.*

In light of the above remark, it is meaningful to ask what "remains" of the state after a measurement. The channel in eq. (7) describes the situation where we discard any remnant after obtaining the measurement result. The following channel keeps as much as possible:

$$\rho \mapsto \sum_{x\in\mathcal{X}} \sqrt{E_x}\rho\sqrt{E_x} \otimes \left|x\right\rangle\!\left\langle x\right|, \tag{14}$$

where the result is in $\mathcal{B}_1(\mathcal{K}\otimes\mathbb{C}^{\mathcal{X}})$.

It is sometimes convenient to relax the normalisation of states and channels. A positive operator with trace at most 1 is a subnormalised state, while a completely positive trace nonincreasing map is a subnormalised channel. The idea is that measurements are inherently random, and a protocol may only give the desired result if we postselect on certain outcomes. The decrease in trace is then interpreted as the probability of failure.

1.3 Quantum entanglement

A system consisting of multiple subsystems is described using the tensor product of the individual Hilbert spaces: $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$. The spaces are assumed to be finite dimensional, so as vector spaces this is the same as the tensor product of the underlying vector spaces and the inner product is the unique extension of $\langle u_1 \otimes \cdots \otimes u_k, v_1 \otimes \cdots \otimes v_k \rangle = \langle u_1, v_1 \rangle \cdots \langle u_k, v_k \rangle$ which is linear in the second and conjugate-linear in the first argument.

When writing products of basis elements using the bra-ket notation, the tensor product sign is often omitted: $|0\rangle \otimes |2\rangle \otimes |1\rangle$ is written as $|0\rangle |2\rangle |1\rangle$ or even $|021\rangle$.

Definition 1.13 (Product state, separable state, entangled state)**.** *A product state on $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ is any state $\rho \in \mathcal{S}(\mathcal{H})$ of the form $\rho_1 \otimes \cdots \otimes \rho_k$ where $\rho_j \in \mathcal{S}(\mathcal{H}_j)$.*

The convex hull of product states is the set of separable states. The complement is the set of entangled states.

The prime example of an entangled state is the Bell state $|\psi\rangle\langle\psi|$ where

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \equiv \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{15}$$

We will be interested in pure states. A pure state is separable iff it is a product.

In entanglement theory we imagine that the parts of the system are far from each other, on each part we still allow arbitrary operations, but between the parts only classical communication is available. The class of channels that arise as the composition of such special channels is denoted LOCC (local operations and classical communication).

Under these restrictions, it is not possible to transform a separable state into an entangled one, which turns entangled states into a resource. Entangled states can be useful in information processing tasks. For example, $2n$ bits of classical communication and n shared Bell states make it possible to communicate n qubits of quantum information. Another example is that shared entanglement increases the amount of classical information that can be transmitted through a quantum channel. Of course this is only interesting if we regard classical communication as a resource as well.

The basic question: given $\rho \in \mathcal{S}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k)$ and $\sigma \in \mathcal{S}(\mathcal{K}_1 \otimes \cdots \otimes \mathcal{K}_k)$, is there a $\Lambda \in \mathrm{LOCC}$ such that $\Lambda(\rho) = \sigma$? It is unlikely that this can be efficiently decided. But it defines a preorder, the self-explanatory notation $\rho \xrightarrow{\mathrm{LOCC}} \sigma$ is sometimes used.

The idea of asymptotically perfect transformations goes back to Shannon. In this spirit we may ask what is

$$R(\rho, \sigma) = \inf \left\{ r \in \mathbb{R} \middle| \lim_{n \to \infty} \inf \left\{ \left\| \Lambda \left(\rho^{\otimes \lceil rn \rceil} \right) - \sigma^{\otimes n} \right\|_1 \middle| \Lambda \in \mathrm{LOCC} \right\} = 0 \right\} \tag{16}$$

for states ρ and σ? This means that roughly $R(\rho, \sigma)n$ copies of ρ are needed if we want n copies of σ and allow LOCC. We may not get the exact state but the approximation gets better as n grows.

What about equivalence? Very little is known in general. Let us restrict to pure states and suppose that $\rho \xrightarrow{\mathrm{LOCC}} \sigma$ and $\sigma \xrightarrow{\mathrm{LOCC}} \rho$. Then this can actually be implemented as a product of partial isometries (unitaries if the local Hilbert spaces have the same dimension). So this question is essentially the same as understanding the orbit spaces $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k / U(\mathcal{H}_1) \times \cdots \times U(\mathcal{H}_k)$, which is still very difficult.

Definition 1.14 (SLOCC)**.** *We say that ρ can be transformed into σ by SLOCC (stochastic LOCC) if there is a $p > 0$ such that $\rho \xrightarrow{\mathrm{LOCC}} p\sigma$ (defined using subnormalised channels). We may write this relation as $\rho \xrightarrow{\mathrm{SLOCC}} \sigma$.*

The idea is that we want a protocol possibly involving measurements and require that there is an outcome with nonzero probability such that, conditioned on this outcome, the resulting state is σ.

Let us restrict to pure states. If $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\varphi\rangle\langle\varphi|$, then it is not very difficult to see that $\rho \xrightarrow{\mathrm{SLOCC}} \sigma$ iff $\psi \geq \varphi$ as tensors! This means that $\omega(\psi, \varphi)$ can be seen as a kind of distant relative to the asymptotically perfect LOCC transformation rate as introduced before. In this context, $\omega(\psi, \varphi)$ is called the SLOCC transformation rate.

1.4 Pure bipartite states

The case of pure bipartite ($k=2$) states is well understood.

Exercise 4. *Prove that, up to local isometries, such a state can be written as $|\psi_P\rangle\langle\psi_P|$ where P is a probability distribution on some finite set $\mathcal{X}$ and*

$$|\psi_P\rangle = \sum_{x\in\mathcal{X}} \sqrt{P(x)}\,|x\rangle \otimes |x\rangle \in \mathbb{C}^{\mathcal{X}} \otimes \mathbb{C}^{\mathcal{X}}. \tag{17}$$

P is uniquely determined up to padding with 0 entries and permutations. (Hint: singular value decomposition.)

Nielsen's theorem characterises the possible exact transformations:

Theorem 1.15 (Nielsen, 1999)**.** $|\psi_P\rangle\langle\psi_P| \xrightarrow{\text{LOCC}} |\psi_Q\rangle\langle\psi_Q|$ *iff Q majorizes P.*

The approximate, asymptotically perfect transformations are characterised by a single parameter, the entropy:

Theorem 1.16 (Bennett, Bernstein, Popescu, Schumacher, 1995)**.**

$$R(|\psi_P\rangle\langle\psi_P|\,, |\psi_Q\rangle\langle\psi_Q|) = \frac{H(Q)}{H(P)} \tag{18}$$

Here $H(P) = -\sum_{x\in\mathcal{X}} P(x)\log P(x)$. (The base of log is 2, but it does not matter here.)

Exercise 5. $|\psi_P\rangle\langle\psi_P| \xrightarrow{\text{SLOCC}} |\psi_Q\rangle\langle\psi_Q|$ *iff* $\operatorname{rk}\psi_P \geq \operatorname{rk}\psi_Q$*, i.e. the number of nonzero entries of P is at least the number of nonzero entries of Q.*

From this follows that the SLOCC transformation rate is

$$\omega(\psi_P, \psi_Q) = \frac{\log\operatorname{rk}\psi_Q}{\log\operatorname{rk}\psi_P}. \tag{19}$$

2 The asymptotic spectrum

2.1 Semiring of tensor classes and the restriction preorder

Consider the set $\bigcup_{n_1,\dots,n_k\in\mathbb{N}} \mathbb{C}^{n_1}\otimes\cdots\otimes\mathbb{C}^{n_k}$ of tensors where $k\geq 2$. Recall that $s \geq t$ iff there are linear maps $A_1,\dots,A_k$ such that $(A_1\otimes\cdots\otimes A_k)s = t$ and $s\sim t$ iff $s\geq t$ and $s\leq t$. Let $T_k = (\bigcup_{n_1,\dots,n_k\in\mathbb{N}} \mathbb{C}^{n_1}\otimes\cdots\otimes\mathbb{C}^{n_k})/\sim$ be the set of equivalence classes. $\leq$ induces a partial order on T_k, which we will also denote by $\leq$. Since a finite dimensional vector space V is isomorphic to $\mathbb{C}^{\dim V}$, every tensor t is equivalent to a representative of a unique element $[t]$ of T_k.

Let s and t be tensors. Then $[s\otimes t]$ only depends on $[s]$ and $[t]$, therefore we get a binary operation $\cdot : T_k\times T_k \to T_k$. This operation turns T_k into a commutative monoid (semigroup with identity), the identity is $[\langle 1\rangle]$. The product and order structures are compatible: if $a,b,c\in T_k$ with $a\leq b$, then $ac\leq bc$. The unit tensors behave like natural numbers, so we may think $\mathbb{N}\subseteq T_k$ and abbreviate $[\langle r\rangle]$ as r.

The asymptotic restriction can be formulated in the ordered commutative monoid T_k:

$$a \lesssim b \iff \exists (x_N)_N \in \mathbb{N}^{\mathbb{N}} : \lim_{N\to\infty} \sqrt[N]{x_N} = 1, \forall N : a^N \leq x_N b^N. \tag{20}$$

This looks different from the asymptotic restriction discussed before, but it is the same. Our goal is to give a characterisation of $\lesssim$

Definition 2.1 (Direct sum)**.** *Let $s\in V_1\otimes\cdots\otimes V_k$ and $t\in W_1\otimes\cdots\otimes W_k$. Their direct sum is $s\oplus t \in (V_1\oplus W_1)\otimes\cdots\otimes(V_k\oplus W_k)$, regarded as a tensor of order k.*

Again, $[s\oplus t]$ only depends on $[s]$ and $[t]$, therefore we get another binary operation on $+ : T_k\times T_k\to T_k$. $(T_k,+,\cdot)$ is a commutative semiring. The order is also compatible with addition: if $a,b,c\in T_k$ with $a\leq b$, then $a+c\leq b+c$.

Definition 2.2 (Preordered commutative semiring). $(S, +, \cdot, \leq)$ *is a preordered commutative semiring if S is a set, $+ : S \times S \to S$ and $\cdot : S \times S \to S$ are binary operations and $\leq \subseteq S \times S$ is a relation, there exist elements $0, 1 \in S$ such that for every $a, b, c \in S$ the following properties are satisfied*

1) $a \leq a$
2) $a \leq b$ *and* $b \leq c$ *implies* $a \leq c$
3) $a + b = b + a$
4) $(a + b) + c = a + (b + c)$
5) $a + 0 = a$
6) $a \cdot b = b \cdot a$
7) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
8) $a \cdot 1 = a$
9) $a \cdot (b + c) = a \cdot b + a \cdot c$
10) $a \leq b$ *implies* $a + c \leq b + c$ *and* $a \cdot c \leq b \cdot c$

From now on we omit the word commutative and every semiring is assumed to be commutative.

To sum up, the set of equivalence classes of tensors is a preordered semiring with the operations and the preorder induced by $\oplus$, $\otimes$ and restriction, the zero is $[0]$ and the unit is $[\langle 1 \rangle]$. $S = T_k$ satisfies two additional properties which will turn out to be important

(S1) $\mathbb{N} \subseteq S$ and $\leq$ restricted to $\mathbb{N}$ is the usual order

(S2) $\forall a, b \in S, b \neq 0 \exists r \in \mathbb{N} : a \leq rb$.

2.2 Asymptotic preorder and its properties

Let S be a preordered semiring. Recall that we define the asymptotic preorder $\lesssim$ as

$$a \lesssim b \iff \exists (x_N)_N \in \mathbb{N}^{\mathbb{N}} : \lim_{N \to \infty} \sqrt[N]{x_N} = 1, \forall N : a^N \leq x_N b^N. \tag{21}$$

Note that $a^{N_1} \leq x_{N_1} b^{N_1}$ and $a^{N_2} \leq x_{N_2} b^{N_2}$ imply $a^{N_1+N_2} \leq x_{N_1} x_{N_2} b^{N_1+N_2}$, therefore any sequence satisfying $\forall N : a^N \leq x_N b^N$ can be replaced with one that is smaller and in addition satisfies $x_{N_1+N_2} \leq x_{N_1} x_{N_2}$. In particular, the formally weaker condition

$$\exists (x_N)_N \in \mathbb{N}^{\mathbb{N}} : \inf_N \sqrt[N]{x_N} = 1, \forall N : a^N \leq x_N b^N. \tag{22}$$

is also equivalent to $a \lesssim b$.

We now show that the asymptotic preorder is also compatible with the semiring operations.

Proposition 2.3. $(S, +, \cdot, \lesssim)$ *is a preordered semiring.*

Proof. If $a \in S$ then $a^N \leq a^N$, so $a \lesssim a$ (choosing $x_N = 1$).

If $a \lesssim b$ and $b \lesssim c$ then there are sequences x_N and y_N with $\sqrt[N]{x_N} \to 1$ and $\sqrt[N]{y_N} \to 1$ such that $a^N \leq x_N b^N$ and $b^N \leq y_N c^N$, so $a^N \leq (x_N y_N) c^N$, which proves $a \lesssim c$ since $\sqrt[N]{x_N y_N}$.

If $a \lesssim b$ and $c \in S$ then $a^N \leq x_N b^N$ for all N for some sequence x_N satisfying $\sqrt[N]{x_N} \to 1$ and using this,

$$\begin{aligned}(a+c)^N &= \sum_{k=0}^{N} \binom{N}{k} a^k c^{N-k} \\ &\leq \sum_{k=0}^{N} \binom{N}{k} x_k b^k c^{N-k} \\ &\leq \max_{0\leq k\leq N} x_k \sum_{k=0}^{N} \binom{N}{k} b^k c^{N-k} \\ &= \max_{0\leq k\leq N} x_k (b+c)^N.\end{aligned} \tag{23}$$

Since $\sqrt[N]{\max_{0\leq k\leq N} x_k} \to 1$, we have $a+c \lesssim b+c$.

Finally, $a^N \leq x_N b^N$ implies $(ac)^N = a^N c^N \leq x_N b^N c^N = x_N (bc)^N$, so $ac \lesssim bc$. □

Any semiring S gives rise to a ring R via the Grothendieck construction. Our next aim is to show that, assuming (S1) and (S2), $\lesssim$ induces a preorder on R (even though $\leq$ in general does not).

Now we sketch a version of the Grothendieck construction. On the set $S \times S$ we define an equivalence relation: (a_1, a_2) is equivalent to (b_1, b_2) if there is an $s \in S$ such that $a_1 + b_2 + s = a_2 + b_1 + s$. (The idea is that we should think of $[(a_1, a_2)]$ as $a_1 - a_2$.) This relation is compatible with the componentwise addition on $S \times S$. Let R be the set of equivalence classes equipped with the operation induced by addition. This turns R into an abelian group: $[(0,0)]$ is the zero and the inverse of $[(a_1, a_2)]$ is $[(a_2, a_1)]$. Finally, multiplication is defined as $[(a_1, a_2)] \cdot [(b_1, b_2)] = [(a_1 b_1 + a_2 b_2, a_1 b_2 + a_2 b_1)]$. We also have a semiring homomorphism $S \to R$ with $a \mapsto [(a, 0)]$, which is injective iff the cancellation law holds in S.

A preorder on a ring is characterised by its cone of positive elements.

Definition 2.4 (Positive cone, Archimedean)**.** *Let R be a ring with unit 1. A subset $P \subseteq R$ is a positive cone if*

1) $P + P \subseteq P$
2) $P \cdot P \subseteq P$
3) $0, 1 \in P$
4) $-1 \notin P$

We also say that b is greater than or equal to a if $b - a \in P$.

The cone is called Archimedean if $\forall a \in R \exists r \in \mathbb{N} : r - a \in P$.

We would like to define the cone $P \subseteq R$ as $[(a_1, a_2)] \in P \iff a_1 \gtrsim a_2$. For this we need to show that the condition does not depend on the representative chosen.

Proposition 2.5. *Let $a_1, a_2, s \in S$ and assume that (S1) and (S2) are satisfied. Then*

$$a_2 \lesssim a_1 \iff a_2 + b \lesssim a_1 + b \tag{24}$$

Proof. If $a_2 \lesssim a_1$ then $a_2 + b \lesssim a_1 + b$ since $(S, +, \cdot, \lesssim)$ is a preordered semiring.

Conversely, suppose that $a_2 + b \lesssim a_1 + b$. We prove by induction that $q a_2 + b \lesssim q a_1 + b$ for any $q \in \mathbb{N}$. The $q = 0$ base case is clear. For $q \geq 1$,

$$\begin{aligned}q a_2 + b &= (q-1)a_2 + a_2 + b \\ &\lesssim (q-1)a_2 + a_1 + b \\ &= (q-1)a_2 + b + a_1 \\ &\lesssim (q-1)a_1 + b + a_1 \\ &= q a_1 + b.\end{aligned} \tag{25}$$

This implies $qa_2 \lesssim qa_1 + b$. There is nothing to prove if $a_2 = 0$, while if $a_2 \neq 0$ then $a_1 \neq 0$, otherwise $qa_2 \lesssim b$ would follow for every q. Choose $r \in \mathbb{N}$ and $q \in \mathbb{N}$ such that $b \leq r$ and $2r \leq qa_2$ (use (S2)). Then there is a sequence x_N with $\sqrt[N]{x_N} \to 1$ and

$$(2r)^N \leq (qa_2)^N \leq x_N b^N \leq x_N r^N, \tag{26}$$

a contradiction.

So we can assume $a_1 \neq 0$. Let $r \in \mathbb{N}$ with $b \leq ra_1$, then for every $q \in \mathbb{N}$ we have $qa_2 \lesssim (q+r)a_1$. Using that $\lesssim$ is compatible with the product, this implies (by induction)

$$a_2^N \lesssim (1+r)a_2^{N-1}a_1 \lesssim (1+2r)a_2^{N-2}a_1^2 \lesssim \cdots \lesssim (1+Nr)a_1^N. \tag{27}$$

Thus there are sequences $(x_{N,M})_{M\in\mathbb{N}}$ with $\lim_{M\to\infty} \sqrt[M]{x_{N,M}} = 1$ and $a_2^{NM} \leq x_{N,M}(1+Nr)^M a_1^{NM}$. Choose a subsequence M_N such that $x_{N,M_N} \leq 2^{M_N}$ Then $\sqrt[NM_N]{(1+Nr)^{M_N} x_{N,M_N}} \leq \sqrt[N]{1+Nr}\sqrt[N]{2} \to 1$ and $a_2^{NM_N} \leq x_{N,M_N}(1+Nr)^{M_N} a_1^{NM_N}$, so $a_2 \lesssim a_1$. □

Now we can see that if (a_1, a_2) is equivalent to (b_1, b_2) and $a_1 \gtrsim a_2$ then also $b_1 \gtrsim b_2$. Indeed, the condition means $a_1 + b_2 + s = a_2 + b_1 + s$ for some $s \in S$. Add $a_2 \lesssim a_1$ to both sides:

$$b_2 + (a_1 + a_2 + s) \lesssim b_1 + (a_1 + a_2 + s), \tag{28}$$

so $b_2 \lesssim b_1$.

Proposition 2.6. *$P = \{[(a_1, a_2)] \in R | a_1 \gtrsim a_2\}$ is an Archimedean positive cone.*

Proof. $0 \lesssim 0$ and $0 \lesssim 1$ but $1 \not\lesssim 0$, so the zero $[(0,0)]$ and the unit $[(1,0)]$ are in P, while $[(0,1)] \notin P$.

Let $[(a_1, a_2)] \in P$ and $[(b_1, b_2)] \in P$. This means $a_1 \gtrsim a_2$ and $b_1 \gtrsim b_2$, which implies $a_1 + b_1 \gtrsim a_2 + b_2$, so $[(a_1, a_2)] + [(b_1, b_2)] = [(a_1 + b_1, a_2 + b_2)] \in P$. For the product, we write

$$\begin{aligned}
(a_1b_2 + a_2b_1)^N &= \sum_{k=0}^{N} \binom{N}{k} a_1^k b_2^k a_2^{N-k} b_1^{N-k} \\
&= \sum_{k\leq N/2} \binom{N}{k} a_1^k a_2^{N-2k} b_1^{N-k} a_2^k b_2^k + \sum_{k>N/2} \binom{N}{k} a_1^k b_1^{N-k} b_2^{2k-N} a_2^{N-k} b_2^{N-k} \\
&\lesssim \sum_{k\leq N/2} \binom{N}{k} a_1^k a_1^{N-2k} b_1^{N-k} a_2^k b_2^k + \sum_{k>N/2} \binom{N}{k} a_1^k b_1^{N-k} b_1^{2k-N} a_2^{N-k} b_2^{N-k} \\
&= \sum_{k\leq N/2} \binom{N}{k} a_1^{N-k} b_1^{N-k} a_2^k b_2^k + \sum_{k>N/2} \binom{N}{k} a_1^k b_1^k a_2^{N-k} b_2^{N-k} \\
&\leq 2(a_1b_1 + a_2b_2)^N,
\end{aligned} \tag{29}$$

and finish the proof as in the previous proposition.

Let $[(a_1, a_2)] \in R$. By (S2), there is an $r \in \mathbb{N}$ such that $a_1 \leq r$. Thus $a_1 \lesssim r \lesssim r + a_2$, so $[(r, 0)] - [(a_1, a_2)] = [(r + a_2, a_1)] \in P$. □

The asymptotic preorder on R will be defined as $a \lesssim b$ iff $b - a \in P$.

Definition 2.7 (Stone cone)**.** *A positive cone P is a Stone cone if*

$$(\forall n \in \mathbb{N} \exists p \in \mathbb{N} \setminus \{0\} : p(1 + na) \in P) \implies a \in P. \tag{30}$$

Proposition 2.8. *The positive cone P defined above is Stone.*

Proof. Let $a = [(a_1, a_2)]$. For $n, p \in \mathbb{N}$, $p(1 + na) \in P$ means $[(p(1 + na_1), pna_2)] \in P$, i.e.

$$pna_2 \lesssim p(1 + na_1). \tag{31}$$

By induction and using $p \geq 1$,

$$(na_2)^N \leq p(na_2)^N \lesssim p(1 + na_1)^N, \tag{32}$$

so $na_2 \lesssim 1 + na_1$. Let $r \in \mathbb{N}$ such that $1 \leq ra_1$ (possible by (S2)). Then $na_2 \lesssim (n + r)a_1$ for every $n \in \mathbb{N}$, which means

$$a_2^N \lesssim (1+r)a_2^{N-1}a_1 \lesssim (1+2r)a_2^{N-2}a_1^2 \lesssim \cdots \lesssim (1+Nr)a_1^N, \tag{33}$$

so as in previous proofs, $a_2 \lesssim a_1$, i.e. $a \in P$. □

2.3 The spectral theorem

Theorem 2.9 (Stone–Kadison)**.** *Let R be a commutative ring, $\lesssim$ a Stone preorder. Let*

$$\Delta = \{f \in \mathrm{Hom}(R, \mathbb{R}) | \forall a \gtrsim 0 : f(a) \geq 0\} \tag{34}$$

with the weakest topology making the evaluation functions $\hat{a} : \Delta \to \mathbb{R}$, $\hat{a}(f) = f(a)$ continuous. Then

1) Δ is a compact Hausdorff space

2) $\{\hat{a} | a \in R\}$ separates the points of Δ

3) $a \lesssim b \iff \forall f \in \Delta : f(a) \leq f(b)$.

We can apply this theorem to the ring of tensor classes, Δ is then the asymptotic spectrum of tensors. Its elements are called universal spectral points. When $f \in \Delta$, we usually want to apply it to an element of S (T_k in the case of tensors). This makes sense by composing with the canonical map $s \mapsto [(s, 0)]$. Conversely, any monotone semiring homomorphism $f : S \to \mathbb{R}_{\geq 0}$ determines a point in Δ as $[(a_1, a_2)] \mapsto f(a_1) - f(a_2)$. So in order to characterise $\lesssim$ on S, it is enough to find all monotone semiring homomorphisms from S to the nonnegative reals! Moreover, a semiring homomorphism is monotone for $\lesssim$ iff it is monotone for $\leq$. Thus we have:

Corollary 2.10 (Strassen, 1988)**.** *Let $(S, +, \cdot, \leq)$ be preordered semiring satisfying (S1) and (S2). Let*

$$\Delta(S) = \{f \in \mathrm{Hom}(R, \mathbb{R}_{\geq 0}) | \forall a \geq b : f(a) \geq f(b)\} \tag{35}$$

be the set of monotone semiring homomorphisms into $\mathbb{R}_{\geq 0}$. Then $a \gtrsim b$ iff for all $f \in \Delta(S)$: $f(a) \geq f(b)$.

Exercise 6. *To appreciate the spectral theorem, consider the following example which does not satisfy (S2). Let $S = \mathbb{N}[x]$ with the usual addition and multiplication of polynomials. Define the partial order as*

$$p \leq q \iff \lim_{x \to \infty} q(x) - p(x) \geq 0. \tag{36}$$

Verify that this turns S into a preordered semiring but (S2) is not satisfied. Describe $\Delta(S)$ and $\lesssim$.

Recall that we introduced the asymptotic restriction rate (or SLOCC transformation rate) $\omega(s, t)$. This can be equivalently written as

$$\omega(s, t) = \inf \left\{ \frac{m}{n} \middle| s^{\otimes m} \gtrsim t^{\otimes n} \right\} = \sup_{f \in \Delta(T_k)} \frac{\log f(t)}{\log f(s)}. \tag{37}$$

2.4 Examples of spectral points

Example 2.11. *Let $k = 2$. Then the equivalence class of a tensor is determined by its rank, which is additive under direct sum and multiplicative under tensor product, therefore $S \simeq \mathbb{N}$ and the unique point in the asymptotic spectrum is the matrix rank, so*

$$\omega(s, t) = \frac{\log \mathrm{rk}\, t}{\log \mathrm{rk}\, s}. \tag{38}$$

For $k \geq 3$ the asymptotic spectrum is not known and finding all the universal spectral points remains a challenging task.

If $S \subseteq T_k$ is any subsemiring, then everything above still applies to it. In particular, $\Delta(S) = \{f \in \mathrm{Hom}(S, \mathbb{R}_{\geq 0}) | \forall a \geq b : f(a) \geq f(b)\}$ characterises the asymptotic preorder on S. It is not hard to see that $\Delta(S)$ is the set of restrictions of the elements of $\Delta(T_k)$.

The simplest type of subsemirings are the ones generated by a single tensor t. In this case any semiring homomorphism is uniquely determined by its value on t, so the spectrum may be identified with a compact subset of $\mathbb{R}_{\geq 0}$ (via $f \mapsto f(t)$).

Example 2.12. *Let* $W = |100\rangle + |010\rangle + |001\rangle$*. Then* $\Delta(W) = [3 \cdot 2^{-2/3}, 2]$*.*

Definition 2.13 (Support)**.** *Let* $t \in V_1 \otimes \cdots \otimes V_k$ *be a tensor and let*

$$C = ((v_{1,1}, v_{1,2}, \ldots, v_{1,d_1}), (v_{2,1}, v_{2,2}, \ldots, v_{2,d_2}), \ldots, (v_{k,1}, v_{k,2}, \ldots, v_{k,d_k})) \tag{39}$$

be a k*-tuple of ordered bases, one for each vector space (*$d_j = \dim V_j$*). Then* t *can be uniquely expressed as*

$$t = \sum_{i_1,\ldots,i_k} t_{i_1 i_2 \ldots i_k} v_{1,i_1} \otimes v_{2,i_2} \otimes \cdots \otimes v_{k,i_k}. \tag{40}$$

The support of t *with respect to* C *is*

$$\operatorname{supp}_C t = \{(i_1, i_2, \ldots, i_k) \in [d_1] \times [d_2] \times \cdots \times [d_k] | t_{i_1 i_2 \ldots i_k} \neq 0\}. \tag{41}$$

Definition 2.14 (Upper support functional)**.** *Let* $t \in V_1 \otimes \cdots \otimes V_k$ *and let* $\theta \in \mathcal{P}([k])$*. We define*

$$\rho^\theta(t) = \min_C \max_{P \in \mathcal{P}(\operatorname{supp}_C t)} \sum_{j=1}^k \theta_j H(P_j), \tag{42}$$

where P_j *denotes the* j*th marginal of* P *(this makes sense as* $\operatorname{supp}_C t \subseteq [d_1] \times [d_2] \times \cdots \times [d_k]$*).*

$$\zeta^\theta(t) = \begin{cases} 2^{\rho^\theta(t)} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases} \tag{43}$$

is the upper support functional (for the parameter value θ*).*

Note that $\rho^\theta(t)$ as a function of θ is the maximum of a family of affine functions, therefore it is convex and so continuous.

Theorem 2.15 (Strassen, 1991)**.** *The upper support functional is an additive and submultiplicative monotone satisfying* $\zeta^\theta(\langle r\rangle) = r$*.*

It is not known whether ζ^θ is multiplicative or not. But it is known that it becomes multiplicative on certain nontrivial subsemirings.

Definition 2.16 (Oblique tensor)**.** *A tensor* $t \in V_1 \otimes \cdots \otimes V_k$ *is oblique if there is a tuple of bases* C *such that* $\operatorname{supp}_C t$ *is an antichain with respect to the product partial order.*

Example 2.17. $W = |100\rangle + |010\rangle + |001\rangle$ *is oblique.*

Theorem 2.18 (Strassen, 1991)**.** *Oblique tensors form a subsemiring of* T_k *and* ζ^θ *is multiplicative on this subsemiring. In particular, they are a family of spectral points for this subsemiring. Also, any tuple of bases* C *in which the support is an antichain satisfies*

$$\rho^\theta(t) = \max_{P \in \mathcal{P}(\operatorname{supp}_C t)} \sum_{j=1}^k \theta_j H(P_j) \tag{44}$$

Exercise 7. *Use this theorem to evaluate* $\zeta^{(1/3,1/3,1/3)}(W)$ *and* $\zeta^{(1,0,0)}(W)$*. Conclude that* $[3 \cdot 2^{-2/3}, 2] \subseteq \Delta(W)$*.*

3 Universal spectral points

3.1 Entanglement polytopes

Definition 3.1 (Marginal of a state)**.** *Let* $\rho \in \mathcal{S}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k)$*. The* j*th marginal of* ρ *is*

$$\rho_j = (\operatorname{Tr}_{\mathcal{H}_1} \otimes \cdots \otimes \operatorname{Tr}_{\mathcal{H}_{j-1}} \otimes \operatorname{id}_{\mathcal{B}_1(\mathcal{H}_j)} \otimes \operatorname{Tr}_{\mathcal{H}_{j+1}} \otimes \cdots \otimes \operatorname{Tr}_{\mathcal{H}_k})\rho \tag{45}$$

The jth marginal is itself a state: $\rho_j \in \mathcal{S}(\mathcal{H}_j)$.

The eigenvalues of a state are nonnegative numbers summing to 1. Let $r(\rho)$ denote the vector of eigenvalues of ρ, arranged in nonincreasing order.

Definition 3.2 (Entanglement polytope). *Let $\psi \in \mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ be a unit vector (so $|\psi\rangle\langle\psi|$ is a pure state). The entanglement polytope of ψ is*

$$\Delta_\psi = \overline{\left\{ (r(|\varphi\rangle\langle\varphi|_1), r(|\varphi\rangle\langle\varphi|_2), \ldots, r(|\varphi\rangle\langle\varphi|_k)) \middle| |\psi\rangle\langle\psi| \xrightarrow{\text{SLOCC}} |\varphi\rangle\langle\varphi| \in \mathcal{S}(\mathcal{H}) \right\}} \tag{46}$$

By definition, Δ_ψ is a compact subset of $\mathbb{R}^{d_1+d_2+\cdots+d_k}$ where $d_j = \dim \mathcal{H}_j$. It is a nontrivial fact that Δ_ψ is a convex polytope. Note that Δ_ψ does not depend on the inner products, so it makes sense to write Δ_t for any tensor $t \in V_1 \otimes \cdots \otimes V_k$.

We will need another characterisation of the entanglement polytope. Recall that $\mathcal{H}^{\otimes n}$ can be decomposed as

$$\mathcal{H}^{\otimes n} \simeq \bigoplus_{\lambda \vdash n} \mathbb{S}_\lambda(\mathcal{H}) \otimes [\lambda], \tag{47}$$

where the sum is over partitions λ of the integer n with at most $\dim \mathcal{H}$ rows, $\mathbb{S}_\lambda(\mathcal{H})$ is an irreducible representation of $\mathrm{GL}(\mathcal{H})$ and $[\lambda]$ is an irreducible representation of S_n. The isomorphism is as representations of $\mathrm{GL}(\mathcal{H}) \times S_n$. The summands on the right hand side are pairwise inequivalent representations, so there is a canonical projection $P^{\mathcal{H}}_\lambda : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ projecting on each subspace.

λ is a tuple of nonnegative integers. Let $\overline{\lambda}$ be the corresponding normalized partition, padded with zeros, i.e. if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $r \leq d$ is a partition of n, then

$$\overline{\lambda} = \left(\frac{\lambda_1}{n}, \ldots, \frac{\lambda_r}{n}, 0, \ldots, 0 \right) \in \mathbb{R}^d. \tag{48}$$

Theorem 3.3 (Brion 1987, Walter, Doran, Gross, Christandl, 2013). *The set*

$$\left\{ (\overline{\lambda_1}, \ldots, \overline{\lambda_2}) \middle| n \in \mathbb{N}, \forall j : \lambda_j \vdash n, (P^{\mathcal{H}_1}_{\lambda_1} \otimes P^{\mathcal{H}_2}_{\lambda_2} \otimes \cdots \otimes P^{\mathcal{H}_k}_{\lambda_k}) t^{\otimes n} \neq 0 \right\} \tag{49}$$

is a dense subset of Δ_t.

3.2 The quantum functionals

Definition 3.4 (Quantum functional). *Let $t \in V_1 \otimes \cdots \otimes V_k$ and $\theta \in \mathcal{P}([k])$. We define*

$$E^\theta(t) = \sup \left\{ \sum_{j=1}^k \theta_j H(\overline{\lambda_j}) \middle| n \in \mathbb{N}, \forall j : \lambda_j \vdash n, (P^{\mathcal{H}_1}_{\lambda_1} \otimes P^{\mathcal{H}_2}_{\lambda_2} \otimes \cdots \otimes P^{\mathcal{H}_k}_{\lambda_k}) t^{\otimes n} \neq 0 \right\}. \tag{50}$$

Its exponential,

$$F^\theta(t) = \begin{cases} 2^{E^\theta(t)} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases} \tag{51}$$

is the quantum functional (for the parameter value θ).

$E^\theta(t)$ only depends on the entanglement polytope. Using the characterisation with marginals, we can write E^θ in the following way. Choose inner products on $V_1, \ldots, V_k$ so they become Hilbert spaces. Then

$$E^\theta(t) = \sup \left\{ \sum_{j=1}^k \theta_j H(\sigma_j) \middle| \sigma = |s\rangle\langle s|, \|s\| = 1, s \leq t \right\} \tag{52}$$

where $H(\rho) = -\operatorname{Tr} \rho \log \rho = H(r(\rho))$ is the von Neumann entropy. Equivalently, we may fix $s = t$ in the above expression and optimize over the choice of the inner products making t a unit vector. This is because the orbit of t is dense in the set of tensors to which t restricts and any two inner products are related by a linear transformation.

Exercise 8. *Use any of the above expressions to show that $s \geq t$ implies $F^\theta(s) \geq F^\theta(t)$.*

3.3 Some properties of entropy

Let $h(p) = -p \log p - (1-p)\log(1-p)$, i.e. the entropy of the probability distribution $(p, 1-p)$.

Exercise 9. *Let $x, y \in \mathbb{R}$. Then*

$$\max_{p\in[0,1]} 2^{px+(1-p)y+h(p)} = 2^x + 2^y. \tag{53}$$

Exercise 10. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, $\rho \in \mathcal{S}(\mathcal{H})$ and $\sigma \in \mathcal{S}(\mathcal{K})$. Then*

1. *$\rho \otimes \sigma \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ and $H(\rho \otimes \sigma) = H(\rho) + H(\sigma)$.*
2. *for any $p \in [0,1]$, $p\rho \oplus (1-p)\sigma \in \mathcal{S}(\mathcal{H} \oplus \mathcal{K})$ and $H(p\rho \oplus (1-p)\sigma) = pH(\rho) + (1-p)H(\sigma) + h(p)$*

Proposition 3.5 (Supermultiplicativity of the quantum functionals)**.** *For any $\theta \in \mathcal{P}([k])$ and tensors s, t the inequality*

$$F^\theta(s \otimes t) \geq F^\theta(s) F^\theta(t) \tag{54}$$

holds.

Proof. We prove $E^\theta(s \otimes t) \geq E^\theta(s) + E^\theta(t)$.

Let $s \in V_1 \otimes \cdots \otimes V_k$, $t \in W_1 \otimes \cdots \otimes W_k$, $\epsilon > 0$. Choose inner products on $V_1, \ldots, V_k, W_1, \ldots, W_k$ such that s and t are unit vectors and $E^\theta(s) - \epsilon \leq \sum_{j=1}^k \theta_j H(|s\rangle\langle s|_j)$ and $E^\theta(t) - \epsilon \leq \sum_{j=1}^k \theta_j H(|t\rangle\langle t|_j)$. Equip $V_j \otimes W_j$ with the induced inner product. Then $s \otimes t$ is also a unit vector and $|s \otimes t\rangle\langle s \otimes t|_j = |s\rangle\langle s|_j \otimes |t\rangle\langle t|_j$, therefore

$$\begin{aligned}
E^\theta(s \otimes t) &\geq \sum_{j=1}^k \theta_j H(|s \otimes t\rangle\langle s \otimes t|_j) \\
&= \sum_{j=1}^k \theta_j \left(H(|s\rangle\langle s|_j) + H(|t\rangle\langle t|_j) \right) \\
&= \sum_{j=1}^k \theta_j H(|s\rangle\langle s|_j) + \sum_{j=1}^k \theta_j H(|t\rangle\langle t|_j) \\
&\geq E^\theta(s) - \epsilon + E^\theta(t) - \epsilon.
\end{aligned} \tag{55}$$

This is true for any $\epsilon > 0$, so $E^\theta(s \otimes t) \geq E^\theta(s) + E^\theta(t)$. □

Proposition 3.6 (Superadditivity of the quantum functionals)**.** *For any $\theta \in \mathcal{P}([k])$ and tensors s, t the inequality*

$$F^\theta(s \oplus t) \geq F^\theta(s) + F^\theta(t) \tag{56}$$

holds.

Proof. We prove $E^\theta(s \otimes t) \geq pE^\theta(s) + (1-p)E^\theta(t) + h(p)$ for all $p \in (0,1)$.

Let $s \in V_1 \otimes \cdots \otimes V_k$, $t \in W_1 \otimes \cdots \otimes W_k$, $\epsilon > 0$, $p \in (0,1)$. Choose inner products on $V_1, \ldots, V_k, W_1, \ldots, W_k$ such that s and t are unit vectors and $E^\theta(s) - \epsilon \leq \sum_{j=1}^k \theta_j H(|s\rangle\langle s|_j)$ and $E^\theta(t) - \epsilon \leq \sum_{j=1}^k \theta_j H(|t\rangle\langle t|_j)$. Equip $V_j \oplus W_j$ with the induced inner product. Then $\sqrt{p}s \oplus \sqrt{1-p}t$ is also a unit vector, equivalent to $s \oplus t$, and $\left|\sqrt{p}s \oplus \sqrt{1-p}t\right\rangle\left\langle\sqrt{p}s \oplus \sqrt{1-p}t\right|_j = p\,|s\rangle\langle s|_j \oplus (1-p)\,|t\rangle\langle t|_j$, therefore

$$\begin{aligned}
E^\theta(s \oplus t) &\geq \sum_{j=1}^k \theta_j H(\left|\sqrt{p}s \oplus \sqrt{1-p}t\right\rangle\left\langle\sqrt{p}s \oplus \sqrt{1-p}t\right|_j) \\
&= \sum_{j=1}^k \theta_j \left(pH(|s\rangle\langle s|_j) + (1-p)H(|t\rangle\langle t|_j) + h(p) \right) \\
&= p\sum_{j=1}^k \theta_j H(|s\rangle\langle s|_j) + (1-p)\sum_{j=1}^k \theta_j H(|t\rangle\langle t|_j) + h(p) \\
&\geq pE^\theta(s) - \epsilon + (1-p)E^\theta(t) - \epsilon + h(p).
\end{aligned} \tag{57}$$

This is true for any $\epsilon > 0$, so $E^\theta(s \oplus t) \geq pE^\theta(s) + (1-p)E^\theta(t) + h(p)$.

By continuity, the inequality extends to $p \in [0,1]$, and then we can use the result of Exercise 9 to conclude

$$\begin{aligned} F^\theta(s \oplus t) &= 2^{E^\theta(s\oplus t)} \\ &\geq \max_{p\in[0,1]} 2^{pE^\theta(s)+(1-p)E^\theta(t)+h(p)} = 2^{E^\theta(s)} + 2^{E^\theta(t)} = F^\theta(s) + F^\theta(t). \end{aligned} \tag{58}$$

□

For the next lemma we use the nontrivial fact that triples of partitions (λ, μ, ν) with nonvanishing Kronecker coefficients form a semigroup, and similarly with Littlewood–Richardson coefficients. More precisely, we only need that for partitions λ, μ, ν and $N \in \mathbb{N}$:

$$g_{\lambda\mu\nu} \neq 0 \implies g_{N\lambda,N\mu,N\nu} \neq 0 \tag{59}$$

and

$$c^\lambda_{\mu\nu} \neq 0 \implies c^{N\lambda}_{N\mu,N\nu} \neq 0. \tag{60}$$

Recall that the dimension estimates

$$\frac{1}{(|\lambda|+1)^{l(\lambda)^2}} 2^{|\lambda| H(\overline{\lambda})} \leq \dim[\lambda] \leq 2^{|\lambda| H(\overline{\lambda})} \tag{61}$$

imply (using $l(\lambda) = l(N\lambda)$)

$$\lim_{N\to\infty} \frac{1}{N} \log \dim[N\lambda] = |\lambda| H(\overline{\lambda}). \tag{62}$$

Lemma 3.7 (Entropic vanishing conditions for Kronecker coefficients and Littlewood–Richardson coefficients)**.** *Let λ, μ, ν be partitions.*

1) *if $g_{\lambda\mu\nu} \neq 0$ then $H(\overline{\lambda}) \leq H(\overline{\mu}) + H(\overline{\nu})$*

2) *if $c^\lambda_{\mu\nu} \neq 0$ then $H(\overline{\lambda}) \leq \frac{|\mu|}{|\mu|+|\nu|} H(\overline{\mu}) + \frac{|\nu|}{|\mu|+|\nu|} H(\overline{\nu}) + h\left(\frac{|\mu|}{|\mu|+|\nu|}\right)$.*

Proof. Suppose $g_{\lambda\mu\nu} \neq 0$ (then of course $|\lambda| = |\mu| = |\nu|$). By the semigroup property, $g_{N\lambda,N\mu,N\nu} \neq 0$, which means that $[N\mu] \otimes [N\nu]$ has a subrepresentation isomorphic to $[N\lambda]$, therefore

$$\dim[N\lambda] \leq \dim([N\mu] \otimes [N\nu]) = \dim[N\mu] \dim[N\nu] \tag{63}$$

and thus

$$\begin{aligned} 0 &\leq \lim_{N\to\infty} \frac{1}{N} \log \frac{\dim[N\mu]\dim[N\nu]}{\dim[N\lambda]} \\ &= |\mu| H(\overline{\mu}) + |\nu| H(\overline{\nu}) - |\lambda| H(\overline{\lambda}) = |\lambda| (H(\overline{\mu}) + H(\overline{\nu}) - H(\overline{\lambda})). \end{aligned} \tag{64}$$

Suppose $c^\lambda_{\mu\nu} \neq 0$ (then $|\lambda| = |\mu| + |\nu|$). By the semigroup property, $c^{N\lambda}_{N\mu,N\nu} \neq 0$, which means that $\mathrm{Ind}^{S_{N|\lambda|}}_{S_{N|\mu|}\times S_{N|\nu|}} [N\mu] \otimes [N\nu]$ contains a subrepresentation isomorphic to $[N\lambda]$, therefore

$$\begin{aligned} \dim[N\lambda] &\leq \dim(\mathrm{Ind}^{S_{N|\lambda|}}_{S_{N|\mu|}\times S_{N|\nu|}} [N\mu] \otimes [N\nu]) \\ &= \frac{|S_{N|\lambda|}|}{|S_{N|\mu|} \times S_{N|\nu|}|} \dim[N\mu] \dim[N\nu] = \binom{N|\lambda|}{N|\mu|} \dim[N\mu] \dim[N\nu] \end{aligned} \tag{65}$$

and thus

$$\begin{aligned} 0 &\leq \lim_{N\to\infty} \frac{1}{N} \log \frac{\binom{N|\lambda|}{N|\mu|} \dim[N\mu]\dim[N\nu]}{\dim[N\lambda]} \\ &= |\lambda| h\left(\frac{|\mu|}{|\nu|}\right) + |\mu| H(\overline{\mu}) + |\nu| H(\overline{\nu}) - |\lambda| H(\overline{\lambda}) \\ &= |\lambda| \left(\frac{|\mu|}{|\mu|+|\nu|} H(\overline{\mu}) + \frac{|\nu|}{|\mu|+|\nu|} H(\overline{\nu}) + h\left(\frac{|\mu|}{|\mu|+|\nu|}\right) - H(\overline{\lambda}) \right). \end{aligned} \tag{66}$$

□

Proposition 3.8 (Submultiplicativity of the quantum functionals)**.** *For any $\theta \in \mathcal{P}([k])$ and tensors s, t the inequality*

$$F^\theta(s \otimes t) \leq F^\theta(s) F^\theta(t) \tag{67}$$

holds.

Proof. We prove $E^\theta(t \otimes s) \leq E^\theta(t) + E^\theta(s)$.

Let $s \in V_1 \otimes \cdots \otimes V_k$, $t \in W_1 \otimes \cdots \otimes W_k$. Let $\lambda_1, \ldots, \lambda_k \vdash n$ be partitions such that

$$(P^{V_1 \otimes W_1}_{\lambda_1} \otimes \cdots \otimes P^{V_k \otimes W_k}_{\lambda_k})(s \otimes t)^{\otimes n} \neq 0. \tag{68}$$

The sum of the isotypic projections over all partitions is the identity, therefore

$$(s \otimes t)^{\otimes n} = s^{\otimes n} \otimes t^{\otimes n} = \sum_{\substack{\mu_1, \ldots, \mu_k \vdash n \\ \nu_1, \ldots, \nu_k \vdash n}} (P^{V_1}_{\mu_1} \otimes \cdots \otimes P^{V_k}_{\mu_k}) s^{\otimes n} (P^{W_1}_{\nu_1} \otimes \cdots \otimes P^{W_k}_{\nu_k}) t^{\otimes n}. \tag{69}$$

Combining with the previous equation we see that there is at least one tuple $\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_k$ such that the following conditions hold:

$$\forall j : P^{V_j \otimes W_j}_{\lambda_j} (P^{V_j}_{\mu_j} \otimes P^{W_j}_{\nu_j}) \neq 0 \tag{70}$$

$$(P^{V_1}_{\mu_1} \otimes \cdots \otimes P^{V_k}_{\mu_k}) s^{\otimes n} \neq 0 \tag{71}$$

$$(P^{W_1}_{\nu_1} \otimes \cdots \otimes P^{W_k}_{\nu_k}) t^{\otimes n} \neq 0 \tag{72}$$

The first one implies $g_{\lambda_j \mu_j \nu_j} \neq 0$. From these we get

$$H(\overline{\lambda_j}) \leq H(\overline{\mu_j}) + H(\overline{\nu_j}) \tag{73}$$

$$E^\theta(s) \geq \sum_{j=1}^k \theta_j H(\overline{\mu_j}) \tag{74}$$

$$E^\theta(t) \geq \sum_{j=1}^k \theta_j H(\overline{\nu_j}) \tag{75}$$

We combine the inequalitites:

$$\begin{aligned} E^\theta(s) + E^\theta(t) &\geq \sum_{j=1}^k \theta_j H(\overline{\mu_j}) + \sum_{j=1}^k \theta_j H(\overline{\nu_j}) \\ &= \sum_{j=1}^k \theta_j \left(H(\overline{\mu_j}) + H(\overline{\nu_j}) \right) \geq \sum_{j=1}^k \theta_j H(\overline{\lambda_j}) \end{aligned} \tag{76}$$

The supremum of the right hand side over the admissible tuples of partitions is $E^\theta(s \otimes t)$. □

Proposition 3.9 (Subadditivity of the quantum functionals)**.** *For any $\theta \in \mathcal{P}([k])$ and tensors s, t the inequality*

$$F^\theta(s \oplus t) \leq F^\theta(s) + F^\theta(t) \tag{77}$$

holds.

Proof. We prove $E^\theta(s \oplus t) \leq p E^\theta(s) + (1-p) E^\theta(t) + h(p)$ for some $p \in [0,1]$.

Let $s \in V_1 \otimes \cdots \otimes V_k$, $t \in W_1 \otimes \cdots \otimes W_k$. Let $\lambda_1, \ldots, \lambda_k \vdash n$ be partitions such that

$$(P^{V_1 \oplus W_1}_{\lambda_1} \otimes \cdots \otimes P^{V_k \oplus W_k}_{\lambda_k})(s \oplus t)^{\otimes n} \neq 0. \tag{78}$$

Decompose $(s \oplus t)^{\otimes n}$ as

$$\sum_{m=0}^n \bigoplus_{\binom{n}{m}} s^{\otimes m} \otimes t^{\otimes n-m}. \tag{79}$$

At least one term is not annihilated by the projectors. Let m be the index of such a term and decompose the powers of s and t:

$$(P^{V_1\oplus W_1}_{\lambda_1}\otimes\cdots\otimes P^{V_k\oplus W_k}_{\lambda_k})\sum_{\substack{\mu_1,\ldots,\mu_k\vdash m\\ \nu_1,\ldots,\nu_k\vdash n-m}}(P^{V_1}_{\mu_1}\otimes\cdots\otimes P^{V_k}_{\mu_k})s^{\otimes m}\otimes(P^{W_1}_{\nu_1}\otimes\cdots\otimes P^{W_k}_{\nu_k})t^{\otimes n-m}\neq 0. \quad (80)$$

We see that there is at least one tuple $\mu_1,\ldots,\mu_k,\nu_1,\ldots,\nu_k$ such that the following conditions hold:

$$\forall j: P^{V_j\oplus W_j}_{\lambda_j}(P^{V_j}_{\mu_j}\otimes P^{W_j}_{\nu_j})\neq 0 \quad (81)$$
$$(P^{V_1}_{\mu_1}\otimes\cdots\otimes P^{V_k}_{\mu_k})s^{\otimes m}\neq 0 \quad (82)$$
$$(P^{W_1}_{\nu_1}\otimes\cdots\otimes P^{W_k}_{\nu_k})t^{\otimes n-m}\neq 0 \quad (83)$$

The first one implies $c^{\lambda_j}_{\mu_j\nu_j}\neq 0$. From these we get

$$H(\overline{\lambda_j})\leq\frac{m}{n}H(\overline{\mu_j})+\frac{n-m}{n}H(\overline{\nu_j})+h\left(\frac{m}{n}\right) \quad (84)$$
$$E^\theta(s)\geq\sum_{j=1}^k\theta_jH(\overline{\mu_j}) \quad (85)$$
$$E^\theta(t)\geq\sum_{j=1}^k\theta_jH(\overline{\nu_j}) \quad (86)$$

We combine the inequalitites and set $p=\frac{m}{n}$:

$$\begin{aligned}pE^\theta(s)+(1-p)E^\theta(t)+h(p)&\geq p\sum_{j=1}^k\theta_jH(\overline{\mu_j})+(1-p)\sum_{j=1}^k\theta_jH(\overline{\nu_j})+h(p)\\&=\sum_{j=1}^k\theta_j\left(pH(\overline{\mu_j})+(1-p)H(\overline{\nu_j})+h(p)\right)=\sum_{j=1}^k\theta_jH(\overline{\lambda_j})\end{aligned} \quad (87)$$

The supremum of the right hand side over the admissible tuples of partitions is $E^\theta(s\oplus t)$. By compactness of $[0,1]$, there is a sequence of tuples of partitions along which the supremum is approached and m/n converges. □

This finishes the proof of the following theorem:

Theorem 3.10 (Christandl, V, Zuiddam, 2017)**.** *For any $\theta\in\mathcal{P}([k])$ the quantum functional F^θ is a universal spectral point for k-tensors.*

Remark 3.11. *For $k=2$, $F^\theta(t)=\operatorname{rk}t$ for any $\theta\in\mathcal{P}([2])$.*

For $k\geq 3$, $F^\theta(t)$ gives a $k-1$-simplex in $\Delta(T_k)$.

For $k=3$ it is not known if there are other universal spectral points.

For $k\geq 4$ there are other universal spectral points. The ranks of the flattenings give $2^{k-1}-1$ points and only k of them are among the quantum functionals, the ones with θ concentrated on one point.

It is known that

$$\zeta^\theta(t)\geq F^\theta(t)=\lim_{n\to\infty}\sqrt[n]{\zeta^\theta(t)}. \quad (88)$$

No example of a tensor t is known with $\zeta^\theta(t)\neq F^\theta(t)$.