Fixed point collisions and tensorial order parameters in Luttinger semimetals and some popular field theories

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Outline:

1) **Physical motivation**: (symmetry-poor real world system)

   Quadratic band touching in three dimensions and the Luttinger Hamiltonian

   Coulomb interaction and the scale-invariant (“non-Fermi liquid”) fixed point

   Fixed point annihilation and the separation of scales

   Spin-2 (tensor) ordering

2) **Conformal spinoffs**

   Chiral symmetry breaking in QED_{d<4} revisited

   UV-complete O(N) models above four (space-time) dimensions
Gapless semiconductors with band inversion (gray tin, HgTe, YPtBi)

Luttinger (spin-orbit) Hamiltonian (p-orbitals, $J=\frac{3}{2}$) (Luttinger 1956)

\[ H = \frac{1}{2m} \left( (\gamma_1 + \frac{5}{2}\gamma_2)k^2 - 2\gamma_2(k \cdot \mathbf{S})^2 \right) \]

with (rotationally symmetric) eigenvalues

\[ E_L(k) = \frac{\gamma_1 + 2\gamma_2}{2m} k^2, \quad E_H(k) = \frac{\gamma_1 - 2\gamma_2}{2m} k^2 \]
Luttinger Hamiltonian \( a \ la \) Dirac:

\[
H(k) = \epsilon(k) + \frac{\gamma_2}{m} d_a \Gamma^a
\]

where,

\[
\epsilon(k) = \frac{\gamma_1}{2m} k^2, \quad d_a(k) = -3 \xi_{ij} k_i k_j
\]

\[
d_1 = -\sqrt{3} k_y k_z, \quad d_2 = -\sqrt{3} k_x k_z, \quad d_3 = -\sqrt{3} k_x k_y
\]

\[
d_4 = -\frac{\sqrt{3}}{2} (k_x^2 - k_y^2),
\]

\[
d_5 = -\frac{1}{2} (2k_z^2 - k_x^2 - k_y^2)
\]

are \( l=2 \) (real) spherical harmonics.

Five 4 \times 4 \text{ Dirac matrices satisfy Clifford algebra:}

\[
\{ \Gamma^a, \Gamma^b \} = 2\delta_{ab}
\]
Long-range Coulomb interaction (1/r):

without the hole band, at ``zero” (low) density:

\[ \text{Wigner crystal} \]

With the hole band filled and particle band empty: the system is critical!

In the RG language, the charge flows with the change of cutoff:

\[
\frac{de^2}{d \ln b} = (z + 2 - d)e^2 - 4e^4
\]

(Abrikosov, ZETF 1974; Moon, Xu, Kim, Balents PRL 2013)
Below and near the upper critical dimension, $d_{up} = 4$, the flow is towards a non-Fermi liquid fixed point, with the charge at

$$e_x^2 = \frac{15\epsilon}{76} + \mathcal{O}(\epsilon^2)$$

$$\epsilon = 4 - d$$

and the dynamical critical exponent $Z < 2$:

$$z = 2 - \frac{16}{15} e^2$$

This implies power-laws in various responses, such as specific heat:

$$c_v \sim T^{d/z} \approx T^{1.7}$$

Emergent scale (conformal?) invariance!

Cheap way to get a non-Fermi liquid phase in 3D !?

\[ L = L_0 + L_a + L_\psi \]

with the free (Luttinger) part,

\[ L_0 = \psi_i^\dagger [\partial_r + H_0(-i\nabla)] \psi_i \]

and long-range (Coulomb) and short-range (Coulomb) interactions

\[ L_a = \frac{1}{2} (\nabla a)^2 + i e a \psi_i^\dagger \psi_i \]

\[ L_\psi = g_1 (\psi_i^\dagger \psi_i)^2 + g_2 (\psi_i^\dagger \gamma_a \psi_i)^2 + g_3 (\psi_i^\dagger \gamma_{ab} \psi_i)^2 \]
The RG flow of all the couplings: (one loop)

\[
\frac{de^2}{d \ln b} = (2 + z - d - \eta_a)e^2,
\]

\[
\frac{dg_1}{d \ln b} = (z - d)g_1 - (e^2 + 2g_1)g_2 - 24g_3^2,
\]

\[
\frac{dg_2}{d \ln b} = (z - d)g_2 + \frac{4(e^2 + 2g_1)g_2}{5} - \frac{(e^2 + 2g_1)^2}{20} - \frac{37 + 16N}{5}g_2^2 + \frac{112}{5}g_2g_3 - \frac{136}{5}g_3^2,
\]

\[
\frac{dg_3}{d \ln b} = (z - d)g_3 - \frac{1}{5}(e^2 + 2g_1)g_3 + g_2^2 - 6g_2g_3 + \frac{4(11 - 4N)}{5}g_3^2
\]

with,

\[
\eta_a = Ne^2 \quad \quad z = 2 - \frac{4}{15}e^2
\]

and the “charge”

\[
e^2 = 2me_{el}^2/(4\pi\hbar^2\varepsilon)
\]
Close to and below $d=4$ there is a (IR stable) NFL fixed point, but also a (UV stable) quantum critical point at strong interaction:

They get closer, but remain separated in the coupling space!
At some “lower critical dimension” NFL and QCP collide:

In one loop calculation, this occurs at \( d_l = 3.26240 \), slightly above three dimensions.
Finally, below the NFL and QCP become complex (unphysical), and there is only a runaway flow left:

The system is unstable towards a gapped (Mott) insulator.

**Scale invariance lost!**
Fixed point collision and annihilation:

(Halperin, Lubensky, Ma, PRL 1974; IH & Tesanovic, PRL 1995; Kaveh & IH, PRB 2005; Gies & Jaeckel 2006; Kaplan, Lee, Son, Stephanov, PRD 2009; Nahum, PRX 2015,.........)
General number of fermions \((N)\) and dimension \((d)\):

Near \(d=2\) the collision occurs in the completely perturbative regime:

\[
N \geq N_c(\epsilon) = \frac{64}{25\epsilon^2} + \mathcal{O}(1/\epsilon)
\]

\[
\epsilon = d - 2
\]
Critical number of fermions in $d=3$:

**TABLE III.** Critical fermion number $N_c$ in $d=3$ spatial dimensions from different approaches.

<table>
<thead>
<tr>
<th>Method</th>
<th>Reference</th>
<th>$N_c(d=3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 + \epsilon$ expansion</td>
<td>Sec. III</td>
<td>2.56</td>
</tr>
<tr>
<td>RG in fixed $d=3$</td>
<td>Sec. IV</td>
<td>2.10</td>
</tr>
<tr>
<td>Functional RG</td>
<td>Sec. VI</td>
<td>1.86</td>
</tr>
<tr>
<td>$1/N$ expansion in $d=3$</td>
<td>Ref. [18]</td>
<td>$\geq 2.6(2)$</td>
</tr>
</tbody>
</table>
Order parameter for \( d < d_{\text{low}} \)

\[
\chi_i = 2g_2 \langle \Psi^\dagger \gamma_i \Psi \rangle
\]

Out of the five \( \chi_1, \ldots, \chi_5 \) not all equivalent:

1. \( \chi_1 \neq 0 \): \( \varepsilon(\vec{p}) \) gapped with minimal gap at two opposite points on equator

2. \( \chi_5 < 0 \): \( \varepsilon(\vec{p}) \) gapless with gap closing at north and south pole

3. \( \chi_5 > 0 \): \( \varepsilon(\vec{p}) \) gapped with minimal gap at entire equator

Energy \( E = \int \frac{d\vec{p}}{(2\pi)^3} \varepsilon(\vec{p}) \) is minimized for (3): \( \chi_5 > 0 \) (modulo \( O(3) \))
The fate of NFL: if $d_\text{F}$ is above but close to $d=3$, the flow becomes slow close to (complex!) NFL fixed point. The RG escape time is long:

$$b_0 = e^{\frac{C}{\sqrt{d_{\text{low}}-d}}} - B + \mathcal{O}(d_{\text{low}} - d)$$

with non-universal constants $C$ and $B$. There is wide crossover region of the NFL behavior within the temperature window

$$(T_c, T_*)$$

with the critical temperature,

$$T_c \approx T_* b_0^{-z}$$

Characteristic energy scale for interaction effects

$$k_B T_* \sim \frac{e_{\text{el}}^2}{\varepsilon L_*} = \frac{\hbar^2}{2m L_*^2} = \frac{4m}{m_{\text{el}} \varepsilon^2} E_0$$

(Sherrington & Kohn, Halperin & Rice, RMP 1968)
Some numbers: (for HgTe)

small mass \[ m/m_{el} \approx 1/50 \]

high dielectric constant \[ \varepsilon \approx 30 \]

still a reasonable \[ T_* \sim 10 \text{ K} - 100 \text{ K} \]

and (maybe) a detectable \[ T_c \approx T_*/100 \]
Cubic symmetry: realistic Luttinger Hamiltonian, cubic symmetry

\[ H = \frac{\hbar^2}{2m^*} \left[ \left( \alpha_1 + \frac{5}{2} \alpha_2 \right) p^2_{14} - 2\alpha_3 (\vec{p} \cdot \vec{J})^2 \right. \\
\left. + 2(\alpha_3 - \alpha_2) \sum_{i=1}^{3} p_i^2 J_i^2 \right], \]

contains particle-hole asymmetry and anisotropy parameters

\[ x = -\frac{\alpha_1}{\alpha_2 + \alpha_3}, \quad \delta = -\frac{\alpha_2 - \alpha_3}{\alpha_2 + \alpha_3} \]

with (generically) particularly slow flow of the anisotropy:

\[ \dot{\delta} \simeq -\frac{8}{105} e^2 \delta. \]
Flow of the charge is now

\[ \dot{e}^2 = \frac{d e^2}{d \log b} = (4 - d - \eta) e^2 - \frac{f_e^2(\delta)}{1 - \delta^2} e^4 \]

which lowers the critical dimension: (Boettcher & IH, PRB 2017)
Chiral symmetry breaking in QED3 revisited

Schwinger-Dyson, large-N, calculation of the mass gap (Appelquist, Nash, Wijewardhana, PRL 1988):

\[ \Sigma(0) = \alpha e^{(\delta+2)} \exp \left[ \frac{-2n\pi}{(32/\pi^2N - 1)^{1/2}} \right] \]

as the number of four-component Dirac fermions \( N \)

\[ N \to 32/\pi^2 \]

from below.

This should also be understandable as a fixed point collision and annihilation.
Consider QED near four space-time dimensions with (generated) quartic terms (Herbut, PRD 2016; Di Pietro et al, PRL 2016)

\[ L = \bar{\Psi}_n i \gamma_\mu (\partial_\mu - ieA_\mu) \Psi_n + \sum_{a=1}^{2} g_a (\bar{\Psi}_n X_a \gamma_\mu \Psi_n)^2 + \frac{F_{\mu\nu}^2}{4} \]

with

\[ X_1 = 1 \]
\[ X_2 = \gamma_5 \]

i. e. with additional (axial) current – (axial) current interactions.
The flow in the IR \((\Lambda \to \Lambda/b)\), one loop:

\[
\beta_1 = (2 - d)g_1 + 4(N + 1)g_1^2 - 8g_1g_2 - 6e^2g_2, \\
\beta_2 = (2 - d)g_2 + 2(2N - 1)g_2^2 + 4g_1g_2 \\
- 6g_1^2 - 6e^2g_1 - \frac{3}{2}e^4, \\
\beta_e = (4 - d)e^2 + \beta_{e0}(e).
\]

and the charge beta-function precisely in \(d=4\) is:

\[
\beta_{e0}(e) = -\frac{4N}{3}e^4 - 4Ne^6 + O(Ne^8, N^2e^8)
\]

(Gorishny, Kataev, Larin 1991 (four loop))
Introducing linear combinations:

\[ g_{\pm} = g_1 \pm g_2 \]

equations (almost) decouple

\[ \beta_+ = (2 - d)g_+ + 2(N - 1)g_+^2 + 2Ng_-^2 - 6g_+e^2 - \frac{3}{2}e^4, \]

\[ \beta_- = (2 - d)g_- + 6g_-^2 + 4(N + 1)g_+g_- + 6g_-e^2 + \frac{3}{2}e^4 \]

When \( N=0 \) the first equation decouples. At zero charge:

1) Gaussian stable FP \( g_{\pm} = 0 \)

2) Critical FP \( g_+ = 0, \ g_- = 1/3 \)

and two more (unimportant) FPs.
Note that

\[ \sum_{a=1}^{2} g_a (\bar{\psi} X_a \gamma_\mu \psi)^2 = -g_- [(\bar{\psi} \psi)^2 - (\bar{\psi} \gamma_5 \psi)^2] \]

So a large positive \( g_- \) indeed favors CSB.

Turning on a small charge by hand FP 1 (conformal phase) and FP 2 (critical point for CSB) approach each other.

At one loop and near \( d=4 \) the fixed points collide at

\[ e_c^2 = 3 - 2\sqrt{2} = 0.17157 \]

At which

\[ g_+ = -e_c^4 / 2 = -0.0147 \quad g_- = e_c^2 / 2 = 0.0857 \]

at least are reasonably small.
Equating the critical and the IR fixed point value of the charge yields

\[ \frac{4 - d}{N_c} = -\lim_{N \to 0} \frac{\beta_{e_0}(e_c)}{Ne_c^2} \]

and finally

\[ N_c = \frac{3(4 - d)}{4(e_c^2 + 3e_c^4)} \approx 2.88596(4 - d) + O((4 - d)^2) \]

Compared well with other analytical approaches; numerically, CSB maybe only at \( N=0 \)? (Karthik and Narayan, PRD 2016)
O(N) critical point above four (space-time) dimensions

Above four dimensions Wilson-Fisher fixed point moves to unphysical region and becomes IR unstable (bicritical):

\[ \epsilon = 4 - d \]

(IH, A modern approach to critical phenomena (CUP 2007), p. 53)

Can it be understood as an IR stable FP of another theory?
Fei, Giombi, Klebanov (PRD 2014): consider

\[ L = \frac{1}{2} (\partial_\mu z)^2 + \frac{1}{2} (\partial_\mu \phi_i)^2 + g z \phi_i \phi_i + \lambda z^3 \]

which is (log) renormalizable at \( d = 6 \).

Below \( d = 6 \) there is an IR stable fixed point for \((d = 6 - \epsilon)\)

\[ N_{\text{crit}} = 1038.266 - 609.840 \epsilon - 364.173 \epsilon^2 + \mathcal{O} (\epsilon^3) \]

(Fei, Giombi, Klebanov, Tarnopolsky, PRD 2015)
Alternative formulation (IH and Janssen, PRD 2016)

Consider XY model \((N=2)\):

\[
(\phi_1^2 + \phi_2^2)^2 = (\phi_1^2 - \phi_2^2)^2 + (2\phi_1\phi_2)^2 = (\phi^T \sigma_3 \phi)^2 + (\phi^T \sigma_1 \phi)^2.
\]

Alternative Hubbard-Stratonovich decoupling

\[-\frac{g^2}{2}(\phi_1^2 + \phi_2^2)^2 = \frac{1}{2} z_a z_a + gz_a \phi^T \sigma_a \phi \quad a \in \{1, 3\}\]

to motivate an another representation of the XY model:

\[
L = \frac{1}{2} z_a (m_z^2 - \partial_\mu) z_a + \frac{1}{2} \phi_i (m_\phi^2 - \partial_\mu) \phi_i + gz_a \phi^T \sigma_a \phi
\]
For a general $N$:

\[
\frac{1}{2}z_a z_a + g z_a \phi^T \Lambda^a \phi = -\frac{g^2}{2} \phi_i \Lambda^a_{ij} \phi_j \phi_k \Lambda^a_{kl} \phi_l
\]

\[a = 1, \ldots, M_N\]

\[M_N = (N - 1)(N + 2)/2\]

is the number of components of second rank irreducible tensor. Completeness of the set of real symmetric $\Lambda^a$ - matrices

\[
\Lambda^a_{ij} \Lambda^a_{kl} = \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{N} \delta_{ij} \delta_{kl}
\]

So that

\[
\frac{1}{2}z_a z_a + g z_a \phi^T \Lambda^a \phi = g^2 \left(\frac{1}{N} - 1\right) (\phi_i \phi_i)^2
\]

is just the original quartic term!
Alternative O(N) model: (IH, Janssen, PRD 2016)

\[ L = \frac{1}{2} (\partial_\mu z_a)^2 + \frac{1}{2} (\partial_\mu \phi_i)^2 + g z_a \phi_i \Lambda^a_{ij} \phi_j + \lambda \text{Tr}[(z_a \Lambda^a)^3]. \]

which is also renormalizable in \( d=6 \). Right below \( d=6 \), one loop:

\[ \frac{d\lambda}{d \ln b} = \frac{1}{2} (\epsilon - 3 \eta_z) \lambda + 36 \left( N + 4 - \frac{24}{N} \right) \lambda^3 + \frac{4}{3} g^3, \]

\[ \frac{dg}{d \ln b} = \frac{1}{2} (\epsilon - \eta_z - 2 \eta_\phi) g + 4 \left( 1 - \frac{2}{N} \right) g^3 + 12 \left( N + 2 - \frac{8}{N} \right) g^2 \lambda, \]

\[ \eta_z = 12 \left( N + 2 - \frac{8}{N} \right) \lambda^2 + \frac{4}{3} g^2, \quad \eta_\phi = \frac{4}{3} \left( N + 1 - \frac{2}{N} \right) g^2. \]
This flow has an **IR stable fixed point** for:

\[ 1 < N < 2.6534 \]

and again for

\[ 2.9991 < N < 3.6846 \]

For \( N=2 \):

\[ \eta_\phi = 2\eta_z = \frac{2}{5}e \]

and for \( N=3 \):

\[ \eta_z = \eta_\phi = \frac{5}{33}e \]

and positive!
Flow for $N=3$:

For $3.6847 < N < 4$ the fixed point A becomes stable, but runs to infinity as $N \to 4$. 
At $N=3$, at the stable fixed point $C$ the theory becomes:

$$L = \frac{1}{2} \text{Tr} \left( \partial_\mu M \right)^2 + g^* \text{Tr} M^3$$

$$M = \sum_{a=1}^{5} Z_a \Lambda^a + \sum_{i=1}^{3} \phi_i S_i$$

$$(S_i^z) = i \varepsilon_{ijk} \phi_j^i$$

and SU(3)-symmetric! (IH, unpublished)
Beyond one-loop: (Roscher and IH, PRD 2018; Gracey, Roscher, IH, in prep.)

\[ N_c = 2.65 - 3.7 \varepsilon^{\frac{n}{2}} - 2.5 \varepsilon + \mathcal{O}(\varepsilon^{3/2}) \]

Anything surviving in $d=5$?
Conclusion:

1) Two possible examples of fixed point collision:
   a) interacting Luttinger fermions in 3D semiconductors,
   b) QED at low N; probably many other examples

2) Characteristic separation of scales; gaps could appear “unnaturally” small

3) Tensor representation of the O(N) models: new IR-stable O(N) fixed points close to d=6. Non-triviality in d=5?
Di Pietro et al PRL 2016: neglect of $e^4$ terms gives

1) Fixed points near $d=4$ are at the line $g_+ = 0$

2) Gaussian FP is pinned at $g_- = 0$

3) Critical point goes through it and destabilizes it at

$$1 - 3e_c^2 = 0$$

4) From the leading order beta function for the charge then

$$N_c = \left(\frac{9}{4}\right)(4 - d)$$
Yukawa-like field theory for the nematic (IR) critical point:
(Janssen & IH, PRB 2015)

\[ L = L_\psi + L_{\psi \phi} + L_\phi \]

\[ L_\psi = \psi^\dagger \left( \partial_\tau + \gamma_a d_a(-i\nabla) \right) \psi, \]
\[ L_{\psi \phi} = g\phi_a \psi^\dagger \gamma_a \psi, \]
\[ L_\phi = \frac{1}{4} T_{ij} \left( -c\partial_\tau^2 - \nabla^2 + r \right) T_{ji} + \lambda T_{ij} T_{jk} T_{ki} + \mathcal{O}(T^4). \]

where the nematic tensorial order parameter is

\[ T_{ij} = \phi_a \Lambda_{a,ij} \quad \langle \phi_a \rangle = \frac{-g}{r} \langle \psi^\dagger \gamma_a \psi \rangle \]

And \( \Lambda_a \) are the five three dimensional Gell-Mann matrices.
RG flow, close to four (spatial) dimensions:

“B”: “classical” nematic critical point  \(^{(Priest \text{ and Lubensky, 1976)}}\)

“F”: new fermionic fixed point