Fixed point collisions and tensorial order parameters in Luttinger semimetals and some popular field theories

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Outline:

1) Physical motivation : (symmetry-poor real world system)

Quadratic band touching in three dimensions and the Luttinger Hamiltonian

Coulomb interaction and the scale-invariant ("non-Fermi liquid") fixed point

Fixed point annihilation and the separation of scales

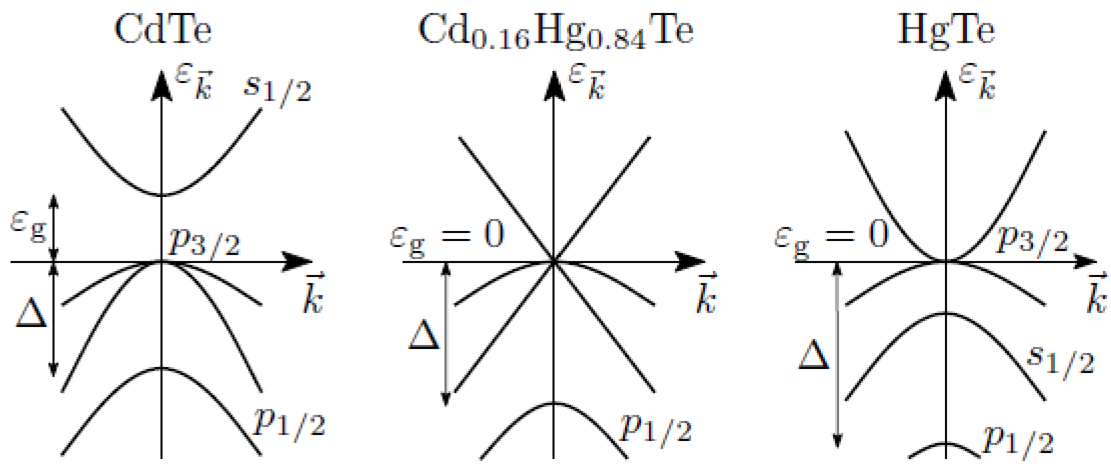
Spin-2 (tensor) ordering

2) Conformal spinoffs

Chiral symmetry breaking in $QED_{d<4}$ revisited

UV-complete O(N) models above four (space-time) dimensions

Gapless semiconductors with band inversion (gray tin, HgTe, YPtBi)



Luttinger (spin-orbit) Hamiltonian (p-orbitals, J= 3/2) (Luttinger 1956)

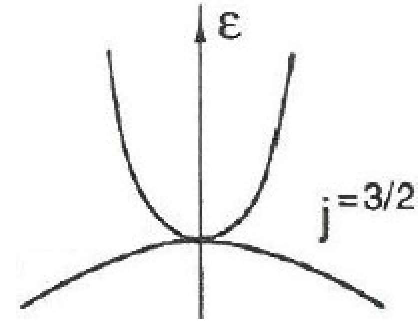
$$H = \frac{1}{2m}\left((\gamma_1 + \frac{5}{2}\gamma_2)k^2 - 2\gamma_2(\mathbf{k}\cdot\mathbf{S})^2\right)$$

with (rotationally symmetric) eigenvalues

$$E_L(k) = \frac{\gamma_1 + 2\gamma_2}{2m}k^2 \quad , \quad E_H(k) = \frac{\gamma_1 - 2\gamma_2}{2m}k^2$$

Luttinger Hamiltonian *à la* Dirac:

$$H(k) = \epsilon(\mathbf{k}) + \frac{\gamma_2}{m} d_a \Gamma^a$$



where,

$$\epsilon(\mathbf{k}) = \frac{\gamma_1}{2m} k^2, \; d_a(\mathbf{k}) = -3\xi_a^{ij} k_i k_j,$$

$$d_1 = -\sqrt{3} k_y k_z, \; d_2 = -\sqrt{3} k_x k_z, \; d_3 = -\sqrt{3} k_x k_y$$

$$d_4 = -\frac{\sqrt{3}}{2}(k_x^2 - k_y^2),$$

$$d_5 = -\frac{1}{2}(2k_z^2 - k_x^2 - k_y^2).$$

are l=2 (real) spherical harmonics.

Five 4 x 4 Dirac matrices satisfy Clifford algebra:

$$\{\Gamma^a, \Gamma^b\} = 2\delta_{ab}$$

Long-range Coulomb interaction (1/r) :

without the hole band, at ``zero" (low) density:

Wigner crystal

With the hole band filled and particle band empty: the system is

critical !

In the RG language, the charge flows with the change of cutoff:

$$\frac{de^2}{d\ln b} = (z + 2 - d)e^2 - 4e^4$$

(Abrikosov, ZETF 1974; Moon, Xu, Kim, Balents PRL 2013)

Below and near the upper critical dimension, $d_{up} = 4$, the flow is towards a non-Fermi liquid fixed point, with the charge at

$$e_*^2 = 15\epsilon/76 + \mathcal{O}(\epsilon^2)$$

$$\epsilon = 4 - d$$

and the dynamical critical exponent $Z < 2$: $z = 2 - \frac{16}{15}e^2$

This implies power-laws in various responses, such as specific heat:

$$c_v \sim T^{d/z} \approx T^{1.7}$$

Emergent scale (conformal ?) invariance!

Cheap way to get a non-Fermi liquid phase in 3D !?

Interplay with short-range components: (IH & Janssen PRL 2014; Janssen & IH PRB 2017, Boettcher & IH PRB 2017)

$$L = L_0 + L_a + L_\psi$$

with the free (Luttinger) part,

$$L_0 = \psi_i^\dagger \left[\partial_\tau + H_0(-\mathrm{i}\nabla)\right] \psi_i$$

and long-range (Coulomb) and short-range (Coulomb) interactions

$$L_a = \frac{1}{2}(\nabla a)^2 + \mathrm{i}ea\psi_i^\dagger \psi_i$$

$$L_\psi = g_1(\psi_i^\dagger \psi_i)^2 + g_2(\psi_i^\dagger \gamma_a \psi_i)^2 + g_3(\psi_i^\dagger \gamma_{ab} \psi_i)^2$$

The RG flow of all the couplings: (one loop)

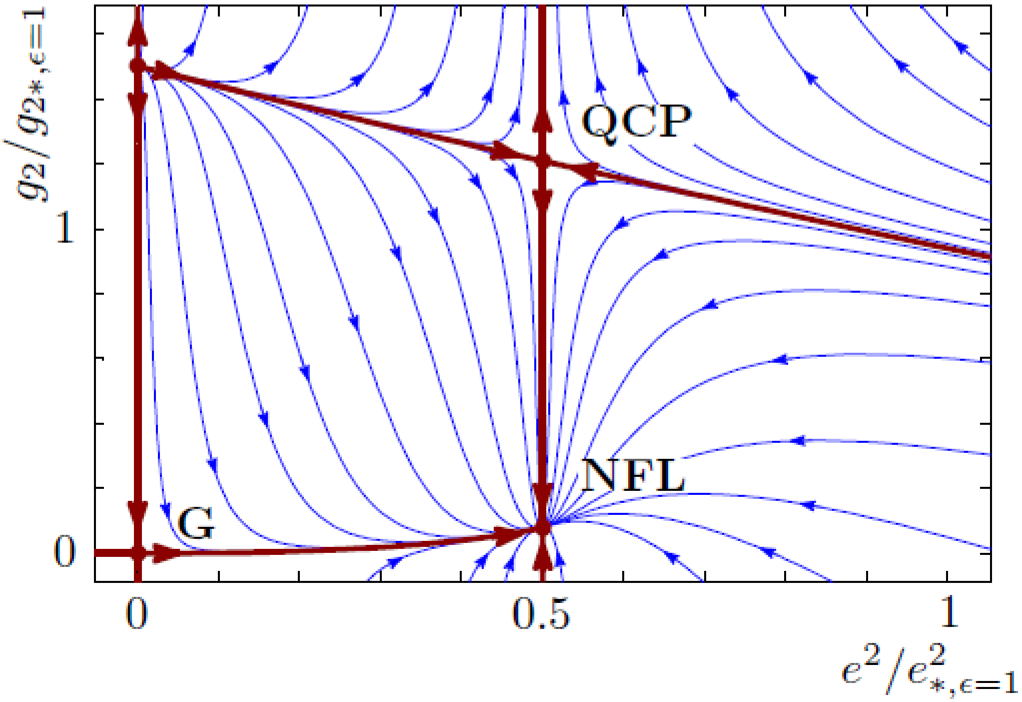
$$\frac{de^2}{d\ln b} = (2 + z - d - \eta_a)e^2,$$
$$\frac{dg_1}{d\ln b} = (z - d)g_1 - (e^2 + 2g_1)g_2 - 24g_3^2,$$
$$\frac{dg_2}{d\ln b} = (z - d)g_2 + \frac{4(e^2 + 2g_1)g_2}{5} - \frac{(e^2 + 2g_1)^2}{20} - \frac{37 + 16N}{5}g_2^2 + \frac{112}{5}g_2 g_3 - \frac{136}{5}g_3^2,$$
$$\frac{dg_3}{d\ln b} = (z - d)g_3 - \frac{1}{5}(e^2 + 2g_1)g_3 + g_2^2 - 6g_2 g_3 + \frac{4(11 - 4N)}{5}g_3^2$$

with,

$$\eta_a = Ne^2 \qquad z = 2 - \frac{4}{15}e^2$$

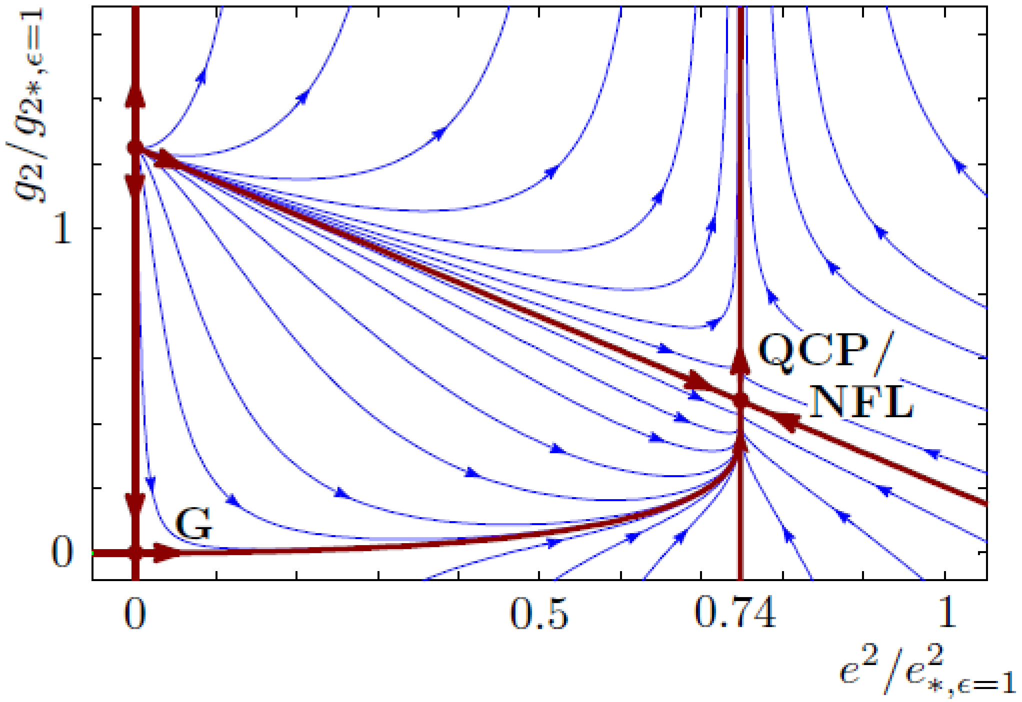
and the “charge” $e^2 = 2me_{\mathrm{el}}^2/(4\pi\hbar^2\varepsilon)$

Close to and below d=4 there is a (IR stable) NFL fixed point, but also a (UV stable) quantum critical point at strong interaction:



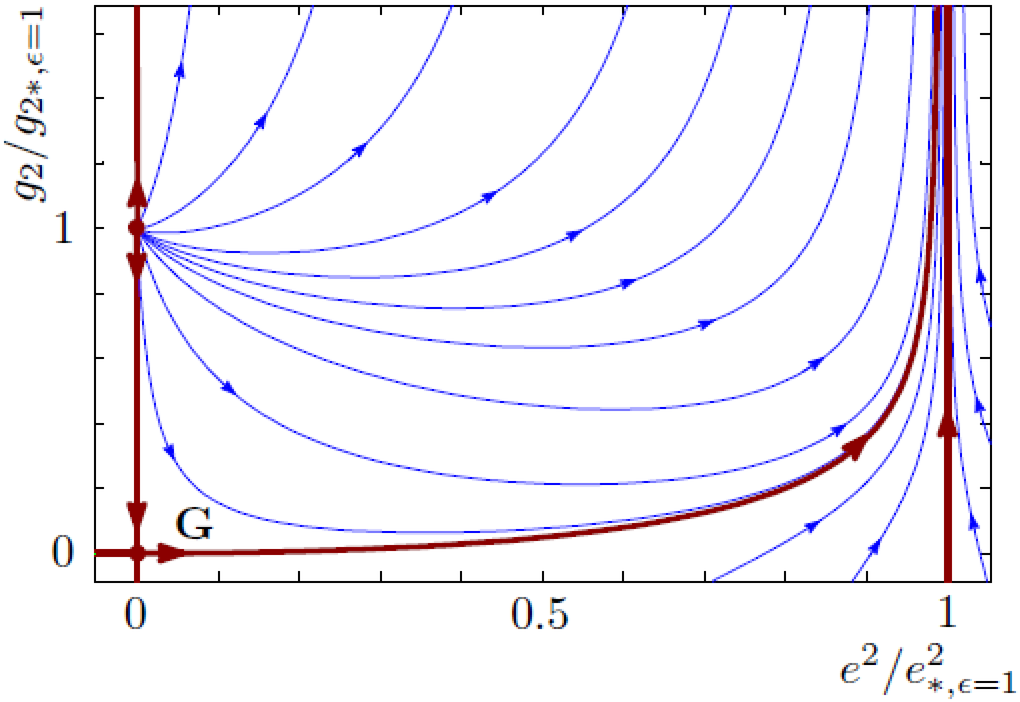
They get closer, but remain separated in the coupling space!

At some “lower critical dimension” NFL and QCP collide:



In one loop calculation, this occurs at $d_l = 3.26240$, slightly above three dimensions.

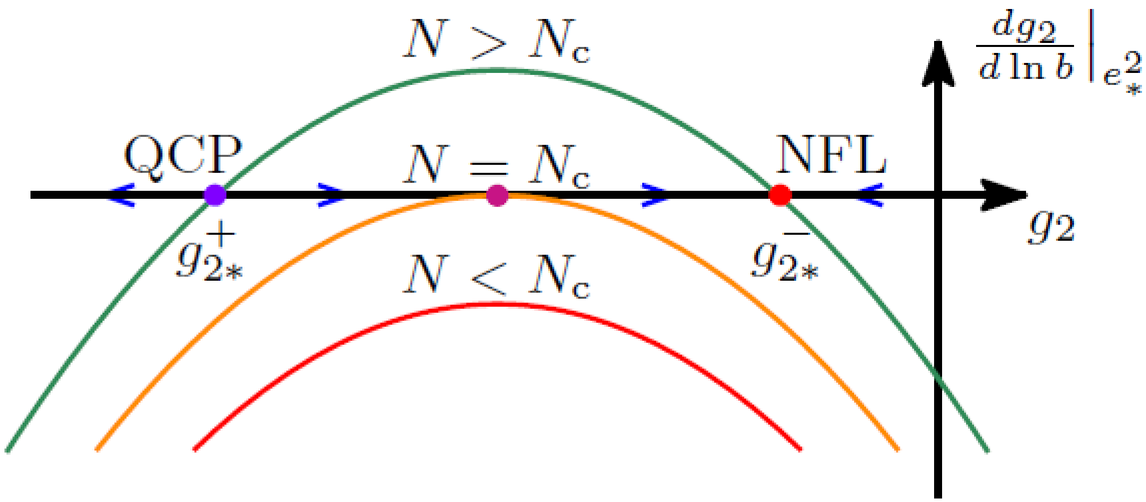
Finally, below d_l the NFL and QCP become complex (unphysical) , and there is only a runaway flow left:



The system is unstable towards a gapped (Mott) insulator.

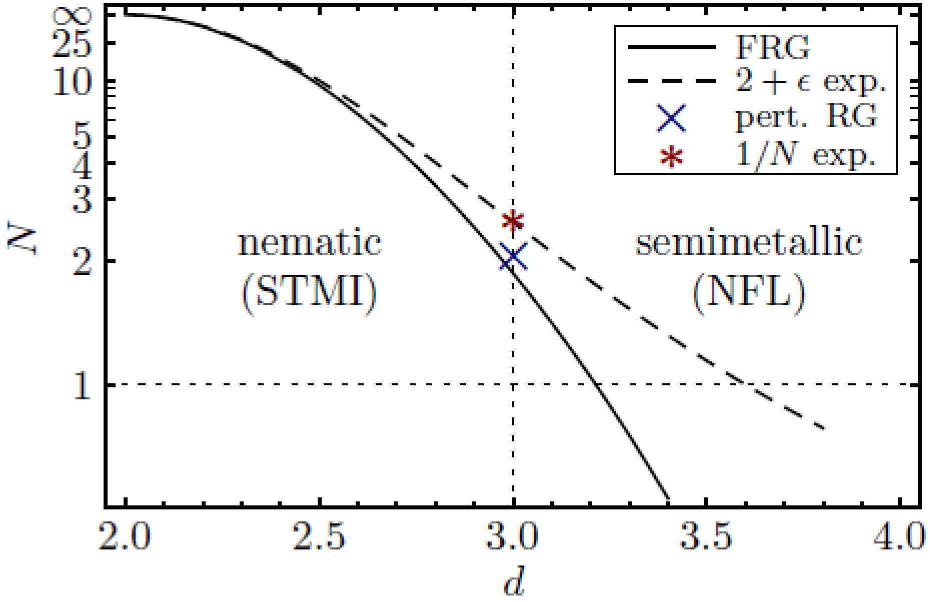
Scale invariance lost!

Fixed point collision and annihilation:



(Halperin, Lubensky, Ma, PRL 1974; IH & Tesanovic, PRL 1995;
Kaveh & IH, PRB 2005; Gies & Jaeckel 2006;
Kaplan, Lee, Son, Stephanov, PRD 2009; Nahum, PRX 2015,……..)

General number of fermions (N) and dimension (d):



Near d=2 the collision occurs in the completely perturbative regime:

$$N \geq N_c(\epsilon) = \frac{64}{25\epsilon^2} + \mathcal{O}(1/\epsilon)$$

$$\epsilon = d - 2$$

Critical number of fermions in d=3:

TABLE III. Critical fermion number N_c in $d = 3$ spatial dimensions from different approaches.

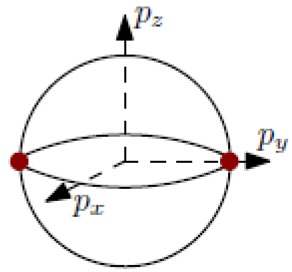
Method	Reference	$N_c(d = 3)$
$2 + \epsilon$ expansion	Sec. III	2.56
RG in fixed $d = 3$	Sec. IV	2.10
Functional RG	Sec. VI	1.86
$1/N$ expansion in $d = 3$	Ref. [18]	$\geq 2.6(2)$

(Janssen & IH, PRB 2017)

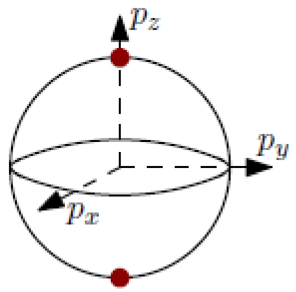
Order parameter for $d < d_{\text{low}}$ $\qquad \chi_i = 2g_2\langle\Psi^\dagger\gamma_i\Psi\rangle$

Out of the five $\chi_1, \ldots, \chi_5$ not all equivalent:

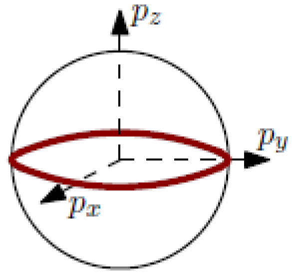
(1) $\chi_1 \neq 0$: $\varepsilon(\vec{p})$ gapped with minimal gap at two opposite points on equator



(2) $\chi_5 < 0$: $\varepsilon(\vec{p})$ gapless with gap closing at north and south pole



(3) $\chi_5 > 0$: $\varepsilon(\vec{p})$ gapped with minimal gap at entire equator



Energy $E = \int \frac{d\vec{p}}{(2\pi)^3}\varepsilon(\vec{p})$ is minimized for (3): $\chi_5 > 0$ (modulo O(3))

The fate of NFL: if d_l is above but close to d=3, the flow becomes slow close to (complex!) NFL fixed point. The RG escape time is long:

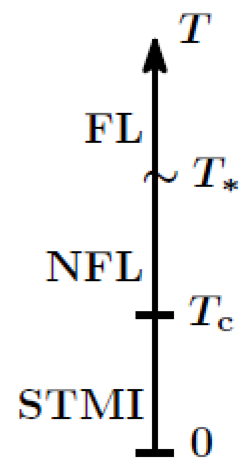
$$b_0 = e^{\frac{C}{\sqrt{d_{\mathrm{low}}-d}} - B + \mathcal{O}(d_{\mathrm{low}}-d)}$$

with non-universal constants C and B. There is wide crossover region of the NFL behavior within the temperature window

$$(T_{\mathrm{c}}, T_*)$$

with the critical temperature,

$$T_{\mathrm{c}} \approx T_* b_0^{-z}$$



Characteristic energy scale for interaction effects

$$k_{\mathrm{B}} T_* \sim \frac{e_{\mathrm{el}}^2}{\varepsilon L_*} = \frac{\hbar^2}{2mL_*^2} = \frac{4m}{m_{\mathrm{el}}\varepsilon^2} E_0$$

(Sherrington & Kohn, Halperin & Rice, RMP 1968)

Some numbers: (for HgTe)

small mass $m/m_{\mathrm{el}} \approx 1/50$

high dielectric constant $\varepsilon \approx 30$

still a reasonable $T_* \sim 10\,\mathrm{K} - 100\,\mathrm{K}$

and (maybe) a detectable $T_{\mathrm{c}} \approx T_*/100$

Cubic symmetry: realistic Luttinger Hamiltonian, cubic symmetry

$$H = \frac{\hbar^2}{2m^*}\Big[\Big(\alpha_1 + \frac{5}{2}\alpha_2\Big)p^2 \mathbb{1}_4 - 2\alpha_3(\vec{p}\cdot\vec{J})^2 + 2(\alpha_3 - \alpha_2)\sum_{i=1}^{3} p_i^2 J_i^2\Big],$$

contains particle-hole asymmetry and anisotropy parameters

$$x = -\frac{\alpha_1}{\alpha_2 + \alpha_3}, \quad \delta = -\frac{\alpha_2 - \alpha_3}{\alpha_2 + \alpha_3}$$

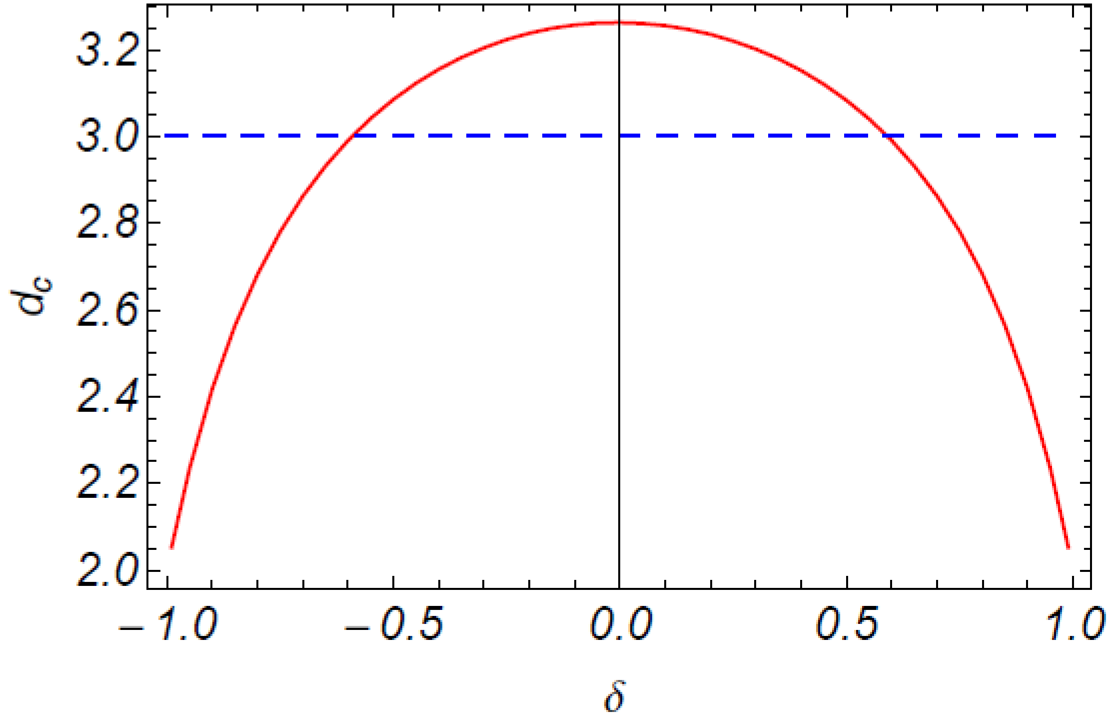
with (generically) particularly slow flow of the anisotropy:

$$\dot{\delta} \simeq -\frac{8}{105} e^2 \delta.$$

Flow of the charge is now

$$\dot{e}^2 = \frac{\mathrm{d}e^2}{\mathrm{d}\log b} = (4-d-\eta)e^2 - \frac{f_{e^2}(\delta)}{1-\delta^2}e^4$$

which lowers the critical dimension: (Boettcher & IH, PRB 2017)



Chiral symmetry breaking in QED3 revisited

Schwinger-Dyson, large-N, calculation of the mass gap
(Appelquist, Nash, Wijewardhana, PRL 1988):

$$\Sigma(0) = \alpha e^{(\delta+2)} \exp\left[\frac{-2n\pi}{(32/\pi^2 N - 1)^{1/2}}\right]$$

as the number of four-component Dirac fermions N

$$N \to 32/\pi^2$$

from below.

This should also be understandable as a fixed point collision and annihilation.

Consider QED near four space-time dimensions with (generated) quartic terms (Herbut, PRD 2016; Di Pietro et al, PRL 2016)

$$L = \bar{\Psi}_n i\gamma_\mu(\partial_\mu - ieA_\mu)\Psi_n + \sum_{a=1}^{2} g_a(\bar{\Psi}_n X_a \gamma_\mu \Psi_n)^2 + \frac{F_{\mu\nu}^2}{4}$$

with

$$X_1 = 1$$
$$X_2 = \gamma_5$$

i. e. with additional (axial) current – (axial) current interactions.

The flow in the IR ($\Lambda \to \Lambda/b$), one loop:

$$\beta_1 = (2-d)g_1 + 4(N+1)g_1^2 - 8g_1 g_2 - 6e^2 g_2,$$
$$\beta_2 = (2-d)g_2 + 2(2N-1)g_2^2 + 4g_1 g_2$$
$$- 6g_1^2 - 6e^2 g_1 - \frac{3}{2}e^4,$$
$$\beta_e = (4-d)e^2 + \beta_{e0}(e).$$

and the charge beta-function precisely in d=4 is:

$$\beta_{e0}(e) = -\frac{4N}{3}e^4 - 4Ne^6 + O(Ne^8, N^2 e^8)$$

(Gorishny, Kataev, Larin 1991 (four loop))

Introducing linear combinations:

$$g_\pm = g_1 \pm g_2$$

equations (almost) decouple

$$\beta_+ = (2-d)g_+ + 2(N-1)g_+^2 + 2Ng_-^2 - 6g_+e^2 - \frac{3}{2}e^4,$$

$$\beta_- = (2-d)g_- + 6g_-^2 + 4(N+1)g_+g_- + 6g_-e^2 + \frac{3}{2}e^4$$

When N=0 the first equation decouples. At zero charge:

1) Gaussian stable FP $g_\pm = 0$

2) Critical FP $g_+ = 0,\ g_- = 1/3$

and two more (unimportant) FPs.

Note that

$$\sum_{a=1}^{2} g_a(\bar{\Psi}X_a\gamma_\mu\Psi)^2 = -g_-[(\bar{\Psi}\Psi)^2 - (\bar{\Psi}\gamma_5\Psi)^2]$$

So a large positive g_- indeed favors CSB.

Turning on a small charge by hand FP 1 (conformal phase) and FP 2 (critical point for CSB) approach each other.

At one loop and near d=4 the fixed points collide at

$$e_c^2 = 3 - 2\sqrt{2} = 0.17157$$

At which

$$g_+ = -e_c^4/2 = -0.0147 \qquad g_- = e_c^2/2 = 0.0857$$

at least are reasonably small.

Equating the critical and the IR fixed point value of the charge yields

$$\frac{4-d}{N_c} = -\lim_{N\to 0} \frac{\beta_{e0}(e_c)}{N e_c^2}$$

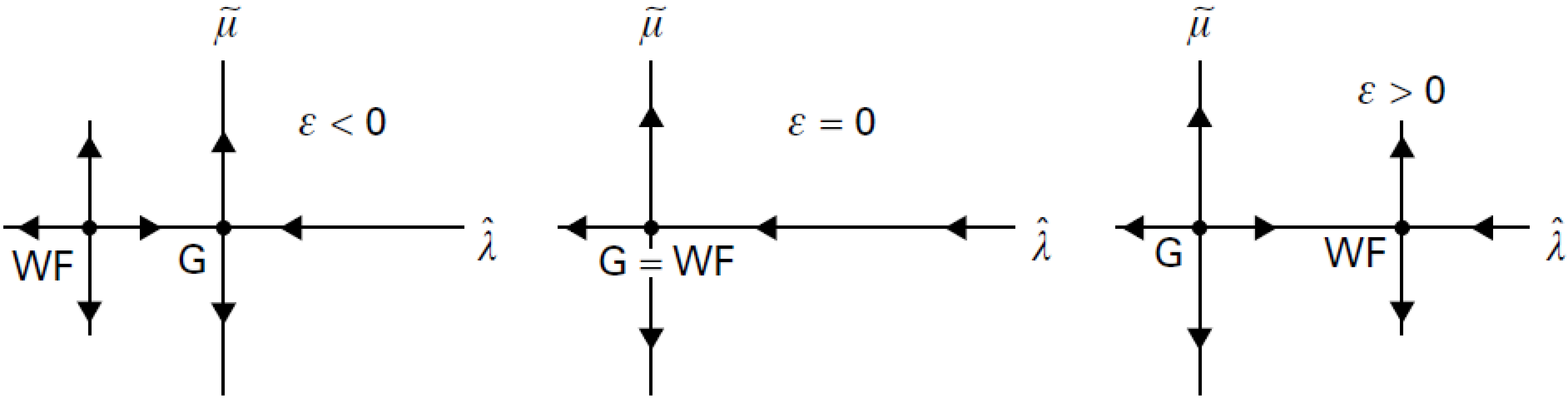
and finally

$$N_c = \frac{3(4-d)}{4(e_c^2 + 3e_c^4)} \approx 2.88596(4-d) + O((4-d)^2)$$

Compares well with other analytical approaches; numerically, CSB maybe only at N=0 ? (Karthik and Narayan, PRD 2016)

O(N) critical point above four (space-time) dimensions

Above four dimensions Wilson-Fisher fixed point moves to unphysical region and becomes IR unstable (bicritical): $\epsilon = 4 - d$



(IH, A modern approach to critical phenomena (CUP 2007), p. 53)

Can it be understood as an IR stable FP of another theory?

Fei, Giombi, Klebanov (PRD 2014): consider

$$L = \frac{1}{2}(\partial_\mu z)^2 + \frac{1}{2}(\partial_\mu \phi_i)^2 + g z \phi_i \phi_i + \lambda z^3$$

which is (log) renormalizable at d=6.

Below d=6 there is an IR stable fixed point for ($d = 6 - \epsilon$)

$$N_{\text{crit}} = 1038.266 - 609.840\epsilon - 364.173\epsilon^2 + \mathcal{O}(\epsilon^3)$$

(Fei, Giombi, Klebanov, Tarnopolsky, PRD 2015)

Alternative formulation (IH and Janssen, PRD 2016)

Consider XY model (N=2):

$$(\phi_1^2 + \phi_2^2)^2 = (\phi_1^2 - \phi_2^2)^2 + (2\phi_1\phi_2)^2$$
$$= (\phi^T \sigma_3 \phi)^2 + (\phi^T \sigma_1 \phi)^2.$$

Alternative Hubbard-Stratonovich decoupling

$$-\frac{g^2}{2}(\phi_1^2 + \phi_2^2)^2 = \frac{1}{2} z_a z_a + g z_a \phi^T \sigma_a \phi \qquad a \in \{1, 3\}$$

to motivate an another representation of the XY model:

$$L = \frac{1}{2} z_a (m_z^2 - \partial_\mu^2) z_a + \frac{1}{2} \phi_i (m_\phi^2 - \partial_\mu^2) \phi_i + g z_a \phi^T \sigma_a \phi$$

For a general N:

$$\frac{1}{2} z_a z_a + g z_a \phi^T \Lambda^a \phi = -\frac{g^2}{2} \phi_i \Lambda^a_{ij} \phi_j \phi_k \Lambda^a_{kl} \phi_l$$

$$a = 1, \ldots, M_N$$

$$M_N = (N-1)(N+2)/2$$

is the number of components of second rank irreducible tensor.
Completeness of the set of real symmetric Λ^a - matrices

$$\Lambda^a_{ij} \Lambda^a_{kl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{N}\delta_{ij}\delta_{kl}$$

So that

$$\frac{1}{2} z_a z_a + g z_a \phi^T \Lambda^a \phi = g^2 \left(\frac{1}{N} - 1 \right) (\phi_i \phi_i)^2$$

is just the original quartic term!

Alternative O(N) model: (IH, Janssen, PRD 2016)

$$L = \frac{1}{2}(\partial_\mu z_a)^2 + \frac{1}{2}(\partial_\mu \phi_i)^2 + g z_a \phi_i \Lambda^a_{ij} \phi_j + \lambda \mathrm{Tr}[(z_a \Lambda^a)^3].$$

which is also renormalizable in d=6. Right below d=6, one loop:

$$\frac{d\lambda}{d \ln b} = \frac{1}{2}(\epsilon - 3\eta_z)\lambda + 36\left(N + 4 - \frac{24}{N}\right)\lambda^3 + \frac{4}{3}g^3,$$

$$\frac{dg}{d \ln b} = \frac{1}{2}(\epsilon - \eta_z - 2\eta_\phi)g + 4\left(1 - \frac{2}{N}\right)g^3 + 12\left(N + 2 - \frac{8}{N}\right)g^2\lambda,$$

$$\eta_z = 12\left(N + 2 - \frac{8}{N}\right)\lambda^2 + \frac{4}{3}g^2, \qquad \eta_\phi = \frac{4}{3}\left(N + 1 - \frac{2}{N}\right)g^2.$$

This flow has an IR stable fixed point for:

$$1 < N < 2.6534$$

and again for

$$2.9991 < N < 3.6846$$

For N=2:

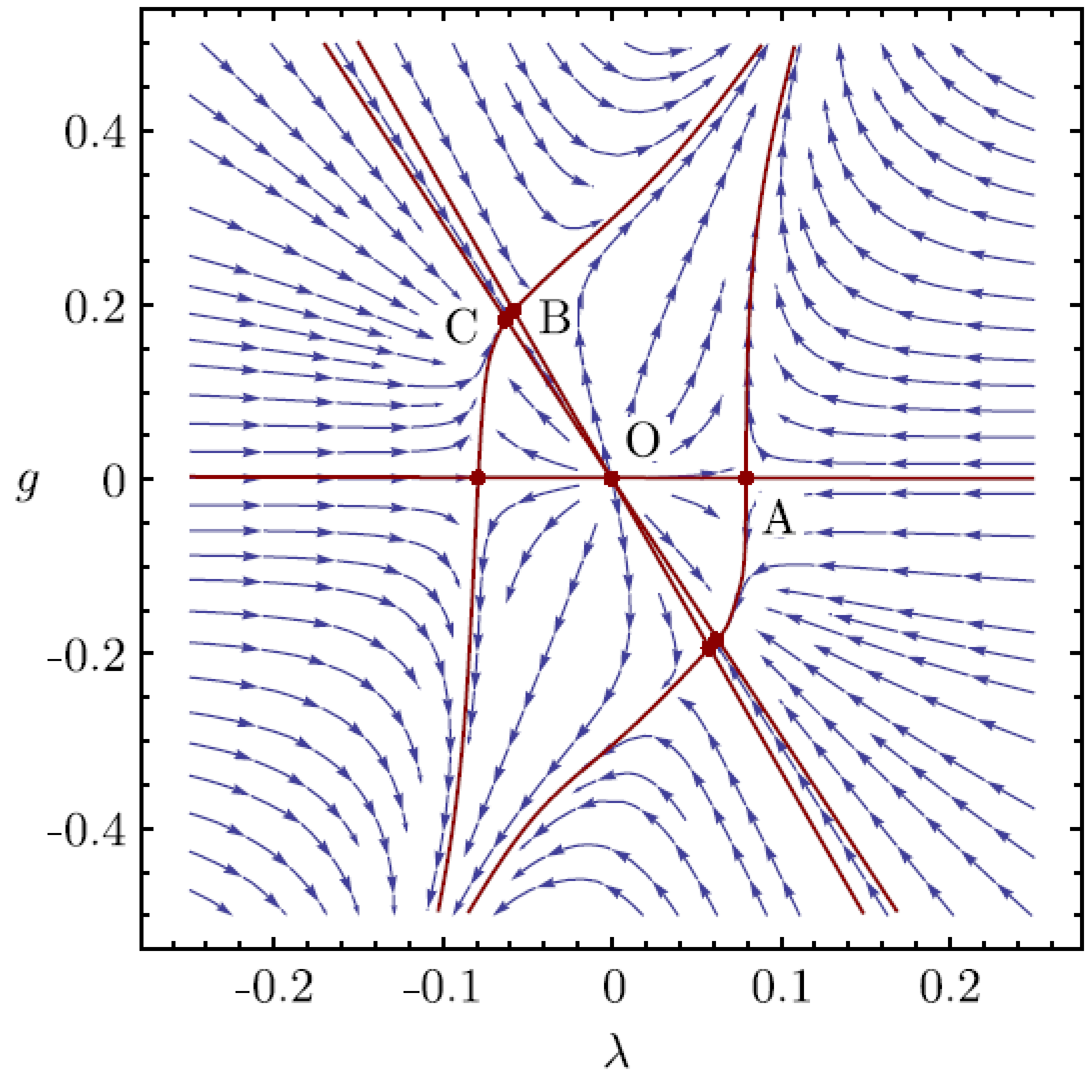
$$\eta_\phi = 2\eta_z = \frac{2}{5}\epsilon$$

and for N=3:

$$\eta_z = \eta_\phi = \frac{5}{33}\epsilon$$

and positive!

Flow for N=3:



For $3.6847 < N < 4$ the fixed point A becomes stable, but runs to infinity as $N \to 4$.

At N=3, at the stable fixed point C the theory becomes:

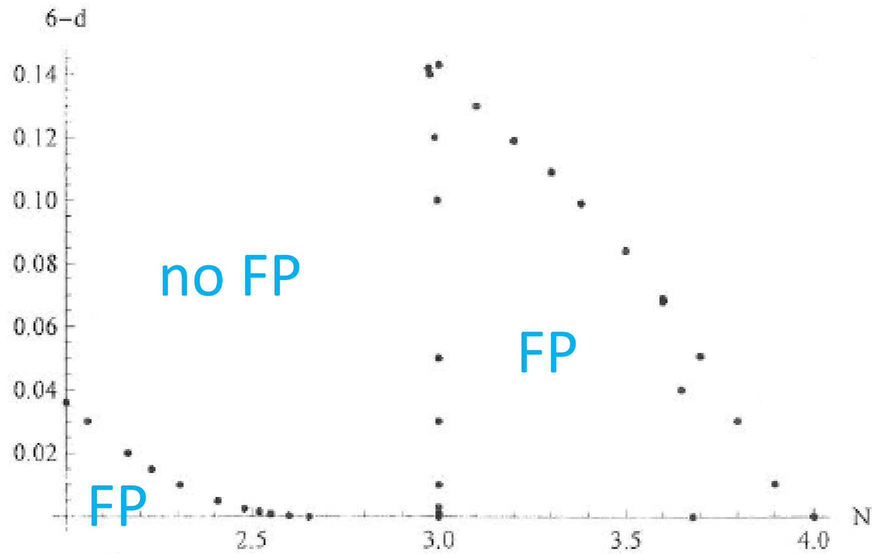
$$L = \frac{1}{2} \mathrm{Tr} (\partial_\mu M)^2 + g_* \mathrm{Tr}\, M^3 ;$$

$$M = \sum_{a=1}^{5} Z_a \Lambda^a + \sum_{i=1}^{3} \phi_i S^i$$

$$(S^i)_{jk} = i\, \varepsilon^{ijk}$$

and SU(3)-symmetric! (IH, unpublished)

Beyond one-loop: (Roscher and IH, PRD 2018; Gracey, Roscher, IH, in prep.)



$$N_c = 2.65 - 3.7\,\varepsilon^{1/2} - 2.5\,\varepsilon + O(\varepsilon^{3/2})$$

Anything surviving in d=5?

Conclusion:

1) Two possible examples of fixed point collision:

    a) interacting Luttinger fermions in 3D semiconductors,

    b) QED at low N; probably many other examples

2) Characteristic separation of scales; gaps could appear "unnaturally" small

3) Tensor representation of the O(N) models: new IR-stable O(N) fixed points close to d=6. Non-triviality in d=5?

Di Pietro et al PRL 2016: neglect of e^4 terms gives

1) Fixed points near d=4 are at the line $g_{+} = 0$

2) Gaussian FP is pinned at $g_{-} = 0$

3) Critical point goes through it and destabilizes it at

$$1 - 3e_c^2 = 0$$

4) From the leading order beta function for the charge then

$$N_c = (9/4)(4 - d)$$

Yukawa-like field theory for the nematic (IR) critical point:
(Janssen & IH, PRB 2015)

$$L = L_\psi + L_{\psi\phi} + L_\phi$$

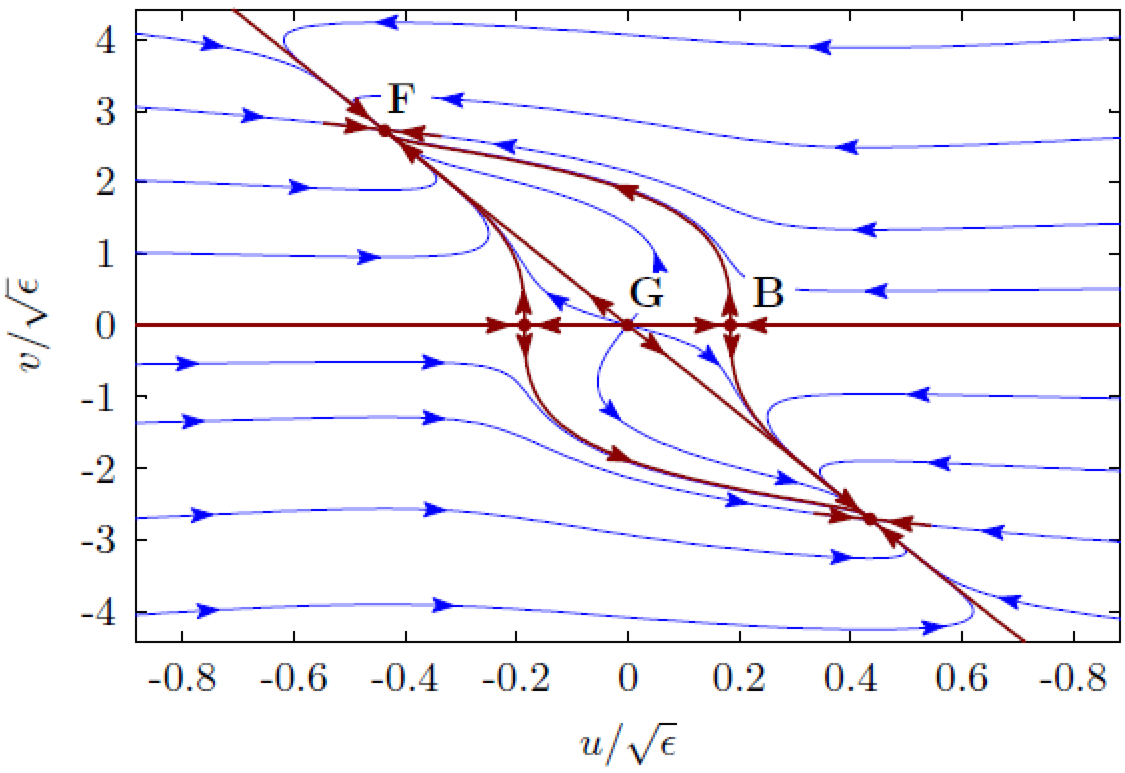
$$\begin{aligned} L_\psi &= \psi^\dagger \left(\partial_\tau + \gamma_a d_a(-i\nabla)\right) \psi, \\ L_{\psi\phi} &= g\phi_a \psi^\dagger \gamma_a \psi, \\ L_\phi &= \frac{1}{4} T_{ij} \left(-c\partial_\tau^2 - \nabla^2 + r\right) T_{ji} + \lambda T_{ij} T_{jk} T_{ki} \\ &\quad + \mathcal{O}(T^4). \end{aligned}$$

where the nematic tensorial order parameter is

$$T_{ij} = \phi_a \Lambda_{a,ij} \qquad \langle \phi_a \rangle = \frac{-g}{r} \langle \psi^\dagger \gamma_a \psi \rangle$$

And Λ_a are the five three dimensional Gell-Mann matrices.

RG flow, close to four (spatial) dimensions:



“B”: “classical” nematic critical point (Priest and Lubensky, 1976)

“F”: new fermionic fixed point