Impurities and holography

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Outline

- Introduction
- Extremal black holes and local critical points
- SYK models
- Magnetic impurities and Wilson loops

AdS/CFT in a nutshell

- Duality (or equivalence) between a field theory and a theory of gravity in higher dimensions
- Global symmetries $\leftrightarrow$ Asymptotic symmetries/isometries
- Same spectrum of physical states
- Different gauge symmetries (not observable)

Example: AdS space

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right)$$

$$x^\mu \to \lambda x^\mu, \quad z \to \lambda z$$

Scale invariance in field theory = isometry of AdS

AdS/CFT in a nutshell

- Field theory UV: conformal boundary of AdS
- RG scale: radial direction

Black holes in AdS/CFT

- Black holes $\sim$ effective descriptions of thermal states
  T = Hawking temperature
- Black hole area = entropy
- Black hole ringing $\sim$ hydrodynamics and dissipative modes
- Interesting behavior at long times: deviations from classical black hole evolution, quantum chaos, etc
- Black hole paradoxes

What's the simplest example?

- There are many non-trivial checks of AdS/CFT, but no general proof yet
- Connections between entanglement and gravity look very promising [Van Raamsdonk '10; Casini, Huerta, Myers '11; Maldacena, Susskind '13]
- Sachdev-Ye-Kitaev (SYK) model provides a simple testing ground for some ideas
- SYK $\to AdS_2/CFT_1$ maybe simplest holographic duality?

Spacetime symmetries of CFTs

- Translations

$$\partial_\mu T^{\mu\nu} = 0$$

- Lorentz transformations

$$T^{[\mu\nu]} = 0$$

- Dilatations and special conformal transformations

$$\boxed{T^\mu_{\ \mu} = 0}$$

Symmetries of AdS_2/CFT_1

- AdS_2 $SL(2,\mathbb{R})$ isometries $\leftrightarrow$ CFT_1 conformal transformations

$$t \to \frac{at+b}{ct+d}, \quad ad - bc = 1$$

- AdS_2 boundary-preserving diffeomorphisms $\leftrightarrow$ CFT_1 reparametrizations ("Virasoro")

$$t \to t(\tau), \quad z \to \dot{t}(\tau)\, z$$

Zero modes: physical states

Peculiarities of AdS_2/CFT_1

AdS: Two-dimensional gravity: topological

AdS: Any finite energy perturbation "destroys" the AdS_2 space: the space is modified at the boundary or the horizon (e.g.[Maldacena,Michelson, Strominger '98])

CFT: CFT_1: trivial Hamiltonian, all states have zero energy

$$T^\mu_{\ \mu} = T^0_{\ 0} = 0 \ \Rightarrow \ H = 0$$

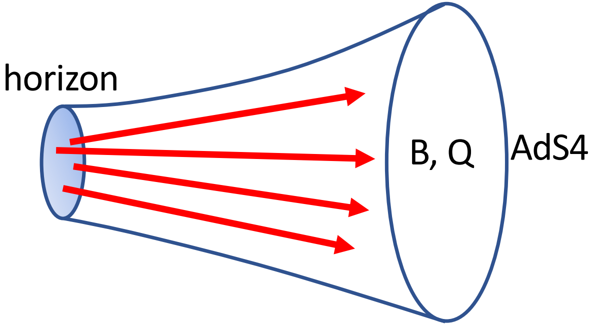
CFT: Any deformation to a dynamical theory will break conformal invariance

Non-trivial dynamics: CFT_1 is part of a larger theory

Extremal black holes and local critical points

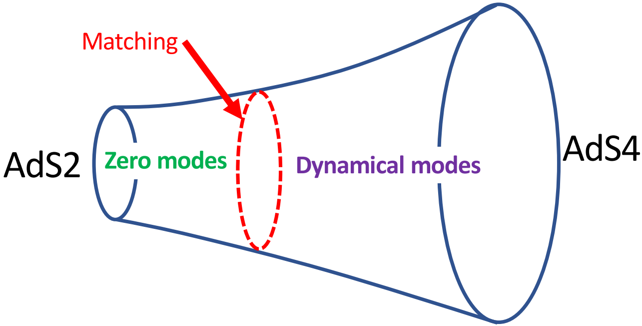
Extremal black hole

- Holographic dual of a $T = 0$ state in a $2 + 1$ dimensional theory
- Charge $\rho = q$, Magnetic field $B = \sqrt{3 - q^2}$
- Finite entropy at zero temperature $s = N^2 s_0 + N^2 s_1 T + \cdots$



Extremal black hole

Near-horizon $AdS_2 \times \mathbb{R}^2$ region: coupled to the rest of the geometry through boundary conditions



“Local criticality”

Near-horizon dynamics

- Generalization of $AdS_2 \times \mathbb{R}^2$ metric: warped $\mathcal{M}_2 \times T^2$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \Phi(x)^2 \delta_{ab} dy^a dy^b$$

- Reduction to two-dimensional Einstein-dilaton gravity

  [Almheiri,Polchinski '14]

$$S = \frac{1}{16\pi G_2} \int d^2x \sqrt{-g} \left(\Phi^2 R + 2(\partial\Phi)^2 - U(\Phi) \right)$$

- $AdS_2 \times T^2$ vacuum

Near-horizon dynamics

- Expansion: $\Phi^2 = \phi_0 + \phi$

$$S \simeq \underbrace{\frac{\phi_0}{16\pi G_2}\int \sqrt{-g}R}_{\text{topological}} + \underbrace{\frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\phi\left(R + \frac{2}{\tilde{L}^2}\right)}_{\text{Jackiw}-\text{Teitelboim 2d gravity}} + O(\phi^2)$$

- Equations of motion

$$\phi : \underbrace{R = -\frac{2}{\tilde{L}^2}}_{AdS_2\,\text{metric}}, \quad g_{\mu\nu} : \quad \underbrace{T^{\phi}_{\mu\nu} = \nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\left(\nabla^2\phi - \frac{1}{\tilde{L}^2}\phi\right) = 0}_{\phi \underset{z\to 0}{\simeq} \frac{1}{z}\phi_0(t)}$$

- Source for $\Delta = 2$ operator $\Rightarrow$ Irrelevant deformation of CFT$_1$

Near-horizon dynamics

We introduce a UV cutoff in the $0+1$ dimensional theory $\Rightarrow$ boundary cutoff in AdS_2

Near-horizon dynamics

- Simplest cutoff: keep $z > \epsilon$. Boundary of space: (t, ϵ)
- More general cutoff: a curve $(t(\tau), z(\tau))$ with fixed proper length

$$-\frac{1}{\epsilon^2} = \frac{-\dot{t}^2 + \dot{z}^2}{z(\tau)^2} \;\Rightarrow\; z \underset{\epsilon\to 0}{\simeq} \epsilon \dot{t} + O(\epsilon^3)$$

- Well-defined variational principle needs a Gibbons-Hawking term

$$S = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}\phi\left(R + \frac{2}{\tilde{L}^2}\right) + 2\int_{bdy}\sqrt{-\gamma}\phi_b\, K$$

$K =$ boundary extrinsic curvature; $\gamma =$ boundary metric
$\phi_b =$ boundary value of scalar

Effective action

- Effective action of field theory dual equals classical on-shell action of gravity

$$S_{\text{on-shell}} = 0 + 2\int \frac{d\tau}{\epsilon}\phi_b\, K\Big|_{AdS_2 \text{ solution}}$$

- Boundary value of scalar: source

$$\phi_b \equiv \frac{1}{\epsilon}J(\tau), \quad J(\tau) = \lim_{\epsilon\to 0}\epsilon\phi\left(t(\tau), \zeta(\tau)\right)$$

- Extrinsic curvature $K = 1 + \epsilon^2\{t(\tau), \tau\} + \cdots$

$$\text{Schwarzian: } \{t(\tau), \tau\} = \frac{\dddot{t}}{\dot{t}} - \frac{3}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2$$

Effective action

$$\boxed{S_{eff} \simeq 2 \int d\tau J(\tau) \{t(\tau), \tau\}}$$

[Maldacena, Stanford,Yang '16; Jensen '16; Engelsy, Mertens, Verlinde '16]

- Invariant under $SL(2, \mathbb{R})$ symmetry

$$t(\tau) \longrightarrow \frac{at(\tau) + b}{ct(\tau) + d}, \quad ad - bc = 1$$

- Broken reparametrization invariance

$$t(\tau) \not\to f(t(\tau))$$

SYK models

Sachdev-Ye-Kitaev model(s)

- $0+1$ dimensions
- N fermions (Majorana)
- Four-fermion interaction

$$H = i \sum_{ijkl} J_{ijkl} \chi_i \chi_j \chi_k \chi_l$$

- Random couplings with Gaussian distribution (disorder)

$$\overline{J_{ijkl}} = 0, \quad \overline{J_{ijkl} J_{i'j'k'l'}} = \frac{3!}{N^3} J^2 \delta_{ii'} \delta_{jj'} \delta_{kk'} \delta_{ll'}$$

Sachdev-Ye-Kitaev model(s)

Many possible generalizations

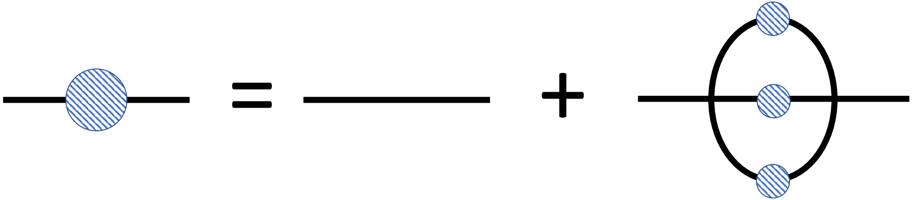
- q-fermion interactions [Kitaev '15; Maldacena, Stanford '16]
- Multiple flavors [Gross, Rosenhaus '17]
- Dirac fermions [Sachdev '15]
- Supersymmetry [Fu,Gaiotto,Maldacena, Sachdev '16]
- Higher dimensions [Gu,Qi Stanford '16; Berkooz,Narayan,Rozali,Simon '16]
- Tensor models: no disorder [Witten '16]

Sachdev-Ye-Kitaev model(s)

- Large-N limit: melonic diagrams dominate [Polchinski,Rosenhaus '16]

$$G(t_1-t_2)=\frac{1}{N}\sum_{i=1}^{N}\langle\chi_i(t_1)\chi_i(t_2)\rangle=\int d\omega e^{-i\omega(t_1-t_2)}G(\omega)$$

$$\frac{1}{G(\omega)}=-i\omega-\Sigma(\omega),\quad \Sigma(t)=J^2\left[G(t)\right]^3$$



Sachdev-Ye-Kitaev model(s)

- RG flow regulated by temperature T
- Strong coupling $J/T \gg 1$
- Emergent $0+1$ conformal symmetry in the IR

$$G(t_1 - t_2) \simeq \frac{1}{|t_1 - t_2|^{2\Delta}}, \quad \Delta = \frac{1}{4}$$

- Zero-temperature entropy and linear growth

$$s \simeq N s_0 + \frac{NT}{J} s_1 \ (+\log T)$$

- Quantum chaos: exponential growth of OTO four-point correlators at early times $\sim e^{\lambda_L t}$

Connections to gravity

- Conformal symmetry + strong coupling: AdS_2/CFT_1
- Zero-temperature entropy: near-horizon geometry of extremal black holes
- Lyapunov exponent maximal $\lambda_L = 2\pi T$ for SYK and AdS/CFT duals of black holes [Kitaev '14; Maldacena, Shenker, Stanford '15]
- **Effective action**

Effective action in SYK

- Action

$$S = \int dt \, [\chi_i \, i\partial_t \chi_i - J_{ijkl}\chi_i\chi_j\chi_k\chi_l]$$

- Gaussian integral over disorder ($\chi_{1,2} = \chi(t_{1,2})$, $\delta_{12} = \delta(t_1 - t_2)$)

$$S \simeq \int dt_1 dt_2 \, \left[\delta_{12}\chi_1 \, i\dot{\chi}_1 - 2N^{-3}J^2(\chi_1\chi_2)^4\right]$$

- Hubbard-Stratonovich with bilocal fields
($\Sigma_{12} = \Sigma(t_1, t_2)$, $\quad G_{12} = G(t_1, t_2)$)

$$S \simeq \int dt_1 dt_2 \, \left[\delta_{12}\chi_1 \, i\dot{\chi}_1 - i\Sigma_{12}\chi_1\chi_2 + \frac{N}{2}\left[\Sigma_{12}G_{12} - \frac{J^2}{4}G_{12}^4\right]\right]$$

Effective action in SYK

- Integrate out the fermions

$$S \simeq -\frac{N}{2}\log\det(\partial_t - \Sigma) + \frac{N}{2}\int dt_1 dt_2 \left[\Sigma_{12}G_{12} - \frac{J^2}{4}G_{12}^4\right]$$

- Strong coupling limit: $\log\det(\partial_t - \Sigma) \simeq \log\det(-\Sigma)$
- Saddle point equations

$$\int dt_2 G_{12}\Sigma_{23} = -\delta_{13}, \quad \Sigma_{12} = J^2 G_{12}^3$$

- Reparametrization invariance $\tau_{1,2} = f(t_{1,2})$

$$G(t_1, t_2) \to \left[f'(t_1)f'(t_2)\right]^{1/4} G\left(f(t_1), f(t_2)\right)$$

$$\Sigma(t_1, t_2) \to \left[f'(t_1)f'(t_2)\right]^{3/4} \Sigma\left(f(t_1), f(t_2)\right)$$

Saddle point solution

- Solution to saddle-point equation (strong coupling)

$$G_c = \frac{B}{|t_1 - t_2|^{1/2}} \operatorname{sign}(t_1 - t_2)$$

- Not invariant under general reparametrizations $t_i \to \tau_i(t_i)$
- Invariant under conformal transformations ($SL(2,\mathbb{R})$)

$$\tau_i = \frac{a t_i + b}{c t_i + d}, \quad ad - bc = 1$$

- At $T \neq 0$ ($\beta = 1/T$)

$$\tau_i \to \tan \frac{\pi \tau_i}{\beta}, \quad t_i \to \tan \frac{\pi t_i}{\beta}$$

Breaking of reparametrization invariance

- Lower derivative terms invariant under $SL(2, \mathbb{R})$ transformations

$$f \longrightarrow \frac{af + b}{cf + d}$$

$\Rightarrow$ Effective action is proportional to the Schwarzian

$$\boxed{S_{SYK} = -\frac{N\alpha}{J} \int dt \, \{f, t\}}$$

- **Same as 2d Einstein-dilaton gravity!**

Magnetic impurities and Wilson loops

Magnetic impurity

- An impurity in a metal with spin that interacts with the spin of the conduction electrons

$$S = \int d^{d+1}x \, \lambda \, \delta^{(d)}(x) \, \vec{S}_{\rm imp} \cdot \vec{S}_e + S_{\rm electrons} + S_{\rm imp}$$

- Spin of electrons (fundamental rep. of $SU(2)$)

$$\vec{S}_e = \psi_e^\dagger \frac{\vec{\sigma}}{2} \psi_e$$

- Spin of impurity: auxiliary fields (Abrikosov/slave fermions)

$$\vec{S}_{\rm imp} = \chi^\dagger \frac{\vec{\sigma}}{2} \chi.$$

The spin of the impurity is fixed by the fixed charge constraint

$$\chi^\dagger \chi = 1.$$

Magnetic impurity

- General case: $SU(N)$ spin

$$S_q^a = \psi_q^\dagger T^a \psi_q$$

$$S_{\rm imp}^a = \chi^\dagger T^a \chi$$

- General representation

$$\chi^\dagger \chi = Q$$

χ can be bosonic or fermionic = rows or columns in a Young tableaux

Magnetic impurity

- Path integral

$$Z = \int D\psi_e^\dagger D\psi_e D\chi^\dagger D\chi\, \delta(\chi^\dagger \chi - 1) e^{iS}$$

- Introduce Lagrange multiplier μ

$$Z = \int D\psi_e^\dagger D\psi_e D\chi^\dagger D\chi D\mu\, e^{iS + i\int dt\, \mu\left(\chi^\dagger \chi - 1\right)}$$

- Effective action for impurity fields

$$S_{\text{imp}} = \int dt\, \chi^\dagger\, i\partial_t \chi + \cdots$$

Lagrange multiplier $\longleftrightarrow$ gauge field in $0+1$ dimensions

$$\mu = a_t$$

Magnetic impurity

- Emergent $U(1)$ gauge symmetry

$$\chi \to e^{i\theta}\chi, \quad a_t \to a_t + \partial_t \theta$$

- Scale invariance $\Delta_\chi = 0$, $\Delta_{a_t} = 1$

$$t \to \lambda^{-1} t, \quad \chi \to \lambda^{\Delta_\chi}\chi, \quad a_t \to \lambda^{\Delta_{a_t}} a_t$$

- Reparametrization invariance

$$t \to t(\tau), \quad \chi(t) \to \chi[t(\tau)], \quad a_t(t) \to \left(\frac{\partial t}{\partial \tau}\right)^{-1} a_\tau[t(\tau)]$$

Wilson loop

- Spin symmetry $\longrightarrow$ gauge symmetry A_μ
- Magnetic impurity with spin in representation $R \to$ Wilson loop in representation R

$$\langle \mathcal{W}_R(\mathcal{C}) \rangle = \left\langle \mathrm{Tr}_R\, e^{i\int_{\mathcal{C}} A} \right\rangle = \int DAD\chi^\dagger D\chi e^{iS_{\mathrm{imp}} + i\int dt\, a_t(\chi^\dagger\chi - Q_R)}$$

- Fundamental representation of $SU(N)$

$$Q_F = 1 - \frac{N}{2}$$

- Impurity fields charged under $SU(N)$

$$S_{\mathrm{imp}} = \int dt\, \chi^\dagger\, i(\partial_t - iA_t)\chi$$

Effective action of defect fields

- Magnetic impurity: integrate out electrons
- Wilson loop: integrate out gauge fields

$$Z = \int D\chi^\dagger D\chi Da_t \exp\left(i \int dt\, \chi^\dagger\, iD_t\chi - iQ_F \int dt\, a_t + i\delta S_{\text{eff}} \right)$$

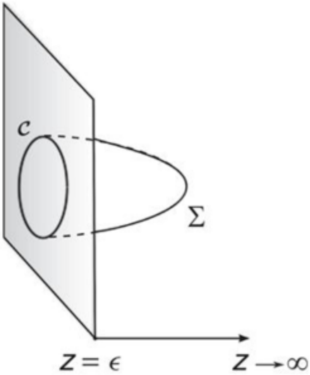
$$D_t\chi = (\partial_t - ia_t)\chi$$

$$\delta S_{\text{eff}} = \int dt|\dot{x}|\delta L_{\text{eff}}\left[\chi^\dagger\chi, \frac{1}{|\dot{x}|}\chi^\dagger D_t\chi, \cdots\right]$$

$x^\mu(t) =$ worldline of defect

Holographic dual of a Wilson loop

- Minimal area two-dimensional surface [Maldacena '98]
- Circular loop $\sim AdS_2$ [Berenstein, Corrado, Fischler '98; Drukker,Gross, Ooguri '99]



AdS_2/CFT_1 for Wilson loops?

- Symmetries of effective action: $U(1)$ gauge and **reparametrization invariance** $t \to f(t)$
- Reparametrization invariance broken by the choice of contour e.g.1/2-BPS loop: breaking to $SL(2,\mathbb{R})$: AdS_2/CFT_1 [Giombi, Roiban, Tseytlin '17]
- 2d gravitational dynamics could emerge naturally in impurities and Wilson loops
- However, no dynamical gravity in AdS_2: no dynamics in CFTs? Consistent with field theory: Matrix model for BPS loops [Erickson, Semenoff, Zarembo '00; Drukker, Gross '00; Pestun '07]

Worth exploring!