Fluid Dynamics for Systems without Boost Invariance

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Introduction

- When do we expect to encounter hydrodynamics without boost symmetries?
- Particles moving through a medium whose interactions break boost symmetries.
- Thermodynamic systems with dynamical scaling for generic values of the dynamical exponent z are not compatible with boost symmetries.
- Most general form of hydrodynamics with spatial rotational symmetries, unifying relativistic and non-relativistic hydrodynamics into one framework.

Introduction: Dynamical scaling

- Dispersion relation $\omega \sim k^z$.
- Integer z: spontaneous symmetry breaking in non-relativistic field theories [Watanabe, Murayama, 2014].
- Non-integer z: weird but not uncommon. Just throw a stone in the water. Capillary waves (ripples on water) have $z = 3/2$.
- Field theories with $z \notin \mathbb{N}$ are expected to be non-local. Explicit example: ripplons with $z = 3/2$ on domain wall between two superfluids [Watanabe, Murayama, 2014].

Introduction: Dynamical scaling

- Biological systems can have dynamical scaling with non-integer values of z: e.g. flocking behavior.
- In CM infrared effective theories can have non-CFT scaling exponents.
- Near quantum critical points electrons may be strongly coupled and thus may form a fluid. See e.g. [Lucas, Fong, 2017] for a review.
- Goal: describe the hydro phase of any field theory with scaling $z > 1$ at finite temperature. Earlier work: [Hoyos, Kim, Oz, 2013] and [Chapman, Hoyos, Oz, 2013] for superfluids.

Outline of Talk

- Thermodynamics
- Ideal fluids and sound mode
- Hydrodynamic modes in non-ideal fluids: viscosities and conductivities

Part I:

Thermodynamics

Thermodynamics

- Assumption: system is homogeneous, isotropic and conserves particles.
- Grand canonical partition function $\mathcal{Z}(T, V, \mu, v^i)$: temperature T, volume V, chemical potential μ (particle number conservation) and velocity v^i.
- Grand potential $\Omega = -k_B T \log \mathcal{Z}(T, V, \mu, v^i)$
- $\Omega = -PV$ and $\delta P = s\delta T + \mathcal{P}_i \delta v^i + n\delta\mu$ with P pressure, s entropy, $\mathcal{P}_i$ momentum, n charge density
- Total energy density: $\mathcal{E} = sT + \mu n + v^i \mathcal{P}_i - P$.

What is mass?

- Isotropy: momentum density $\mathcal{P}_i = \rho v^i$.
- ρ is the kinetic mass density (not conserved).
- n is the particle number density (conserved).
- Example: Ideal gas of particles with dispersion $\omega \propto k^z$.

Ideal gas of Lifshitz particles

- Boltzmann gas of N identical free Lifshitz particles with single-particle Hamiltonian $H_1 = \lambda|\vec{p}|^z$. Canonical partition function

$$
\begin{aligned}
Z(N,T,V,v^i) &= \frac{1}{N!}\left[Z_1(T,V,v^i)\right]^N , \\
Z_1(T,V,v^i) &= \frac{V}{h^d}\int d^d\vec{p}e^{-\beta H_1-\beta\vec{v}\cdot\vec{p}} = \lambda_{\text{th}}^{-d}V\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+2n}{z}\right)}{\Gamma\left(\frac{d}{z}\right)\Gamma\left(\frac{d+2n}{2}\right)}\alpha^n \\
\alpha &= \#\frac{h^2}{\lambda_{\text{th}}^2}v^2\beta^2
\end{aligned}
$$

- Approximation valid when $\lambda_{\text{th}} \ll \left(\frac{V}{N}\right)^{\frac{1}{d}}$ with $H_1 = \lambda p_{\text{th}}^z \sim k_B T$ and $p_{\text{th}} = \frac{h}{\lambda_{\text{th}}}$.

Ideal gas of Lifshitz particles

- Equation of state:

$$P(T,\mu,v^2) = \lambda_{\text{th}}^{-d}\beta^{-1}e^{\beta\mu}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d+2n}{z}\right)}{\Gamma\left(\frac{d}{z}\right)\Gamma\left(\frac{d+2n}{2}\right)}\alpha^n$$

- Small α means v is small compared to speed of sound.
- Gibbs–Duhem: $\delta P = s\delta T + n\delta\mu + \frac{1}{2}\rho\delta v^2$
- Ideal gas law for any z: $P = nk_B T$
- Mass/particle: $\frac{\rho}{n} = \#\frac{p_{\text{th}}^{2-z}}{\lambda} + \mathcal{O}(v^2)$
- Speed of sound: $v_s^2 = \gamma\frac{P}{\rho}$ with $\gamma = \frac{C_P}{C_V} = 1 + \frac{z}{d}$

Part II:

Ideal Fluids and Sound Mode

Energy-Momentum Tensor and Particle Current

- Hydrodynamics is a theory for fluctuations of the chemical potentials (on large length and time scales): T, μ and v^i that are conjugate to conserved charges, whose equations of motion are the charge conservation equations.

- There is a conserved energy-momentum tensor $T^\mu{}_\nu$ and a conserved current J^μ.

- The charges are $H = -\int d^dx T^0{}_0$, $P_i = \int d^dx T^0{}_i$ and $Q = \int d^dx J^0$.

- The objects $T^\mu{}_\nu$, J^μ transform in a representation of the full spacetime symmetry algebra (incl. e.g. $SO(d)$).

Energy-Momentum Tensor and Charge Current

- $T^0{}_0$ is the energy density
- $T^i{}_0$ is the energy flux
- $T^0{}_i$ is the momentum density
- $T^i{}_j = T^j{}_i$ is the stress (rotational symmetries J_{ij})
- J^0 is the charge density
- J^i is the charge flux
- Energy (H), momentum (P_i), charge conservation (Q):

$$\partial_\mu T^\mu{}_\nu = 0\,, \qquad \partial_\mu J^\mu = 0$$

- Lorentz boosts: $T^0{}_i = -T^i{}_0$
  Galilean boosts: $T^0{}_i = J^i$.

Perfect fluid

- In LAB frame we have

$$T^0{}_0 = -\mathcal{E}\,, \quad T^i{}_0 = -\left(\mathcal{E} + P\right) v^i\,, \quad T^0{}_i = \rho v^i\,, \quad T^i{}_j = P\delta^i_j + \rho v^i v^j$$
$$J^0 = n\,, \quad J^i = n v^i$$

- The most general energy-momentum tensor and particle current with $SO(d)$ symmetry obeying the thermodynamic condition: flux $=$ charge $\times\, v^i$.
- Constant velocity: $\exists$ a frame in which all fluxes vanish:

$$\tilde{T}^0{}_0 = -\tilde{\mathcal{E}}\,, \quad \tilde{T}^i{}_0 = 0\,, \quad \tilde{T}^0{}_i = \rho v^i\,, \quad \tilde{T}^i{}_j = P\delta^i_j\,, \quad \tilde{J}^0 = n\,, \quad \tilde{J}^i = 0$$

- Internal energy: $\tilde{\mathcal{E}} = \mathcal{E} - \mathcal{P}_i v^i = Ts + \mu n - P$
- v^i breaks rotations $SO(d) \to SO(d-1)$ spontaneously.

Boost invariant cases

- Lorentz boost invariance: ${T^0}_i = -{T^i}_0$ means $\rho = \frac{\tilde{\mathcal{E}}+P}{1-v^2}$.
- Galilei boost invariance: ${T^0}_i = J^i$ implies $\rho = n$. In this case the thermodynamics becomes:

$$\tilde{\mathcal{E}} = Ts - P + \mu n\,, \qquad d\tilde{\mathcal{E}} = Tds - \frac{1}{2}ndv^2 + \mu dn$$

- We can remove the dv^2 terms by defining the variables $\hat{\mathcal{E}} = \tilde{\mathcal{E}} + \frac{1}{2}nv^2$ and $\hat{\mu} = \mu + \frac{1}{2}v^2$ [Jensen, 2014] so that

$$\hat{\mathcal{E}} = Ts - P + \hat{\mu} n\,, \qquad d\hat{\mathcal{E}} = Tds + \hat{\mu} dn$$

Here $\hat{\mathcal{E}}$ is the internal energy.

Scale invariance: a nogo result

- Dilatation current: $\partial_\mu \left(T^\mu{}_\nu D^\nu\right) = zT^0{}_0 + T^i{}_i = 0$ where $D = zt\partial_t + x^i\partial_i$.

- The scale Ward identity implies that $dP = z\tilde{\mathcal{E}} + (z-1)\rho v^2$.

- Lorentz: $\rho = \frac{\tilde{\mathcal{E}}+P}{1-v^2}$ with P, $\tilde{\mathcal{E}}$ independent of $v^2 \to z = 1$.

- Galilei: $\rho = n$ with P, $\hat{\mathcal{E}} = \tilde{\mathcal{E}} + \frac{1}{2}nv^2$ and n independent of v^2, so $dP = z\hat{\mathcal{E}} + \frac{z-2}{2}nv^2 \to z = 2$.

- In Galilean case we can add scale with any z to the algebra but we cannot form a perfect fluid with $z \neq 2$. See also [Grinstein, Pal, 2018].

Speed of sound

- Modified Euler equation:

$$\rho(\partial_t + v^i\partial_i)v^j = -\partial_j P - v^j\left[\partial_t\rho + \partial_i(\rho v^i)\right]$$

- Fluctuate the eom around a constant background:

$$\rho = \rho_0 + \delta\rho\,, \qquad \mathcal{E} = \mathcal{E}_0 + \delta\mathcal{E}\,, \qquad P = P_0 + \delta P\,, \qquad v^i = v_0^i + \delta v^i$$

- If we assume scale invariance and $v_0^i = 0$ we have $z\delta\mathcal{E} = d\delta P$ so

$$(\omega^2 - v_s^2 k^2)\delta P = 0\,, \qquad v_s^2 = \frac{z}{d}\frac{\mathcal{E}_0 + P_0}{\rho_0}$$

- For $v_0^i \neq 0$ the speed of sound depends on the angle between v_0^i and k^i.

Part V:

Hydrodynamic modes in non-ideal fluids: viscosities and conductivities

Hydro Frame

- We will work up to first order in derivatives and first order in fluctuations around global equilibrium at rest.
- Landau frame: energy current and charge density are those of a perfect fluid.

$$
\begin{aligned}
T^0{}_0 &= -\tilde{\mathcal{E}}_0 - \delta\tilde{\mathcal{E}}\,, \qquad && T^i{}_0 = -\left(\tilde{\mathcal{E}}_0 + P_0\right)\delta v^i \\
T^0{}_j &= \rho_0 \delta v^j + T^0_{(1)j}\,, && T^i{}_j = (P_0 + \delta P)\,\delta^i_j + T^i_{(1)j} \\
J^0 &= n_0 + \delta n\,, && J^i = n_0 \delta v^i + J^i_{(1)}
\end{aligned}
$$

$$
\begin{aligned}
T^i_{(1)j} &= -\zeta_0 \delta_{ij}\partial_k \delta v^k - \eta_0 \left(\partial_i \delta v^j + \partial_j \delta v^i - \frac{2}{d}\delta_{ij}\partial_k \delta v^k\right) \\
T^0_{(1)j} &= -\pi_0 \partial_t \delta v^j + T_0\left(\alpha_0 + \gamma_0\right)\partial_j \delta\frac{\mu}{T} \\
J^i_{(1)} &= \left(\alpha_0 - \gamma_0\right)\partial_t \delta v^i - T_0 \sigma_0 \partial_i \delta\frac{\mu}{T}
\end{aligned}
$$

Entropy Current

- Entropy current $S^\mu = S^\mu_{\text{can}} + S^\mu_{\text{non-can}}$ where

$$S^\mu_{\text{can}} = -\frac{1}{T} T^\mu{}_\nu u^\nu + \frac{P}{T} u^\mu - \frac{\mu}{T} J^\mu$$

- One shows (using Gibbs–Duhem), (0)=perfect fluid:

$$\partial_\mu S^\mu = -\left(T^\mu{}_\nu - T^\mu_{(0)\nu}\right) \partial_\mu \frac{u^\nu}{T} - \left(J^\mu - J^\mu_{(0)}\right) \partial_\mu \frac{\mu}{T}$$

- Split in dissipative and non-dissipative parts:

$$T^\mu{}_\nu = T^\mu_{\text{D}\nu} + T^\mu_{\text{ND}\nu}\,, \quad J^\mu = J^\mu_{\text{D}} + J^\mu_{\text{ND}}\,, \quad \partial_\mu S^\mu_{\text{non-can}} = T^\mu_{\text{ND}\nu} \partial_\mu \frac{u^\nu}{T} + J^\mu_{\text{ND}} \partial_\mu \frac{\mu}{T}$$

- $S^\mu_{\text{non-can}}$ is the most general current obeying the above as well as $\partial_\mu S^\mu \geq 0$ for all fluid configurations.

Entropy Current

$$\zeta_0 = \bar{\zeta}_0 + a_T T_0^2 \left(\frac{\partial P_0}{\partial \tilde{\mathcal{E}}_0}\right)_{n_0} + a_{\frac{\mu}{T}} \left(\frac{\partial P_0}{\partial n_0}\right)_{\tilde{\mathcal{E}}_0}$$

$$\pi_0 = \bar{\pi}_0 - a_T \frac{\rho_0 T_0^2}{\tilde{\mathcal{E}}_0 + P_0}$$

$$\alpha_0 = \bar{\alpha}_0 + \frac{a_T}{2} \frac{n_0 T_0^2}{\tilde{\mathcal{E}}_0 + P_0} - \frac{a_{\frac{\mu}{T}}}{2}$$

- Bars are dissipative and $a(T, \mu/T)$ is non-dissipative.
- Positive entropy production:

$$\bar{\zeta}_0 \geq 0\,, \quad \eta_0 \geq 0\,, \quad \bar{\pi}_0 \geq 0\,, \quad \sigma_0 \geq 0\,, \quad \bar{\alpha}_0^2 \leq \bar{\pi}_0 \sigma_0$$

- Onsager relations: $\gamma_0 = 0$.

Hydrodynamic Modes

$$\begin{aligned} \omega_{\text{shear}} &= -i\frac{\eta_0}{\rho_0}k^2 && \text{multiplicity } d-1 \\ \omega_{\text{sound}} &= \pm v_s k - i\Gamma k^2 && \text{multiplicity } 2 \\ \omega_{\text{diff}} &= -iDk^2 && \text{multiplicity } 1 \end{aligned}$$

- Diffusion constant: $D = \frac{(\tilde{\mathcal{E}}_0+P_0)^2}{n_0^3 T_0 c_P}\sigma_0$
- $\Gamma \geq 0$ is the sound attenuation constant:

$$\Gamma = \frac{1}{2\rho_0 v_s^2}\left[\left[\bar{\zeta}_0 + \frac{2}{d}(d-1)\eta_0\right]v_s^2 + \bar{\pi}_0 v_s^4 + \sigma_0\left(\left(\frac{\partial P_0}{\partial n_0}\right)_{\tilde{\mathcal{E}}_0}\right)^2 + 2\bar{\alpha}_0 v_s^2\left(\frac{\partial P_0}{\partial n_0}\right)_{\tilde{\mathcal{E}}_0}\right]$$

- Lorentz: $\bar{\pi}_0 = \bar{\alpha}_0 = 0$ and Galilei: $\bar{\pi}_0 = \sigma_0 = -\bar{\alpha}_0$.

Outlook

- One formalism that only assumes H, P_i, J_{ij} and Q symmetries which includes all boost invariant cases.
- At perfect fluid level it just requires the specification of one extra function: kinetic mass density ρ.
- Scaling with $z > 1$ and $z \neq 2$ cannot have Galilean boost symmetry. Hence systems in condensed matter with $z > 1$ scaling and a hydro regime are non-boost invariant.

Outlook

- At first order in derivatives and fluctuations around rest we find 5 dissipative (2 viscosities and 3 conductivities) and 1 non-dissipative transport coefficients.
- Full result at first order in derivatives is under construction.
- Find examples of more interesting equations of state and compute transport coefficients in simple models.
- Applications to CM.

Geometry

- In general the geometry is some absolute space-time with metrics $\tau_\mu \tau_\nu$ and $h_{\mu\nu}$ where $h_{\mu\nu}$ has signature $(0, 1, \ldots 1)$ and some $U(1)$ potential m_μ.

- Local symmetries are diffeos and a $U(1)$.

- Special cases of this are:
  - Newton–Cartan geometry: local (Galilean) boosts acting on $h_{\mu\nu}$ and m_μ
  - Lorentzian geometry: $g_{\mu\nu} = -\tau_\mu \tau_\nu + h_{\mu\nu}$
  - Other non-Lorentzian geometries

Energy-Momentum Tensor and Charge Current

- Energy-momentum tensor $T^\mu{}_\nu$ as response to varying non-Lorentzian geometry:

$$\delta S = \int d^{d+1}xe\left(-T^\mu \delta\tau_\mu + \frac{1}{2}T^{\mu\nu}\delta h_{\mu\nu} + J^\mu \delta A_\mu\right)$$

$$e = \sqrt{-\det\left(-\tau_\mu\tau_\nu + h_{\mu\nu}\right)}\,, \quad T^\mu{}_\nu = -T^\mu\tau_\nu + h_{\nu\rho}T^{\mu\rho}$$

- Hydrostatic partition function: $\log\mathcal{Z} = -S$ with

$$S = \int d^{d+1}xeP(T,u^2)\,, \qquad u^2 = h_{\mu\nu}u^\mu u^\nu$$

T is the local temperature and $\beta^\mu = T_0 u^\mu/T$ with $u^\mu\tau_\mu = 1$ is a Killing vector of τ_μ and $h_{\mu\nu}$ [Jensen, 2014] where T_0 is the global temperature.

Hyperscaling violation and anomalous dimension

- Hyperscaling violation exponent θ: replace d by $d-\theta$ in scaling of variables, e.g. $\delta P = (d-\theta+z)\lambda P$

- Anomalous dimension α : $\quad \delta\mu = (z+\alpha)\lambda\mu$

- Scaling of T and v^2 : $\quad \delta T = z\lambda T\,, \quad \delta v^2 = 2(z-1)\lambda v^2$.

- Equation of state:

$$(d-\theta)P = z\hat{\mathcal{E}} + \alpha\hat{\mu}n + \frac{z-2-\alpha}{2}nv^2$$

- Compatible with Galilean boost invariance if $\alpha = z-2$.