Holography beyond conformality

Marika Taylor

Mathematical Sciences and
STAG research centre, Southampton

June 21, 2018

Introduction

- The original example of holography in string theory is the famous AdS/CFT conjecture of Maldacena:

  - String theory on a background with $(d+1)$-dimensional Anti-de Sitter asymptotics is dual to a d-dimensional conformal field theory.

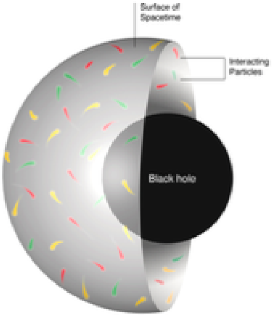
- Many examples of gauge/gravity dualities involving various spacetime asymptotics.

Holographic dualtiies

- **Top-down** models postulate a complete relationship between string theory in a given background and a specific QFT e.g. $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM.
- In **bottom-up** models, we instead engineer the gravity theory to capture defining features of the QFT.
- The latter approach is useful in determining universal features.

STAG RESEARCH CENTRE

Introduction

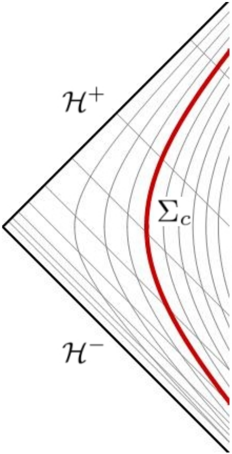


- Original argument for holography: maximum entropy associated with a given spacetime volume scales as the surface area in Planck units.
- Follows from black holes being the most entropic objects for a given mass.
- No dependence on spacetime asymptotics!

('t Hooft, Susskind 1992-1994)

Introduction

- Consider a timelike **hypersurface** Σ_c, in a spacetime with generic asymptotics.
- Interpret radius as RG scale.
- Can we define a QFT on Σ_c, holographically dual to the interior of the spacetime?



STAG RESEARCH CENTRE

Outline

- **Review of holography for UV conformal theories**
- Holography for non-conformal theories - running couplings
- Towards generic holography

Holographic RG flow

- Consider an **RG flow to a UV fixed point**, driven by a single operator $\mathcal{O}$.
- The minimal ingredients required to describe this holographically are:

$$S = \int d^{d+1}x\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 + V(\phi)\right)$$

where ϕ is the bulk scalar dual to $\mathcal{O}$ and the potential is such that the action admits AdS_{d+1} extrema.

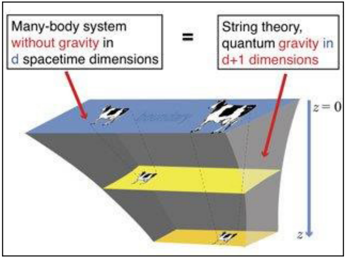
Holographic RG flow

- Consider solutions

$$ds^2 = dr^2 + \exp(2A(r))dx^i dx_i \qquad \phi(r)$$

- The radius corresponds to the **RG scale** of the dual QFT.
- As $r \to \infty$, $A(r) \to r$ and theory approaches AdS/critical point.
- Interior behavior captures IR of QFT.

Spacetime reconstruction



- Given information about the *d*-dimensional theory, we reconstruct spacetime layer by layer.
- This process can be viewed as **coarse-graining**: taking a set of data and representing it by a smaller set of data.

Holographic dictionary

- More precisely, one can expand solutions near the conformal **boundary** $\rho = 0$ as

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}\left(g_{(0)ij} + \rho^2 g_{(2)ij} \cdots + \rho^d g_{(d)ij} \cdots\right) dx^i dx^j$$

and

$$\phi = \rho^{d-\Delta}(\phi_{(d-\Delta)}(x) + \cdots) + \rho^{\Delta}(\phi_{(\Delta)}(x) + \cdots)$$

where Δ relates to the **mass** of the scalar field, and is the **scaling dimension** of $\mathcal{O}$.

STAG RESEARCH CENTRE

Holographic dictionary

- We read off the expectation values of the dual operators as

$$\langle T_{ij} \rangle \sim g_{(d)ij} \qquad \langle \mathcal{O} \rangle \sim \phi_{(\Delta)}$$

where T_{ij} is the d-dimensional QFT stress tensor.

- The **dilatation Ward identity** is then

$$\langle T^i_i \rangle + \phi_{(d-\Delta)} \langle \mathcal{O} \rangle \sim \mathcal{A}.$$

Conformal symmetry **explicitly broken** by source $\phi_{(d-\Delta)}$ for operator $\mathcal{O}$; $\mathcal{A}$ is conformal anomaly.

STAG RESEARCH CENTRE

Essential ingredients:

1. Stress tensor $\leftrightarrow$ bulk metric
2. QFT operators $\leftrightarrow$ other bulk fields
3. Defining behaviour of QFT captured by Ward identities e.g.

$$\partial_i \langle T^{ij} \rangle = 0 \qquad \langle T_i^i \rangle + \sum_a \phi_a \langle \mathcal{O}_a \rangle \sim 0$$

maps into asymptotic behaviour of bulk fields/constraint equations in bulk.

Defining relation of gauge/gravity duality

The bulk partition function acts as the generating functional for QFT correlation functions.

- In the weak coupling limit of the bulk theory: the onshell gravity action gives the functional for QFT correlation functions.
- Hence for AdS gravity calculate

$$S_E = \int d^{d+1}x\sqrt{g}(R + 2\Lambda) - \int d^d h\sqrt{h}K$$

for a solution of the Einstein equations, with boundary condition $g_{(0)}$.

Defining relation of gauge/gravity duality

- S_E is the generating functional in the CFT, and $g_{(0)}$ is the source for the stress energy tensor:

$$\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{g_{(0)}}} \frac{\delta S_E[g_{(0)}]}{\delta g_{(0)}^{\mu\nu}}$$

- However, S_E diverges, due to the infinite volume of AdS!
- Equivalent to the UV divergences of the CFT.

Holographic renormalization

- Convent to use radial foliation near the conformal boundary
$$ds^2 = dr^2 + \gamma_{ij}(r,x)dx^i dx^j$$
where for AAdS $\gamma_{ij}(r,x) \sim e^{2r} g_{(0)ij} + \cdots$ as $r \to \infty$.
- The conjugate momentum to γ is the **Brown-York quasi-local stress tensor**
$$\mathcal{T}_{ij} = (K_{ij} - K\gamma_{ij})$$
where the extrinsic curvature $K_{ij} = \frac{1}{2}\partial_r \gamma_{ij}$.

Holographic renormalization

- $\mathcal{T}_{ij}$ is not finite as $r \to \infty$.
- **Boundary counterterms** added to the Einstein-Hilbert action
$$S_{\rm ct} = -\int d^d x \sqrt{-h} \left((d-1) + \cdots \right)$$
render the onshell action finite and give additional contributions to the quasi-local stress tensor:
$$T_{ij} = \left(K_{ij} - K\gamma_{ij} + (d-1)\gamma_{ij} + \cdots \right)$$
(Balasubramanian and Kraus; de Haro, Skenderis and Solodukhin)

STAG RESEARCH CENTRE

Holographic renormalization

- T_{ij} does have a finite limit as $r \to \infty$:

$$\mathcal{L}_{r\to\infty}\left(T_{ij}\right) = \langle T_{ij}\rangle \sim g_{(d)ij}.$$

- The renormalized stress tensor satisfies the expected CFT identities e.g. for $d = 2$

$$\langle T^i_i \rangle = \frac{c}{6}\mathcal{R}(g_{(0)})$$

where c is the central charge.

STAG RESEARCH CENTRE

Outline

- Review of holography for UV conformal theories
- **Holography for non-conformal theories (running couplings)**
- Towards generic holography

Beyond UV conformality

- Framework described above suffices for spontaneous and explicit breaking of conformal symmetry.
- Asymptotically free: weak UV coupling $\leftrightarrow$ high curvature (not Einstein gravity).
- What kind of UV behaviour can we accommodate in Einstein gravity?

Einstein-dilaton theories

- To describe a **holographic RG flow**, we can still use

$$S = \int d^{d+1}x\sqrt{-g}e^{\gamma\phi}\left(R - \frac{1}{2}(\partial\phi)^2 + V(\phi)\right)$$

  where ϕ is the bulk scalar dual to operator $\mathcal{O}$ driving flow.
- The potential is chosen to capture required **asymptotic behaviour**.

Example: dimensionally driven flow

- **Exponential potential**; choose γ such that equations admit anti-de Sitter solutions

$$ds^2 = dr^2 + e^{2r} dx^i dx_i \qquad \phi = \alpha r$$

- Dual QFT has **generalized conformal structure**: scale invariance is broken only by a dimensionally running coupling.
- Scalar field diverges at boundary (UV) or deep interior (IR): classical gravity description breaks down.

Example 1: dimensionally driven flow

- Prototypical example of dual QFT: (super) Yang-Mills in $d \neq 4$

$$S \sim \int d^d x \mathrm{Tr} \left(-\phi_s (F_{ij} F^{ij}) + (D_i X^a D^i X^a) + \frac{1}{\phi_s} [X^a, X^b]^2 + \cdots \right)$$

- Here the scalar is dual to operator $\mathcal{O}$

$$\mathcal{O} = \mathrm{Tr}(F_{ij} F^{ij}) + \frac{1}{\phi_s^2} \mathrm{Tr}[X, X]^2$$

and the generalized conformal structure is respected at the quantum level (due to supersymmetry).

Generalized conformal structure

- Not conformally invariant but symmetric under scaling transformations provided that we also transform coupling i.e. invariant under

$$\phi_s \to e^{-(d-\Delta_\phi)\sigma}\phi_s \qquad g_{(0)ij} \to e^{2\sigma} g_{(0)ij} \qquad \cdots$$

- This **generalized conformal structure** is captured by Ward identities, which imply an infinite set of relations for correlation functions.
- Associated with ϕ_s is a dimensionless coupling

$$g^2_{\mathrm{eff}}(x) = |x|^{\Delta_\phi - d}\phi_s^{-1}$$

STAG RESEARCH CENTRE

Two point functions

- A general scalar operator $\mathcal{O}$ then has a two point function

$$\langle \mathcal{O}(x) \mathcal{O}(0) \rangle = \frac{f(g^2_{\rm eff}(x), N, \cdots)}{|x|^{2\Delta}}$$

  where *any* function of the dimensionless quantities is consistent with the conformal structure. (Kanitscheider, Skenderis, M.T.)

- In particular, in certain limits we may find

$$f(g^2_{\rm eff}(x), N, \cdots) \sim |x|^{\beta}$$

  i.e. a Taylor/Laurent series dominated by one single term.

Example 2: The Sachdev-Ye-Kitaev (SYK) model

- The Lagrangian for N fermions in 1d is

$$\mathcal{L} = \frac{1}{2} \sum_{i} \psi^i \partial_\tau \psi^i + \sum_{ijkl} \lambda_{ijkl} \psi^i \psi^j \psi^k \psi^l$$

- SYK: **couplings** λ_{ijkl} are randomly taken from a Gaussian distribution with zero mean and width $\mathcal{J}/N^{3/2}$, where $\mathcal{J}$ is dimensionful.
- SYK has **generalized conformal structure.**

STAG RESEARCH CENTRE

Example 2: The Sachdev-Ye-Kitaev (SYK) model

- The bilocal field $G(\tau_1, \tau_2)$

$$G(\tau_1, \tau_2) = \frac{1}{N} \sum_i \langle \psi^i(\tau_1) \psi^i(\tau_2) \rangle$$

is classical in the large N limit and

$$G(\tau_1, \tau_2) \sim \frac{1}{|\tau_1 - \tau_2|^{\frac{1}{2}}} + \cdots$$

for $(\tau_1 - \tau_2) \gg 1/\mathcal{J}$.

- $\mathcal{J}$ is the dimensionful scale.

STAG RESEARCH CENTRE

Holography beyond Einstein gravity

SYK should not be describable by two derivative gravity/finite number of bulk fields:

- No gap in spectrum.
- No truncation to finite set of operators $\psi_{i_1}\psi_{i_2}\psi_{i_3}\psi_{i_4}\cdots$.

(Maldacena and Stanford; Gross and Rosenhaus;......)

Why does Einstein-dilaton gravity capture SYK features?

Generalized conformal structure

- Consider a bulk action

$$S \propto - \int d^{d+1}x \sqrt{g} e^{\gamma\phi} \left(R + \beta(\partial\phi)^2 + C \right)$$

- Constants (γ, β, C) related to give solutions

$$ds^2 = dr^2 + e^{2r} dx \cdot dx \qquad \phi = \alpha r$$

i.e AdS_{d+1} with running coupling.

(Kanitscheider, Skenderis, M.T)

STAG RESEARCH CENTRE

Holographic dictionary

- Holographic dictionary: metric dual to stress energy tensor T_{ij} and scalar dual to scalar operator $\mathcal{O}$.
- Dual operators indeed satisfy Ward identity:

$$\langle T^i_i \rangle + (d - \Delta)\phi_s \mathcal{O} \sim 0$$

capturing dimensionally driven flow.

STAG RESEARCH CENTRE

Parent CFT

- All such theories can be viewed as a formal dimensional reduction from an AdS theory:

$$S \propto - \int d^{2\sigma+1}x\sqrt{g}\,(R + 2\Lambda)$$

over a torus of dimension

$$(2\sigma - d) = -2\alpha\gamma$$

- In special cases, $-2\alpha\gamma$ is a positive integer but formal structure persists even for non-integer values.

(Kanitscheider and Skenderis)

STAG RESEARCH CENTRE

Holographic dictionary

- Two independent parameters control **dimension of coupling** and **thermodynamics.**
- For SYK, $d = 2$, $\beta = 0$, $\gamma = 1$ gives

$$\log Z = -\frac{E_0}{T} + S_0 + \frac{1}{2} c T$$

  E_0 and S_0 ground state energy/entropy and c is specific heat, plus required Lyapunov exponent.
- **Parent AdS** theory is AdS_3.

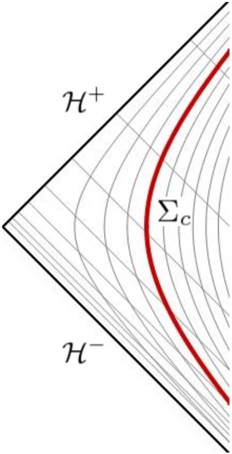
Outline

- Review of holography for UV conformal theories
- Holography for non-conformal theories - running couplings
- **Towards generic holography**

STAG RESEARCH CENTRE

Finite radius hypersurface

- Natural to ask about duality for finite radius hypersurface.
- From QFT perspective: radial evolution is RG flow.
- In presence of horizons, one obtains a fluid/gravity relation.



(Minwalla et al; Polchinski et al; Strominger et al; Compère, McFadden, Skenderis and Taylor; ....)

Gauss-Codazzi relations

- Defining the quasilocal stress energy tensor at finite radius as before,

$$T^i_i = -4\pi G \left(T_{ij} T^{ij} - \frac{1}{(d-1)} (T^i_i)^2 \right)$$

  on flat hypersurfaces.
- Follows from Einstein equations with negative cosmological constant in Gauss-Codazzi form.

QFT interpretation

- We view this relation as a Ward identity:

$$T_i^i = -\lambda \mathcal{T}$$

where

$$\mathcal{T} = \left(T_{ij} T^{ij} - \frac{1}{(d-1)} (T_i^i)^2 \right)$$

- In $d = 2$, $\mathcal{T}$ is the $T\bar{T}$ operator explored by Zamoldchikov.
- Holographic relation in $d = 2$ proposed by (McGough et al).

STAG RESEARCH CENTRE

$T\bar{T}$ operator in 2d

- Zamoldchikov showed that this operator has a remarkable **OPE structure** as $x \to y$:

$$T\bar{T}(x, y) = \mathcal{T}(y) + \sum_{\alpha} A_{\alpha}(x - y)\nabla_y \mathcal{O}_{\alpha}(x)$$

  i.e. we can identify the operator as local, modulo derivatives of other local operators.

- Smirnov and Zamoldchikov also explored the behaviour of a CFT under deformations by $\mathcal{T}$ i.e.

$$S_{\rm CFT} \to S_{\rm CFT} + \lambda \int d^2x \, \mathcal{T}.$$

STAG RESEARCH CENTRE

Energy spectrum

- Consider the (Euclidean) theory on a cylinder of radius R.
- In a **stationary** state such that

$$\langle T_{\tau\tau} \rangle = -\frac{E}{R}$$

the defining relation for the family of QFTs implies that

$$\frac{\partial E}{\partial \lambda} + 2E \frac{\partial E}{\partial R} = 0$$

STAG RESEARCH CENTRE

Energy spectrum

- This can be re-expressed in terms of dimensionless quantities (ϵ, α) using

$$\alpha = \frac{\lambda}{R^2} \qquad E = \frac{1}{R}\epsilon$$

with

$$\partial_\alpha \epsilon = 2\epsilon \left(\epsilon + 2\alpha \partial_\alpha \epsilon\right)$$

- This is the defining ODE for the **energy spectrum** $\epsilon(\alpha)$.

STAG RESEARCH CENTRE

Generalization to $d > 2$

In general dimensions:

$$\mathcal{T} = \left(T_{ij}T^{ij} - \frac{1}{(d-1)}(T_i^i)^2 \right)$$

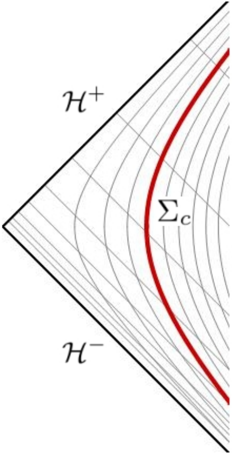
- Definite of composite operator more subtle; **renormalization** required as operators approach each other.
- Details of operator definition not required for energy spectrum, but would be needed for **correlation functions, entanglement entropy** etc.

Back to gravity

- The conjectured holographic theory dual for finite radius is

$$S_{\text{CFT}} \to S_{\text{CFT}} + \lambda \int d^{D+1}x \, \mathcal{T}.$$

- Identifying the quasi-local stress tensor as the dual stress tensor, **Ward identity** matches by construction.
- Can we also reproduce **energy spectrum** in gravity?



STAG RESEARCH CENTRE

Black brane solutions

- Consider a static **black brane** in $(D+2)$ dimensions

$$ds^2 = (\rho^2 - \frac{\mu}{\rho^{D-1}})d\tau^2 + \frac{d\rho^2}{(\rho^2 - \frac{\mu}{\rho^{D-1}})} + \rho^2 dx^a dx_a$$

We can then read off from the quasi local stress tensor the dimensionless energy:

$$\epsilon = \frac{D\rho^d}{2\lambda}\left(1 - \left(1 - \frac{\lambda M}{\rho^d}\right)^{\frac{1}{2}}\right)$$

where $\mu = 4\pi GM$.

STAG RESEARCH CENTRE

Black brane solutions

- In terms of dimensionless coupling $\alpha = \lambda/\rho^d$,

$$\epsilon = \frac{D}{2\alpha}\left(1 - (1 - \alpha M)^{\frac{1}{2}}\right)$$

- Note that the CFT energy is

$$\epsilon(0) = \frac{D}{4}M$$

and $\epsilon(\alpha)$ indeed satisfies:

$$\partial_\alpha \epsilon = \left(1 + \frac{1}{D}\right)(\epsilon + 2\alpha\epsilon\partial_\alpha\epsilon)$$

Comments

1. Trivial to generalize to **boosted (spinning) branes.**
2. Addition of extra **bulk fields** (gauge fields, scalars etc) modifies CFT deformation e.g.

$$T^i_i = -\lambda \left(T^{ij} T_{ij} - \frac{1}{D} (T^i_i)^2 + 2 \mathcal{J}^i \mathcal{J}_i \right)$$

Also noticed in $d = 2$ by (Bzowski and Guica; Kraus et al).

STAG RESEARCH CENTRE

Conclusions and outlook

- Examples of holography beyond conformality, from bootstrapping dilatation Ward identities.
- Dimensionful coupling plus thermodynamics of SYK captured by AdS_3 gravity.
- Weakly coupled theories cannot be described by Einstein gravity (no go for asymptotic freedom).

Conclusions and outlook

- At finite radius, bootstrap dual QFT from dilatation Ward identity.
- For AdS, suggests family of Zamoldchikov TT theories.
- Beyond AdS: proposals at finite temperature for gravity/fluid relations.