Fish gotta swim, Birds gotta fly, I gotta do Feynmann Graphs 'til I die:

A continuum theory of flocking

Original idea: T. Vicsek, Eötvos Institute Budapest, Hungary

Collaborators: $\underbrace{\text{Yu-hai Tu, IBM, Research}}_{\text{Everything}}$

$\underbrace{\text{Markus Ulm}}_{\text{simulations}}$

Inspiration: A. Hitchcock, E. Fields

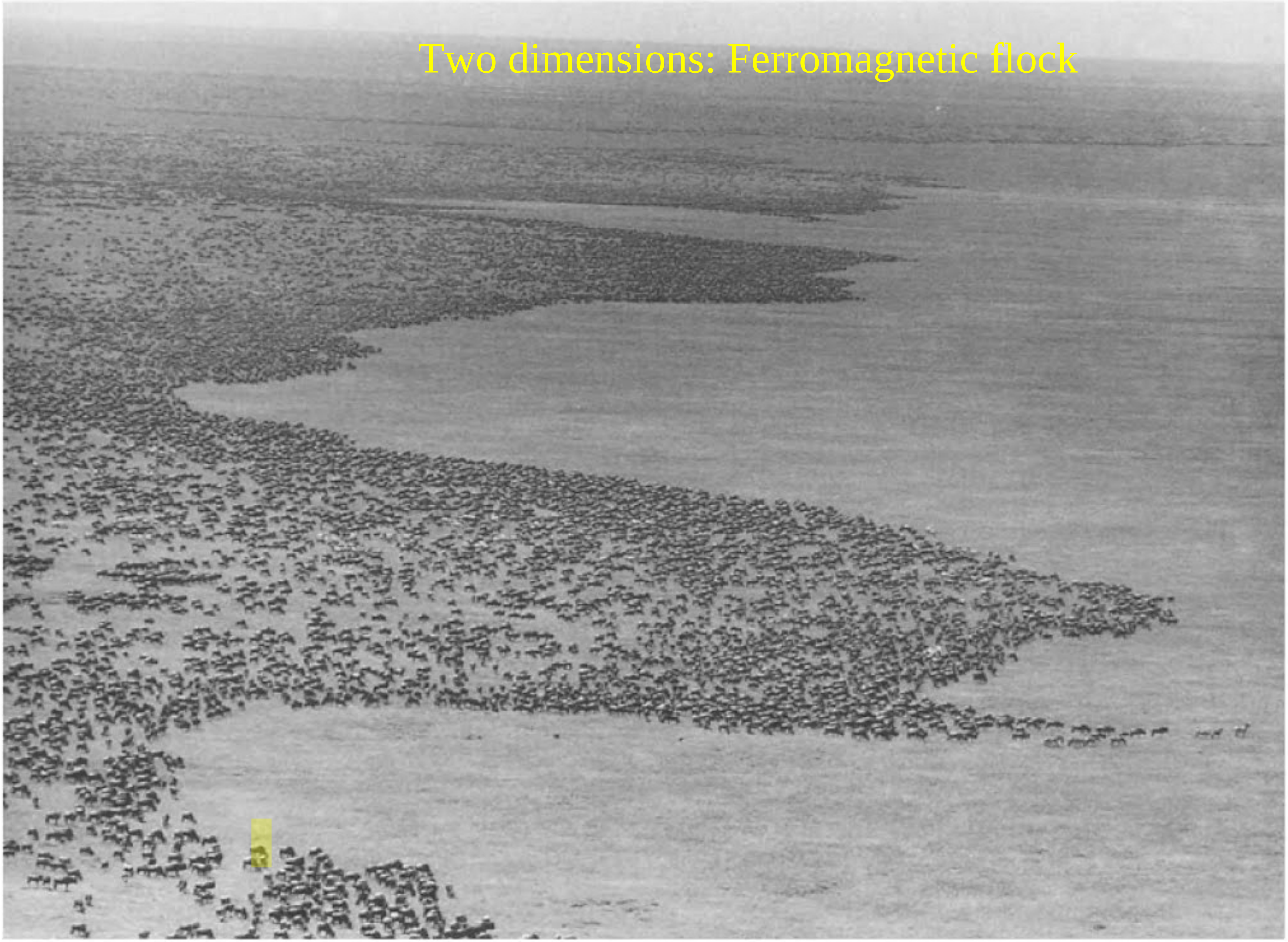


Figure 1: Several hundred thousand wildbeest in a moving front, grazing on the Serengeti. From A.R.E. Sinclair (1977), reproduced by permission of the author and publisher.

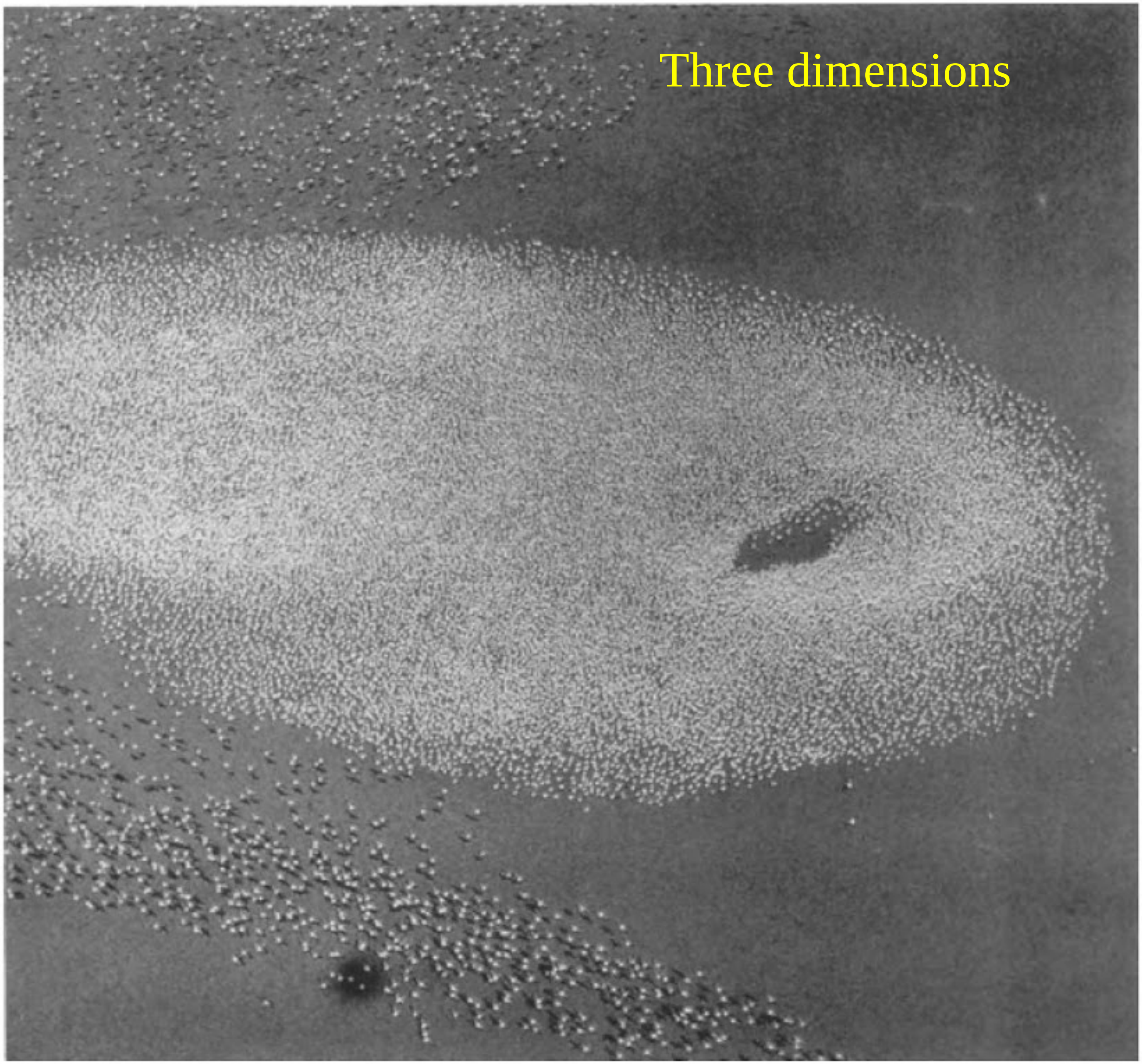


Figure 2: A flock of flamingos on an African lake. Note variations in density and in the orientations of the birds. A fairly distinct front is maintained.

Bacteria:
(10^9 of 'em)

All cases Spontaneously Broken Continuous Symmetry

arbitrary direction But same for all

Short ranged interactions $\Rightarrow$ Long ranged order

Violates Mermin-Wagner Thm in $d=2$!

How? Non-Equilibrium (Motion)

$\Downarrow$

Breakdown of linearized hydrodynamics

$\Downarrow$

Stabilizes Long ranged Order

Ferromagnetic Flocks
Outline

I) Intro: What's Flocking?

II) Microscopic Models - Vicsek

Important points: Rotation Invariance (Lost sheep)

Locality:

III) Mermin-Wagner Theorem:

Are Birds smarter than Nerds?

IV) Continuum model:
- Analogy with Navier-Stokes (Fluid mechanics)
- Use symmetries
- Bury our ignorance

V) No, Birds ain't smarter:

How motion beats Mermin-Wagner

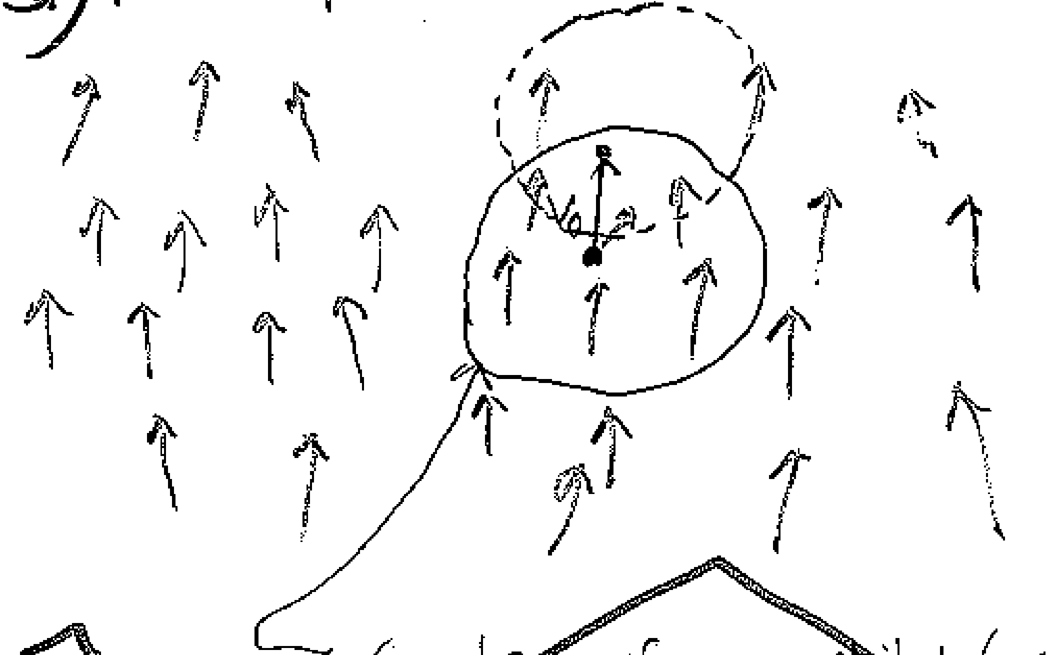
VI) Our other predictions (4a

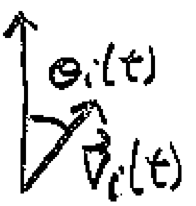
VII) Experiments we'd like to see.

VIII) Future theory (+ some already done)

~~IX~~

II) "Microscopic Models": Vicsek's (2d)





$|\vec{v}_i(t)| = v_0$ (all i, t)

Circle of visibility (radius $\equiv 1$) C

$$\vec{V}_i(t+1) = \langle \vec{V}(t) \rangle_C + \underbrace{\vec{\eta}_i(t)}_{\text{Mistake (random)}}$$

$\langle \vec{\eta}_i(t) \rangle = 0$ (~~right~~ correct, on average)

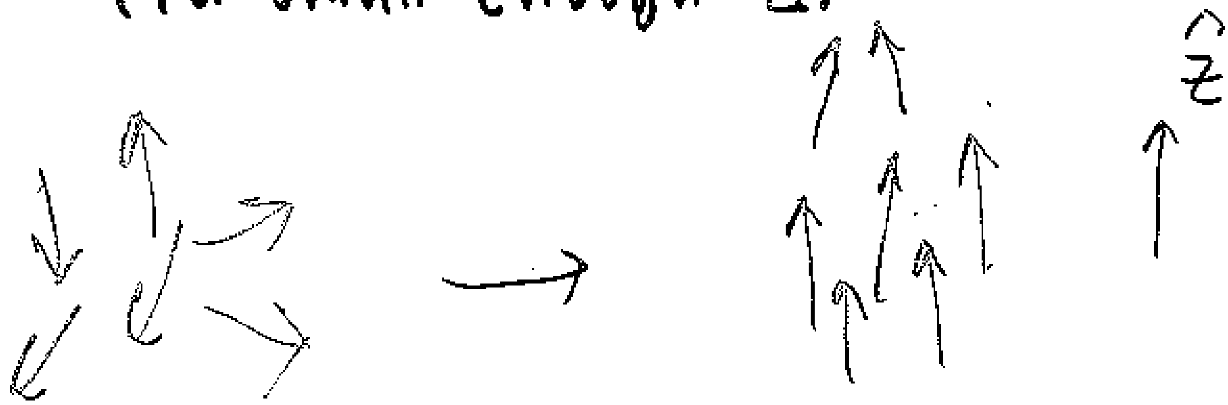
$$\langle \eta_{i\alpha}(t)\, \eta_{j\beta}(t') \rangle = \begin{cases} 0, & i \neq j \text{ (different birds' errors uncorrelated)} \\ 0, & t \neq t' \text{ (no memory)} \\ \Delta, & i=j,\ t=t',\ \alpha=\beta \text{ (noise strength)} \end{cases}$$

Note: Rotation invariant

$\Rightarrow$ "lost sheep"

Vicsek simulates this, finds:

Flock Spontaneously orders
(For small enough Δ)



$t = 0$ $\qquad\qquad$ $t \gg 1$

$\langle \vec{v} \rangle = 0$ $\qquad\qquad$ $\langle \vec{v} \rangle = v_0 \hat{z} \neq 0$

Arbitrary direction

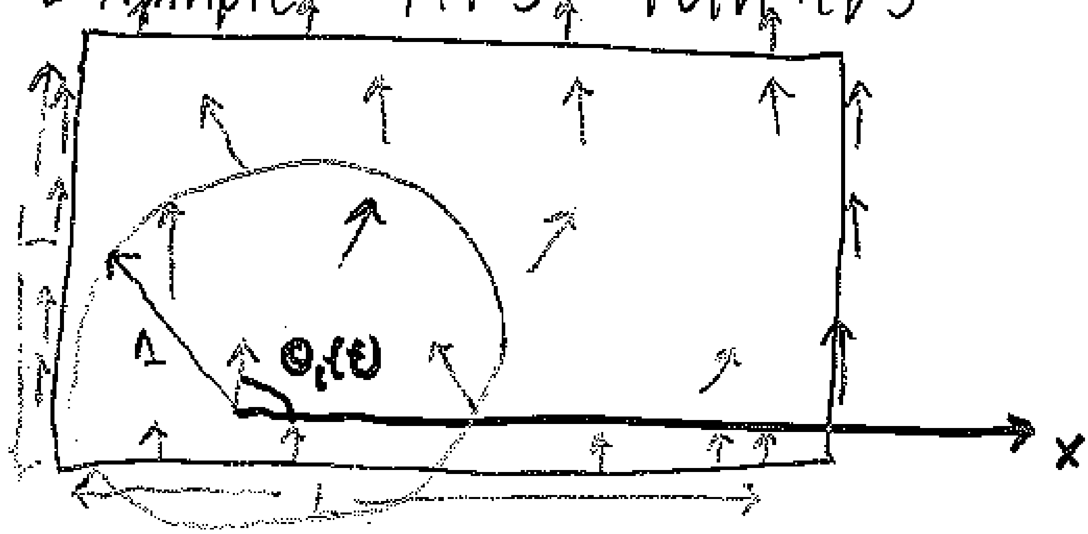
In $d=2$, this is <u>Astonishing</u>!

Violates Mermin-Wagner Thm!

III) Mermin-Wagner Thm: $\boxed{\text{Are birds smarter than nerds?}}$

Spontaneously Broken Continuous symmetry Impossible in **d=2**:

Example: APS Pointers



Same orientational rule:

$$\theta_i(t+1) = \langle \theta_j(t) \rangle_{j \in O} + \eta_i(t)$$

$$\langle \eta_i \rangle = 0, \quad \langle \eta_i(t)\, \eta_j(t') \rangle = \Delta\, \delta_{ij}\, \delta_{tt'}$$

No motion $\Rightarrow$ 2d XY model
Equilibrium

$$\Rightarrow \langle \theta_i^2 \rangle \propto \ln L \to \infty$$

$\Rightarrow$ No LRO

Why? Monte Carlo for Equilibrium

2d XY model $\Rightarrow$ Mermin-Wagner Theorem

Physics: In $d \leq 2$, errors generated faster than dissipated

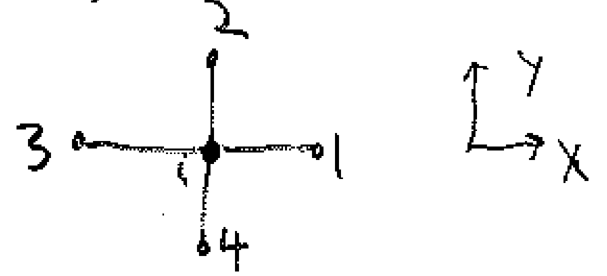
Rewrite evolution rule: subtract $\theta_i(t)$ from both sides

$$\underbrace{\theta_i(t+1) - \theta_i(t)}_{\partial_t \theta} = \underbrace{\langle \theta_j(t) \rangle - \theta_i(t)}_{D(\partial_x^2 \theta + \partial_y^2 \theta)} + \eta_i$$

$\longleftarrow$ continuum limit $\longrightarrow$

$$\partial_t \theta = D(\partial_x^2 \theta + \partial_y^2 \theta) + \eta$$

To see this, consider lattice:



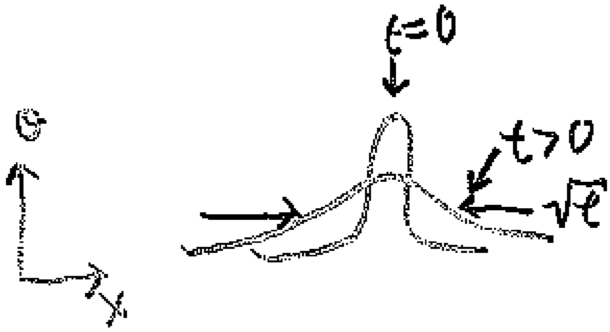
$$\langle \theta_j \rangle - \theta_i = \frac{1}{4}(\theta_1 + \theta_2 + \theta_3 + \theta_4) - \theta_i$$

$$= \frac{1}{4}\Big[\overbrace{\underbrace{(\theta_1 - \theta_i)}_{\partial_x \theta(x,y)} - \underbrace{(\theta_i - \theta_3)}_{\partial_x \theta(x-1,y)}}^{\partial_x(\partial_x \theta)} + \overbrace{\underbrace{(\theta_2 - \theta_i)}_{\partial_y \theta(x,y)} - \underbrace{(\theta_i - \theta_4)}_{\partial_y \theta(x,y-1)}}^{\partial_y(\partial_y \theta)} \Big]$$

$$\partial_t \Theta = D\nabla^2\Theta + \eta$$

$\boxed{\text{Diffusion}}$ equation: Like Ink in water

Properties: (1) Slow



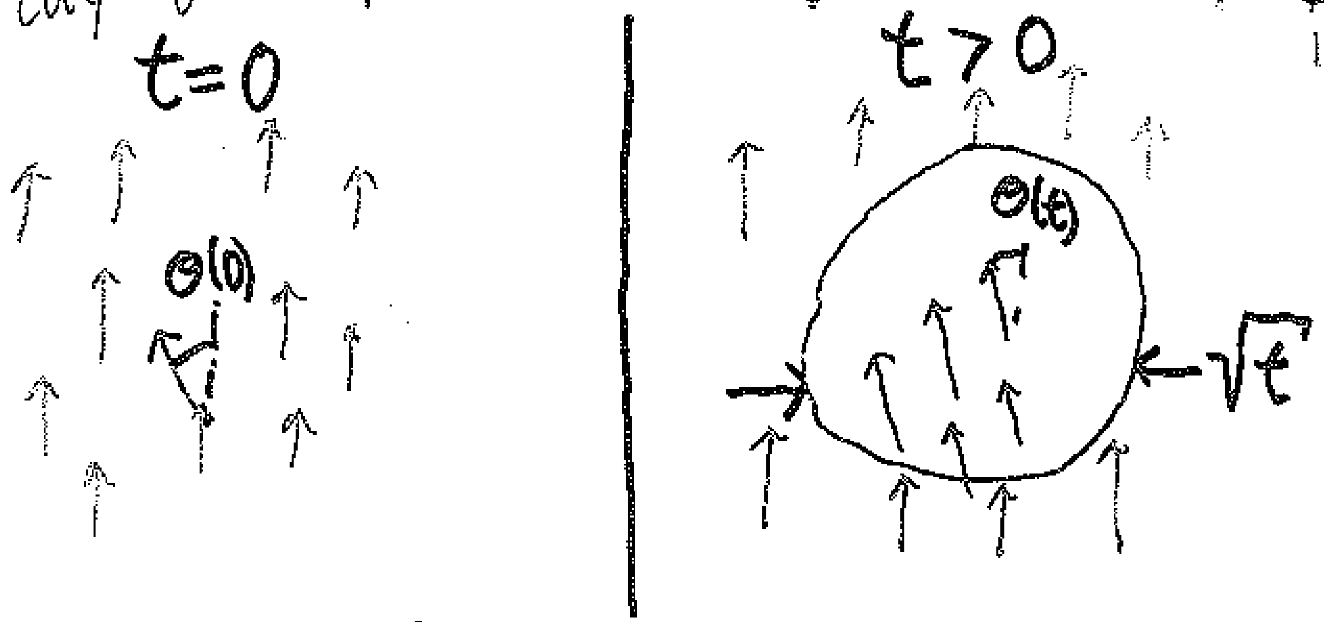
$$\frac{\Theta}{t} \sim \frac{\Theta}{x^2} \Rightarrow x^2 \propto t \Rightarrow x \propto \sqrt{t}$$

2) Conserves:

$$\partial_t \int \Theta \, d^d r = D\int \nabla^2 \Theta \, d^d r = D\int \vec{\nabla}\Theta \cdot dS$$

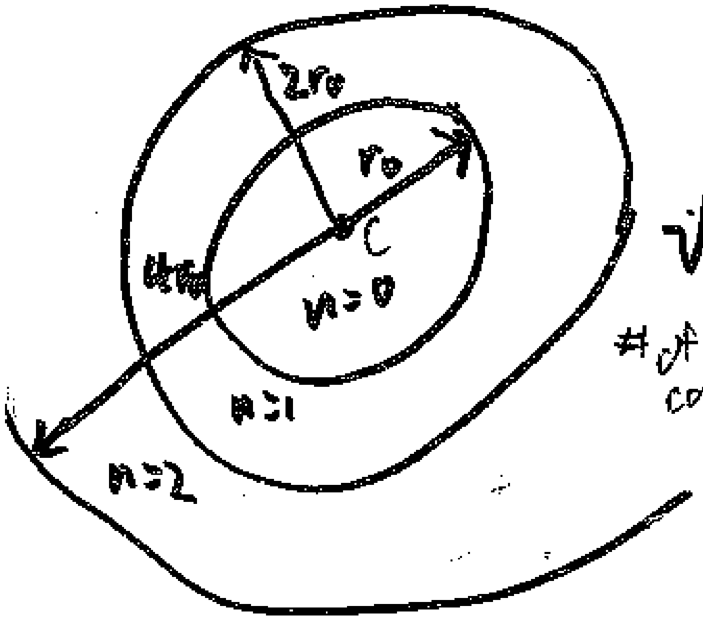
$\int \vec{\nabla}\Theta \cdot \vec{dS} = 0$ if flock size $\gg \sqrt{t}$

Decay of One error: $d\uparrow \Rightarrow$ slower dec[illegible]



Θ conserved $\Rightarrow \Theta(t) \approx \dfrac{\Theta_0}{N} \approx \dfrac{\Theta_0}{t^{d/2}}$ (d dimensions)

Meanwhile: More errors:



Total error at center:

$$\sqrt{\delta\Theta_0^2} = \sqrt{\underbrace{r_0^d}_{\text{\# of error committers}}\ \underbrace{t_0}_{\text{\# of errors each makes}}} \Big/ r_0^d$$

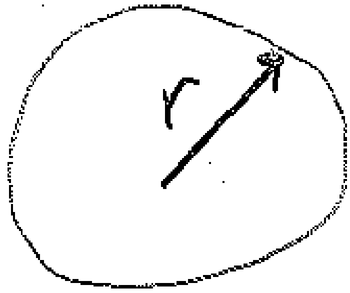
$$t_0 \sim r_0^2 \Rightarrow \delta\Theta_0^2 \sim r_0^{2-d}$$

$$\delta\Theta_1^2 \sim (2r_0)^{2-d} = \delta\Theta_0^2\, 2^{2-d}$$

$\vdots$

[redacted]

Meanwhile: More errors



Diffusion time $t_D(r) \propto r^2$

\# of errors/spin $\propto t_D \propto r^2$

\# of spins $N \propto r^d$

$\Rightarrow$ # of errors $\propto N t_D \propto r^{d+2}$

$\Rightarrow$ RMS error/spin $\delta\theta \propto \frac{\sqrt{r^{d+2}}}{r^d}$

$$\Rightarrow \delta\theta \propto r^{1-\frac{d}{2}} \xrightarrow[r\to\infty]{} \begin{cases} 0, & d>2 \Rightarrow \text{LRO} \\ \infty, & d<2 \Rightarrow \text{No LRO} \\ \ln L \to \infty, & d=2 \Rightarrow \text{No LRO} \end{cases}$$

How do birds beat this?

IV) Continuum theory of birds:

- Hard (impossible) to solve microscopic model with $\sim 10^6$ birds
- Harder to figure out what happens if you change model

Historical analog: Fluid mechanics
Navier-Stokes equation
@ 1840

- No theory of atoms, molecules, etc.
- No statistical Physics
- No 37" TV
- No computers

IV) Back to birds: Continuum model: Phenomenological

How do you do this?

Arbitrary spatial dimension d

Fields: $\vec{v}(\vec{r},t)$: Coarse grained velocity

$\rho(\vec{r},t)$: " " number density

Length scales: $L \gg$ Interbird distance

time scales $t \gg$ microscopic time scale

- Expand in powers of spacetime gradients
- Expand " " " fields
- keep all terms consistent with symmetries

: Rotation invariance: $\vec{v} \to \underset{\sim}{R}\vec{v}$

$\vec{r} \to \underset{\sim}{R}\vec{r}$

$\Rightarrow \vec{\nabla} \to \underset{\sim}{R}\vec{\nabla}$

⇒ Equations of Motion: for Birds

$F = ma$

$$\partial_t \vec{v} + \underbrace{\lambda_1 (\vec{v}\cdot\vec{\nabla})\vec{v}}_{\text{Convection}} + \underbrace{\lambda_2 \vec{v}(\vec{\nabla}\cdot\vec{v})}_{\text{crunch anxiety}} + \underbrace{\lambda_3 \vec{\nabla}(|\vec{v}|^2)}_{\substack{\text{speed kills } (\lambda_3>0)\\ \text{speed thrills } (\lambda_3<0)}}$$

$$= \underbrace{\alpha \vec{v} - \beta |\vec{v}|^2 \vec{v}}_{\text{keep moving: } v_0 = \sqrt{\alpha/\beta} = |\vec{v}|} - \vec{\nabla} P(\rho)$$

→ Pressure Forces

$$+ \underbrace{D_B \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + D_1 \nabla^2 \vec{v} + D_2 (\vec{v}\cdot\vec{\nabla})^2 \vec{v}}_{\text{Follow your neighbors}} + \underbrace{\vec{f}}_{\text{mistakes}}$$

Bird conservation:

$$\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

we $P(\rho) \equiv \sum_{n=1}^{\infty} \alpha_n (\rho - \langle\rho\rangle)^n$: Only $n=1,2$ important

$\vec{f}$ Gaussian, white

$\langle \vec{f} \rangle = 0$

$\langle f_i(\vec{r},t)\, f_j(\vec{r}\,',t') \rangle = \Delta\, \delta_{ij}\, \delta(\vec{r}-\vec{r}\,')\, \delta(t-t')$

Solve using fluctuating hydrodynamics techniques (Dynamical P.T., RG, etc)

Results: Negative Feedback stabilizes order:

Fluctuations $\uparrow \Rightarrow$ Diffusion $\uparrow \Rightarrow$ Fluctuations $\downarrow$

(anomalous diffusion: $D_\perp(L) \propto L^{4/5}$)

seen in simulations by Yu-hai Tu

(like "breakdown of hydrodynamics" in $d=2$ fluid, but much stronger)

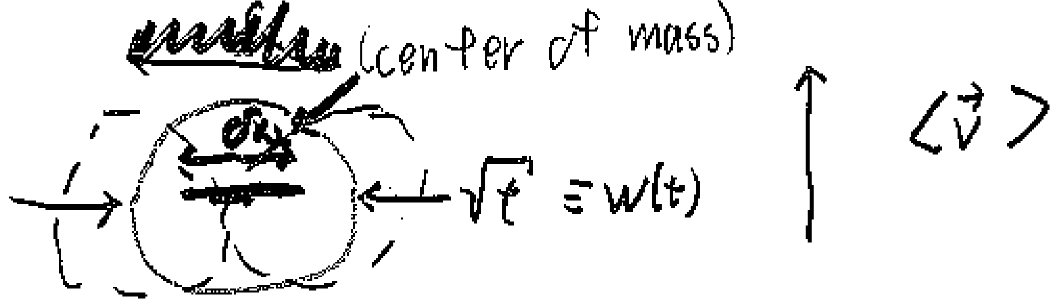
$\Downarrow$

LRO $\underline{\underline{\text{is}}}$ possible in $d=2$

~~Mermin-Wagner~~

V) How motion beats Mermin-Wagner

Hand waving:



How far $\delta x_\perp$ does blob move $\perp$ $\langle \vec{v} \rangle$?

$$\delta x_\perp \sim \sqrt{v_\perp^2}\, t \sim v_0 \sqrt{\theta^2}\, t$$

remember: $\sqrt{\theta^2} \propto r^{1-\frac{d}{2}} \propto t^{\frac{1}{2}-\frac{d}{4}}$

$$\Rightarrow \boxed{\delta x_\perp \propto t^{\frac{3}{2}-\frac{d}{4}}}$$

Compare with width of blob:

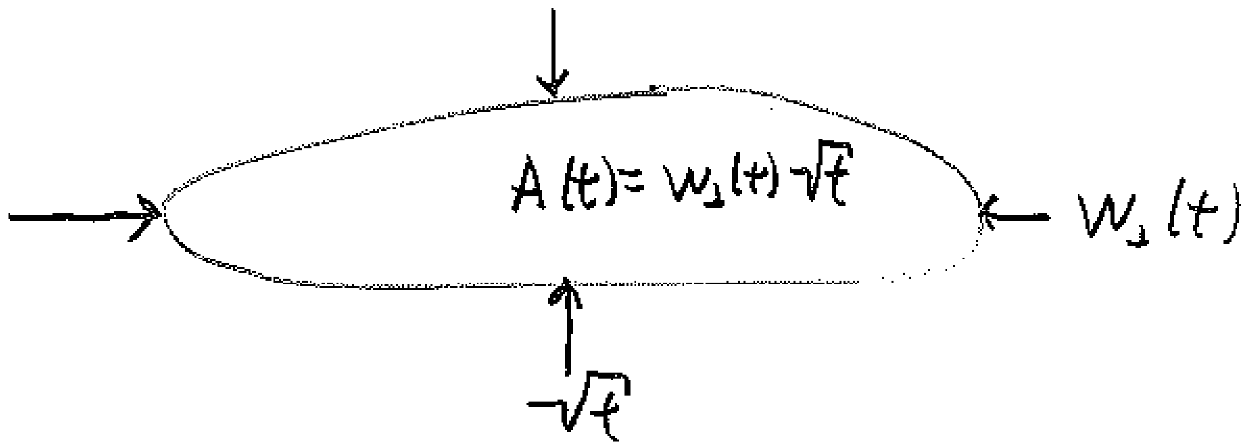
$$\frac{\delta x_\perp}{w(t)} \propto \frac{t^{\frac{3}{2}-\frac{d}{4}}}{t^{1/2}} \propto t^{1-\frac{d}{4}} \xrightarrow[t\to\infty]{} \begin{cases} 0 & , d>4 \\ \infty & , d<4 \end{cases}$$

$\Rightarrow$ Bird motion dominates diffusion for $d<4$

$\Rightarrow$ Physics changes in $d<4$

("Breakdown of linearized hydrodynamics")

So what does happen in $d < 4$?
Blobs spread anisotropically



# of errors/spin $\propto t$
# of spins $\propto A(t) \propto W_\perp(t)\sqrt{t}$

$\Rightarrow$ # of errors $\propto A(t)\, t \propto W_\perp(t)\, t^{3/2}$

$\Rightarrow$ RMS error/spin $\delta\theta \propto \dfrac{\sqrt{W_\perp(t)\, t^{3/2}}}{W_\perp(t)\sqrt{t}} \propto \dfrac{t^{1/4}}{W_\perp^{1/2}}$

self-consistency:

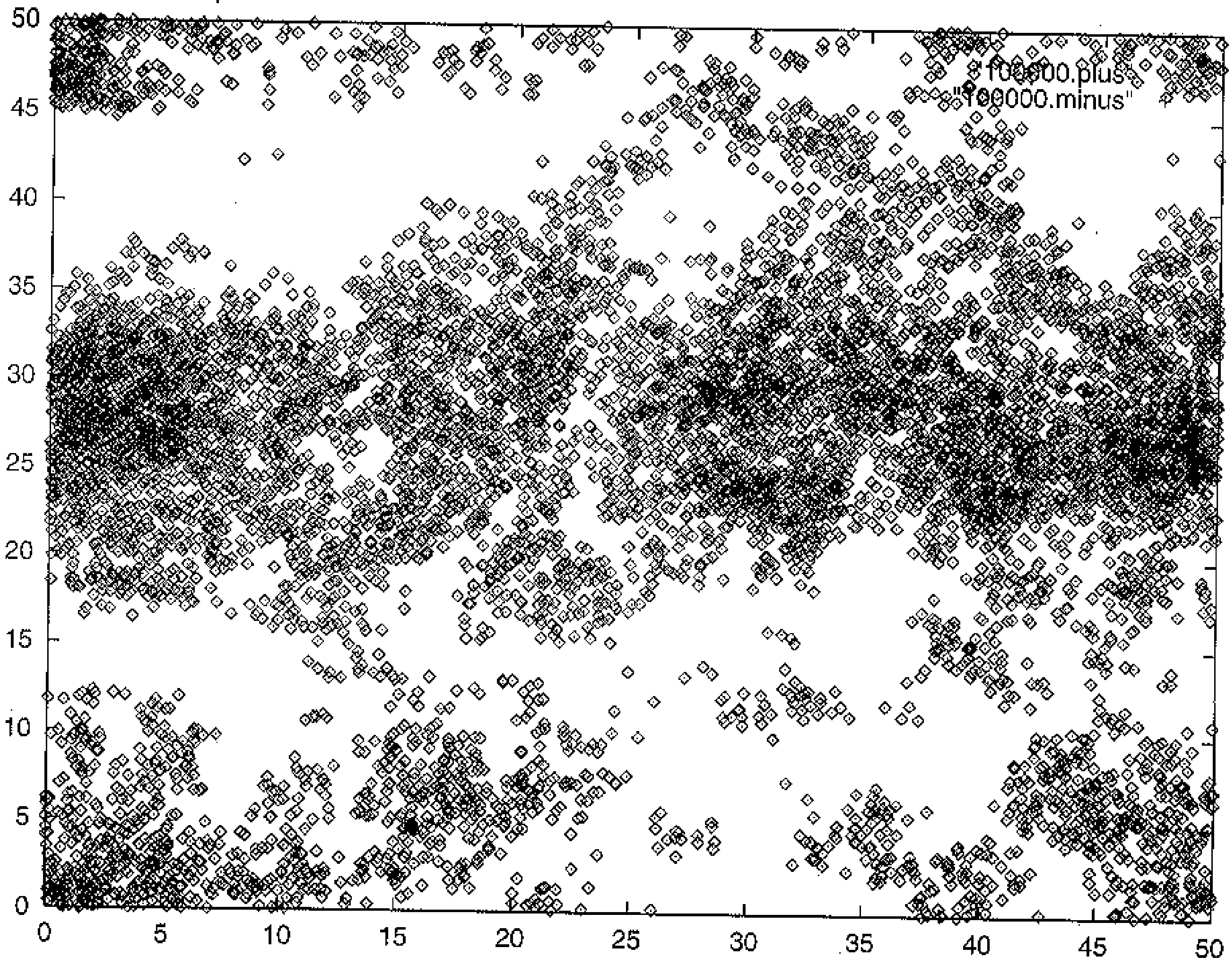
$$W_\perp(t) \sim v_0\, \delta\theta\, t \propto \frac{t^{5/4}}{W_\perp^{1/2}} \Rightarrow W_\perp^{3/2} \propto t^{5/4} \Rightarrow W_\perp \propto t^{5/6} \gg t^{1/2}$$

$$\Rightarrow \delta\theta \propto \frac{t^{1/4}}{t^{5/12}} \propto t^{-1/6} \to 0 \text{ as } t \to \infty$$

$\Rightarrow$ LRO!!

~~Mermin-Wagner~~

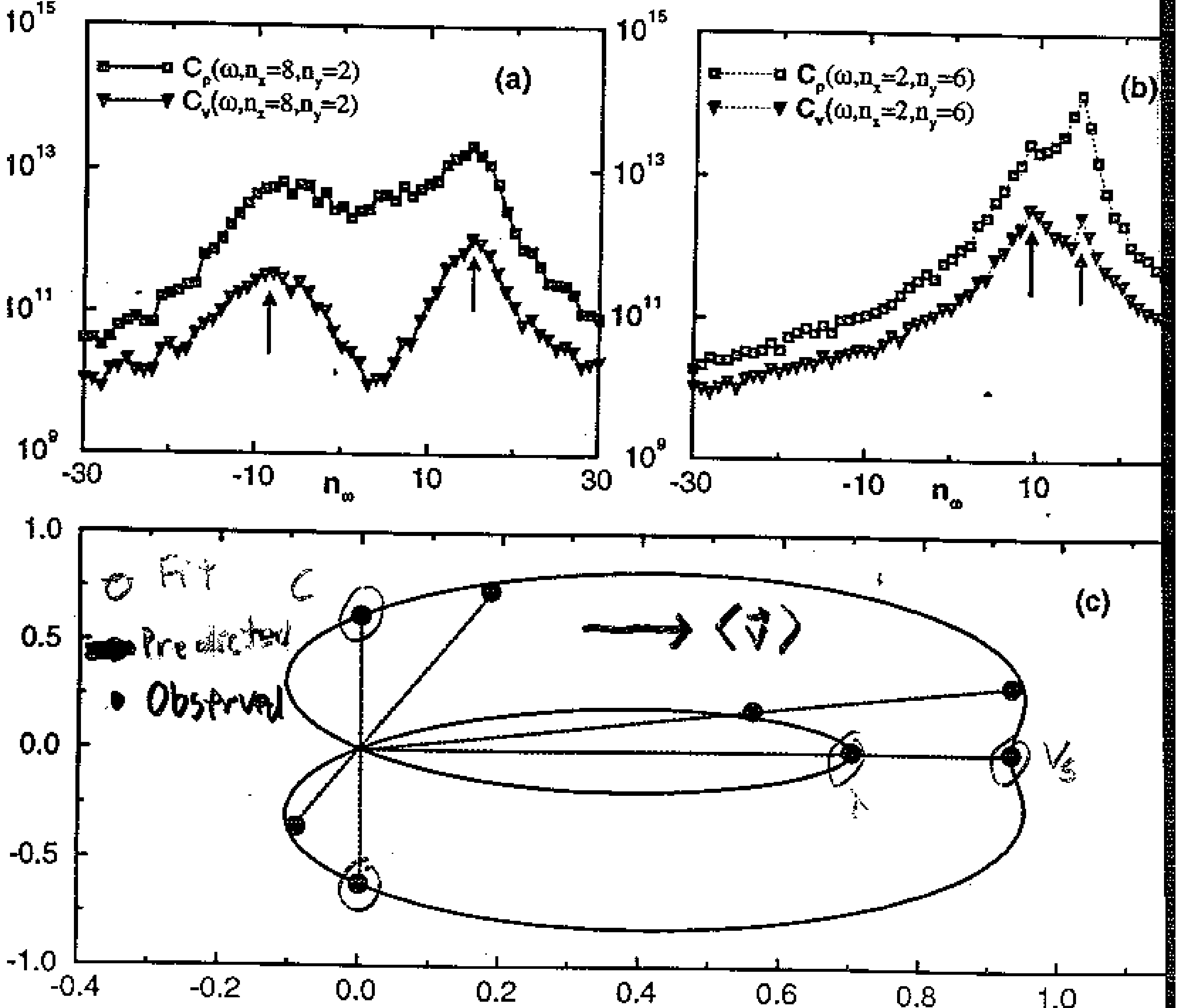
Simulation by Markus Ulm



1) 0 Our (other) Predictions: - Anisotropic sound speeds

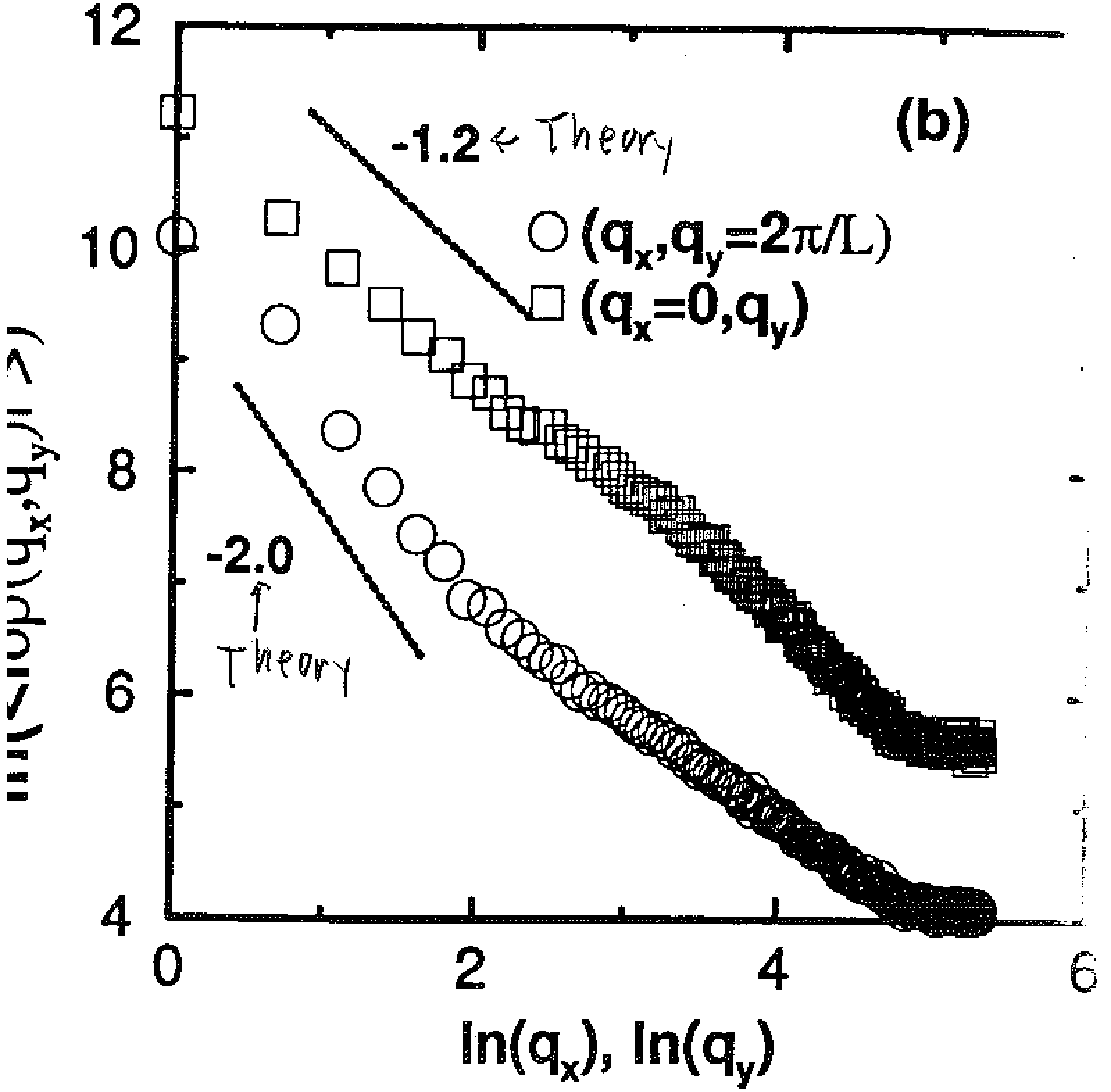
Simple fluid:  - 2 sound modes

Verify in simulations:

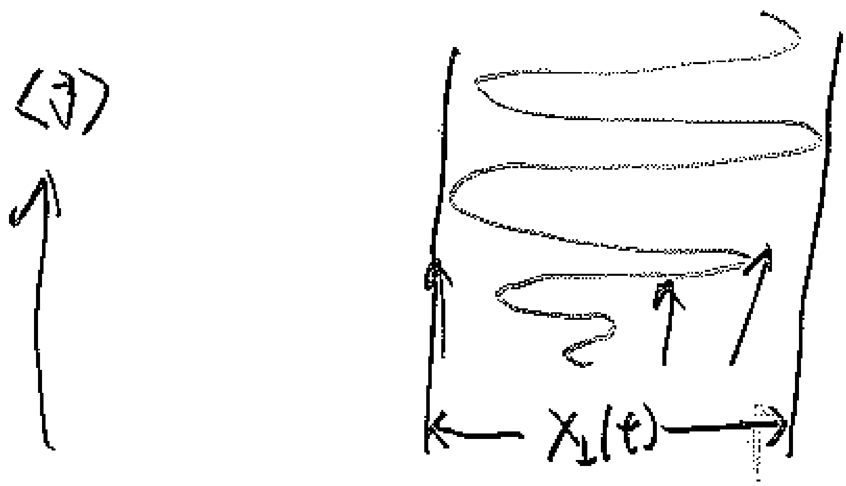


Big, Anisotropic Density Fluctuations:

Simulation results:

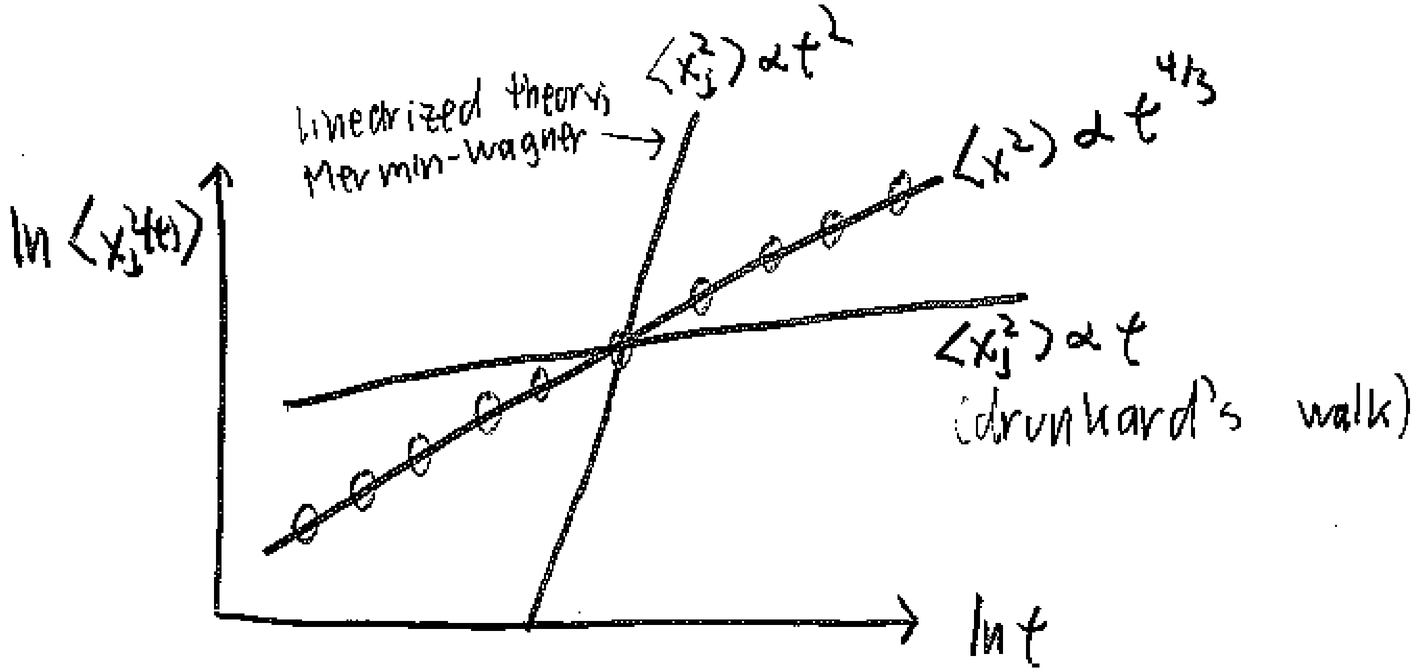


Motion of individual bird in flock:



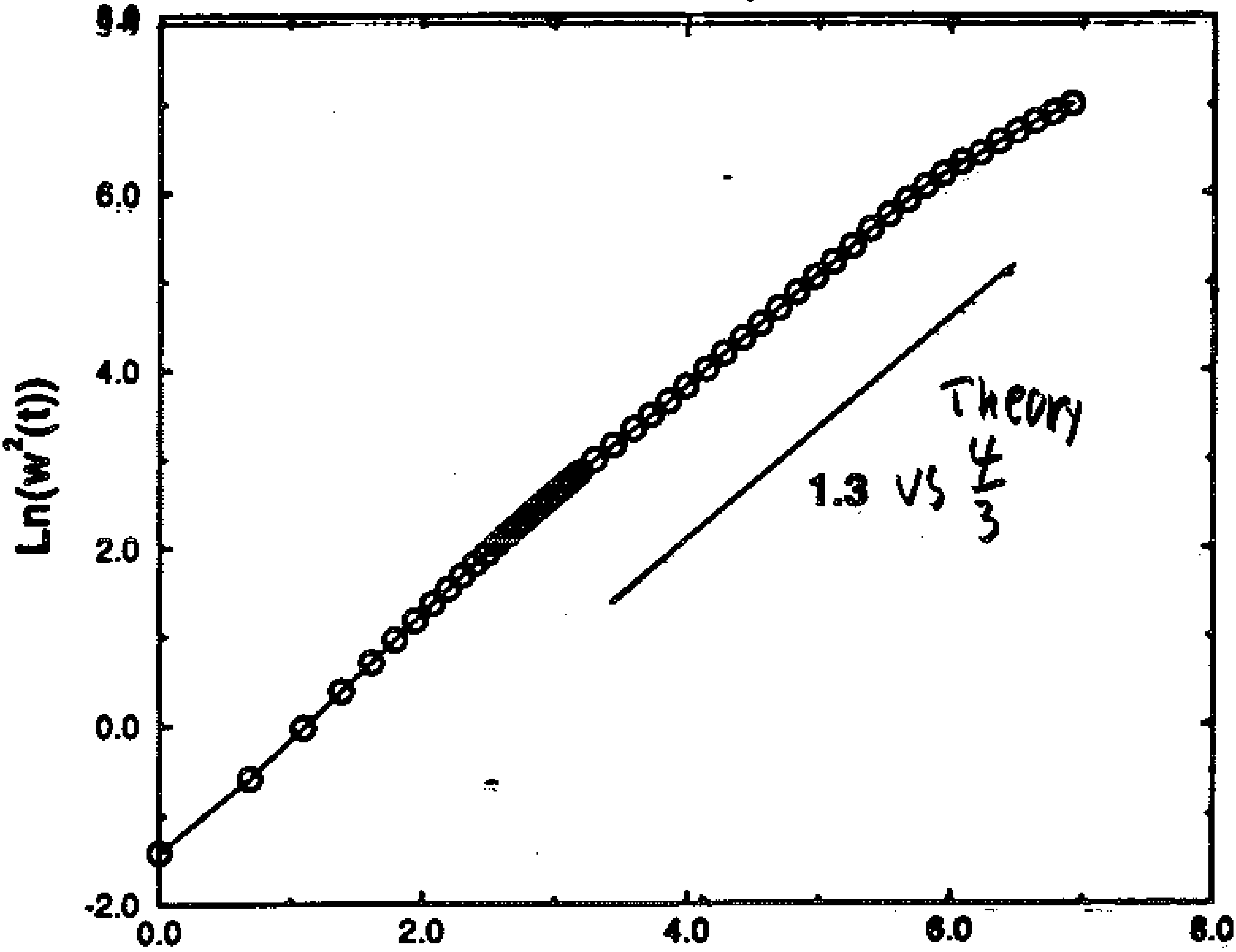
We predict: $\langle x_\perp^2(t) \rangle \propto t^{4/3}$

Simulations by Vim: Viczek algorithm



Same plot, different "microscopic rules" (avoid collisions, etc)
(Oil vs. water)

Universality:



VII) The future:

What we have so far:

1) Powerful analytic theory
2) $\boxed{\text{Numerical}}$ experiments that confirm analytic theory

So what's missing?

Real Experiments.

What sort of experiments?
" " " data?
" kind of analysis?

Basically, repeat our numerical experiments with real critters

E.g.: Video images (microscopy, film, etc.

Data? trajectories $\vec{r}_i(t)$

labels birds

$\Rightarrow$ Fourier transformed density

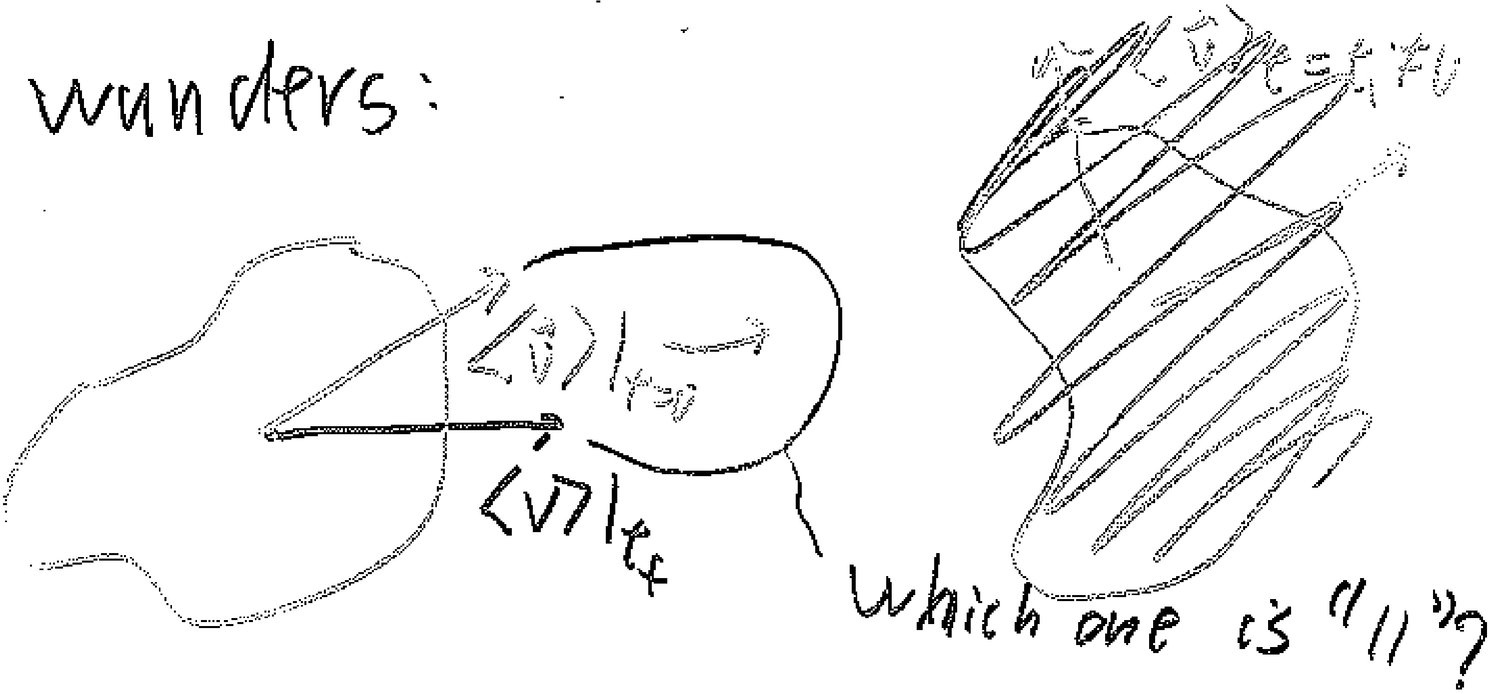
$$\rho(\vec{q},t) = \sum_i e^{i\vec{q}\cdot\vec{r}_i(t)}$$

$$\Downarrow$$

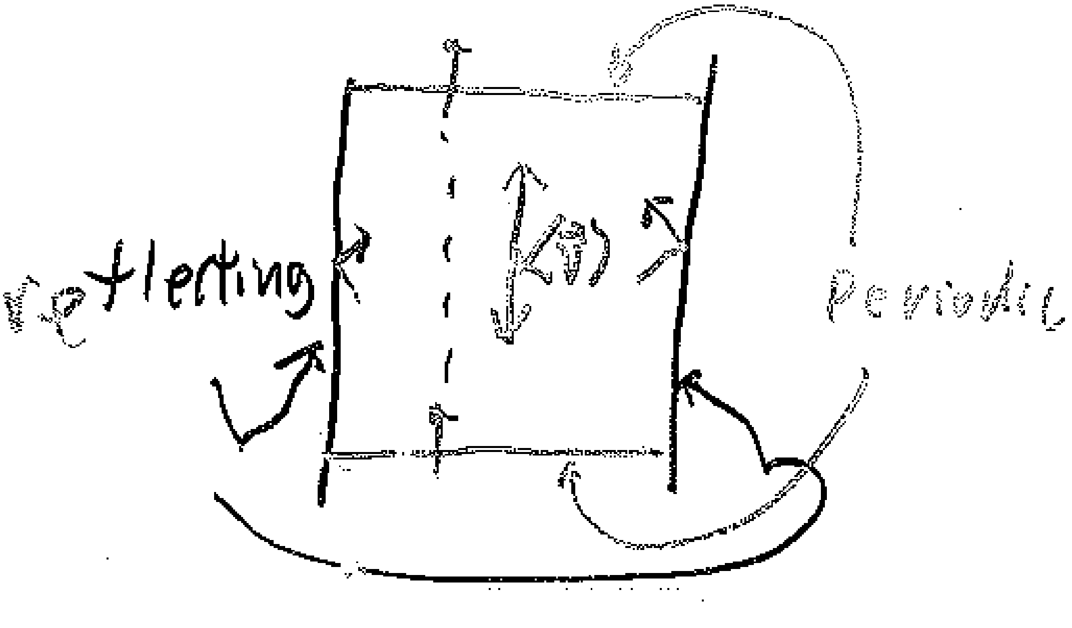
$$\underbrace{\langle |\rho(\vec{q},t)|^2 \rangle}_{\text{time average}} \propto \begin{cases} \dfrac{1}{q_{\parallel}^2}, & \vec{q} \text{ along flock motion} \\ \dfrac{1}{q_{\perp}^{6/5}}, & \vec{q} \perp \text{ flock motion} \end{cases}$$

Caveat: Very important to know which way flock moves.

In free flock, this direction wanders:

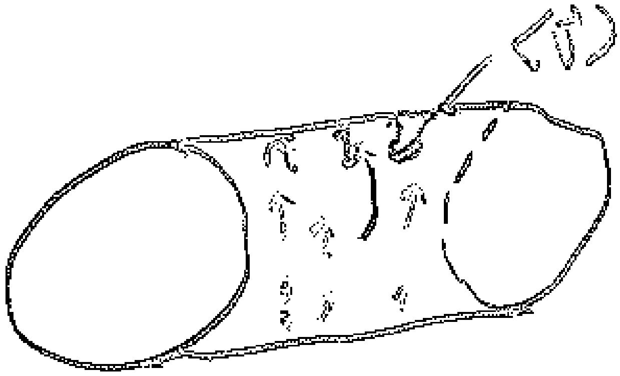


In numerical experiment, we fixed $\parallel$ with boundary conditions:

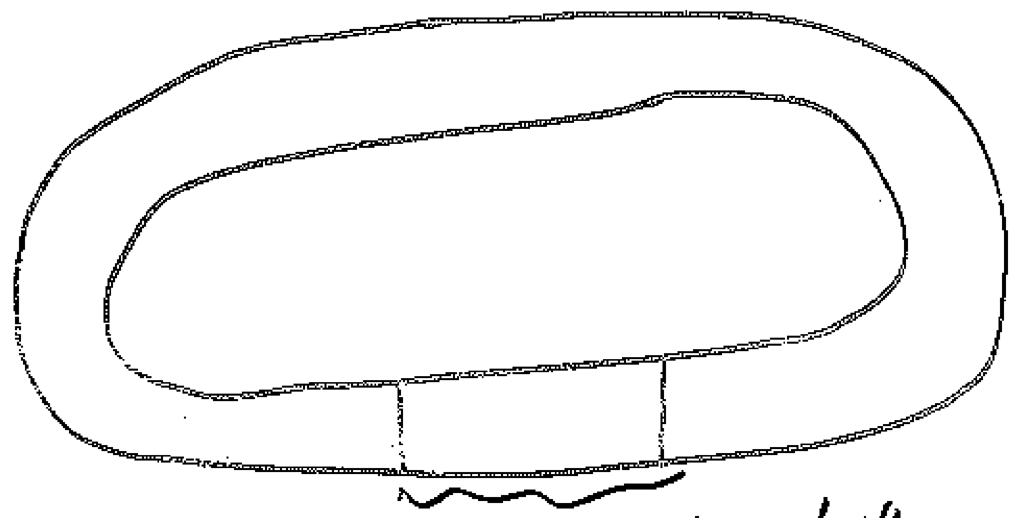


Doing this in real experiment:

1) "Tin can" (cylinder):



2) "Straightaway":



only take data
here.

Other caveats:

1) Need huge herds:

hydrodynamics valid for $L \gg a$

$$\Rightarrow \quad N \sim \left(\frac{L}{a}\right)^d \gg 1$$

---

2) Respect symmetries!

No compasses!

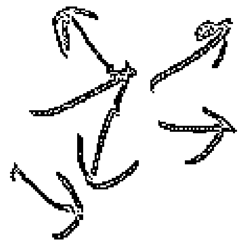
Creatures must spontaneously pick their direction of motion.

Future theoretical directions:

I "Ferromagnetic flocks" are only one possible "phase" of flocks.

<u>Many</u> others possible:

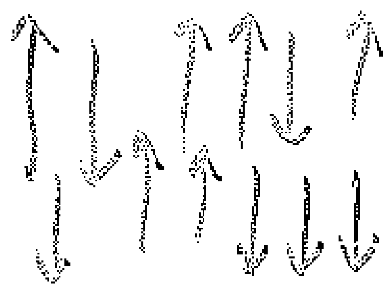
Disordered phase



nematic phase

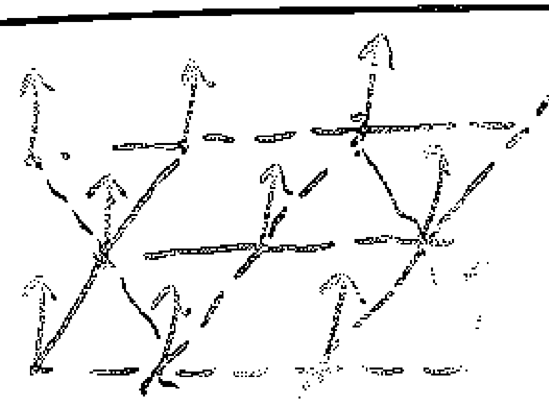
~~[illegible]~~, S. Ramaswamy, JT

Simha



Found experimentally in melanocytes (skin pigment)

"Flying crystal"



Hydrodynamics of background fluid: (Simha + Ramaswamy):

Instability at long wavelengths, low Reynolds number.

May have been seen in experiment

Phase transitions between the phases

Boundaries of flocks: Surface tension? shapes?