Classical and Quantum Gravitational Scattering and the General Relativistic Two-Body Problem

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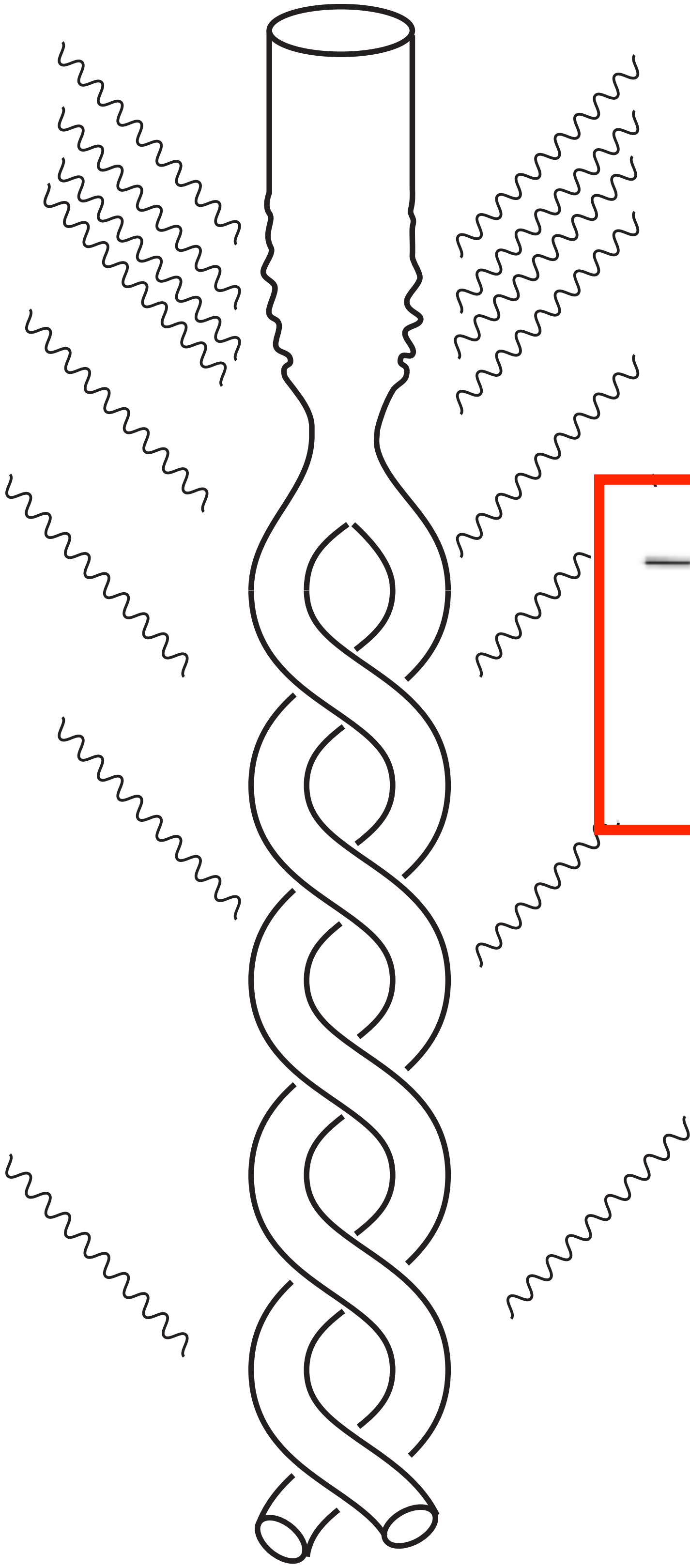
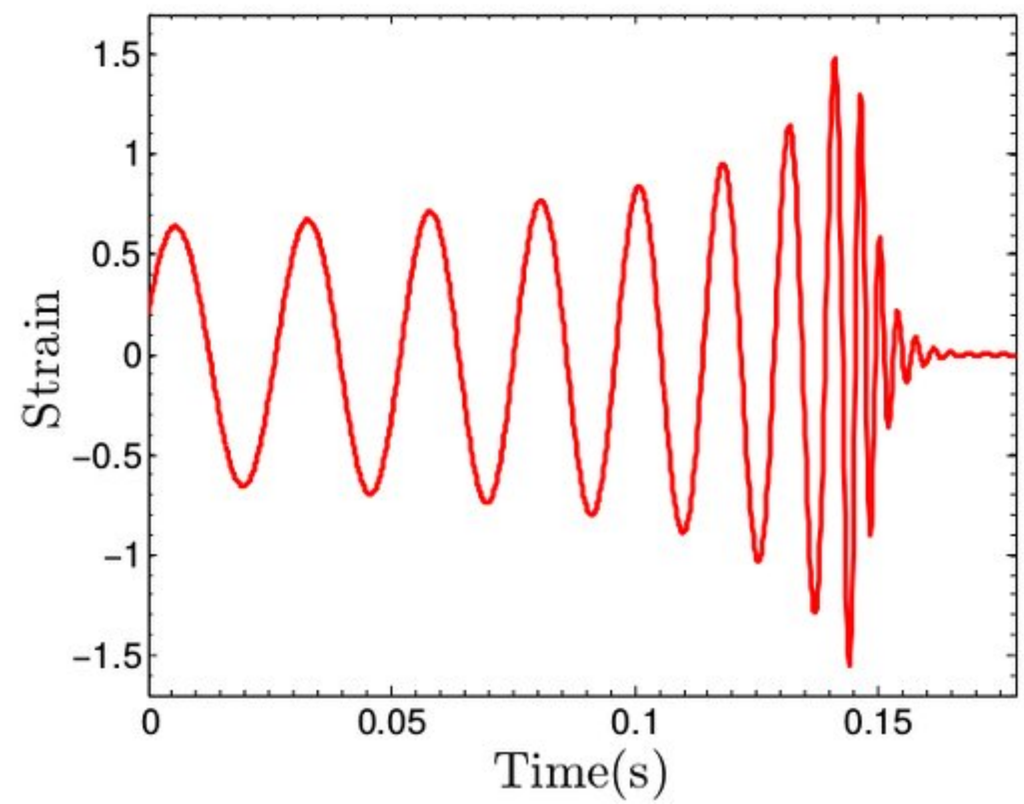
$$ds^2 = g_{\mu\nu}(x^\lambda)\, dx^\mu dx^\nu$$

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$



$$R_{\mu\nu} = 0$$

$$-g^{\mu\nu} g_{\alpha\beta,\,\mu\nu} + g^{\mu\nu} g^{\rho\sigma} \big(g_{\alpha\mu,\,\rho}\, g_{\beta\nu,\,\sigma} - g_{\alpha\mu,\,\rho}\, g_{\beta\sigma,\,\nu} + g_{\alpha\mu,\,\rho}\, g_{\nu\sigma,\,\beta} + g_{\beta\mu,\,\rho}\, g_{\nu\sigma,\,\alpha} - \frac{1}{2} g_{\mu\rho,\,\alpha}\, g_{\nu\sigma,\,\beta}\big) = 0$$



Basic Physico-Mathematical Tools Having Allowed one to Predict GW signals from Coalescing Binary Black Holes

Matched Asymptotic Expansion approach to the motion of strongly self-gravitating bodies -> UV finiteness up to 5PN=5 loops

Post-Newtonian (PN) approximation theory to the motion of binary systems -> fully known up to 4 PN=4 loops; « static » terms computed at 5PN [Foffa et al.19; Bluemlein et al 19]

Post-Minkowskian (PM) (classical) approximation theory (in G) to the motion of binary systems -> known to 2 PM= O(G^2)=1 loop; led to the first derivation of the 2PN dynamics [Damour-Deruelle'81,'82]

PN-matched Multipolar post-Minkowskian (MPM) approximation theory to the GW emission of binary systems -> known up to 3 loops

Effective One-Body (EOB) Approach to coalescing binary black holes (and binary neutron stars) -> known up to 4PN; and recently extended to 5PN = 5 loops [Bini-Damour-Geralico'19]

Gravitational Self-Force (SF) -> compute some terms up to 22 loops

Numerical Relativity (NR) simulations of coalescing binary black holes (and binary neutron stars)

[Quantum amplitude —> classical PM approximation theory
need a dictionary amplitude <—> classical Hamiltonian
need efficient computation of gravitational amplitude
recent claim at 3PM= 2 loops [Bern-Cheung-Roiban-Shen-Solon-Zeng'19]]

MATCHED FILTERING SEARCH AND DATA ANALYSIS

O1: precomputed bank of ~ 200 000 EOB templates for inspiralling and coalescing BBH GW waveforms: m1, m2, chi1=S1/m1^2, chi2=S2/m2^2 for m1+m2> 4Msun; + ~ 50 000 PN inspiralling templates for m1+m2< 4 Msun;
O2: ~ 325 000 EOB templates + 75 000 PN templates

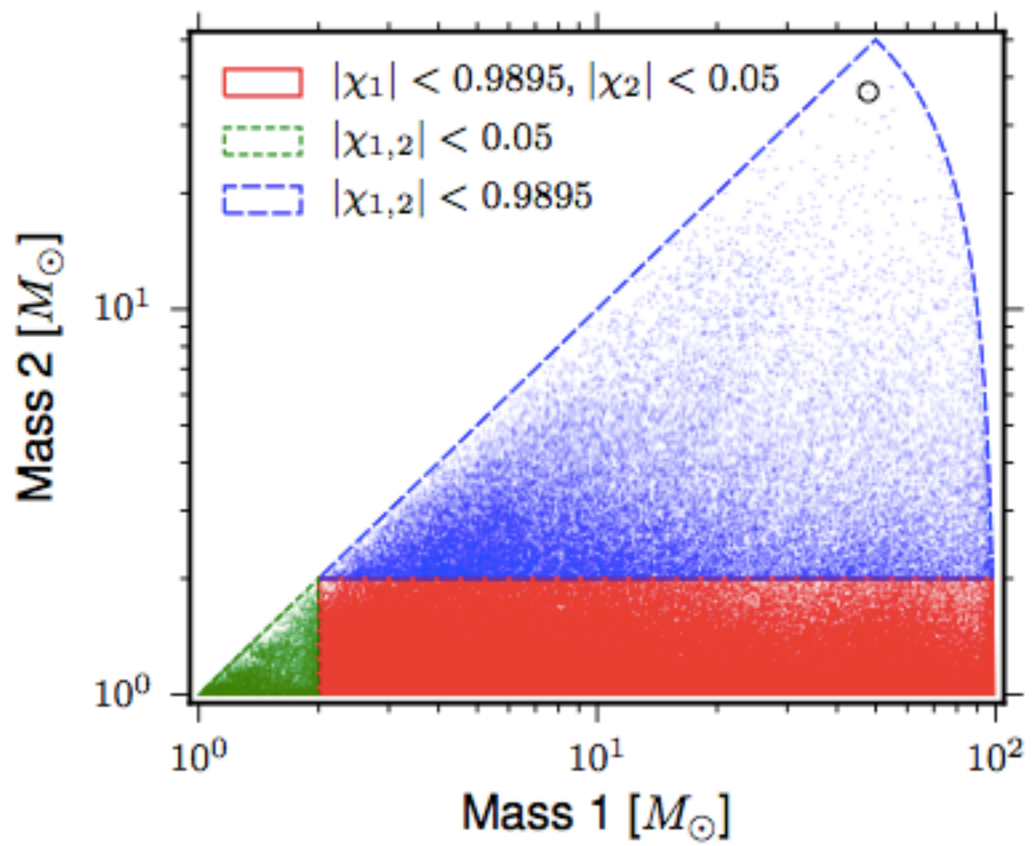
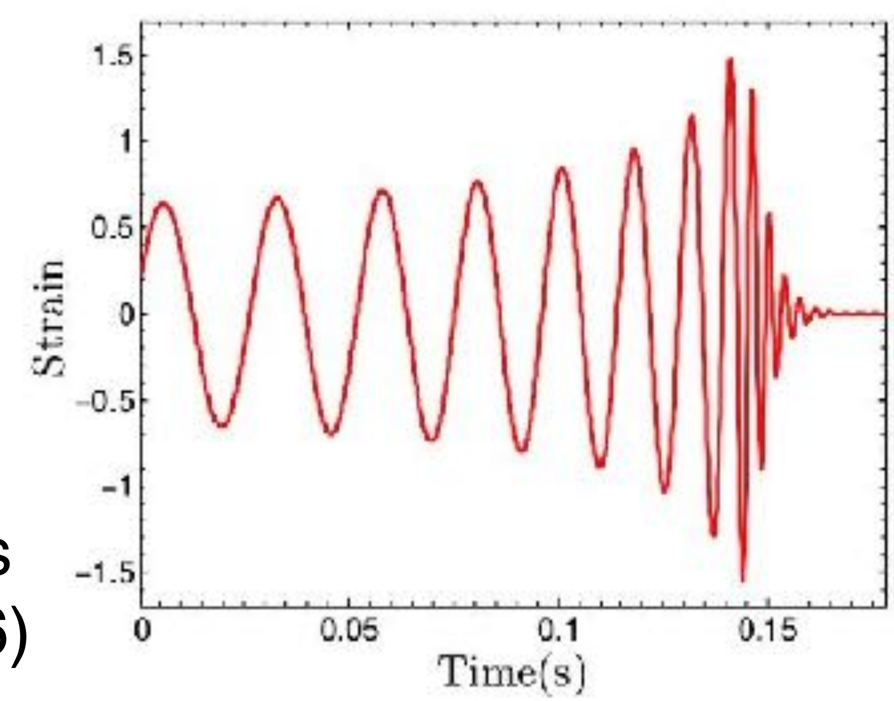


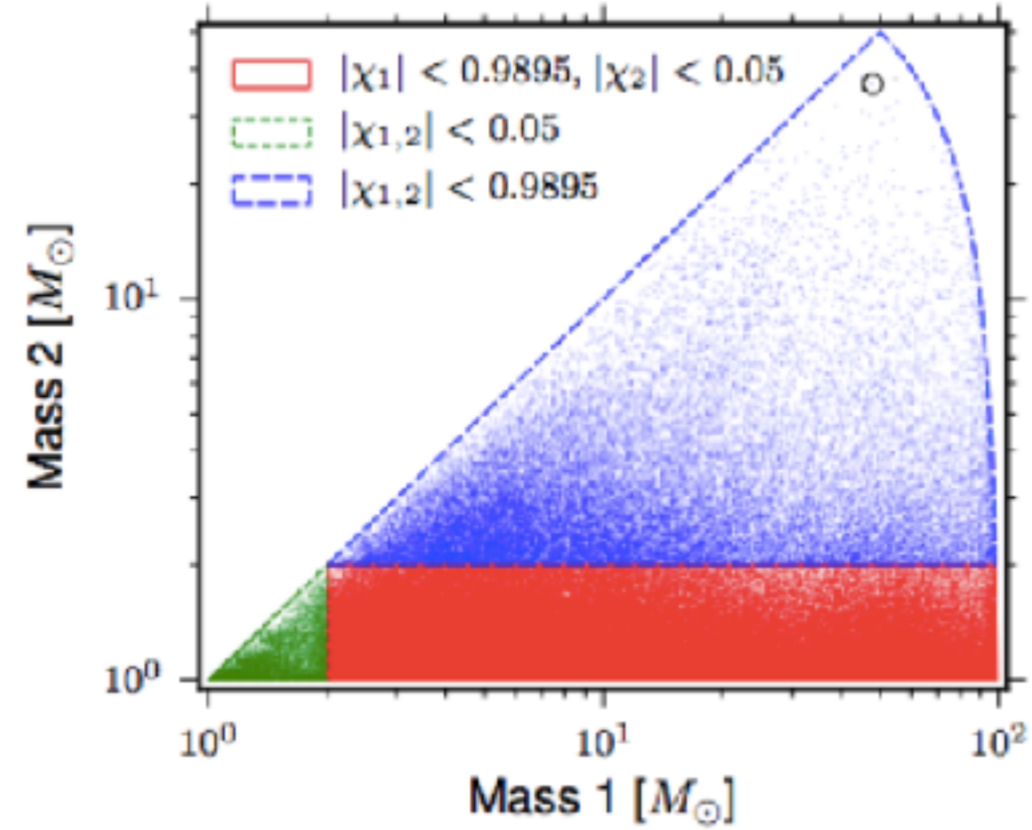
FIG. 1. The four-dimensional search parameter space covered by the template bank shown projected into the component-mass plane, using the convention $m_1 > m_2$. The lines bound mass regions with different limits on the dimensionless aligned-spin parameters χ_1 and χ_2. Each point indicates the position of a template in the bank. The circle highlights the template that best matches GW150914. This does not coincide with the best-fit parameters due to the discrete nature of the template bank.

Search template bank made of spinning EOB[NR] templates (Buonanno-Damour99,Damour'01…,Taracchini et al. 14) in ROM form (Puerrer et al.'14);
Recently improved (Bohé et al '17) by including leading 4PN terms (Bini-Damour '13), spin-dependent terms (Pan-Buonnano et al. '13), and calibrating against 141 NR simulations.
[post-computed NR waveform for GW151226 took three months and 70 000 CPU hours !]



+ auxiliary bank of Phenom[EOB+NR] templates (Ajith…'07, Hannam…'14, Husa…'16, Khan…'16)

LIGO's bank of search templates
O1: 200 000 EOB + 50 000 PN
O2: 325 000 EOB + 75 000 PN
(Taracchini et al.'14, Bohé et al.'17)



PN

$v \ll c$

$R \gg GM/c^2$

EOB

LISA's templates via EOB[SF] ?

PM

$R \gg GM/c^2$

NR

$v \sim c$

$R \sim GM/c^2$

QFT perturbation theory

STRING perturbation theory

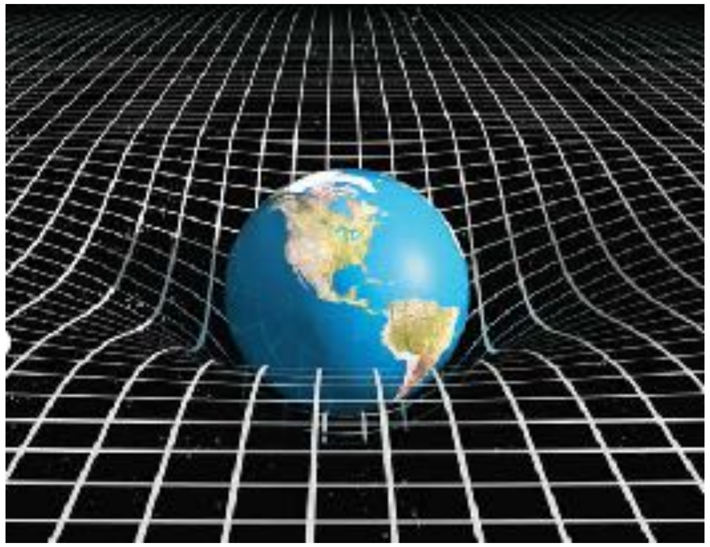
SF

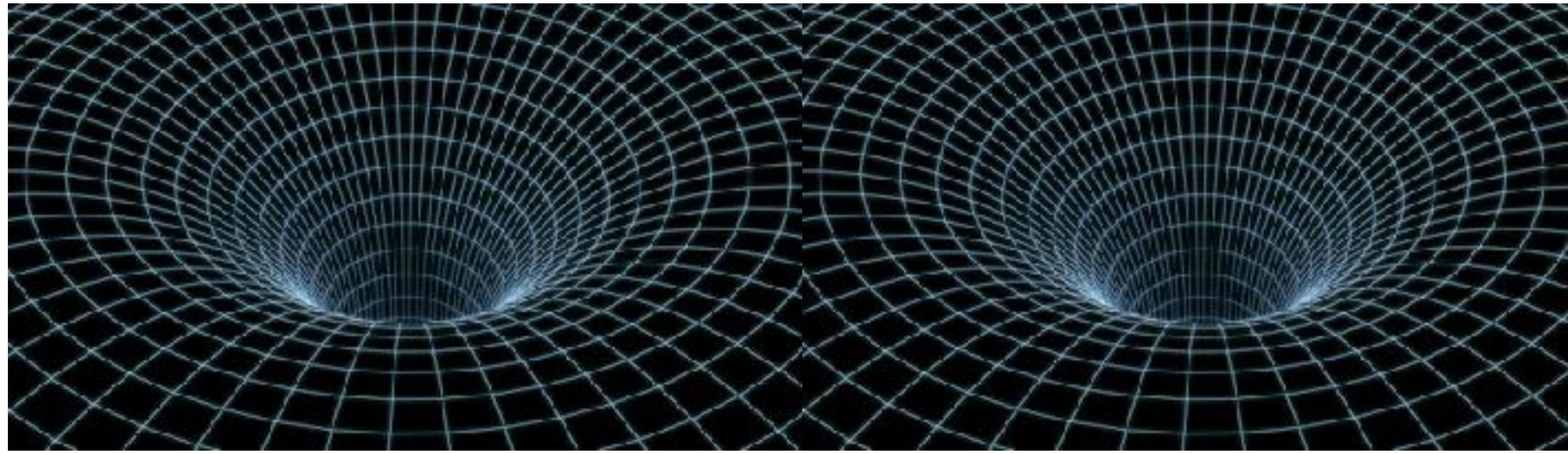
$m_1 \ll m_2$

Quantum Scattering Amplitudes

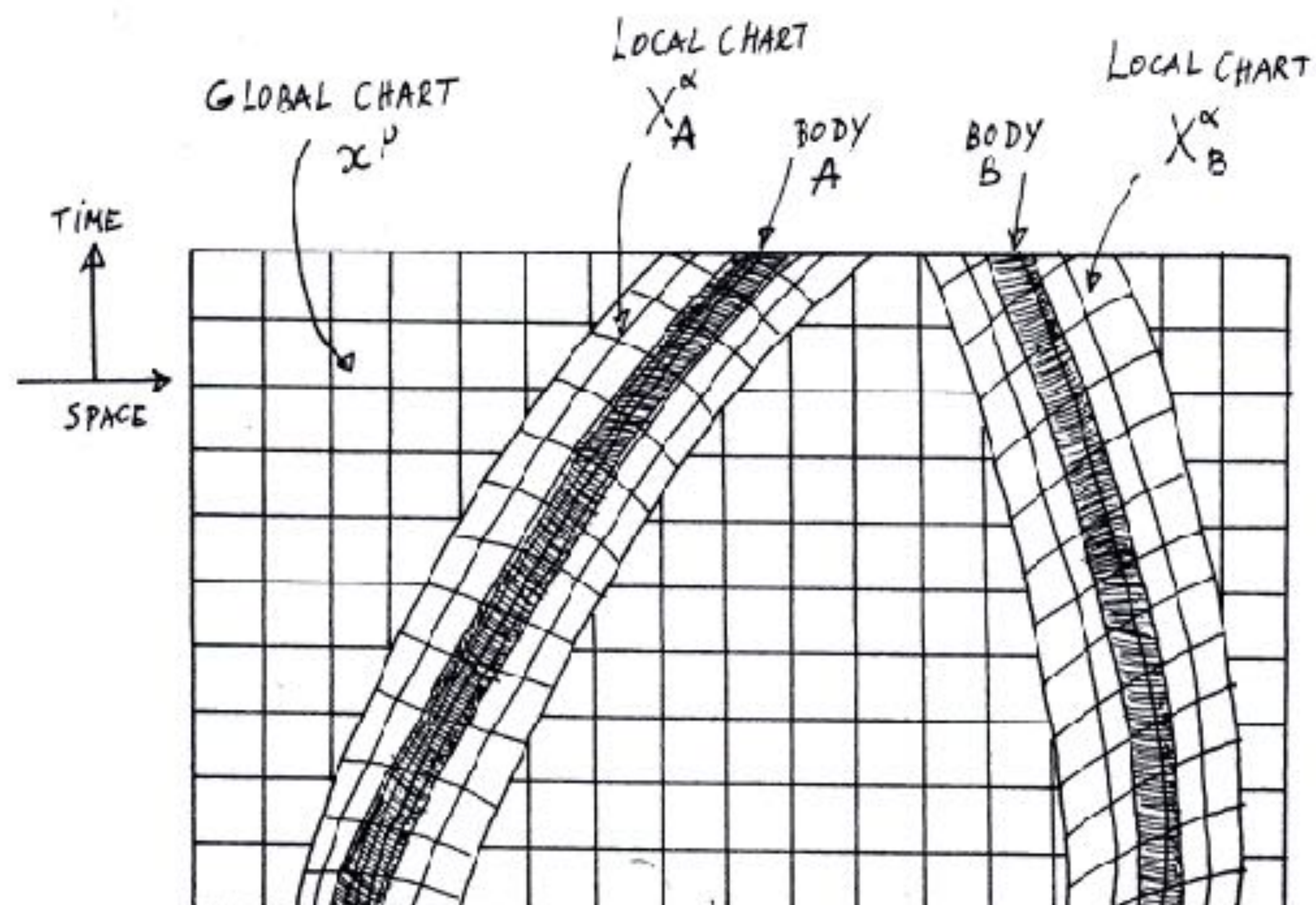
EOB= Effective-One-Body
Buonanno-Damour 1999, 2000;
Damour-Jaranowski-Schaefer 2000;
Damour 2001

Motion of Strongly Self-gravitating Bodies (NS, BH)





Multi-chart approach to motion of strong-self-gravity bodies, and matched asymptotic expansions [EIH ’38], Manasse ’63, Demianski-Grishchuk ’74, D’Eath’75, Kates ’80, Damour ’82



Useful even for weakly self-gravitating bodies, i.e.“relativistic celestial mechanics”, Brumberg-Kopeikin ’89, Damour-Soffel-Xu ‘91-94

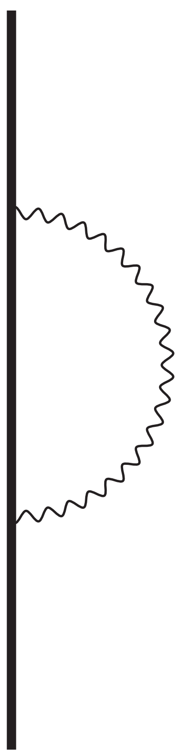
Combine two expansions in two charts:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + G h^{(1)}_{\mu\nu}(x) + G^2 h^{(2)}_{\mu\nu}(x) + \dots \quad G_{\alpha\beta}(x) = G^{(0)}_{\alpha\beta}(x) + G^{(1)}_{\alpha\beta}(x) + \cdots$$

Practical Techniques for Computing the Motion of Compact Bodies (NS or BH)

Skeletonization : $T_{\mu\nu} \longrightarrow$ point-masses (Mathisson '31)

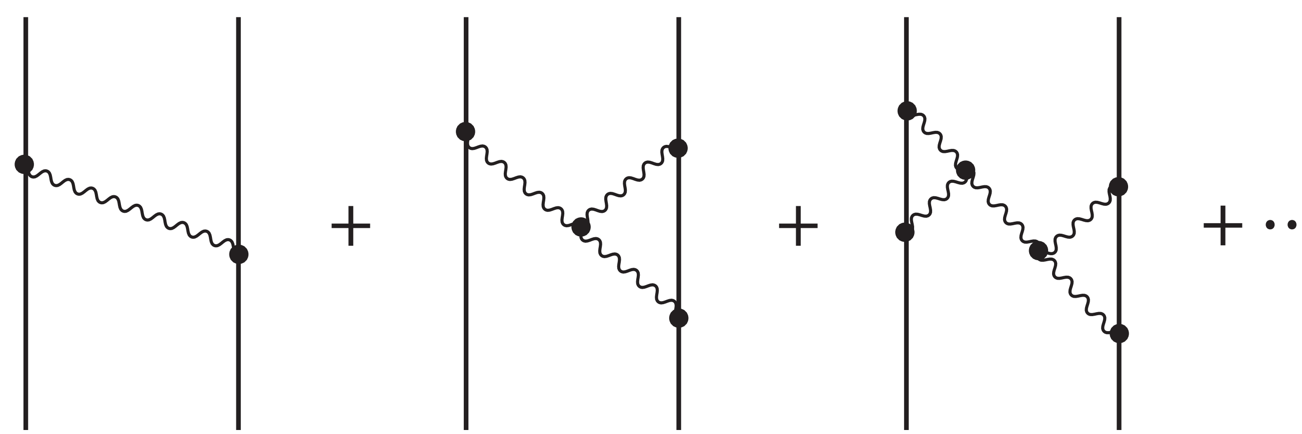
delta-functions in GR : Infeld '54, Infeld-Plebanski '60



justified by Matched Asymptotic Expansions (« Effacing Principle » Damour '83 possible internal-structure dependence in strong self-gravity objects (NSs, BHs) only arise at 5PN= 5-loop level)

UV divergences linked to self-field effects (loops on external lines) [Dirac, 1938]

QFT's analytic (Riesz '49) or dimensional regularization (Bollini-Giambiagi '72, t'Hooft-Veltman '72) imported in GR (Damour '80, Damour-Jaranowski-Schäfer '01, …)



Feynman-like diagrams and

« Effective Field Theory » techniques

Bertotti-Plebanski '60,

Damour-Esposito-Farèse '96,

Goldberger-Rothstein '06, Porto '06, Gilmore-Ross' 08, Levi '10,

Foffa-Sturani '11 '13, Levi-Steinhoff '14, '15

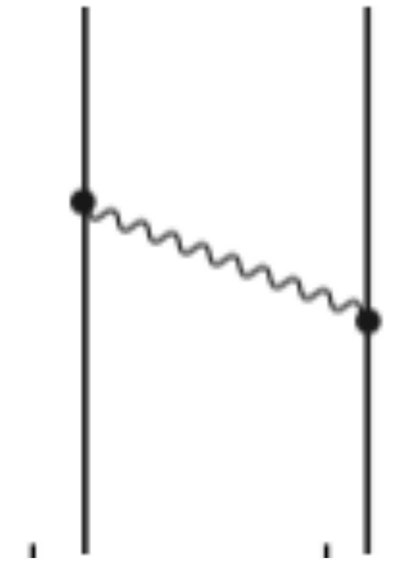
Reduced Worldline Action in Electrodynamics (Fokker 1929)

$$S_{\rm tot}[x_a^\mu, A_\mu] = -\sum_a \int m_a ds_a + \sum_a \int e_a dx_a^\mu A_\mu(x_a) - \int d^D x \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + S_{\rm gf}$$

« Integrate out » the field A_mu in the total (particle+field) action

$$S_{\rm eff}^{\rm class}[x_a(s_a)] = -\sum_a m_a \int {\rm d}s_a + \frac{1}{2} \sum_{a,b} e_a e_b \iint {\rm d}x_a^\mu \, {\rm d}x_{b\mu} \, \delta\left((x_a - x_b)^2\right).$$

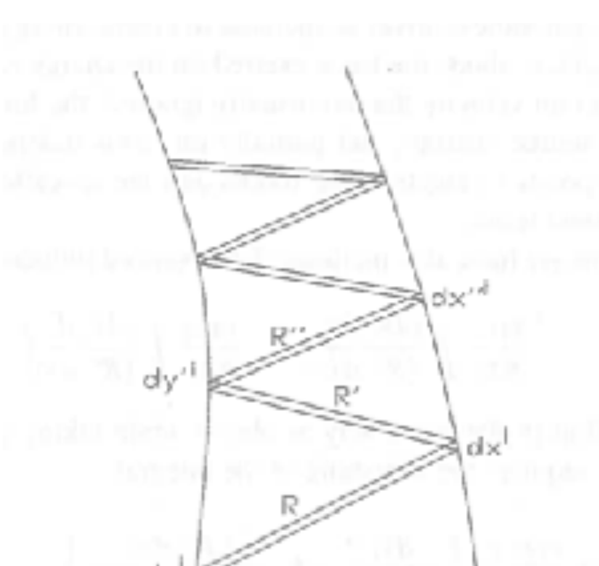
One-photon-exchange diagram

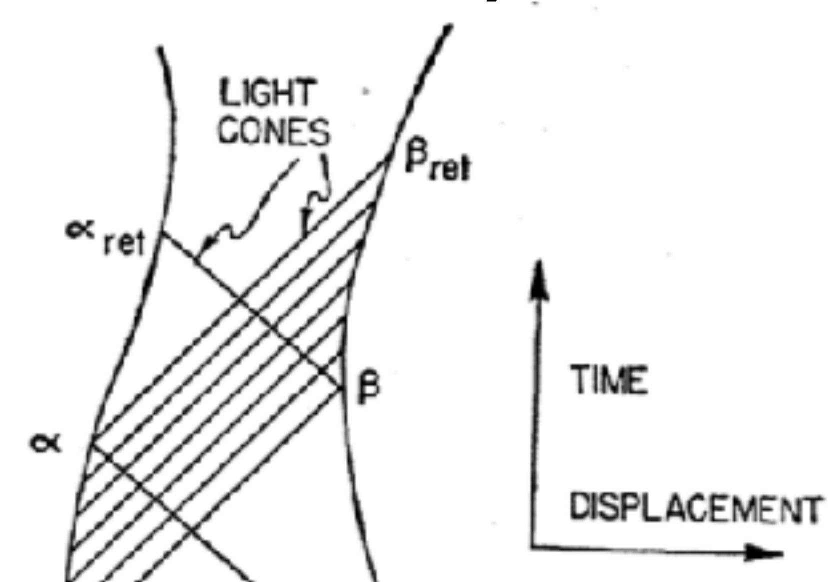


time-symmetric Green function G.

$$G(x) = \delta(-\eta_{\mu\nu} x^\mu x^\nu) = \frac{1}{2r}\left(\delta(t-r) + \delta(t+r)\right) \; ; \; \Box G(x) = -4\pi\delta^4(x)$$

The effective action S_eff(x_a) was heavily used in the (second) Wheeler-Feynman paper (1949) together with similar diagrams to those used by Fokker





Reduced Action in Gravity and its Diagrammatic Expansion

Damour-Esposito-Farese ‘96

$$S_{\rm eff}^{\rm class}[x_a(s_a)] = [S_{\rm pm} + S_{\rm EH} + S_{\rm gf}]_{g_{\mu\nu}(x)\to g^{\rm gf}_{\mu\nu}[x_a(s_a)]}$$

Needs gauge-fixed* action and time-symmetric Green function G.
*E.g. Arnowitt-Deser-Misner Hamiltonian formalism or harmonic coordinates.
Perturbatively solving (in dimension D=4 + eps) Einstein’s equations
to get the equations of motion and the action for the conservative dynamics

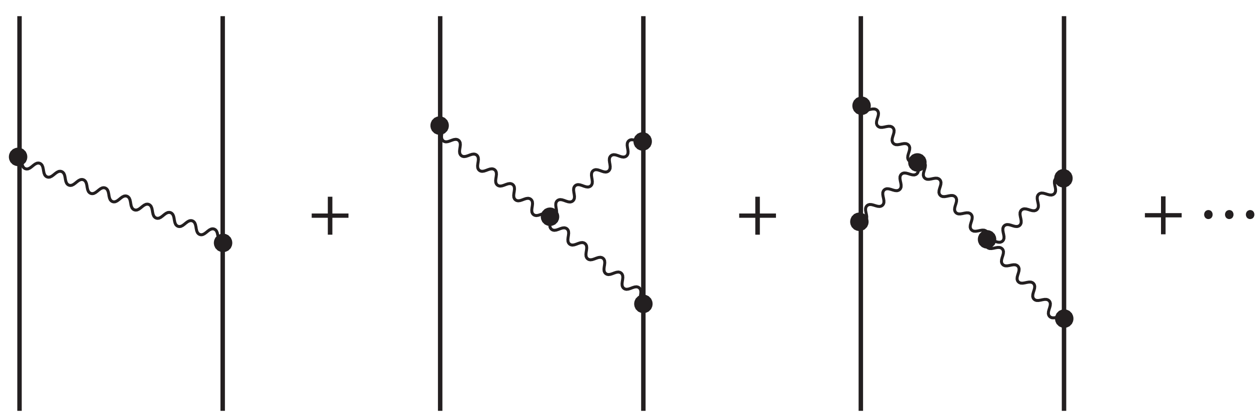
$$g = \eta + h$$

$$S(h,T) = \int \left(\frac{1}{2} h \Box h + \partial\partial hhh + \ldots + (h + hh + \ldots) T\right)$$

$$\Box h = -T + \ldots \to h = G\,T + \ldots$$

$$S_{\rm red}(T) = \frac{1}{2} T\,G\,T + V_3(G\,T, G\,T, G\,T) + \ldots$$

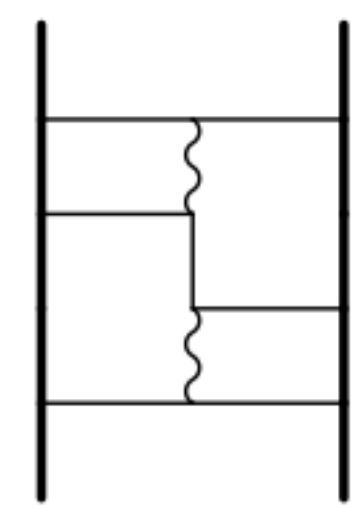
time-symmetric Green function G



O(G)= 1PM=
Newtonian
+ (v/c)^n corrections

O(G^2)=2PM
=1 loop -> 1PN

O(G^3)=3PM
=2 loop-> 2PN



O(G^5)=5PM
=4 loop-> 4PN

Alternative (Equivalent) Computation of Reduced Action

Instead of on-shell replacing the field dofs

$$S_{\rm eff}^{\rm class}[x_a(s_a)] = [S_{\rm pm} + S_{\rm EH} + S_{\rm gf}]_{g_{\mu\nu}(x) \to g_{\mu\nu}^{\rm gf}[x_a(s_a)]}$$

Formal functional integral over the field (QED: Feynman '50; …;
GR: « Effective Field Theory » approach
(Goldberger-Rothstein '06, Porto '06, Gilmore-Ross' 08, Levi '10, Foffa-Sturani '11 '13, Levi-Steinhoff '14, '15; Foffa-Mastrolia-Sturani-Sturm'16, Damour-Jaranowski '17)

$$e^{\frac{i}{\hbar} S_{\rm eff}^{\rm quant}} = \int D g_{\mu\nu}\, e^{\frac{i}{\hbar}(S_{\rm pm} + S_{\rm EH} + S_{\rm gf})}.$$

Saddle-point estimation: $S_{\rm eff}^{\rm quant}[x_a(s_a)] = S_{\rm eff}^{\rm class}[x_a(s_a)] + O(\hbar).$

However, the explicit computations are done differently: by means of Wick's theorem, and p-space integrations

$$e^{\frac{i}{\hbar} S_{\rm eff}} = \int D\varphi\, e^{\frac{i}{\hbar}(\int[\frac{1}{2}\varphi\mathcal{K}\varphi + \varphi s + g\varphi^3 + \cdots]}.$$

$$\int D\varphi\, e^{\frac{i}{\hbar}\int[\frac{1}{2}\varphi\mathcal{K}\varphi]} \sum_n \frac{(i/\hbar)^n}{n!} \left(\int(\varphi s + g\varphi^3 + \cdots)\right)^n$$

$$\langle\varphi(x)\varphi(y)\rangle = \int D\varphi\, e^{\frac{i}{\hbar}\int[\frac{1}{2}\varphi\mathcal{K}\varphi]}\varphi(x)\varphi(y) = i\hbar\mathcal{K}^{-1}_{x,y},$$

Slow-Motion (PN) computation of the Reduced Gravity Action

Beyond 1-loop order efficient to use PN-expanded Green function for explicit computations.

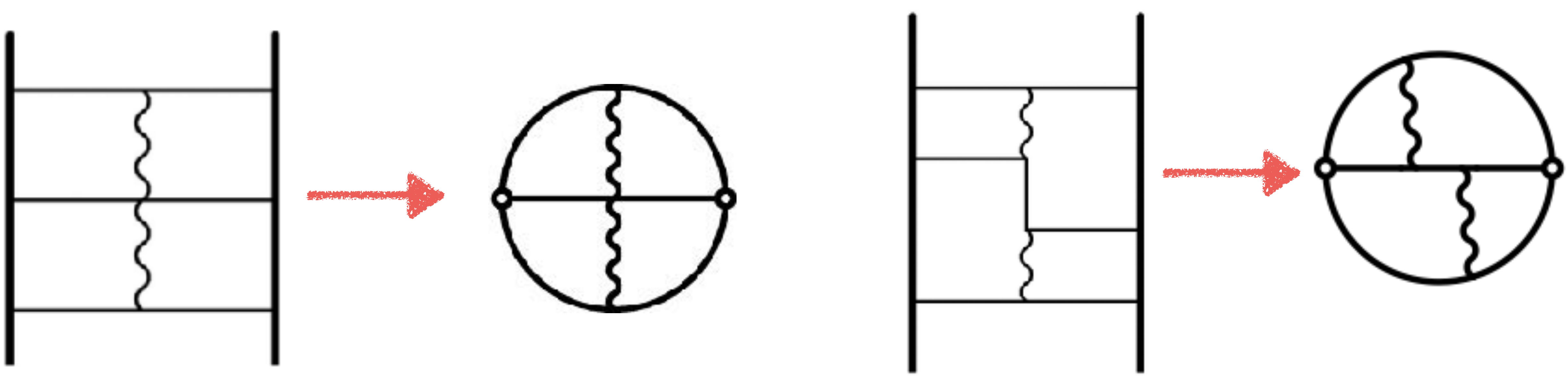
$$\Box^{-1} = (\Delta - \frac{1}{c^2}\partial_t^2)^{-1} = \Delta^{-1} + \frac{1}{c^2}\partial_t^2\Delta^{-2} + \ldots$$

PN expansion: in powers of 1/c^2: 1PN= (v/c)^2; 2PN= (v/c)^4, etc nPN=(v/c)^(2n)

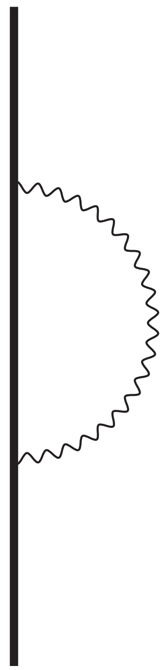
$$G = \delta(-\eta_{\mu\nu}x^\mu x^\nu) \sim \mathrm{PP}\left(\frac{4\pi}{p^2}\right) \to \frac{\delta(x^0)}{|\mathbf{x}|} + \cdots \sim \frac{4\pi\delta(x^0)}{\mathbf{p}^2} + \cdots$$

This transforms spacetime diagrams (between two worldlines) into (massless) two-point space diagrams (in three dimensions)

E.g. at 4PN, some 4-loop space diagrams ~ G^5 m_1^3 m_2^3 among 515 4PN-level diagrams (Damour-Jaranowski-Schaefer '14, Bernard et al '16, Foffa et al '17)



Regularization and Renormalization



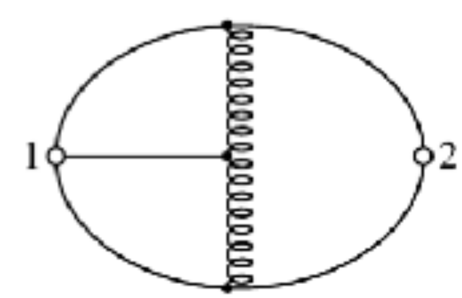
Like in Dirac 1938, the use of point-particles (delta functions) introduces UV divergences (linked to self-field effects).

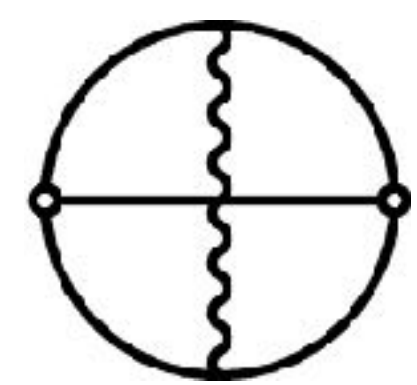
It has been shown (« Effacing property » Damour '83) that possible internal-structure dependence in strong self-gravity objects (NSs, BHs) only arise at 5PN= 5-loop level

Below 5PN (5-loop) point-masses in GR are renormalizable
(Damour '83, Goldberger-Rothstein '06)

One still needs a regularization method that respects gravity's gauge invariance:
dimensional regularization D=4+ epsilon ('t Hooft-Veltman '72)

It has been explicitly shown that S_eff was UV finite
(in ADM gauge, and in dim.reg.) at
3 loops (Damour-Jaranowski-Schäfer '01) and
4 loops (Damour-Jaranowski-Schäfer '14, Jaranowski-Schäfer '15)
There appear UV (1/epsilon) divergences in harmonic gauge
starting at 3PN that can be renormalized away
(Blanchet-Damour-EspositoFarese'04,…,Foffa-Porto-Rothstein-Sturani'19)





There appear IR divergences at 4PN (4 loop) linked to non-locality (Blanchet-Damour '88).

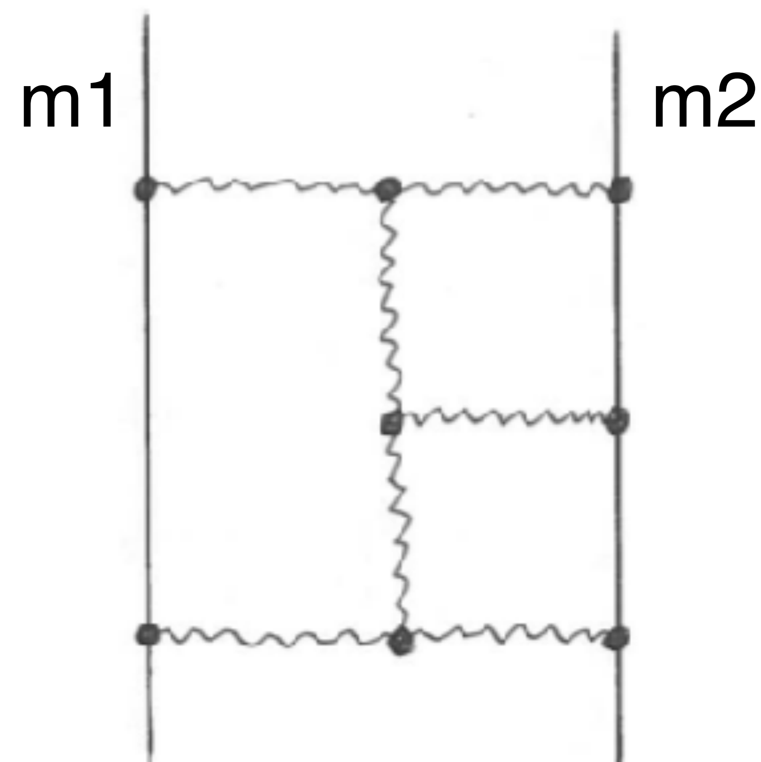
Nonlocal-in-time (IR divergent) 4PM and 4PN contribution

Blanchet-Damour'88: nonlocality in dynamics arises at 4PN via tail term
Damour-Jaranowski-Schaefer'14: Nonlocal contribution to 4PN reduced action

$$H_{4PN}^{\rm nonloc}(t) = -\frac{1}{5}\frac{G^2 M}{c^8} I_{ij}^{(3)}(t) \times \mathrm{Pf}_{2r_{12}/c} \int_{-\infty}^{+\infty} \frac{dv}{|v|} I_{ij}^{(3)}(t+v), \quad + C\, \frac{2}{5}\frac{G^2 M}{c^8} (I_{ij}^{(3)})^2.$$

Constant C obtained by matching the PN-ambiguous near-zone result to a well-defined computation using a global Green function valid both in near-zone and wave-zone (Bini-Damour'13)

$$C = -\frac{1681}{1536}.$$



4PM (3-loop) diagrams leading to problematic 4PN ones are O(G^4 m1^2 m2^3) = 1SF

Bernard et al.'18 and Foffa et al'19 claim to recover our result without explicit near-zone-wave-zone matching

Post-Newtonian Equations of Motion [2-body, wo spins]

- 1PN (including v^2/c^2) [Lorentz-Droste '17], Einstein-Infeld-Hoffmann '38

- 2PN (inc. v^4/c^4) Ohta-Okamura-Kimura-Hiida '74, Damour-Deruelle '81
  Damour '82, Schäfer '85, Kopeikin '85

- 2.5 PN (inc. v^5/c^5) Damour-Deruelle '81, Damour '82, Schäfer '85,
  Kopeikin '85

- 3 PN (inc. v^6/c^6) Jaranowski-Schäfer '98, Blanchet-Faye '00,
  Damour-Jaranowski-Schäfer '01, Itoh-Futamase '03,
  Blanchet-Damour-Esposito-Farèse' 04, Foffa-Sturani '11

- 3.5 PN (inc. v^7/c^7) Iyer-Will '93, Jaranowski-Schäfer '97, Pati-Will '02,
  Königsdörffer-Faye-Schäfer '03, Nissanke-Blanchet '05, Itoh '09

- 4PN (inc. v^8/c^8) Jaranowski-Schäfer '13, Foffa-Sturani '13,'16
  Bini-Damour '13, Damour-Jaranowski-Schäfer '14, Bernard et al'16-18, Foffa et al.'19)
  New feature : non-locality in time (Blanchet-Damour'88)

- Static two-body potential in harmonic coords at 5PN (Foffa-Mastrolia-Sturani-Sturm-TorresBobadilla'19, Bluemlein-Maier-Marquard'19)

Inclusion of spin-dependent effects: Barker-O' Connell'75, Faye-Blanchet-Buonanno'06, Damour-Jaranowski-Schaefer'08, Porto-Rothstein '06, Levi '10, Steinhoff-Hergt-Schaefer '10, Steinhoff'11, Levi-Steinhoff'15-18

2-body Taylor-expanded N + 1PN + 2PN Hamiltonian

$$H_{\mathrm{N}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{1}{2}\frac{Gm_1m_2}{r_{12}} + (1 \leftrightarrow 2)$$

$$c^2 H_{1\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(-12\frac{\mathbf{p}_1^2}{m_1^2} + 14\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + 2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right) + \frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}} + (1 \leftrightarrow 2),$$

$$c^4 H_{2\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(5\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} - \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + 5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} - 6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} - \frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right) + \frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2} + 19\frac{\mathbf{p}_2^2}{m_2^2}\right) - \frac{1}{2}(m_1+m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2) + 6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right) - \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2 + 5m_1m_2 + m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2),$$

2-body Taylor-expanded 3PN Hamiltonian [DJS 01]

$$
\begin{aligned}
c^6 H_{3\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = & -\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32}\frac{G m_1 m_2}{r_{12}}\left(-14\frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 4\mathbf{p}_1^2\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4 m_2^2} + 6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4 m_2^2}\right.\\
& - 10\frac{(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2 + \mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4 m_2^2} + 24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4 m_2^2}\\
& + 2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3 m_2^3} + \frac{(7\mathbf{p}_1^2\mathbf{p}_2^2 - 10(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2^3}\\
& + \frac{(\mathbf{p}_1^2\mathbf{p}_2^2 - 2(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2^3} + 15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3 m_2^3}\\
& \left. - 18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3 m_2^3} + 5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3 m_2^3}\right) + \frac{G^2 m_1 m_2}{r_{12}^2}\left(\frac{1}{16}(m_1 - 27 m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}\right.\\
& - \frac{115}{16} m_1 \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2} - \frac{1}{48} m_2 \frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 371\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2} + \frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3} + \frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}\\
& - \frac{1}{8} m_1 \frac{(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2) + 11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3 m_2} - \frac{3}{2} m_1 \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2}\\
& + \frac{125}{12} m_2 \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2 m_2^2} + \frac{10}{3} m_2 \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\\
& \left. - \frac{1}{48}(220 m_1 + 193 m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\right) + \frac{G^3 m_1 m_2}{r_{12}^3}\left(-\frac{1}{48}\left(425 m_1^2 + \left(473 - \frac{3}{4}\pi^2\right) m_1 m_2 + 150 m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\right.\\
& + \frac{1}{16}\left(77(m_1^2 + m_2^2) + \left(143 - \frac{1}{4}\pi^2\right) m_1 m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \frac{1}{16}\left(20 m_1^2 - \left(43 + \frac{3}{4}\pi^2\right) m_1 m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}\\
& \left. + \frac{1}{16}\left(21(m_1^2 + m_2^2) + \left(119 + \frac{3}{4}\pi^2\right) m_1 m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\right)\\
& + \frac{1}{8}\frac{G^4 m_1 m_2^3}{r_{12}^4}\left(\left(\frac{227}{3} - \frac{21}{4}\pi^2\right) m_1 + m_2\right) + (1 \leftrightarrow 2).
\end{aligned}
$$

2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014]

$$
\begin{aligned}
c^8 H^{\rm local}_{\rm 4PN}(\mathbf{x}_a,\mathbf{p}_a) &= \frac{7(\mathbf{p}_1^2)^5}{256 m_1^9} + \frac{G m_1 m_2}{r_{12}} H_{48}(\mathbf{x}_a,\mathbf{p}_a) + \frac{G^2 m_1 m_2}{r_{12}^2} m_1 H_{46}(\mathbf{x}_a,\mathbf{p}_a) \\
&+ \frac{G^3 m_1 m_2}{r_{12}^3}\left(m_1^2 H_{441}(\mathbf{x}_a,\mathbf{p}_a) + m_1 m_2 H_{442}(\mathbf{x}_a,\mathbf{p}_a)\right) \\
&+ \frac{G^4 m_1 m_2}{r_{12}^4}\left(m_1^3 H_{421}(\mathbf{x}_a,\mathbf{p}_a) + m_1^2 m_2 H_{422}(\mathbf{x}_a,\mathbf{p}_a)\right) \\
&+ \frac{G^5 m_1 m_2}{r_{12}^5} H_{40}(\mathbf{x}_a,\mathbf{p}_a) + (1 \leftrightarrow 2), \qquad \text{(A3)}
\end{aligned}
$$

$$
\begin{aligned}
H_{48}(\mathbf{x}_a,\mathbf{p}_a) &= \frac{45(\mathbf{p}_1^2)^4}{128 m_1^8} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64 m_1^6 m_2^2} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16 m_1^6 m_2^2} \\
&- \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{35(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{256 m_1^5 m_2^3} \\
&+ \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{128 m_1^5 m_2^3} + \frac{33(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256 m_1^5 m_2^3} - \frac{85(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3} \\
&- \frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2^3} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3} + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3} \\
&+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3} - \frac{3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3} + \frac{55(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} \\
&- \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128 m_1^5 m_2^3} - \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256 m_1^5 m_2^3} - \frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} \\
&+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{128 m_1^5 m_2^3} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{64 m_1^4 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64 m_1^4 m_2^4} \\
&- \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{4 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^4 m_2^4} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64 m_1^4 m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64 m_1^4 m_2^4} \\
&- \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{4 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{16 m_1^4 m_2^4} \\
&- \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64 m_1^4 m_2^4} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32 m_1^4 m_2^4} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128 m_1^4 m_2^4}, \qquad \text{(A4a)}
\end{aligned}
$$

$$
\begin{aligned}
H_{46}(\mathbf{x}_a,\mathbf{p}_a) &= \frac{369(\mathbf{n}_{12}\cdot\mathbf{p}_1)^6}{160 m_1^6} - \frac{889(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_1^2}{192 m_1^6} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16 m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64 m_1^6} - \frac{549(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{128 m_1^5 m_2} \\
&+ \frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{16 m_1^5 m_2} - \frac{167(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2}{128 m_1^5 m_2} + \frac{1547(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2} - \frac{851(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2} \\
&+ \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2} + \frac{3263(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2} + \frac{1067(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{480 m_1^4 m_2^2} - \frac{4567(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840 m_1^4 m_2^2} \\
&- \frac{3571(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{320 m_1^4 m_2^2} + \frac{3073(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{480 m_1^4 m_2^2} + \frac{4349(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2} \\
&- \frac{3461\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{3840 m_1^4 m_2^2} + \frac{1673(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{1920 m_1^4 m_2^2} - \frac{1999(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2} - \frac{13(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8 m_1^3 m_2^3} \\
&+ \frac{191(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{192 m_1^3 m_2^3} - \frac{19(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3} \\
&+ \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{192 m_1^3 m_2^3} + \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{96 m_1^3 m_2^3} + \frac{233(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{96 m_1^3 m_2^3} - \frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32 m_1^3 m_2^3} \\
&+ \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{4 m_1^2 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{4 m_1^2 m_2^4} \\
&- \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{2 m_1^2 m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^2 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{6 m_1^2 m_2^4} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48 m_1^2 m_2^4} \\
&- \frac{133(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{24 m_1^2 m_2^4} - \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{96 m_1^2 m_2^4} + \frac{197(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96 m_1^2 m_2^4} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48 m_1^2 m_2^4} + \frac{13(\mathbf{p}_2^2)^3}{8 m_2^6}, \qquad \text{(A4b)}
\end{aligned}
$$

$$
\begin{aligned}
H_{441}(\mathbf{x}_a,\mathbf{p}_a) &= \frac{5027(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{384 m_1^4} - \frac{22993(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{960 m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152 m_1^4} - \frac{3191(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{640 m_1^3 m_2} \\
&+ \frac{28561(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920 m_1^3 m_2} + \frac{8777(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2} - \frac{752969\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{28800 m_1^3 m_2} \\
&- \frac{16481(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{960 m_1^2 m_2^2} + \frac{94433(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{4800 m_1^2 m_2^2} - \frac{103957(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{2400 m_1^2 m_2^2} \\
&+ \frac{791(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{400 m_1^2 m_2^2} + \frac{26627(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{1600 m_1^2 m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800 m_1^2 m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32 m_2^4}, \qquad \text{(A4c)}
\end{aligned}
$$

$$
\begin{aligned}
H_{442}(\mathbf{x}_a,\mathbf{p}_a) &= \left(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\right)\frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^4} \\
&+ \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\right)\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2} \\
&+ \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2 m_2^2} + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^2 m_2^2} \\
&- \left(\frac{30383}{960} + \frac{36405\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\right)\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2} \\
&+ \left(\frac{2369}{60} + \frac{35655\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2} + \left(\frac{4310\pi^2}{16384} - \frac{39711}{6400}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{m_1^3 m_2} \\
&+ \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2}, \qquad \text{(A4d)}
\end{aligned}
$$

$$
H_{421}(\mathbf{x}_a,\mathbf{p}_a) = \frac{64861\mathbf{p}_1^2}{4800 m_1^2} - \frac{91(\mathbf{p}_1\cdot\mathbf{p}_2)}{8 m_1 m_2} + \frac{105\mathbf{p}_2^2}{32 m_2^2} - \frac{9841(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600 m_1^2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2 m_1 m_2}, \qquad \text{(A4e)}
$$

$$
\begin{aligned}
H_{422}(\mathbf{x}_a,\mathbf{p}_a) &= \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\right)\frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\right)\frac{\mathbf{p}_2^2}{m_2^2} \\
&+ \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} - \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2} \\
&+ \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2}, \qquad \text{(A4f)}
\end{aligned}
$$

$$
H_{40}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\right)m_1^3 m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\right)m_1^2 m_2^2. \qquad \text{(A4g)}
$$

$$
H^{\rm nonloc}_{\rm 4PN}(t) = -\frac{1}{5}\frac{G^2 M}{c^8} I^{(3)}_{ij}(t) \times \mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{dv}{|v|} I^{(3)}_{ij}(t+v),
$$

Effective One Body (EOB) Method)

Buonanno-Damour 1999, 2000; Damour-Jaranowski-Schaefer 2000; Damour 2001; Damour-Nagar 2007; Damour-Iyer-Nagar 2009

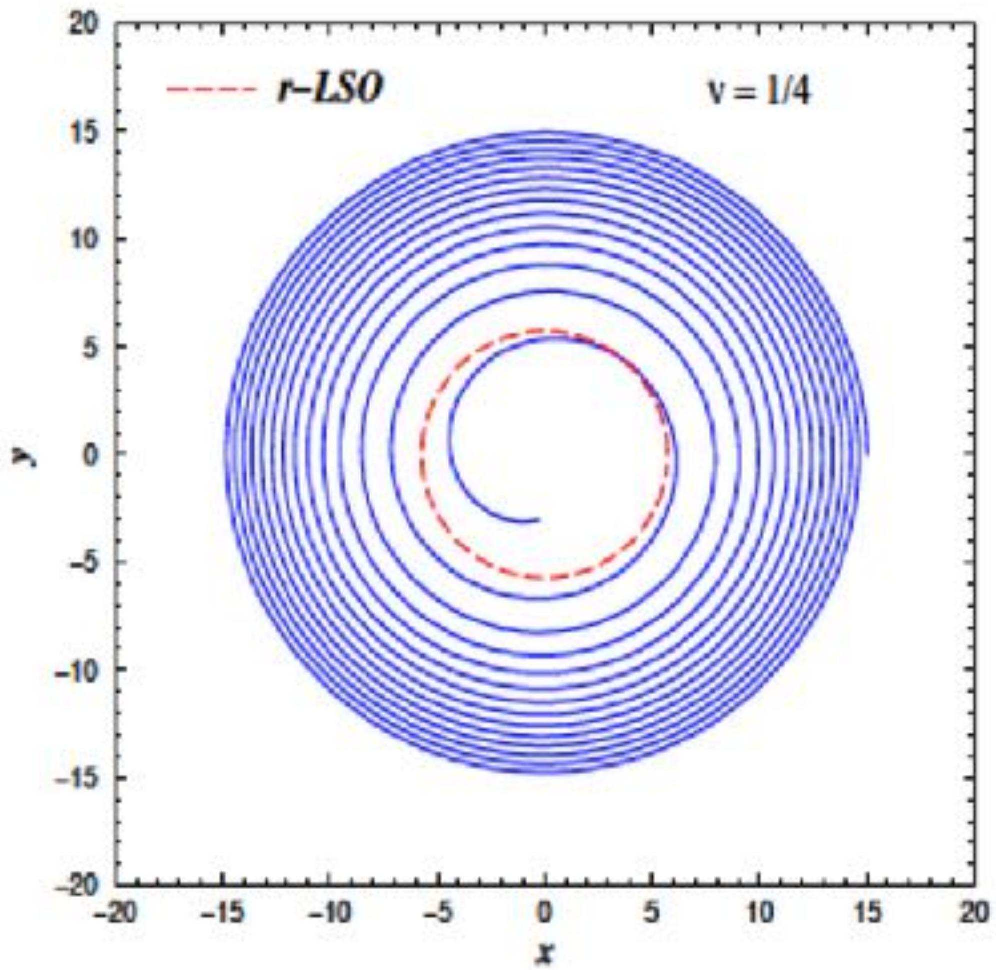
Inability of PN expansions to describe the late inspiral and the merger:

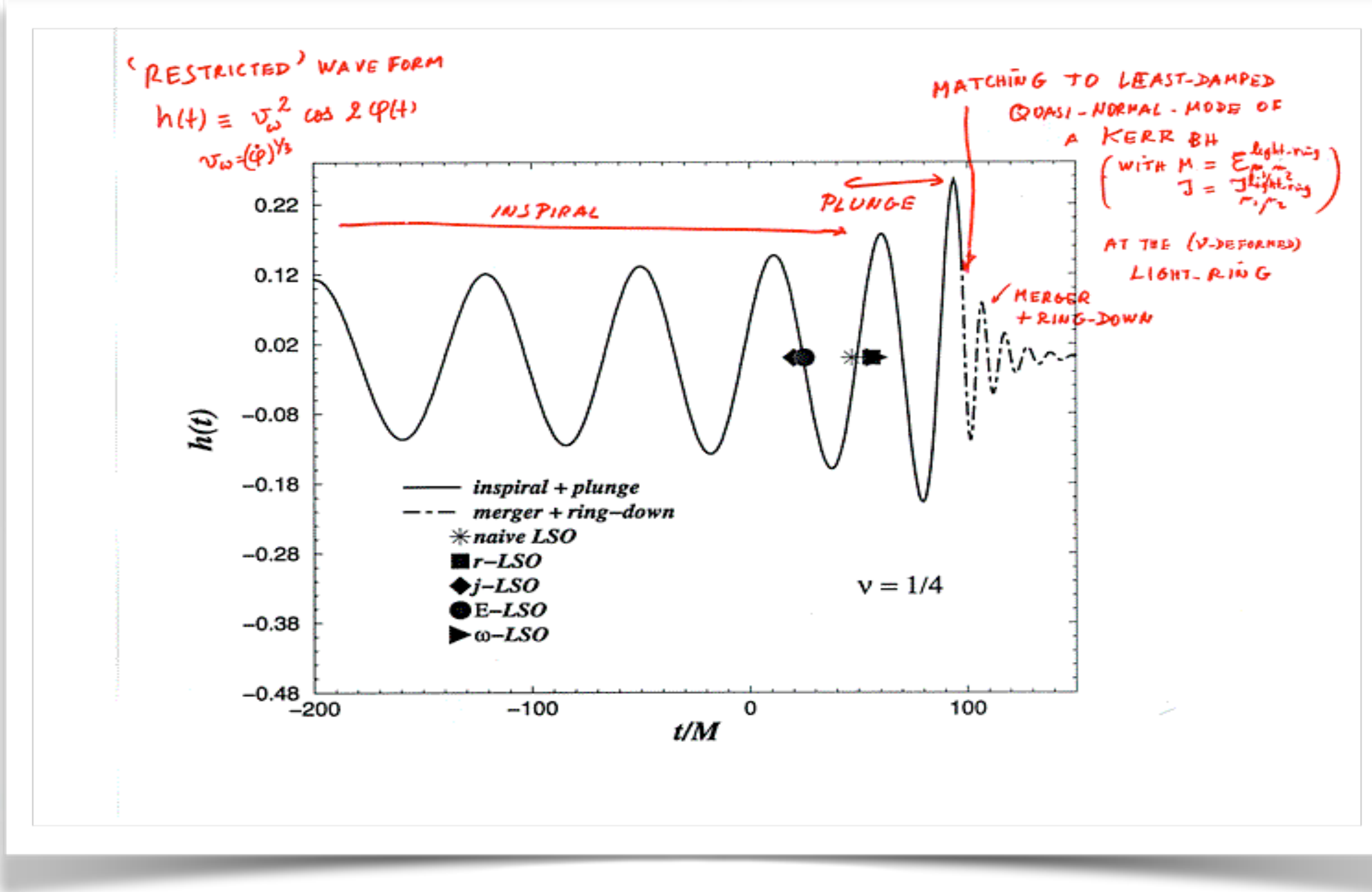
the PN expansions for both the flux and the dynamics reach their radius of convergence

RESUMMATION of both the Hamiltonian, the waveform and radiation-reaction

—> description of the coalescence + addition of ringdown (Vishveshwara 70, Davis-Ruffini-Tiomno 72)

Buonanno-Damour 2000





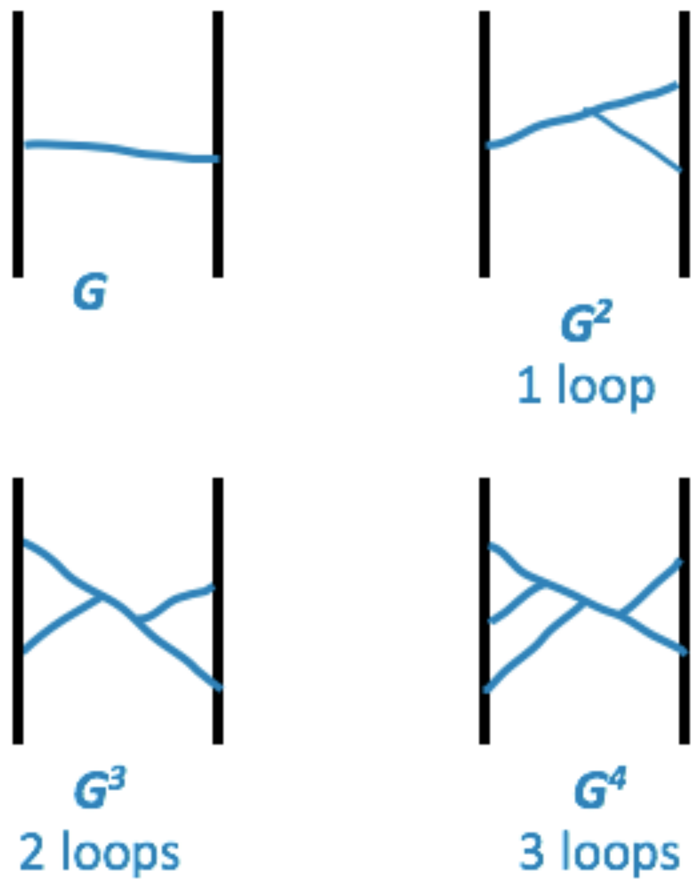
Predictions as early as 2000 :

continued transition, non adiabaticity, first complete waveform, final spin (OK within 10%), final mass

TWO-BODY/EOB "CORRESPONDENCE":

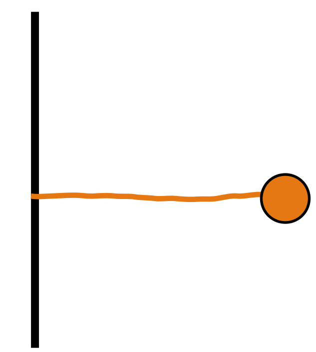
THINK QUANTUM-MECHANICALLY (J.A. WHEELER)

Real 2-body system
(in the c.o.m. frame)
(m_1, m_2)



An effective particle of mass mu in some effective metric

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$



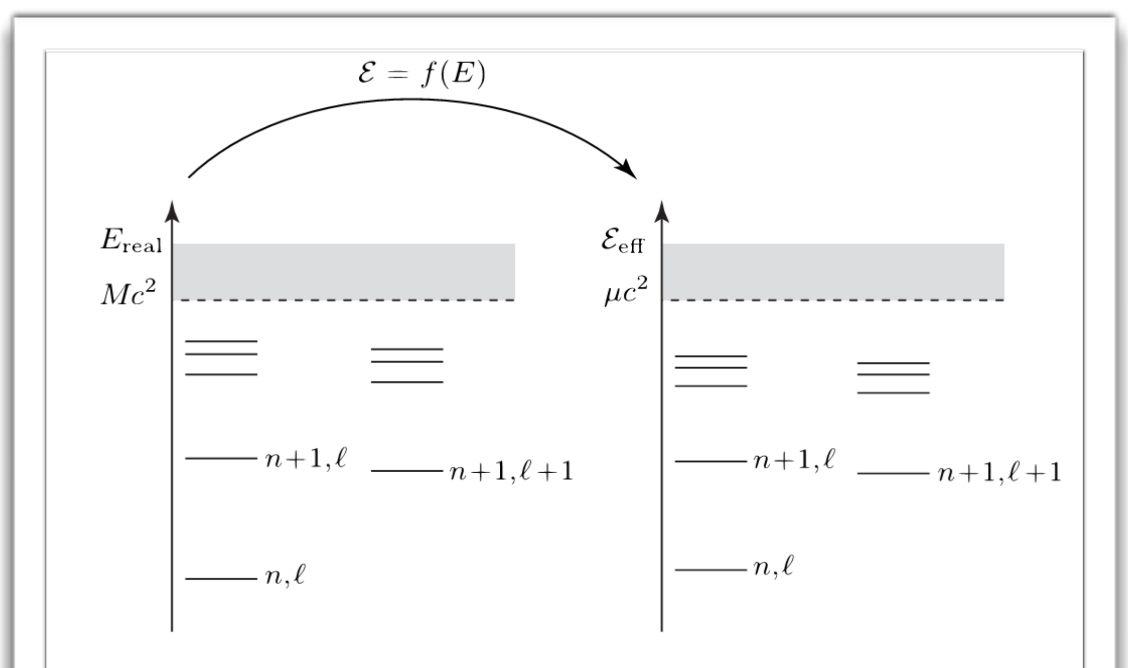
$$g_{\rm eff}^{\mu\nu}(X)$$

mass-shell constraint

$$0 = g_{\rm eff}^{\mu\nu}(X) P_\mu P_\nu + \mu^2 + Q(X,P)$$

Level correspondence
in the semi-classical limit:
Bohr-Sommerfeld ->
identification of
quantized action variables

$$\begin{aligned} J &= \ell\hbar = \frac{1}{2\pi}\oint p_\varphi d\varphi \\ N &= n\hbar = I_r + J \\ I_r &= \frac{1}{2\pi}\oint p_r dr \end{aligned}$$



Crucial energy map

$$\mathcal{E} = f(E)$$

2-body Taylor-expanded N + 1PN + 2PN+ 3PN Hamiltonian

$$H_{\rm N}(\mathbf{x}_a,\mathbf{p}_a)=\frac{\mathbf{p}_1^2}{2m_1}-\frac{1}{2}\frac{Gm_1m_2}{r_{12}}+(1\leftrightarrow 2)$$

$$c^2H_{\rm 1PN}(\mathbf{x}_a,\mathbf{p}_a)=-\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(-12\frac{\mathbf{p}_1^2}{m_1^2}+14\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)+\frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}}+(1\leftrightarrow 2),$$

$$\begin{aligned}c^4H_{\rm 2PN}(\mathbf{x}_a,\mathbf{p}_a)=&\frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(5\frac{(\mathbf{p}_1^2)^2}{m_1^4}-\frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}-\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}+5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right.\\
&\left.-6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}-\frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right)\\
&+\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2}+19\frac{\mathbf{p}_2^2}{m_2^2}\right)-\frac{1}{2}(m_1+m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2)+6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)\\
&-\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2+5m_1m_2+m_2^2)}{r_{12}^2}+(1\leftrightarrow 2),\end{aligned}$$

$$\begin{aligned}c^6H_{\rm 3PN}(\mathbf{x}_a,\mathbf{p}_a)=&-\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7}+\frac{1}{32}\frac{Gm_1m_2}{r_{12}}\left(-14\frac{(\mathbf{p}_1^2)^3}{m_1^6}+4\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2+4\mathbf{p}_1^2\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4m_2^2}+6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2}\right.\\
&-10\frac{(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2+\mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4m_2^2}+24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2}\\
&+2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}+\frac{(7\mathbf{p}_1^2\mathbf{p}_2^2-10(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3}\\
&+\frac{(\mathbf{p}_1^2\mathbf{p}_2^2-2(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3}+15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}\\
&\left.-18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}+5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\right)+\frac{G^2m_1m_2}{r_{12}^2}\left(\frac{1}{16}(m_1-27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}\right.\\
&-\frac{115}{16}m_1\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2}+\frac{1}{48}m_2\frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2+371\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}+\frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3}+\frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}\\
&-\frac{1}{8}m_1\frac{(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)+11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2}-\frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2}\\
&+\frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}+\frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\\
&\left.-\frac{1}{48}(220m_1+193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right)+\frac{G^3m_1m_2}{r_{12}^3}\left(-\frac{1}{48}\left(425m_1^2+\left(473-\frac{3}{4}\pi^2\right)m_1m_2+150m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\right.\\
&+\frac{1}{16}\left(77(m_1^2+m_2^2)+\left(143-\frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+\frac{1}{16}\left(20m_1^2-\left(43+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}\\
&\left.+\frac{1}{16}\left(21(m_1^2+m_2^2)+\left(119+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)\\
&+\frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left(\left(\frac{227}{3}-\frac{21}{4}\pi^2\right)m_1+m_2\right)+(1\leftrightarrow 2).\end{aligned}$$

Explicit 3PN EOB dynamics (Damour-Jaranowski-Schaefer '01)

A **simple**, but crucial transformation between the real energy and the effective one:

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1+m_2)} .$$

A **simple post-geodesic** effective mass-shell:

$$g_{\rm eff}^{\mu\nu}\, P'_\mu\, P'_\nu + \mu^2\, c^2 + Q(P'_\mu) = 0\,,$$

$$ds^2_{\rm eff} = -A(R;\nu)dt^2 + B(R;\nu)dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$M = m_1 + m_2\,; \qquad \mu = \frac{m_1 m_2}{m_1+m_2}\,, \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2} = \frac{\mu}{M}$$

$$u \equiv \frac{GM}{R\,c^2}$$

$$A^{\rm 3PN}(u) = 1 - 2u + 2\nu\, u^3 + \left(\frac{94}{3} - \frac{41}{32}\,\pi^2\right)\nu u^4,$$

$$\overline{D}^{\rm 3PN}(u) = 1 + 6\nu\, u^2 + (52\nu - 6\nu^2)\, u^3,$$

$$\widehat{Q}^{\rm 3PN} \equiv \frac{Q}{\mu^2 c^2} = (8\nu - 6\nu^2)\, u^2\, \frac{p_r^4}{c^4}.$$

Spinning EOB effective Hamiltonian

$$H_{\rm eff} = H_{\rm orb} + H_{\rm so} \qquad \rightarrow H_{\rm EOB} = Mc^2 \sqrt{1 + 2\nu \left(\frac{H_{\rm eff}}{\mu c^2} - 1 \right)}$$

$$\hat{H}^{\rm eff}_{\rm orb} = \sqrt{A\left(1 + B_p \boldsymbol{p}^2 + B_{np}(\boldsymbol{n}\cdot\boldsymbol{p})^2 - \frac{1}{1 + \frac{(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2}{r^2}} \frac{(r^2 + 2r + (\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2)}{\mathcal{R}^4 + \Delta(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2} ((\boldsymbol{n}\times\boldsymbol{p})\cdot\boldsymbol{\chi}_0)^2 + Q_4\right)}.$$

$$H_{\rm so} = G_S \boldsymbol{L}\cdot\boldsymbol{S} + G_{S^*}\boldsymbol{L}\cdot\boldsymbol{S}^*,$$

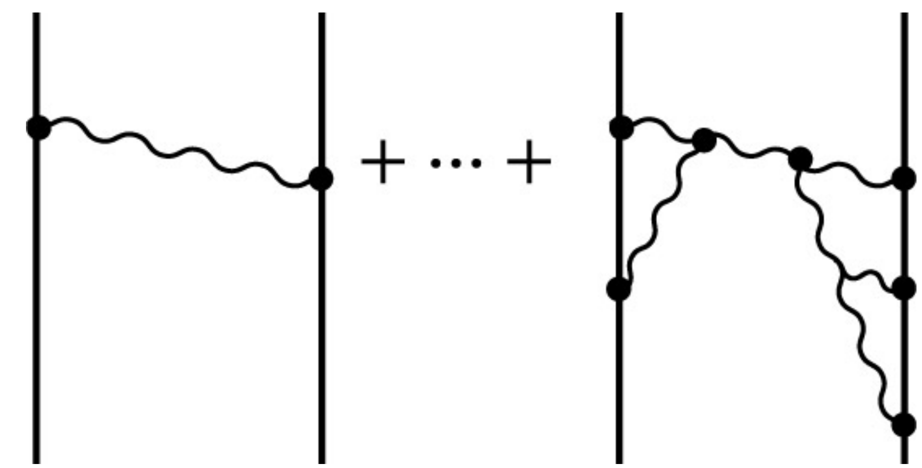
$$\mathbf{S} = \mathbf{S_1} + \mathbf{S_2}\,;\; \mathbf{S}_* = \frac{m_2}{m_1}\mathbf{S_1} + \frac{m_1}{m_2}\mathbf{S_2}\,,$$

Gyrogravitomagnetic ratios (when neglecting spin^2 effects)

$$r^3 G_S^{\rm PN} = 2 - \frac{5}{8}\nu u - \frac{27}{8}\nu p_r^2 + \nu\left(-\frac{51}{4}u^2 - \frac{21}{2}u p_r^2 + \frac{5}{8}p_r^4\right) + \nu^2\left(-\frac{1}{8}u^2 + \frac{23}{8}u p_r^2 + \frac{35}{8}p_r^4\right)$$

$$r^3 G_{S_*}^{\rm PN} = \frac{3}{2} - \frac{9}{8}u - \frac{15}{8}p_r^2 + \nu\left(-\frac{3}{4}u - \frac{9}{4}p_r^2\right) - \frac{27}{16}u^2 + \frac{69}{16}u p_r^2 + \frac{35}{16}p_r^4 + \nu\left(-\frac{39}{4}u^2 - \frac{9}{4}u p_r^2 + \frac{5}{2}p_r^4\right) + \nu^2\left(-\frac{3}{16}u^2 + \frac{57}{16}u p_r^2 + \frac{45}{16}p_r^4\right)$$

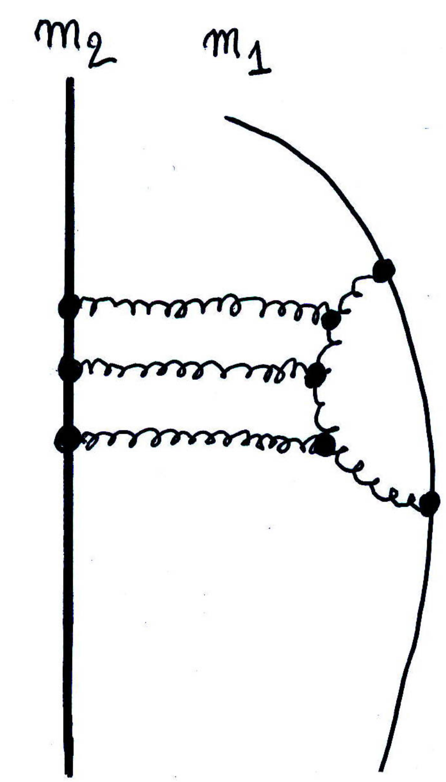
EOB AND GSF



Comparable-mass case: $m_1 \sim m_2$

Gravitational Self-Force Theory : $m_1 << m_2$

based on BH perturbation theory:
Regge-Wheeler-Zerilli-Teukolsky



• Analytical high-PN results : Blanchet-Detweiler-LeTiec-Whiting '10, Damour '10, Blanchet et al '10, LeTiec et al '12, Bini-Damour '13-15, Kavanagh-Ottewill-Wardell '15 Bini-Damour-Geralico'16, Hopper-Kavanagh-Ottewill'16

• (gauge-invariant) Numerical results : Detweiler '08, Barack-Sago '09, Blanchet-Detweiler-LeTiec-Whiting '10, Barack-Damour-Sago '10, Shah-Friedman-Keidl '12, Dolan et al '14, Nolan et al '15, Akcay-van de Meent '16

• Analytical PN results from high-precision (hundreds to thousands of digits !) numerical results : Shah-Friedman-Whiting '14, Johnson-McDaniel-Shah-Whiting '15

GSF : ANALYTICAL HIGH-ORDER_PN RESULTS (22 LOOPS)

Bini-Damour 15

$$A(u;\nu) = 1 - 2u + \nu a(u) + O(\nu^2)$$

Kavanagh et al 15

$$a^c_{10} = \frac{18605478842060273}{7079830758000}\ln(2) - \frac{1619008}{405}\zeta(3) - \frac{21339873214728097}{1011404394000}\gamma + \frac{27101981341}{100663296}\pi^6 - \frac{6236861670873}{125565440}\ln(3) + \frac{360126}{49}\ln(2)\ln(3) + \frac{180063}{49}\ln(3)^2 - \frac{121494974752}{9823275}\ln(2)^2 - \frac{24229836023352153}{549755813888}\pi^4 + \frac{1115369140625}{124540416}\ln(5) + \frac{96889}{2779} + \frac{75437014370623318623299}{186907532011200000} - \frac{60648244288}{9823275}\ln(2)\gamma + \frac{200706848}{280665}\gamma^2 + \frac{11980569677139}{2306867200}\pi^2 + \frac{360126}{49}\gamma\ln(3),$$

$$a^{\ln}_{10} = -\frac{21275143333512097}{2022808788000} + \frac{200706848}{280665}\gamma - \frac{30324122144}{9823275}\ln(2) + \frac{180063}{49}\ln(3),$$

$$a^{\ln^2}_{10} = \frac{50176712}{280665},$$

$$a^c_{10.5} = -\frac{185665618769828101}{24473489040000}\pi + \frac{377443508}{77175}\ln(2)\pi + \frac{2414166668}{1157625}\pi\gamma - \frac{5846788}{11025}\pi^3 - \frac{24}{}$$

$$a^{\ln}_{10.5} = \frac{1207083334}{1157625}\pi.$$

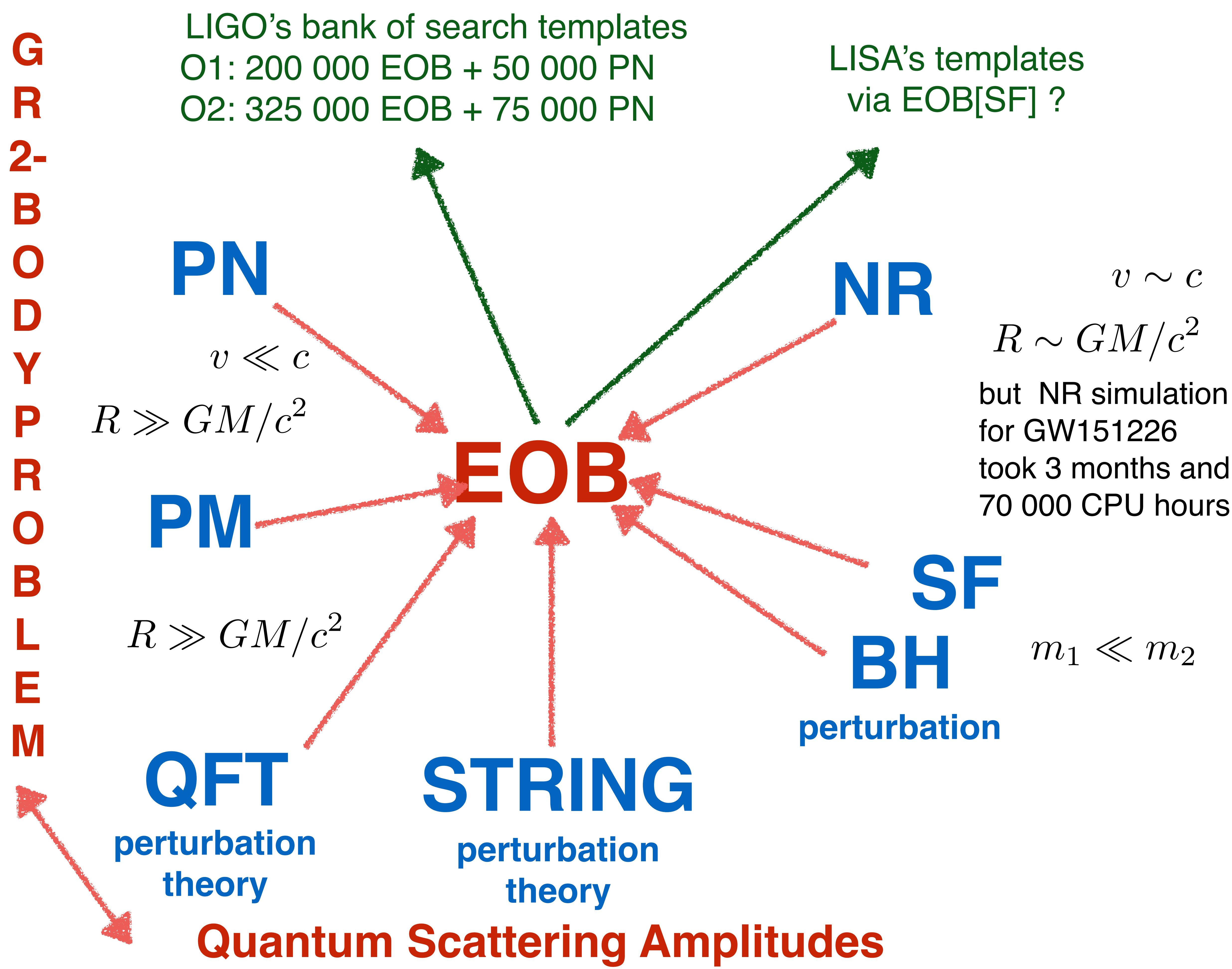
$$c_{15} = -\frac{2069543450583769619340376724}{32547744208650606084375}\zeta(3) + \frac{6519502629824500793 6}{22370298575625}\gamma\zeta(3) - \frac{5049442304}{25725}\gamma^2\zeta(3) + \frac{1262360576}{15435}\pi^2\zeta(3)$$
$$+ \frac{171722752}{441}\zeta(3)^2 + \frac{1613866959570176}{496621125}\zeta(5) - \frac{343445504}{441}\gamma\zeta(5) - \frac{146997248}{105}\zeta(7) + \frac{56314978304}{385875}\zeta(3)\log^2(2)$$
$$- \frac{106445664}{343}\zeta(3)\log^2(3) + \frac{1516709982448497 97696}{22370298575625}\zeta(3)\log(2) - \frac{190336581632}{1157625}\gamma\zeta(3)\log(2)$$
$$+ \frac{28863591064624341}{4909804900}\zeta(3)\log(3) - \frac{212891328}{343}\gamma\zeta(3)\log(3) - \frac{212891328}{343}\zeta(3)\log(2)\log(3) - \frac{77186767578125}{19876428}\zeta(3)\log$$
$$- \frac{2039263232}{3675}\zeta(5)\log(2) - \frac{49128768}{49}\zeta(5)\log(3) + \frac{298267427515018397019736592175289419501391539444290849}{65876122225446532261424684050319173195312 50}$$
$$- \frac{68076617684536377683132869480603290875014 19}{70431094812480372256260772954406250 0}\gamma + \frac{15983469444126032478310062898293 88}{526171715038677033591890625}\gamma^2 - \frac{1007647146215971027644}{335890033113009375}$$
$$+ \frac{461219496448}{72930375}\gamma^4 - \frac{2833827508207759158785506345027630379006576290724 3197}{99970315584514341811574404579275571200000 0}\pi^2 + \frac{2519117865539927569 1104}{67178006622601875}\gamma\pi^2$$
$$- \frac{230609748224}{14586075}\gamma^2\pi^2 + \frac{10548032335775722689471378776039118077624803624 1}{30424535483131602802509905532026880000 0}\pi^4 + \frac{1262360576}{385875}\gamma\pi^4$$
$$- \frac{620847283961296697269145713114 3}{266930151354100246118400}\pi^6 + \frac{357317878192092911828132 9}{151996487423754240}\pi^8 - \frac{10136323685888}{72930375}\log^4(2) + \frac{38438712}{2401}\log^4(3)$$
$$- \frac{177896086126482679647872}{549638236003106 25}\log^3(2) - \frac{896860131061 76}{364651875}\gamma\log^3(2) + \frac{153754848}{2401}\log^3(2)\log(3)$$
$$- \frac{131463845322790269123}{245735735245000}\log^3(3) + \frac{153754848}{2401}\gamma\log^3(3) + \frac{153754848}{2401}\log(2)\log^3(3) + \frac{11933074267578125}{51161925672}\log^3(5)$$
$$+ \frac{38782586741666289745954206352002 04}{18942181741392373209308062 5}\log^2(2) - \frac{344085637991460169215116 8}{1007670099339028125}\gamma\log^2(2) - \frac{16582891400192}{121550625}\gamma^2\log^2(2)$$
$$+ \frac{41457228500 48}{72930375}\pi^2\log^2(2) - \frac{5236971633734839056 09}{245735735245000}\log^2(2)\log(3) + \frac{461264544}{2401}\gamma\log^2(2)\log(3)$$
$$+ \frac{454545357661890658883022992617 59}{65697282267898830348800 00}\log^2(3) - \frac{394391535968370807369}{245735735245000}\gamma\log^2(3) + \frac{230632272}{2401}\gamma^2\log^2(3)$$
$$- \frac{96096780}{2401}\pi^2\log^2(3) - \frac{437493411770075173449}{245735735245000}\log(2)\log^2(3) + \frac{461264544}{2401}\gamma\log(2)\log^2(3)$$
$$+ \frac{230632272}{2401}\log^2(2)\log^2(3) + \frac{11933074267578125}{17053975224}\log^2(2)\log(5) - \frac{2505842696993145943705498046875}{40213632089533222243123 2}\log^2(5)$$
$$+ \frac{11933074267578125}{17053975224}\gamma\log^2(5) + \frac{11933074267578125}{17053975224}\log(2)\log^2(5) + \frac{479295083164704151420102 51}{5646463517021184000 0}\log^2(7)$$
$$- \frac{181636067216895220421537747685253699734494659}{6338798533123233503063469565896562500}\log(2) + \frac{742036621552191085437995316530101 36}{47355454353480933023270156 25}\gamma\log(2)$$
$$- \frac{14821693265224925154993 92}{1007670099339028125}\gamma^2\log(2) - \frac{4905667647488}{364651875}\gamma^3\log(2) + \frac{3712281154906676684511 68}{604602059603416875}\pi^2\log(2)$$
$$+ \frac{1226416911872}{72930375}\gamma\pi^2\log(2) + \frac{23792072704}{17364375}\pi^4\log(2) - \frac{41411583753971803023870951249358557477 27}{108266631596274488880198656000000}\log(3)$$
$$+ \frac{945935800113157545433205527623 9}{69155033966209295104000 0}\gamma\log(3) - \frac{394391535968370807369}{245735735245000}\gamma^2\log(3) + \frac{153754848}{2401}\gamma^3\log(3)$$
$$+ \frac{131463845322790269123}{196588588196000}\pi^2\log(3) - \frac{192193560}{2401}\gamma\pi^2\log(3) + \frac{8870472}{1715}\pi^4\log(3)$$
$$+ \frac{214411501060211389845962927148381}{131394564535797660697600 00}\log(2)\log(3) - \frac{437493411770075173449}{122867867622500}\gamma\log(2)\log(3)$$
$$+ \frac{461264544}{2401}\gamma^2\log(2)\log(3) - \frac{192193560}{2401}\pi^2\log(2)\log(3) + \frac{97861294850170985327709557611886523437 5}{179427491919561270211323849031 68}\log(5)$$
$$- \frac{2505842696993145943705498046875}{20106816044766611121561 6}\gamma\log(5) + \frac{11933074267578125}{17053975224}\gamma^2\log(5) - \frac{5966537133789062 5}{204647702688}\pi^2\log(5)$$
$$- \frac{2505842696993145943705498046875}{20106816044766611121561 6}\log(2)\log(5) + \frac{11933074267578125}{8526987612}\gamma\log(2)\log(5)$$
$$- \frac{585800617379230891566511301391464808 1}{323919193207512802977792000000}\log(7) + \frac{479295083164704151420102 51}{28232317585105920000}\gamma\log(7)$$
$$+ \frac{479295083164704151420102 51}{28232317585105920000}\log(2)\log(7) + \frac{740024994425816010121 1}{65676344832000000}\log(11),$$

Numerical GSF computation of the O(nu) A(u;nu)-potential: with singularity at u=1/3 (Akcay et al 2012)

$$a(u) \simeq 0.25(1-3u)^{-1/2}$$

GSF can also bring scattering information

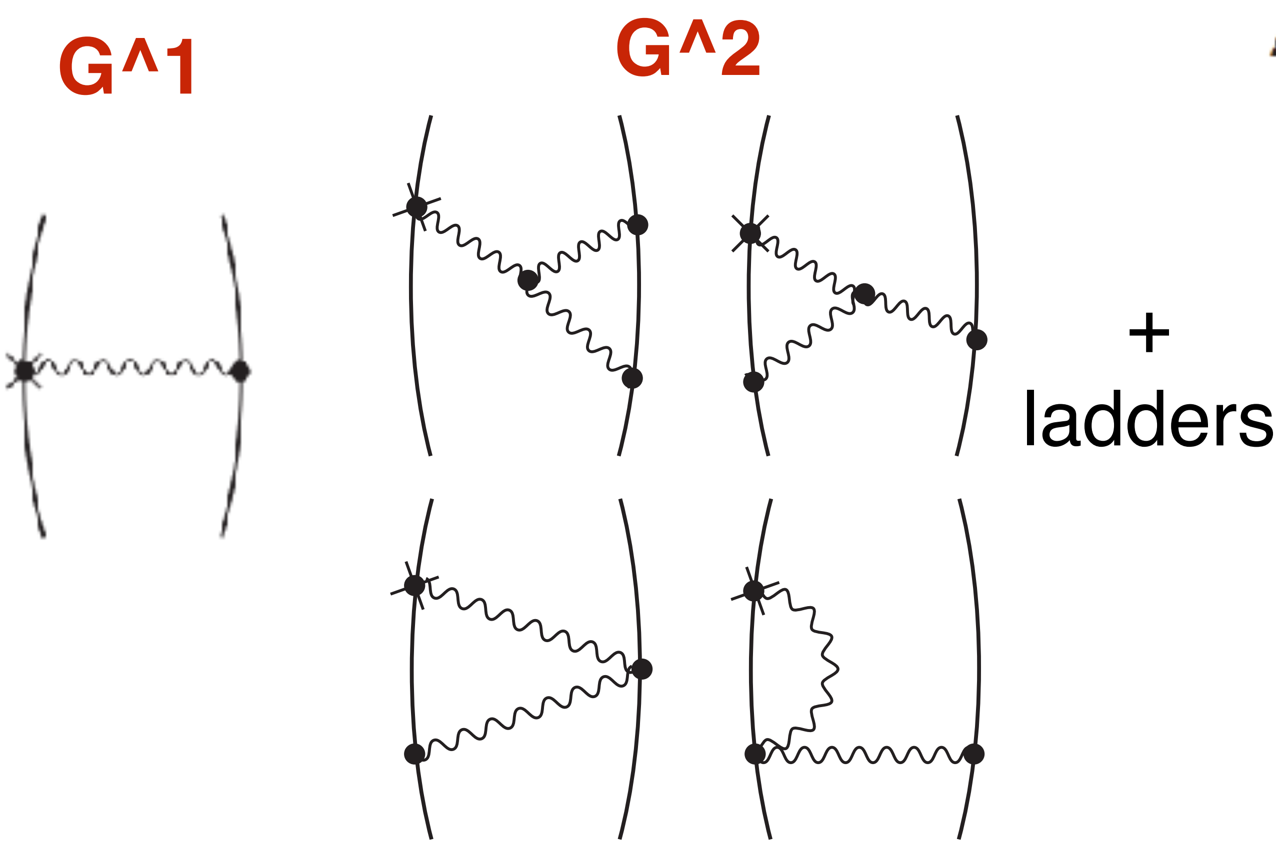
(Damour 2010, Barack et al, in prepar)



LIGO's bank of search templates
O1: 200 000 EOB + 50 000 PN
O2: 325 000 EOB + 75 000 PN
LISA's templates
via EOB[SF] ?
GR 2-BODY PROBLEM
PN
$v \ll c$
$R \gg GM/c^2$
NR
$v \sim c$
$R \sim GM/c^2$
but NR simulation
for GW151226
took 3 months and
70 000 CPU hours
EOB
PM
$R \gg GM/c^2$
SF
BH
perturbation
$m_1 \ll m_2$
QFT
perturbation
theory
STRING
perturbation
theory
Quantum Scattering Amplitudes

PM Perturbation Theory for Classical Gravitational Scattering

Damour'18, using Bel-Martin '75-'81, Portilla '79,Westpfahl-Goller '79, Portilla '80, Bel-Damour-Deruelle-Ibanez-Martin'81,Westpfahl '85



$$\Delta p_{1\mu} = 2G \int d\sigma_1 d\sigma_2 p_{1\alpha} p_{1\beta} \times \partial_\mu \mathcal{P}^{\alpha\beta;\alpha'\beta'}(x_1(\sigma_1) - x_2(\sigma_2)) p_{2\alpha'} p_{2\beta'} + 2G \int d\sigma_1 d\sigma_1' p_{1\alpha} p_{1\beta} \partial_\mu \mathcal{P}^{\alpha\beta;\alpha'\beta'}(x_1(\sigma_1) - x_1(\sigma_1')) p_{1\alpha'} p_{1\beta'} + O(G^2)$$

$$\Delta p_{1\mu} = 8\pi G \int \frac{d^4k}{(2\pi)^4} i k_\mu p_1^\alpha p_1^\beta \frac{P_{\alpha\beta;\alpha'\beta'}}{k^2} p_2^{\alpha'} p_2^{\beta'} \times \int d\sigma_1 \int d\sigma_2 e^{ik.(x_1(\sigma_1) - x_2(\sigma_2))}.$$

$$\frac{1}{2}\chi_{\text{class}}(E, J) = \frac{1}{j}\chi_1(\hat{E}_{\text{eff}}, \nu) + \frac{1}{j^2}\chi_2(\hat{E}_{\text{eff}}, \nu) + O(G^3)$$

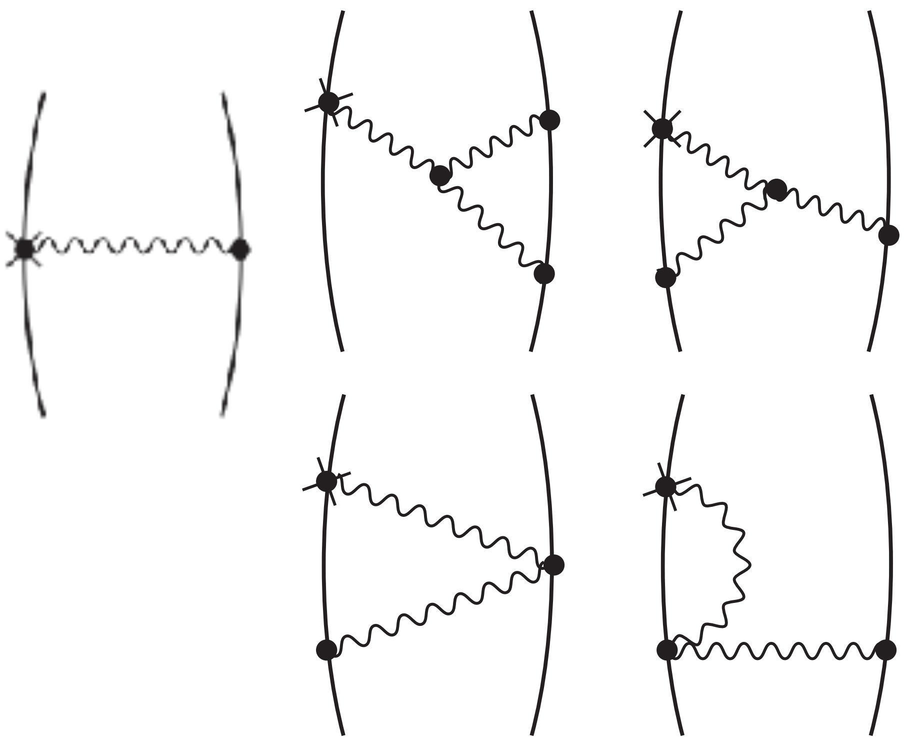
$$j \equiv \frac{J}{G m_1 m_2}$$

$$\frac{1}{2}\chi_{1PM}^{\text{real}} = \frac{G}{J} \frac{2(p_1.p_2)^2 - p_1^2 p_2^2}{\sqrt{(p_1.p_2)^2 - p_1^2 p_2^2}}.$$

$$\hat{\mathcal{E}}_{\text{eff}} \equiv \frac{\mathcal{E}_{\text{eff}}}{\mu} \equiv \frac{(E_{\text{real}})^2 - m_1^2 - m_2^2}{2m_1 m_2} = \frac{s - m_1^2 - m_2^2}{2m_1 m_2}.$$

$$\chi_1(\hat{\mathcal{E}}_{\text{eff}}, \nu) = \frac{2\hat{\mathcal{E}}_{\text{eff}}^2 - 1}{\sqrt{\hat{\mathcal{E}}_{\text{eff}}^2 - 1}}, \qquad \chi_2(\hat{\mathcal{E}}_{\text{eff}}, \nu) = \frac{3\pi}{8} \frac{5\hat{\mathcal{E}}_{\text{eff}}^2 - 1}{\sqrt{1 + 2\nu(\hat{\mathcal{E}}_{\text{eff}} - 1)}}.$$

Classical Gravitational Scattering and the EOB description of the GR 2-body problem



$$0 = g_{\rm eff}^{\mu\nu} P_\mu P_\nu + \mu^2 + Q$$

Using

$$\frac{1}{2}\chi = \Phi(E_{\rm real}, J; m_1, m_2, G)$$

At each order in G: EOB transcription of the gauge-invariant scattering function

as a simple energy-dependent modification of a Schwarzschild-metric mass-shell condition

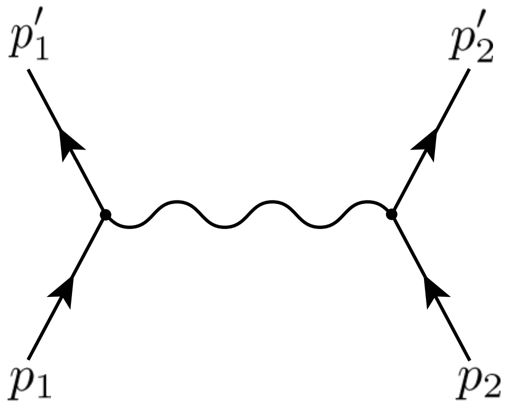
$g_{\rm eff}^{\mu\nu}$

Schwarzschild metric M=m1+m2

$$Q = \left(\frac{GM}{R}\right)^2 q_2(E) + \left(\frac{GM}{R}\right)^3 q_3(E) + O(G^4)$$

2PM **3PM**

Very simple EOB result at the 1PM order (linear in G)



$$\frac{1}{2}\chi^{\text{real}}_{1PM} = \frac{G}{J}\frac{2(p_1.p_2)^2 - p_1^2 p_2^2}{\sqrt{(p_1.p_2)^2 - p_1^2 p_2^2}}.$$

to order G^1, the relativistic dynamics of a two-body system (of masses m_1 , m_2) is equivalent to the relativistic dynamics of an effective test particle of mass $\mu = m_1 m_2/(m_1 + m_2)$ moving in a Schwarzschild metric of mass $M = m_1 + m_2$, i.e. the rather complicated 1PM Hamiltonian of Ledvinka-Schaefer-Bicak2010: with

$$\begin{aligned} H_{\text{lin}} = {} & \sum_a \overline{m}_a + \frac{1}{4} G \sum_{a,b\neq a} \frac{1}{r_{ab}} \left(7\, \mathbf{p}_a\cdot\mathbf{p}_b + (\mathbf{p}_a\cdot\mathbf{n}_{ab})(\mathbf{p}_b\cdot\mathbf{n}_{ab})\right) - \frac{1}{2} G \sum_{a,b\neq a} \frac{\overline{m}_a \overline{m}_b}{r_{ab}} \\ & \times \left(1 + \frac{p_a^2}{\overline{m}_a^2} + \frac{p_b^2}{\overline{m}_b^2}\right) - \frac{1}{4} G \sum_{a,b\neq a} \frac{1}{r_{ab}} \frac{(\overline{m}_a \overline{m}_b)^{-1}}{(y_{ba}+1)^2 y_{ba}} \Bigg[2\Big(2(\mathbf{p}_a\cdot\mathbf{p}_b)^2 (\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 \qquad (6) \\ & - 2(\mathbf{p}_a\cdot\mathbf{n}_{ba})(\mathbf{p}_b\cdot\mathbf{n}_{ba})(\mathbf{p}_a\cdot\mathbf{p}_b)\mathbf{p}_b^2 + (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2 \mathbf{p}_b^4 - (\mathbf{p}_a\cdot\mathbf{p}_b)^2 \mathbf{p}_b^2 \Big) \frac{1}{\overline{m}_b^2} + 2\Big[-\mathbf{p}_a^2 (\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 \\ & + (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2 (\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 + 2(\mathbf{p}_a\cdot\mathbf{n}_{ba})(\mathbf{p}_b\cdot\mathbf{n}_{ba})(\mathbf{p}_a\cdot\mathbf{p}_b) + (\mathbf{p}_a\cdot\mathbf{p}_b)^2 - (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2 \mathbf{p}_b^2 \Big] \\ & + \Big[-3\mathbf{p}_a^2 (\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 + (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2 (\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 + 8(\mathbf{p}_a\cdot\mathbf{n}_{ba})(\mathbf{p}_b\cdot\mathbf{n}_{ba})(\mathbf{p}_a\cdot\mathbf{p}_b) \\ & + \mathbf{p}_a^2 \mathbf{p}_b^2 - 3(\mathbf{p}_a\cdot\mathbf{n}_{ba})^2 \mathbf{p}_b^2 \Big] y_{ba} \Bigg], \qquad y_{ba} = \frac{1}{\overline{m}_b}\sqrt{m_b^2 + (\mathbf{n}_{ba}\cdot\mathbf{p}_b)^2}. \end{aligned}$$

$$\overline{m}_a = (m_a^2 + \mathbf{p}_a^2)^{\frac{1}{2}}$$

is fully described by the EOB energy map applied to

$$\mathcal{E}_{\text{eff}} = \frac{(\mathcal{E}_{\text{real}})^2 - m_1^2 - m_2^2}{2(m_1 + m_2)}$$

$$ds^2_{\text{lin}} = -(1 - 2\frac{GM}{r})dt^2 + (1 + 2\frac{GM}{r})dr^2 + r^2 d\Omega^2$$

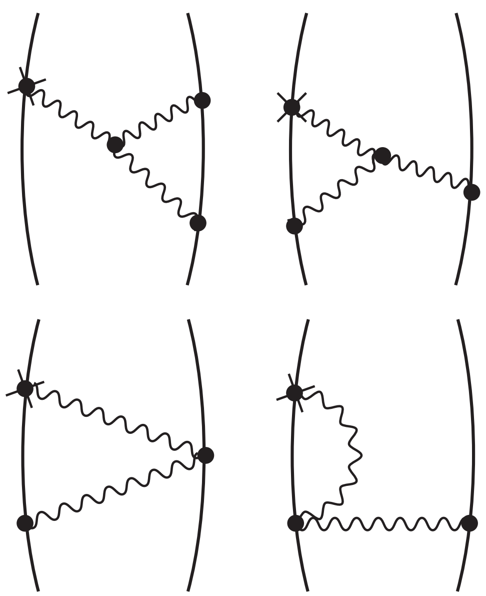
Simple 2PM (one-loop) EOB Potential

Damour'18, using Westpfahl-Goller '79, Bel-Damour-Deruelle-IbanezMartin'81,Westpfahl '85

G^1

$$0 = g^{\mu\nu}_{\mathrm{Schw}} P_\mu P_\nu + \mu^2 + q_2(E)(GM/R)^2$$

G^2



Schwarzschild metric M=m1+m2

$$\mathcal{E}_{\mathrm{eff}} = \frac{(\mathcal{E}_{\mathrm{real}})^2 - m_1^2 - m_2^2}{2(m_1 + m_2)}.$$

$$\chi_2(\hat{\mathcal{E}}_{\mathrm{eff}}, \nu) = \frac{3\pi}{8} \frac{5\hat{\mathcal{E}}_{\mathrm{eff}}^2 - 1}{\sqrt{1 + 2\nu(\hat{\mathcal{E}}_{\mathrm{eff}} - 1)}}.$$

$$q_2(\hat{H}_{\mathrm{Schw}}, \nu) = \frac{3}{2}(5\hat{H}_{\mathrm{Schw}}^2 - 1)\left[1 - \frac{1}{\sqrt{1 + 2\nu(\hat{H}_{\mathrm{Schw}} - 1)}}\right]$$

Equivalent Potential of Cheung-Rothstein-Solon’18

$$H(\mathbf{P},\mathbf{X}) = \sqrt{m_1^2+\mathbf{P}^2} + \sqrt{m_2^2+\mathbf{P}^2} + V(R,\mathbf{P}^2)$$

$$V(R,\mathbf{P}^2) = G\frac{c_1(\mathbf{P}^2)}{R} + G^2\frac{c_2(\mathbf{P}^2)}{R^2} + G^3\frac{c_3(\mathbf{P}^2)}{R^3} + \cdots$$

$$c_1(\boldsymbol{p}^2) = \frac{1}{E^2\xi}[m_A^2 m_B^2 - 2(p_1\cdot p_2)^2], \qquad \xi = \frac{E_A E_B}{(E_A+E_B)^2}$$

$$c_2(\boldsymbol{p}^2) = \frac{1}{32E^2\xi}[2E(\xi-1)c_1^2(\boldsymbol{p}^2) - 16E(p_1\cdot p_2)c_1(\boldsymbol{p}^2) + 3(m_A+m_B)(m_A^2 m_B^2 - 5(p_1\cdot p_2)^2)], \qquad (26)$$

equivalent to EOB result

$$0 = g^{\mu\nu}_{\rm Schw}P_\mu P_\nu + \mu^2 + q_2(E)(GM/R)^2$$

$$q_2(\hat{H}_{\rm Schw},\nu) = \frac{3}{2}(5\hat{H}^2_{\rm Schw}-1)\left[1 - \frac{1}{\sqrt{1+2\nu(\hat{H}_{\rm Schw}-1)}}\right]$$

Predicted Regge-like Behavior of High-Energy Unstable Circular Bound States

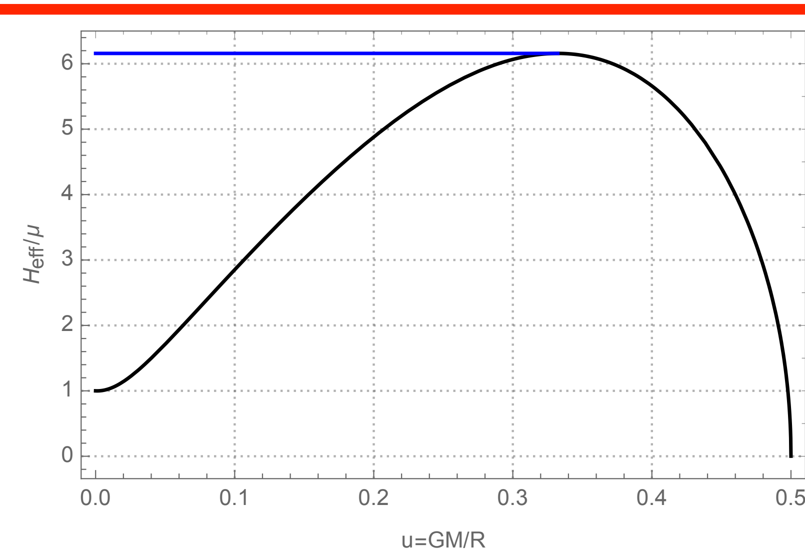
$$u \equiv \frac{GM}{R}$$

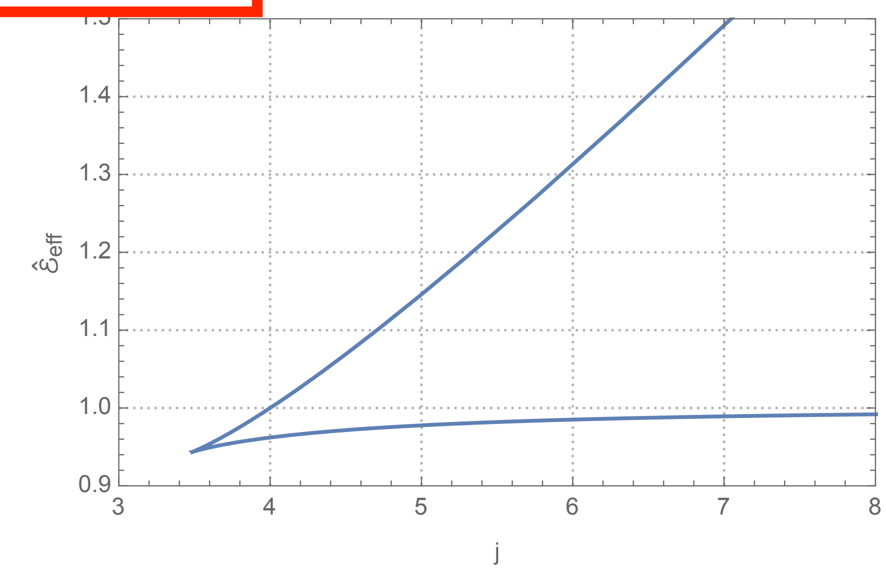
$$\widehat{H}^2_{\rm eff}(p_r, r, p_\varphi; \nu) = \widehat{H}^2_{\rm Schw} + \frac{3}{2}(1-2u)u^2\left(5\,\widehat{H}^2_{\rm Schw} - 1\right)\left(1 - \frac{1}{\sqrt{1+2\nu(\widehat{H}_{\rm Schw}-1)}}\right)$$

$$H^2_{\rm Schw} = \left(1 - \frac{2GM}{R}\right)\left[\mu^2 + (1-\frac{2GM}{R})P_R^2 + \frac{J^2}{R^2}\right]$$

High J —> H^2_eff ~ B(u) J^2
but

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1+m_2)}.$$





HE unstable circular bound states:
asymptotic constant Regge slope

string-like
$$s = E^2_{\rm real} \propto J$$

$$\frac{ds}{dJ} = \frac{2}{G}\frac{d\hat{\mathcal{E}}_{\rm eff}}{dj} \overset{\rm HE}{\approx} \frac{0.719\,964}{G} \qquad E^2_{\rm real} \overset{\rm HE}{=} C\,\frac{J}{G},$$

Self-Force Expansion , Light-Ring Behavior

Small mass-ratio expansion: \nu ->0

$$\widehat{H}^2_{\rm eff}(p_r, r, p_\varphi; \nu) = \widehat{H}^2_{\rm Schw} + \frac{3}{2}(1-2u)u^2\left(5\,\widehat{H}^2_{\rm Schw} - 1\right)\left(1 - \frac{1}{\sqrt{1+2\nu(\widehat{H}_{\rm Schw}-1)}}\right)$$

$$1 - \frac{1}{\sqrt{1+2\nu(\hat{H}_{\rm Schw}-1)}} = \nu(\hat{H}_{\rm Schw}-1) - \frac{3}{2}\nu^2(\hat{H}_{\rm Schw}-1)^2 + \frac{5}{2}\nu^3(\hat{H}_{\rm Schw}-1)^3 + \cdots$$

$$\hat{H}^2_{\rm eff} = \hat{H}^2_{\rm Schw} + \frac{3}{2}\nu(1-2u)u^2(5\hat{H}_{\rm Schw}{}^2 - 1)(\hat{H}_{\rm Schw}-1) \times \left[1 - \frac{3}{2}\nu(\hat{H}_{\rm Schw}-1) + \frac{5}{2}\nu^2(\hat{H}_{\rm Schw}-1)^2 + \cdots\right] \quad (8.7$$

Singular (but without logs at LO) Light-Ring Behavior of Self-Force expansion in DJS gauge (Akcay-Barack-Damour-Sago'12)

$$\bar{A}^{\rm SF}(\bar{u};\nu) = 1 - 2\bar{u} + \nu a_{1\rm SF}(\bar{u}) + \nu^2 a_{2\rm SF}(\bar{u}) + O(\nu^3).$$

$$a_{1\rm SF}(\bar{u}) \underset{\bar{u}\to\frac{1}{3}}{\sim} \frac{1}{4}\zeta(1-3\bar{u})^{-1/2}, \quad \text{with} \quad \zeta \approx 1.$$

General Properties of Q(P;nu) Potential (Damour'18,19)

Mass-ratio dependence: nu=m1m2/(m1+m2)^2

$$h(\gamma;\nu) = \sqrt{s}/(m_1+m_2) = \sqrt{1+2\nu(\gamma-1)}$$

$$\begin{aligned}
q_2(\gamma,\nu) &= \hat{q}_2^{(1)}(\gamma)\left(1-\frac{1}{h(\gamma,\nu)}\right)\\
q_3(\gamma,\nu) &= \hat{q}_3^{(1)}(\gamma)\left(1-\frac{1}{h(\gamma,\nu)}\right)+\hat{q}_3^{(2)}(\gamma)\left(1-\frac{1}{h^2(\gamma,\nu)}\right)\\
q_4(\gamma,\nu) &= \hat{q}_4^{(1)}(\gamma)\left(1-\frac{1}{h(\gamma,\nu)}\right)+\hat{q}_4^{(2)}(\gamma)\left(1-\frac{1}{h^2(\gamma,\nu)}\right)\\
&+ \hat{q}_4^{(3)}(\gamma)\left(1-\frac{1}{h^3(\gamma,\nu)}\right) \qquad (9.10)
\end{aligned}$$

Some functions already known

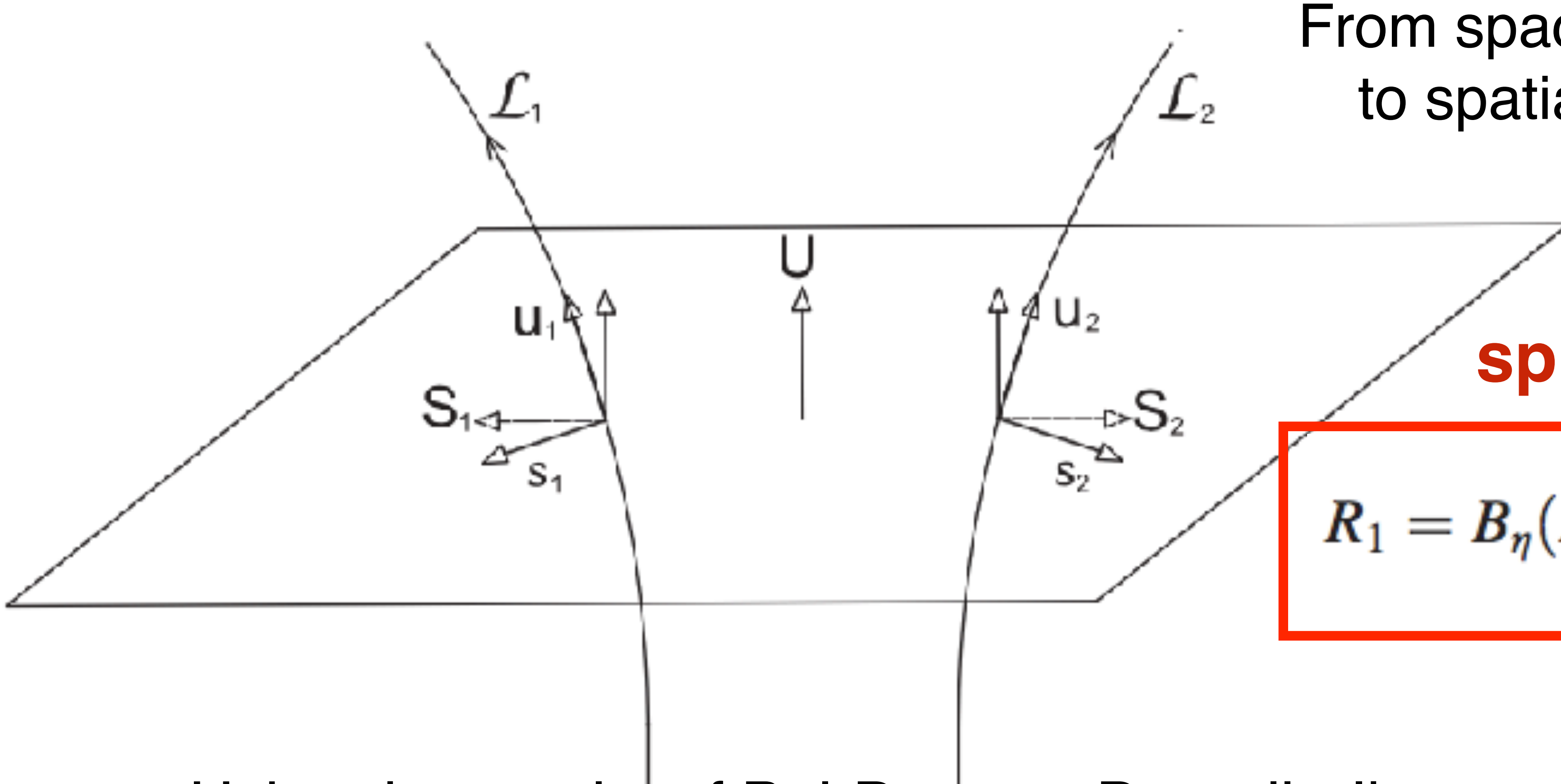
$$\hat{q}_2^{(1)}(\gamma) = \frac{3}{2}(5\gamma^2-1) \qquad \hat{q}_3^{(1)}(\gamma) = -\frac{2\gamma^2-1}{\gamma^2-1}\hat{q}_2^{(1)}(\gamma) = -\frac{3}{2}\frac{(2\gamma^2-1)(5\gamma^2-1)}{\gamma^2-1}$$

High-energy Behavior (following from SF Akcay-Barack-Damour-Sago'12)

$$\hat{q}_n^{(p)}(\gamma) \overset{\mathrm{HE}}{\sim} \gamma^2$$

without logs!

1PM and 2PM-accurate (one-loop) spin-orbit couplings



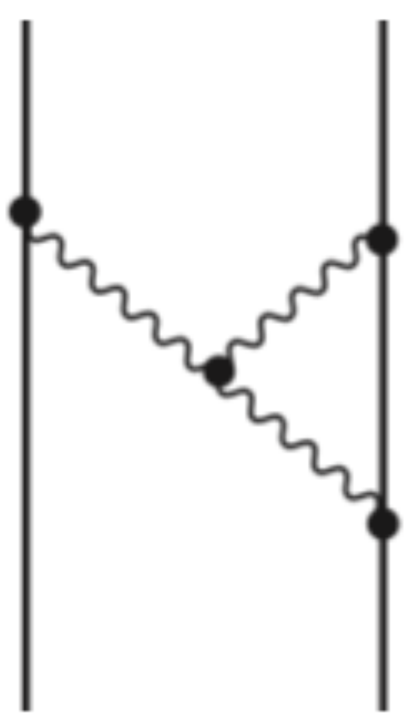
From spacetime spin four-vectors s_1,s_2 to spatial spin three-vectors S_1, S_2

Bini-Damour '17,'18 (see also Vines'18,Vines-Steinhoff-Buonanno'18, Guevara-Ochirov-Vines'18)

spin-scattering holonomy

$$R_1 = B_\eta(\Lambda_1 u_1^- \to U^{\rm as})\Lambda_1[B_\eta(u_1^- \to U^{\rm as})]^{-1}.$$

Using the results of Bel-Damour-Deruelle-Ibanez-Martin'81 for the 2PM metric, one gets the 2PM-accurate value of the integrated spin rotation (spin holonomy)



$$\theta_1 = -\frac{2}{hj\sqrt{\gamma^2-1}}\left[\gamma X_2 + (2\gamma^2-1)(X_1-h)\right] + \frac{\pi}{4h^2j^2}\left[-3(5\gamma^2-1)(X_1-h) - 6\gamma X_2 + \gamma(5\gamma^2-3)X_1X_2\right]\,. \quad (\text{5}$$

EOB transcription of the 2PM-accurate spin-rotation

energy spin-gauge
instead of DJS gauge

$$g_S = g_S^{1PM}(H_{\rm eff}) + g_S^{2PM}(H_{\rm eff})\, u + O(u^2)\,,$$
$$g_{S*} = g_{S*}^{1PM}(H_{\rm eff}) + g_{S*}^{2PM}(H_{\rm eff})\, u + O(u^2)\,,$$

$$\boldsymbol{\theta}_1^{\rm EOB} = G \int \frac{\mathbf{L}}{R^3}\left(g_S + \frac{m_2}{m_1} g_{S*}\right)\frac{B}{A} E_{\rm eff} \frac{dR}{P_R}\,,$$

$$g_S^{1PM}(\gamma,\nu) = \frac{(2\gamma+1)(2\gamma+h)-1}{h(h+1)\gamma(\gamma+1)} = \frac{1}{h(h+1)}\left[4 + \frac{h-1}{\gamma+1} + \frac{h-1}{\gamma}\right]$$

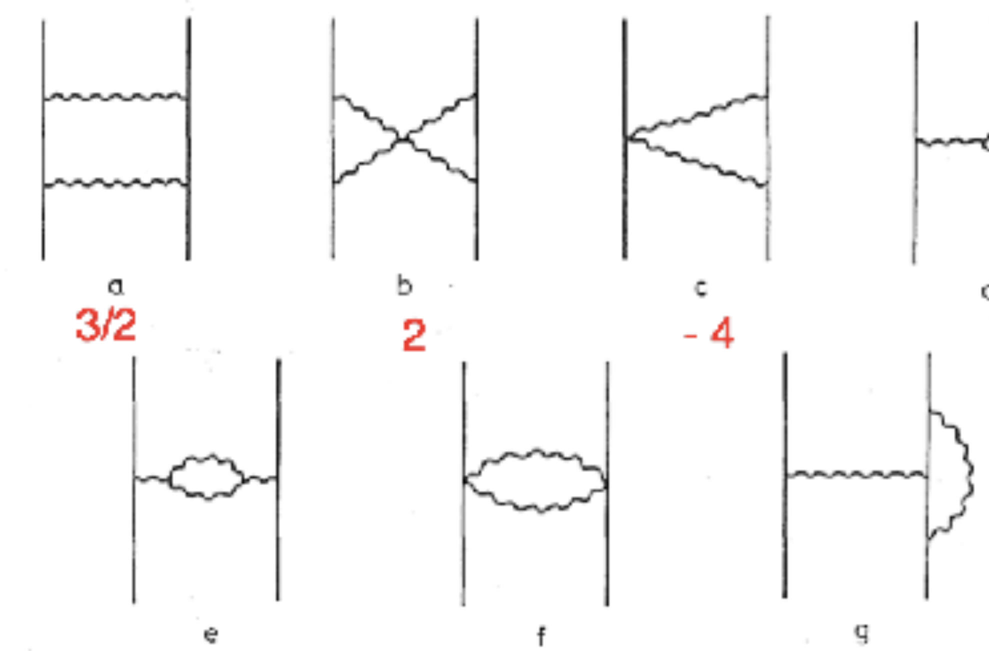
$$g_{S*}^{1PM}(\gamma,\nu) = \frac{2\gamma+1}{h\gamma(\gamma+1)} = \frac{1}{h}\left[\frac{1}{\gamma+1} + \frac{1}{\gamma}\right].$$

$$\gamma = \hat{H}_{\rm eff} \qquad h = \sqrt{1+2\nu(\gamma-1)}$$

$$g_S^{2PM}(\gamma,\nu) = -\frac{\nu}{\gamma(\gamma+1)^2 h^2 (h+1)^2}\left[2(2\gamma+1)(5\gamma^2-3)h + (\gamma+1)(35\gamma^3-15\gamma^2-15\gamma+3)\right]$$
$$= \frac{\nu}{h^2(h+1)^2}\left[-5(7\gamma+4h-10) + \frac{8(3h-4)}{\gamma+1} - \frac{4h}{(\gamma+1)^2} + \frac{3(2h-1)}{\gamma}\right]$$

$$g_{S*}^{2PM}(\gamma,\nu) = -\frac{1}{2\gamma(\gamma+1)^2 h^2 (h+1)}\left[(5\gamma^2+6\gamma+3)(h+1) + 4\nu(1+2\gamma)(5\gamma^2-3)\right]$$
$$= \frac{1}{h^2(h+1)}\left[-20\nu + \frac{24\nu-h-1}{\gamma+1} + \frac{h+1-4\nu}{(\gamma+1)^2} - \frac{3}{2}\frac{h+1-4\nu}{\gamma}\right]$$
$$= \frac{1}{h^2(h+1)}\left[-\frac{20\gamma\nu}{\gamma+1} + (h+1-4\nu)\left(\frac{1}{(\gamma+1)^2} - \frac{1}{\gamma+1} - \frac{3}{2}\frac{1}{\gamma}\right)\right].$$

Quantum Scattering Amplitudes and 2-body Dynamics



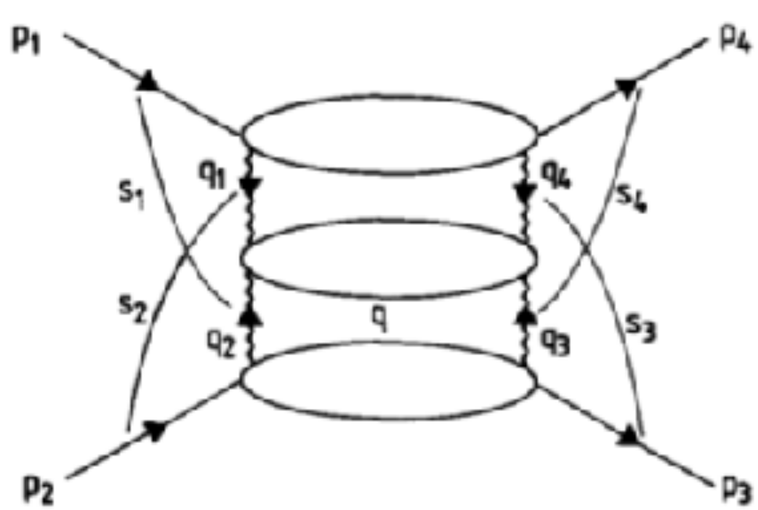


Fig. 3. The "H" diagram that provides the leading correction to the eikonal.

- Quantum Scattering Amplitudes —> Potential

one-graviton exchange :
Corinaldesi '56 '71,
Barker-Gupta-Haracz 66,
Barker-O'Connell 70, Hiida-Okamura72

Nonlinear: Iwasaki 71 [First post-Newtonian approx.],
Okamura-Ohta-Kimura-Hiida 73[2 PN]

Amati-Ciafaloni-Veneziano 1987-2008
Ultra-High-Energy (s >> M_Planck^2)
Four-graviton Scattering at 2 loops
Eikonal phase \delta in D=4
with one- and two-loop corrections
using the Regge-Gribov approach

confirmed by DiVecchia+'19

$$\delta = \frac{Gs}{\hbar}\left(\log\left(\frac{L_{IR}}{b}\right) + \frac{6\ell_s^2}{\pi b^2} + \frac{2G^2 s}{b^2}(1 + \frac{2i}{\pi}\log(\cdots))\right)$$

New technique: use EOB as a scattering-> Hamiltonian translation device

Progress in gravity amplitudes (Bern, Carrasco et al., Cachazo et al., Bjerrum-Bohr et al., Cachazo-Guevara,…) can be used (Damour '17CheungRothsteinSolon'18) to improve the classical 2-body dynamics: need a quantum/classical dictionary.

High-energy limit of 2-body scattering and 2-body dynamics

Using the (eikonal) ultra-high-energy results of Amati-Ciafaloni-Veneziano: get HE information up to G^4

$$\sin\frac{1}{2}\chi^{\rm ACV} \overset{\rm HE}{=} 2\alpha + (2\alpha)^3 + O(\alpha^5),$$

with $$\alpha_{\rm HE} = \frac{G E_{\rm real}}{b}$$

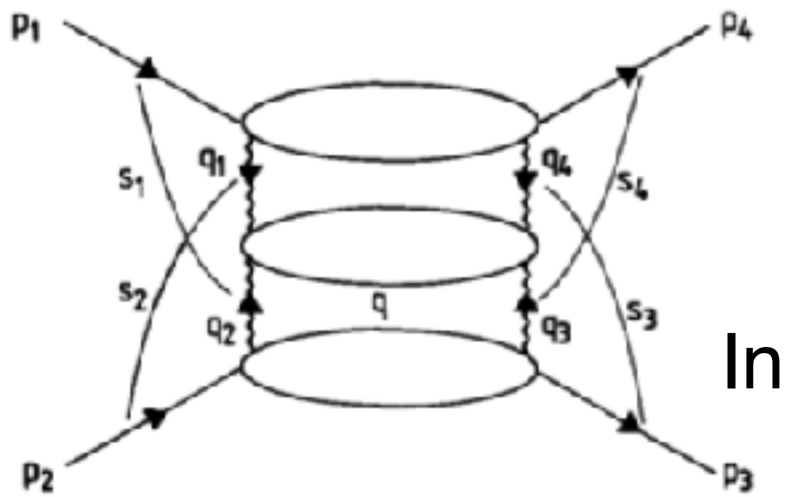


Fig. 3. The "H" diagram that provides the leading correction to the eikonal.

In HE limit the EOB energy map is such that

$$\alpha = \frac{G M E_{\rm eff}}{J} = \frac{G}{2}\frac{(E_{\rm real})^2 - m_1^2 - m_2^2}{J} \approx_{\rm HE} \frac{G E_{\rm real}}{b}$$

The masses disappear and the HE scattering is equivalent to a null geodesic in the « effective HE metric »

$$ds^2 = -A_{\rm HE}(u) dT^2 + \frac{dR^2}{1-2u} + R^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

$$A_{\rm HE}(u) = (1-2u)\left(1 + \frac{15}{2}u^2 - 3u^3 + \frac{1749}{16}u^4 + O(u^5)\right)$$

2PM 3PM 4PM

Translating quantum scattering amplitudes into classical dynamical information

How to translate a scattering amplitude into a classical Hamiltonian ?

$$\mathcal{M}(s,t) = \mathcal{M}^{(\frac{G}{\hbar})}(s,t) + \mathcal{M}^{(\frac{G^2}{\hbar^2})}(s,t) + \cdots$$

$$\mathcal{M}^{(\frac{G}{\hbar})}(s,t) = 16\pi \frac{G}{\hbar} \frac{2(p_1 \cdot p_2)^2 - p_1^2 p_2^2}{-t} .$$

NB: The domain of validity of the Born expansion is GE_1 E_2/(hbar v) << 1, while the domain of validity of the classical scattering is GE_1 E_2/(hbar v) >> 1!

Proposed quantum/classical transcriptions:

Damour'17: EOB potential Q(R,E)

Cheung-Rothstein-Solon'18: different potential V(R,P^2)

$$Q^E(u, \mathcal{E}_{\rm eff}) = u^2 q_2(\mathcal{E}_{\rm eff}) + u^3 q_3(\mathcal{E}_{\rm eff}) + u^4 q_4^E(\mathcal{E}_{\rm eff}) + O(G^5)$$

$$H(\mathbf{P},\mathbf{X}) = \sqrt{m_1^2 + \mathbf{P}^2} + \sqrt{m_2^2 + \mathbf{P}^2} + V(R, \mathbf{P}^2)$$

$$V(R, \mathbf{P}^2) = G \frac{c_1(\mathbf{P}^2)}{R} + G^2 \frac{c_2(\mathbf{P}^2)}{R^2} + G^3 \frac{c_3(\mathbf{P}^2)}{R^3} + \cdots$$

EOB map: quantize the classical EOB Hamiltonian dynamics.

2PM, O(G^2) EOB classical mass-shell condition

$$\mathbf{p}^2 = p_\infty^2 + w(r, p_\infty)$$

$$w(r, p_\infty) = \frac{w_1(\gamma)}{r} + \frac{w_2(\gamma)}{r^2} + \frac{w_3(\gamma)}{r^3} + \frac{w_4(\gamma)}{r^4} + \cdots$$

$$p_\infty^2 = \hat{\mathcal{E}}_{\rm eff}^2 - 1,$$

$$w_1 = 2(2\hat{\mathcal{E}}_{\rm eff}^2 - 1),$$

$$w_2 = \frac{3}{2}\frac{5\hat{\mathcal{E}}_{\rm eff}^2 - 1}{h(\hat{\mathcal{E}}_{\rm eff})}.$$

Quantized version:

$$-\widehat{\hbar}^2 \Delta_{\mathbf{x}} \psi(\mathbf{x}) = \left[p_\infty^2 + \frac{w_1}{r} + \frac{w_2}{r^2} + \frac{w_3}{r^3} + O\left(\frac{1}{r^4}\right)\right] \psi(\mathbf{x})$$

Scattering amplitude for this potential scattering computable by higher-order Born approximation

$$\psi_a^+ \underset{r\to\infty}{\approx} e^{i\mathbf{k}_a\cdot\mathbf{x}} + f_{\mathbf{k}_a}^+(\Omega)\frac{e^{ikr}}{r}$$

$$f_{\mathbf{k}_a}(\widehat{\mathbf{k}}_b) = +\frac{1}{4\pi\widehat{\hbar}^2}\langle\varphi_b|w|\psi_a^+\rangle$$

Classical/quantum dictionary: one-loop

from EOB q_2(E) <—> w_2(E) (classical)

$$f^{+\mathrm{B}1}_{\mathbf{k}_a}(\mathbf{k}_b) = \frac{1}{\hat{\hbar}^2}\left[e^{\delta_C}\frac{w_1}{q^2} + \frac{\pi}{2}\frac{w_2}{q}\right],$$

$\mathcal{M}^{G^2}/\mathcal{M}^{G^1}$

$$\frac{f^{+}_{(1/q)}}{f^{+}_{(1/q^2)}} = \frac{3\pi}{8}\frac{5\hat{\mathcal{E}}^2_{\mathrm{eff}}-1}{2\hat{\mathcal{E}}^2_{\mathrm{eff}}-1}\frac{G(m_1+m_2)\sqrt{-t}}{\hbar} + O(G^2).$$

OK with one-loop result of Guevara 1706.02314;
and Bjerrum-Bohr et al. 2018

Breakthrough: Bern et al.1901.04424: 2-loop amplitude
? gives 3PM O(G^3) EOB Hamiltonian: q_3 u^3

higher-order Born

$$f^{+}_{k_a}(k_b) = \frac{1}{\hat{\hbar}^2}\left[e^{\delta_C}(\frac{w_1}{q^2} + \frac{\pi}{2}\frac{w_2}{q}) + (w_3 \,\&\, w_1 w_2)\log q\right],$$

What is known at 3PM (G^3=2-loops) ?

$$q_3(\gamma,\nu) = A(\gamma) + \frac{B(\gamma)}{h(\gamma,\nu)} + \frac{C(\gamma)}{h^2(\gamma,\nu)}$$
$$= A(\gamma) + \frac{B(\gamma)}{h(\gamma,\nu)} - \frac{A(\gamma)+B(\gamma)}{h^2(\gamma,\nu)}$$

$$B(\gamma) = -\hat{q}_3^{(1)}(\gamma) = +\frac{3}{2}\frac{(2\gamma^2-1)(5\gamma^2-1)}{\gamma^2-1}$$
$$= +\frac{3}{2}\frac{(1+2\,p_{\rm eob}^2)(4+5\,p_{\rm eob}^2)}{p_{\rm eob}^2}$$
$$= \frac{6+\frac{39}{2}\,p_{\rm eob}^2+15\,p_{\rm eob}^4}{p_{\rm eob}^2}$$

$$A(\gamma) = A^{4PN}(\gamma) + O(p_{\rm eob}^4)$$

$$p_{\rm eob}^2 \equiv \gamma^2 - 1$$

$$A^{4PN}(\gamma) = \frac{2+\frac{37}{2}\,p_{\rm eob}^2+\frac{117}{10}\,p_{\rm eob}^4}{p_{\rm eob}^2}$$

$$A(\gamma) \overset{\rm HE}{=} a_3\, p_{\rm eob}^2$$ a_3^ACV=-3

Bern et al 19.

$$A^{B\,7PN}(\gamma) = \frac{1}{p_{\rm eob}^2}\Big(2+\frac{37}{2}p_{\rm eob}^2+\frac{117}{10}p_{\rm eob}^4+\frac{219}{140}p_{\rm eob}^6 - \frac{7079}{10080}p_{\rm eob}^8+\frac{989}{2240}p_{\rm eob}^{10}+O(p_{\rm eob}^{12})\Big).$$

Bini-Damour-Geralico'19

5PN

$$A^{3PN}(\gamma) = \frac{1}{p_{\rm eob}^2}\Big(2+\frac{37}{2}p_{\rm eob}^2+\frac{117}{10}p_{\rm eob}^4+\frac{219}{140}p_{\rm eob}^6\Big)$$

? non uniqueness from 6PN ?incompatible with HE behavior following both from SF result and ACV

Bern et al 19.+ Antonelli et al.19

$$q_{3PM} = -\frac{2\hat{H}_S^2-1}{\hat{H}_S^2-1}q_{2PM} + \frac{4}{3}\nu\hat{H}_S\frac{14\hat{H}_S^2+25}{1+2\nu(\hat{H}_S-1)} + \frac{8\nu}{\sqrt{\hat{H}_S^2-1}}\frac{4\hat{H}_S^4-12\hat{H}_S^2-3}{1+2\nu(\hat{H}_S-1)}\sinh^{-1}\sqrt{\frac{\hat{H}_S-1}{2}}.$$

From 2-loop amplitude to 3PM O(G^3) EOB Hamiltonian i.e. q_3 u^3

$$\hat{Q} = u^2 q_2(\hat{H}^{\rm Schw}_{\rm eff}) + u^3 q_3(\hat{H}^{\rm Schw}_{\rm eff}) + O(u^4).$$

$$\mathbf{p}^2 = p_\infty^2 + \bar{W}(\bar{u}) = p_\infty^2 + w_1 \bar{u} + w_2 \bar{u}^2 + w_3 \bar{u}^3 + \cdots$$

Potential scattering

$$-\hat{\hbar}^2 \Delta_{\mathbf{x}} \psi(\bar{x}) = W\left(\frac{1}{\bar{r}}\right) \psi(\bar{x})$$

$$\bar{W}(\bar{u}) = w_1 \bar{u} + w_2 \bar{u}^2 + w_3 \bar{u}^3 + \cdots$$

3PM

$$p_\infty^2 = \hat{\mathcal{E}}_{\rm eff}^2 - 1,$$

$$w_1 = 2(2\hat{\mathcal{E}}_{\rm eff}^2 - 1),$$

$$w_2 = \frac{3}{2} \frac{5\hat{\mathcal{E}}_{\rm eff}^2 - 1}{h(\hat{\mathcal{E}}_{\rm eff})}.$$

$$w_3(\hat{\mathcal{E}}_{\rm eff}) = -q_3(\hat{\mathcal{E}}_{\rm eff}) \,\&\, q_2(\hat{\mathcal{E}}_{\rm eff}) \,\&\, 1$$

Born expansion of the scattering amplitude

(Coulomb + second-order in w_1 w_2)

$$f^+_{k_a}(k_b) = \frac{1}{\hat{\hbar}^2}\left[e^{\delta_C} \frac{w_1}{q^2} + \frac{\pi}{2}\frac{w_2}{q} + (w_3 \,\&\, w_1 w_2) \log q \right],$$

directly determines q_3(E_eff)

Checks: **4PN**

$$q_3^{\rm PN}(\hat{H}_{\rm Schw}, \nu) = 5\nu + \frac{1}{4}(108\nu - 23\nu^2)(\hat{H}^2_{\rm Schw} - 1)$$ + 4PN

and EOB[ACV]

$$q_3(\hat{\mathcal{E}}_{\rm eff}; \nu) \overset{\rm HE}{=} -3\, \hat{\mathcal{E}}_{\rm eff}^2$$

Conjectural generalization of one-loop scattering of spinning particles to nonlinear-in-spin effects from exponentiated soft factors (Guevara-Ochirov-Vines'19)

$$\mathcal{M}_3^{(s)}(p_1, p_2, k^-) = \left(-\frac{\kappa}{2}\right) \times \frac{2(p\cdot\varepsilon)^2}{m^{2s}} \langle 2|^{2s} \exp\left(i\frac{k_\mu \varepsilon_\nu J^{\mu\nu}}{p\cdot\varepsilon}\right)|1\rangle^{2s},$$

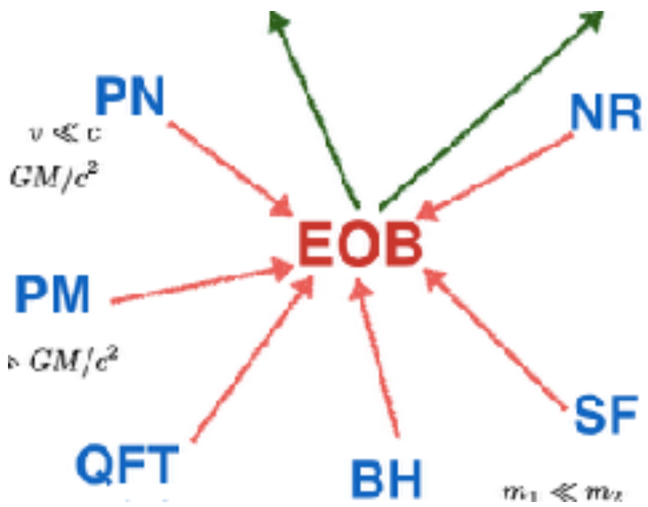
$$\theta = -\frac{E}{(2m_a m_b \gamma v)^2}\frac{\partial}{\partial b}\int \frac{d^2\boldsymbol{k}}{(2\pi)^2} e^{i\boldsymbol{k}\cdot\boldsymbol{b}} \lim_{s_a, s_b\to\infty} \langle \mathcal{M}_4^{(s_a,s_b)}\rangle + \mathcal{O}(G^3)$$

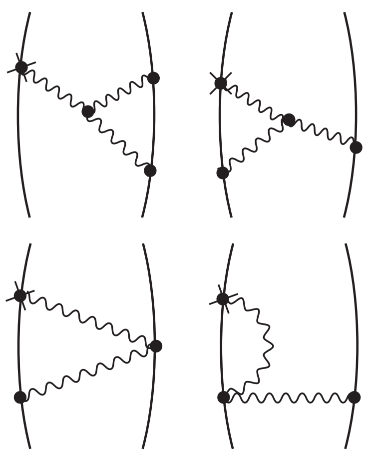
$$\theta = \frac{GE}{v^2}\left[\frac{(1+v)^2}{b+a_a+a_b} + \frac{(1-v)^2}{b-a_a-a_b}\right] - \pi G^2 E \frac{\partial}{\partial b}\Big[m_b f(a_a, a_b) + m_a f(a_b, a_a)\Big] + \mathcal{O}(G^3),$$

where $E = \sqrt{m_a^2 + m_b^2 + 2m_a m_b\gamma}$ with $\gamma = (1-v^2)^{-1/2}$, and

$$f(\sigma, a) = \frac{1}{2a^2}\left(-b + \frac{(\jmath + \varkappa - 2a)^5}{4v\varkappa\big[(\jmath+\varkappa)^2 - (2va)^2\big]^{3/2}}\right) + \mathcal{O}(\sigma^5),$$

Summary





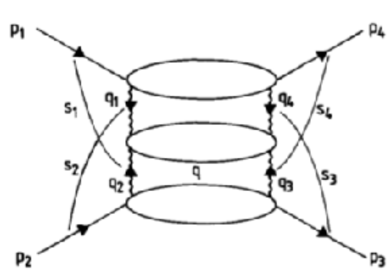


Fig. 3. The "H" diagram that provides the leading correction to the eikonal.

$$-\hbar^2\Delta_{\mathbf{x}}\psi(\mathbf{x}) = \left[p_\infty^2 + \frac{w_1}{\bar{r}} + \frac{w_2}{\bar{r}^2} + O\left(\frac{1}{\bar{r}^3}\right)\right]\psi(\mathbf{x}).$$

The EOB formulation of 2-body dynamics is a useful tool for transcribing classical and quantum scattering information into bound-state information.

The classical one-loop (G^ 2) scattering has been transcribed in EOB theory thereby giving new vistas on high-energy gravitational interactions.

The Amati-Ciafaloni-Veneziano 2-loop HE result has been transcribed in EOB theory.

A quantum/classical dictionary has been established in EOB theory (equivalent to the Cheung-Rothstein-Solon one)

The EOB-based dictionary allows for an easy and simple translation of two-loop quantum scattering amplitude of gravitationally interacting into a 3PM EOB Hamiltonian.

The proposed 3PM (2-loop) result of Bern-Cheung-Roiban-Shen-Solon-Zeng'19 has been confirmed at the 5PN level

Doubts (because of SF results) about the correctness (and uniqueness) of the conjectured resummed 3PM (2-loop) result of Bern-Cheung-Roiban-Shen-Solon-Zeng'19

EOB offers also a useful framework to transcribe spin effects from classical to quantum

.