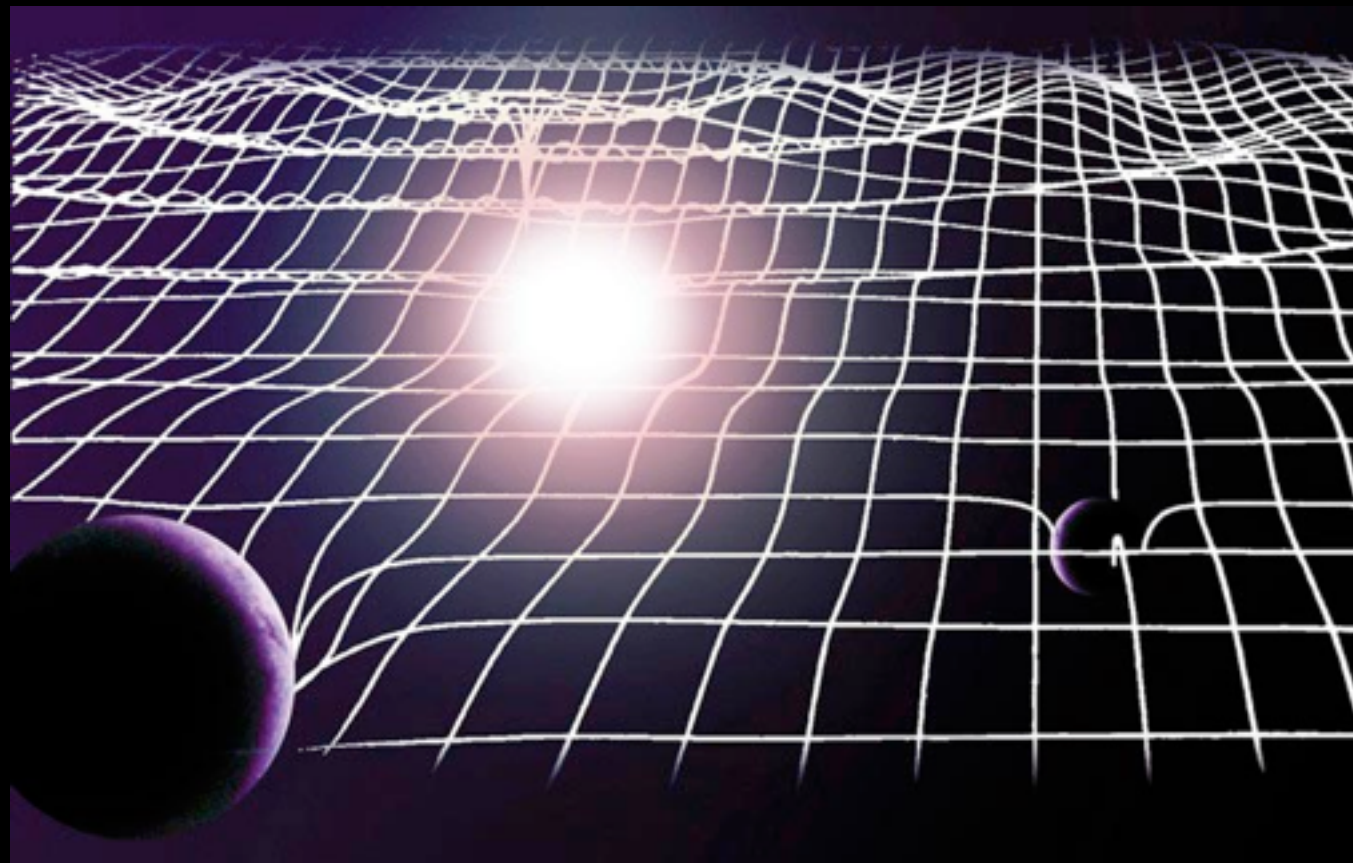
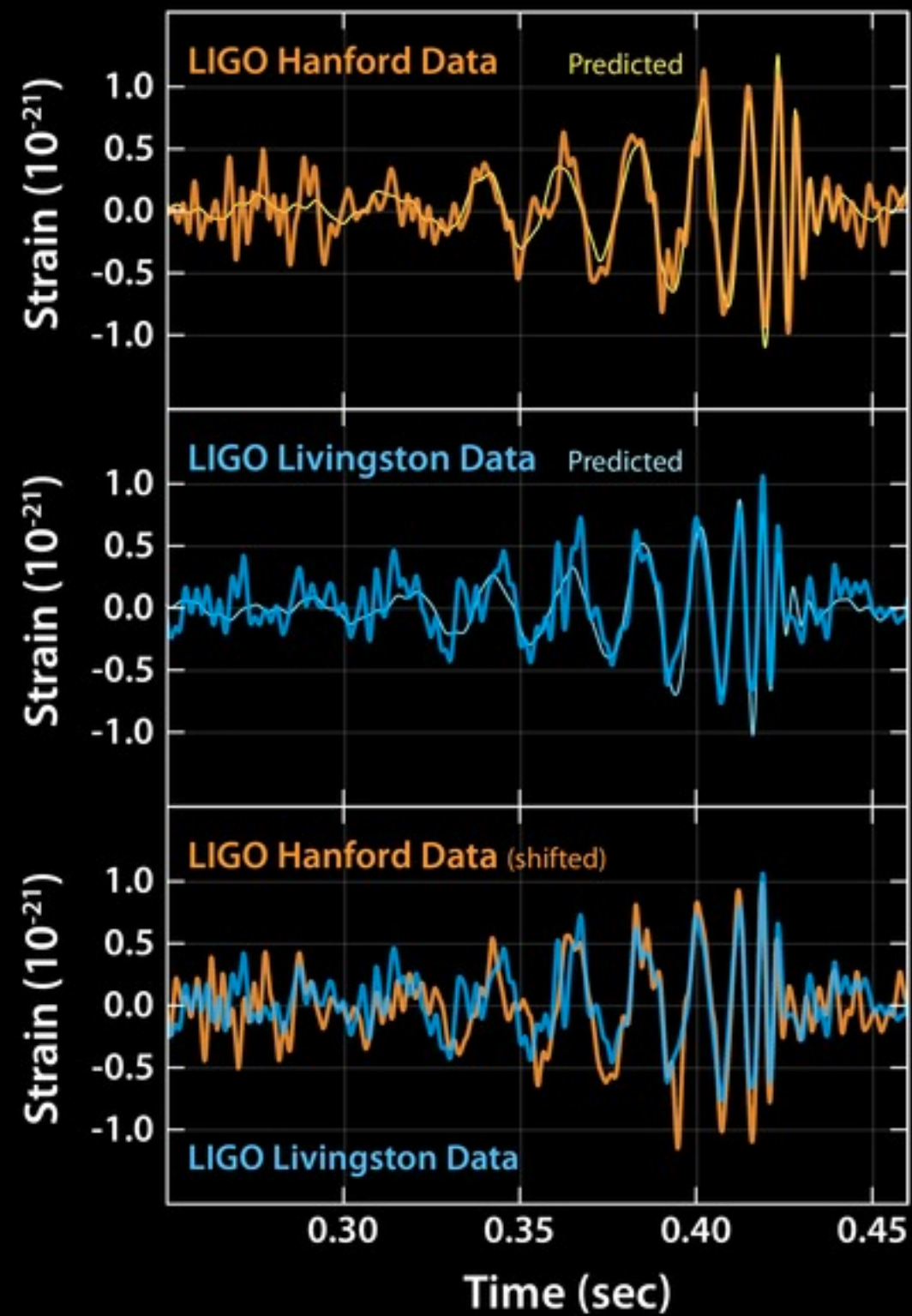


Dynamical Formation of Black Hole Mergers

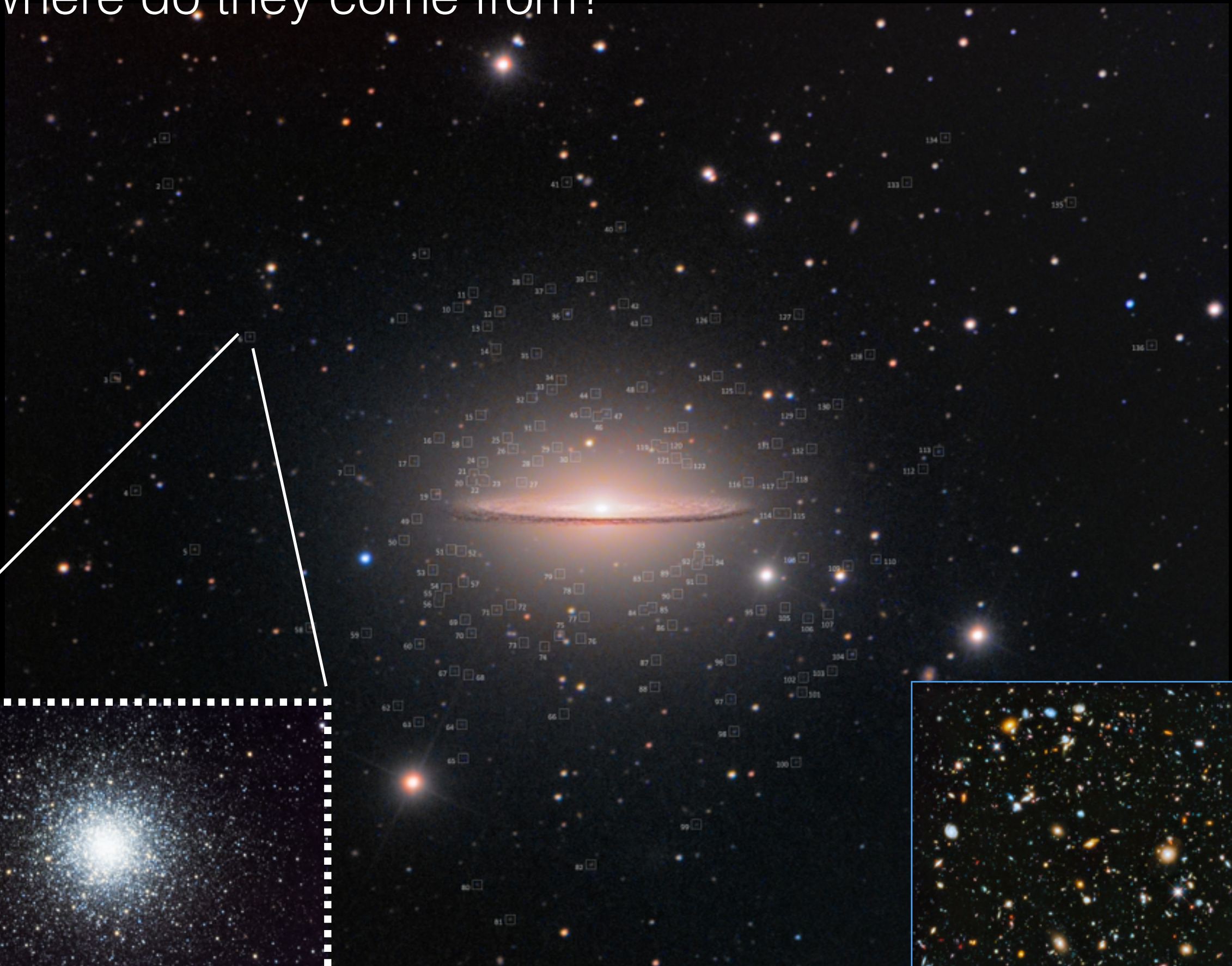
Johan Samsing
Princeton University



This is what we observe:

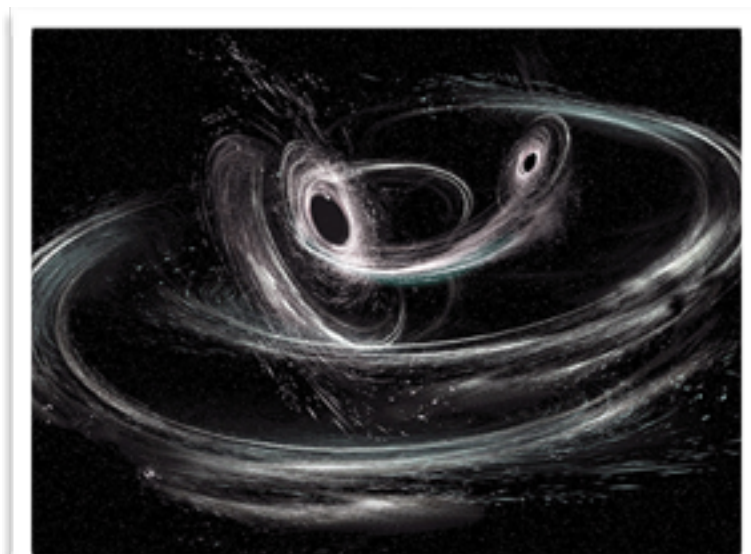


But where do they come from?

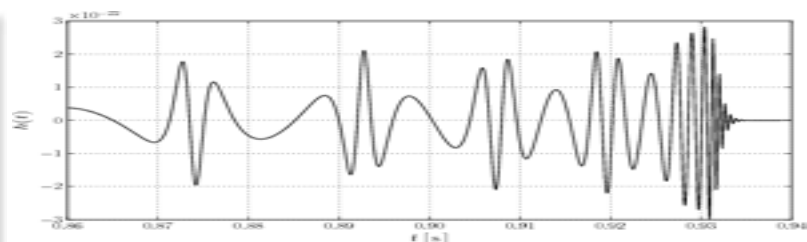


What is the origin of BBH mergers?

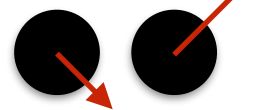
Clusters



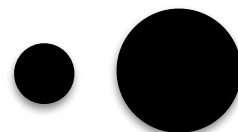
parameters



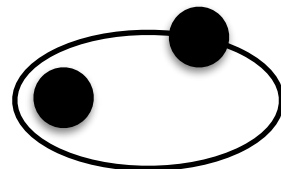
Spin



Mass



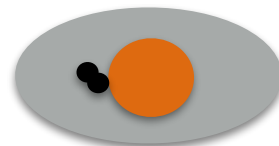
Eccentricity



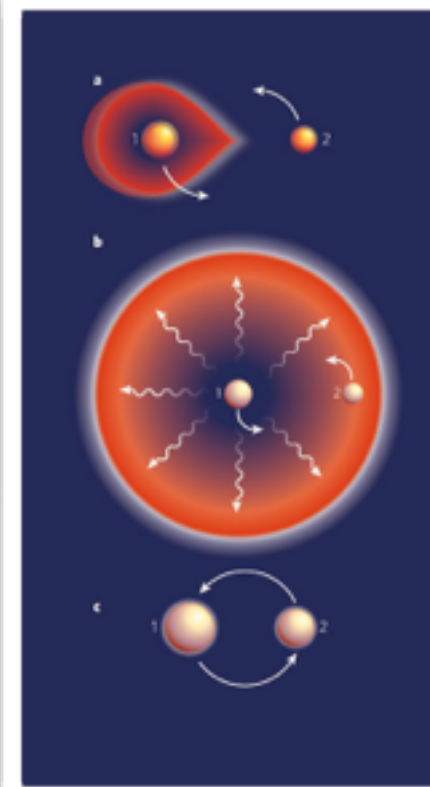
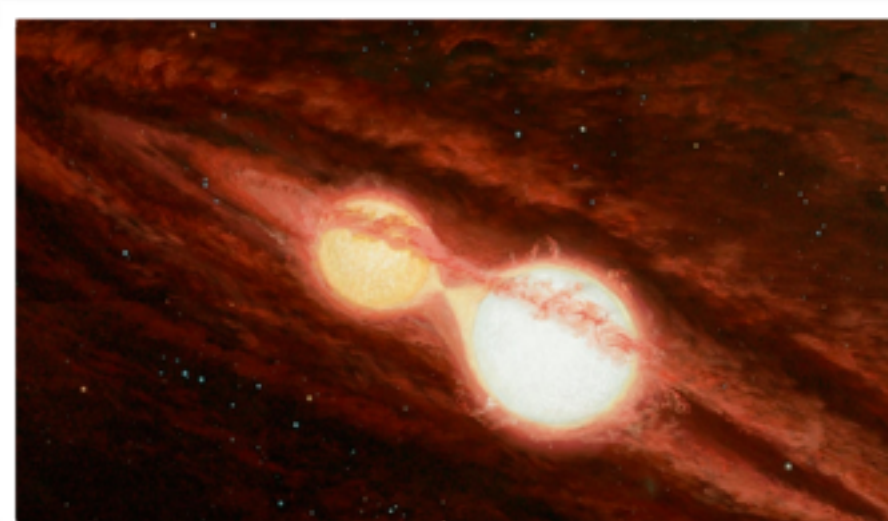
Rates



Location

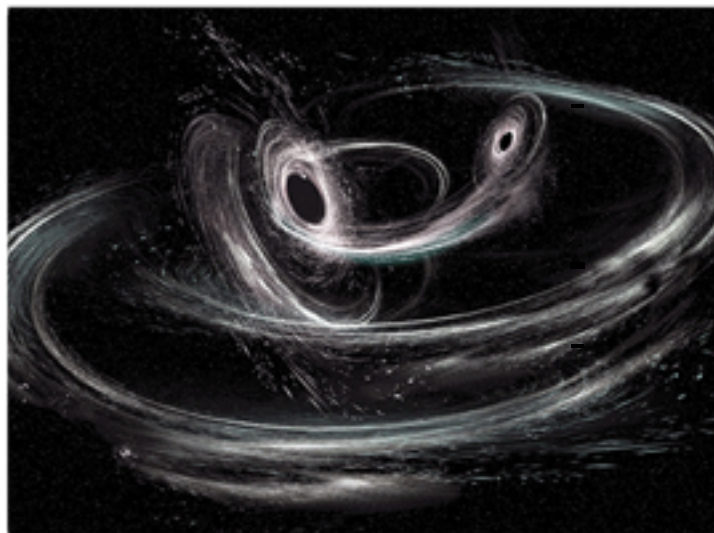


Field



What is the origin of BBH mergers?

Clusters



Previous studies:

Classical studies (global evolution.):

Piet Hut. John Bahcall. Lyman Spitzer.

Relativistic Studies (sin-sin GW cap.):

G. Quinlan. S. Shapiro. H.B. Lee.

Black Hole Mergers (analytical):

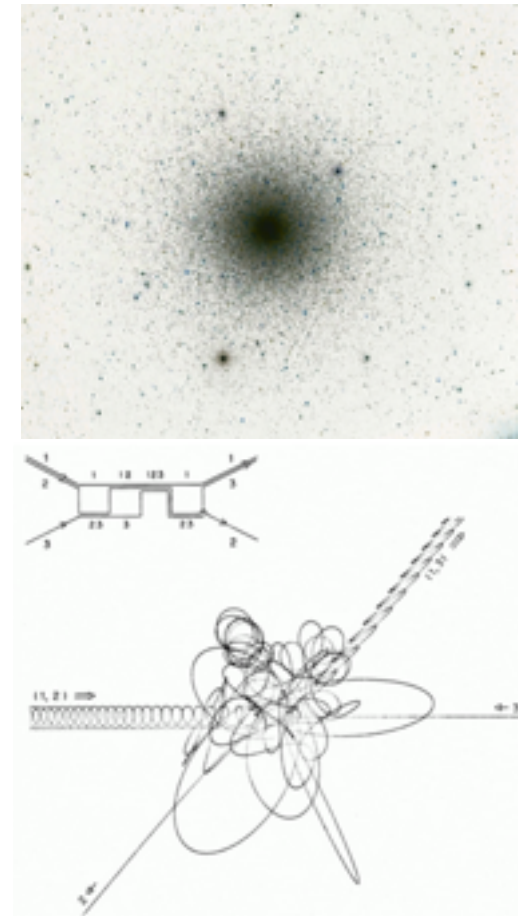
S. McMillan. S.P. Zwart.

Black Hole Mergers (Numerical):

Fred Rasio. Mirek Giersz.

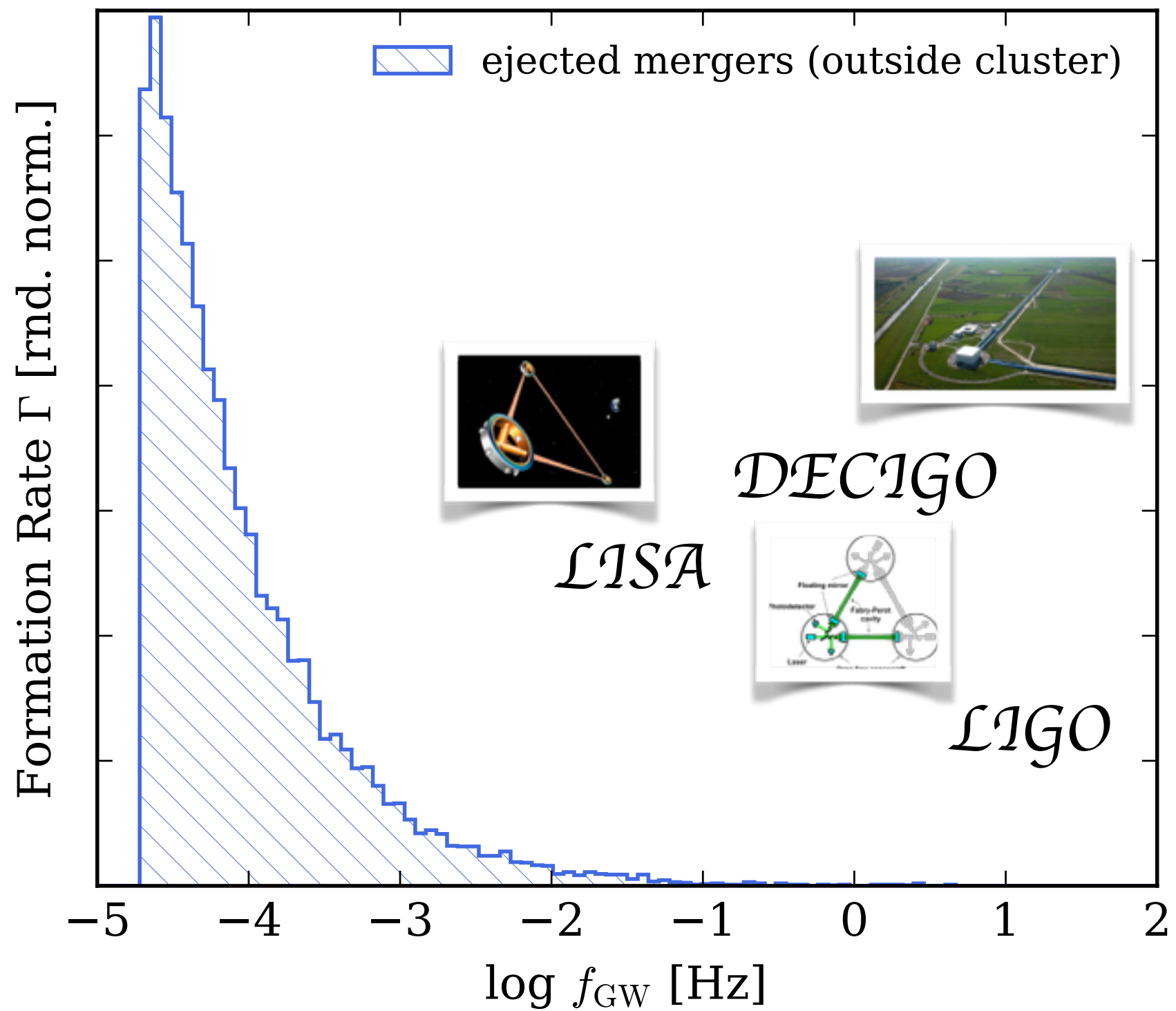
Black Hole Mergers (Relativistic):

Ongoing studies...

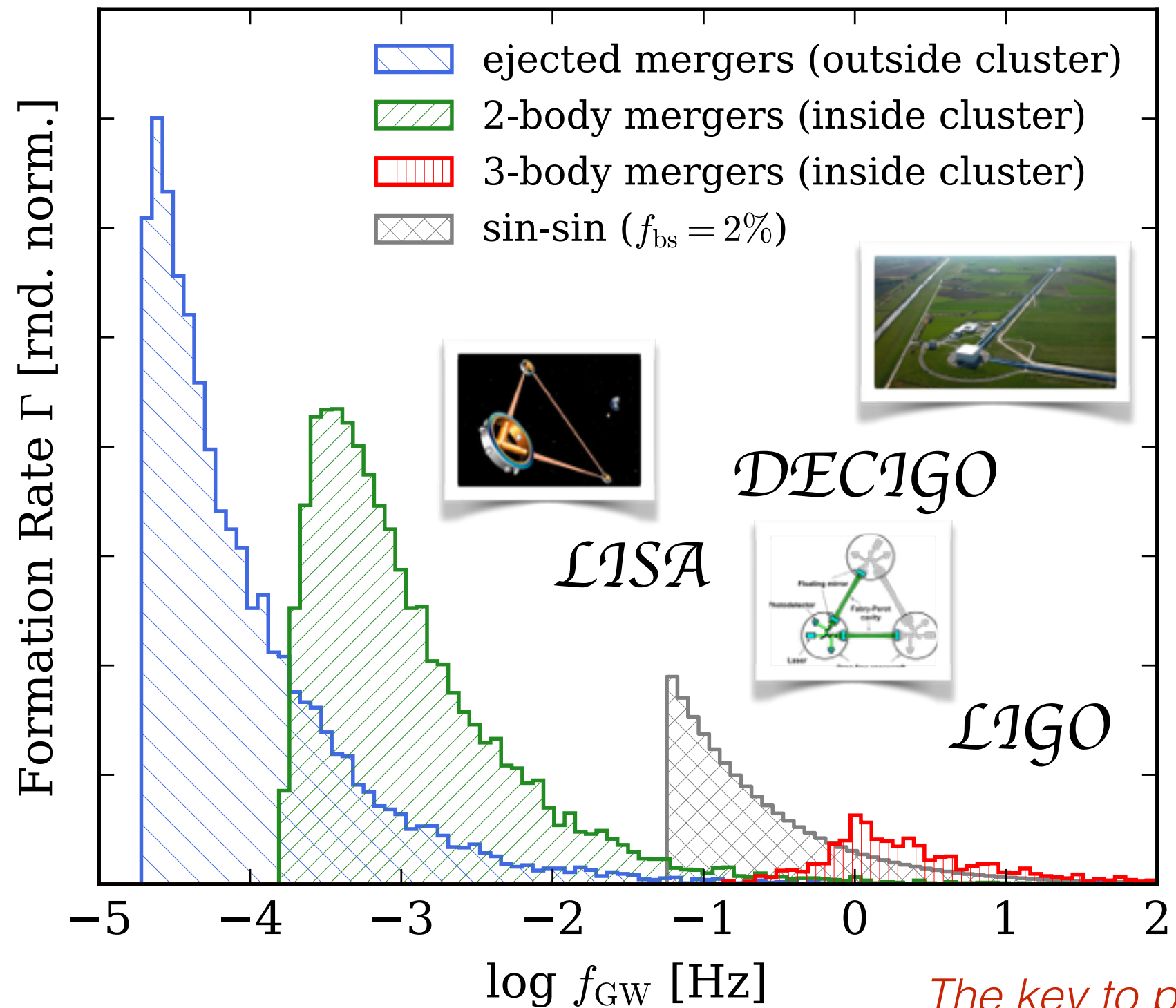


← a completely new picture is under development!

Old Newtonian Studies



Our new PN Studies



*The key to probe the
origin of BBH mergers!*

Background

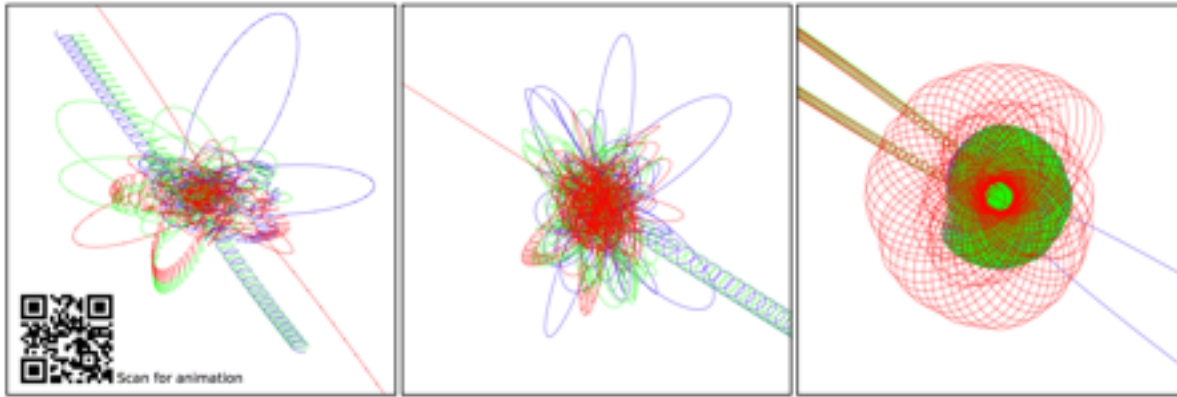
THE FORMATION OF ECCENTRIC COMPACT BINARY INSPIRALS AND THE ROLE OF GRAVITATIONAL WAVE EMISSION IN BINARY-SINGLE STELLAR ENCOUNTERS

JOHAN SAMSING¹, MORGAN MACLEOD², ENRICO RAMIREZ-RUIZ²

Draft version October 29, 2018

ABSTRACT

The inspiral and merger of eccentric binaries leads to gravitational waveforms distinct from those generated by circularly merging binaries. Dynamical environments can assemble binaries with high eccentricity and peak frequencies within the *LIGO* band. In this paper, we study binary-single stellar scatterings occurring in dense



$$\mathbf{a}_5 = \frac{4}{5} \frac{G^2 m_1 m_2}{r_{12}^3} \left[\left(\frac{2Gm_1}{r_{12}} - \frac{8Gm_2}{r_{12}} - v_{12}^2 \right) \mathbf{v}_{12} + (\hat{\mathbf{r}}_{12} \cdot \mathbf{v}_{12}) \left(\frac{52Gm_2}{3r_{12}} - \frac{6Gm_1}{r_{12}} + 3v_{12}^2 \right) \hat{\mathbf{r}}_{12} \right]$$

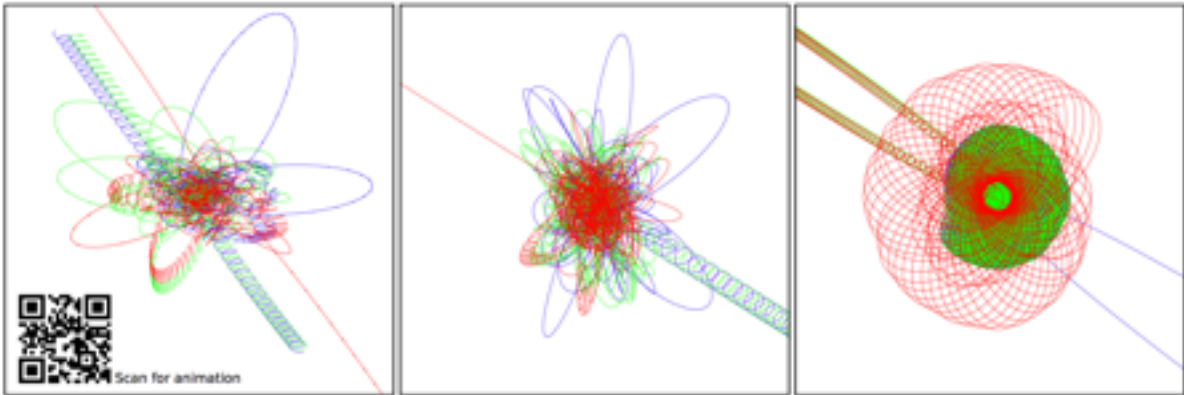
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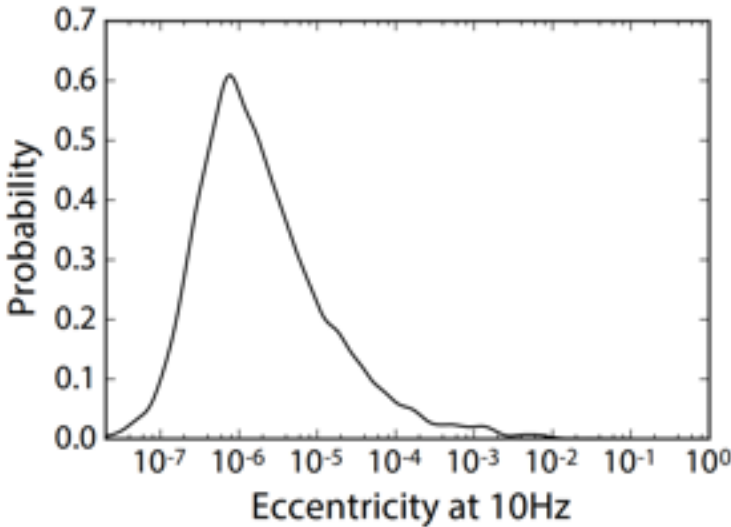


Binary Black Hole Mergers from Globular Clusters: Masses, Merger Rates, and the Impact of Stellar Evolution

Carl L. Rodriguez,¹ Sourav Chatterjee,¹ and Frederic A. Rasio¹

¹*Center for Interdisciplinary Exploration and Research in Astrophysics (CIERA) and Dept. of Physics and Astronomy, Northwestern University, 2145 Sheridan Rd, Evanston, IL 60208, USA*
(Dated: March 25, 2016)

The recent discovery of GW150914, the binary black hole merger detected by Advanced LIGO,



Background

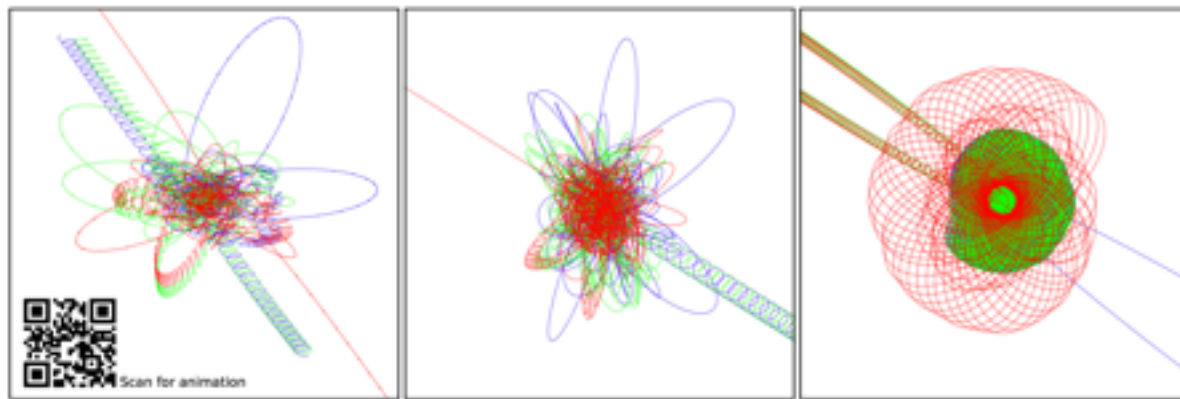
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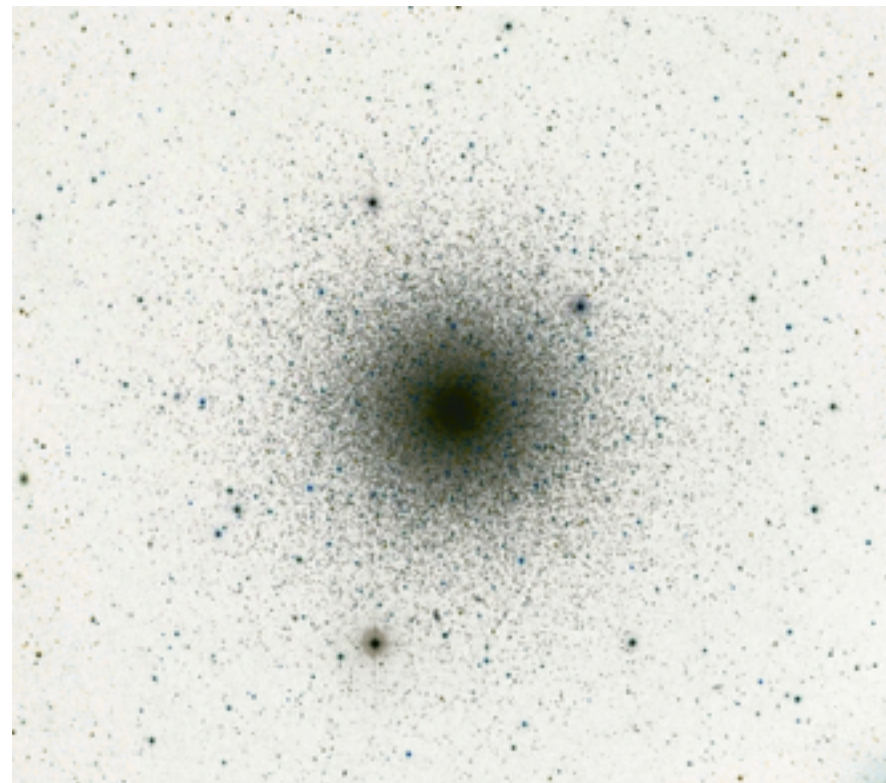
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Why is it so difficult?



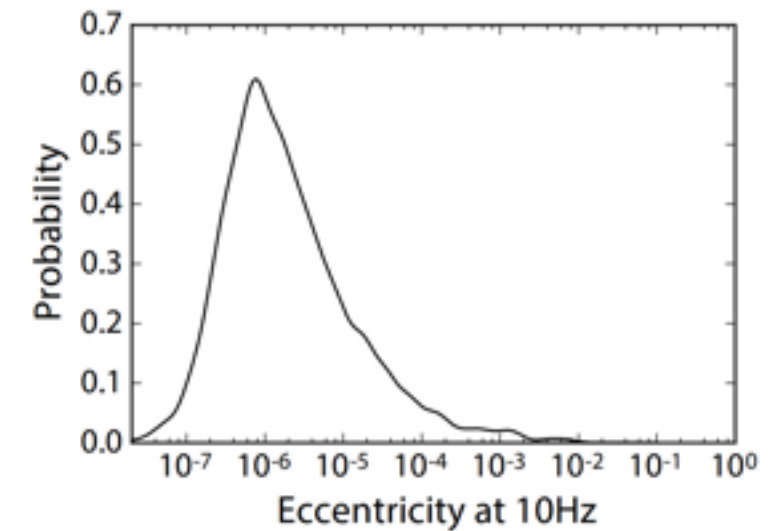
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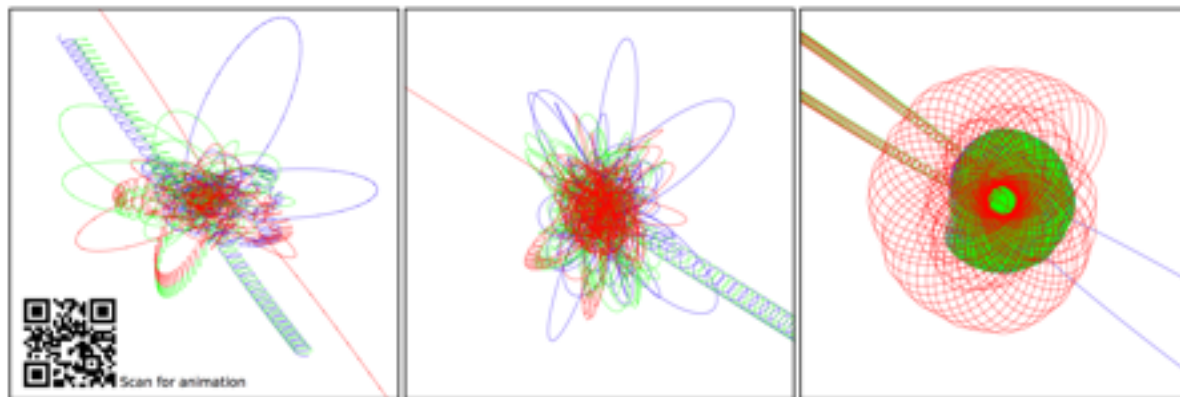
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Eccentric Black Hole Mergers Forming in Globular Clusters

Johan Samsing*

*Department of Astrophysical Sciences, Princeton University,
Peyton Hall, 4 Ivy Lane, Princeton, NJ 08544, USA.*

We derive the probability for a newly formed binary black hole (BBH) to undergo an eccentric gravitational wave (GW) merger during binary-single interactions inside a stellar cluster. By integrating over the hardening interactions such a BBH must undergo before ejection, we find that the observable rate of BBH mergers with eccentricity > 0.1 at 10 Hz relative to the rate of circular mergers can be as high as $\sim 5\%$ for a typical globular cluster (GC). This further suggests that BBH mergers forming through GW captures in binary-single interactions, eccentric or not, are likely to constitute $\sim 10\%$ of the total BBH merger rate from GCs. Such GW capture mergers can only

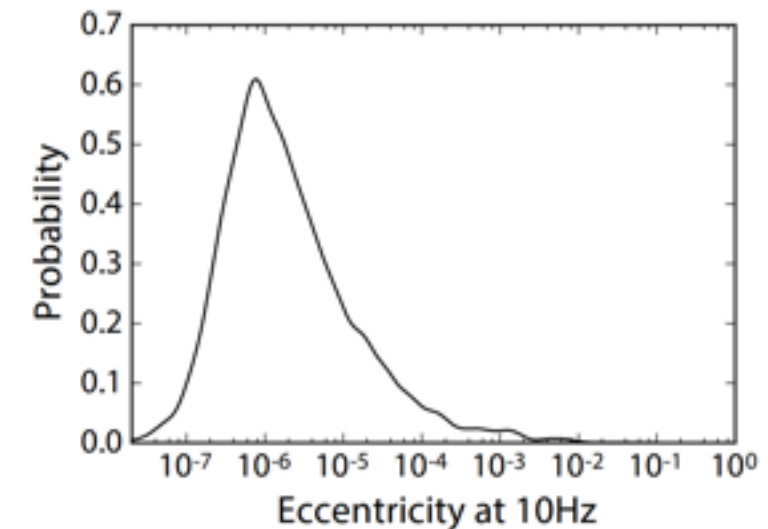
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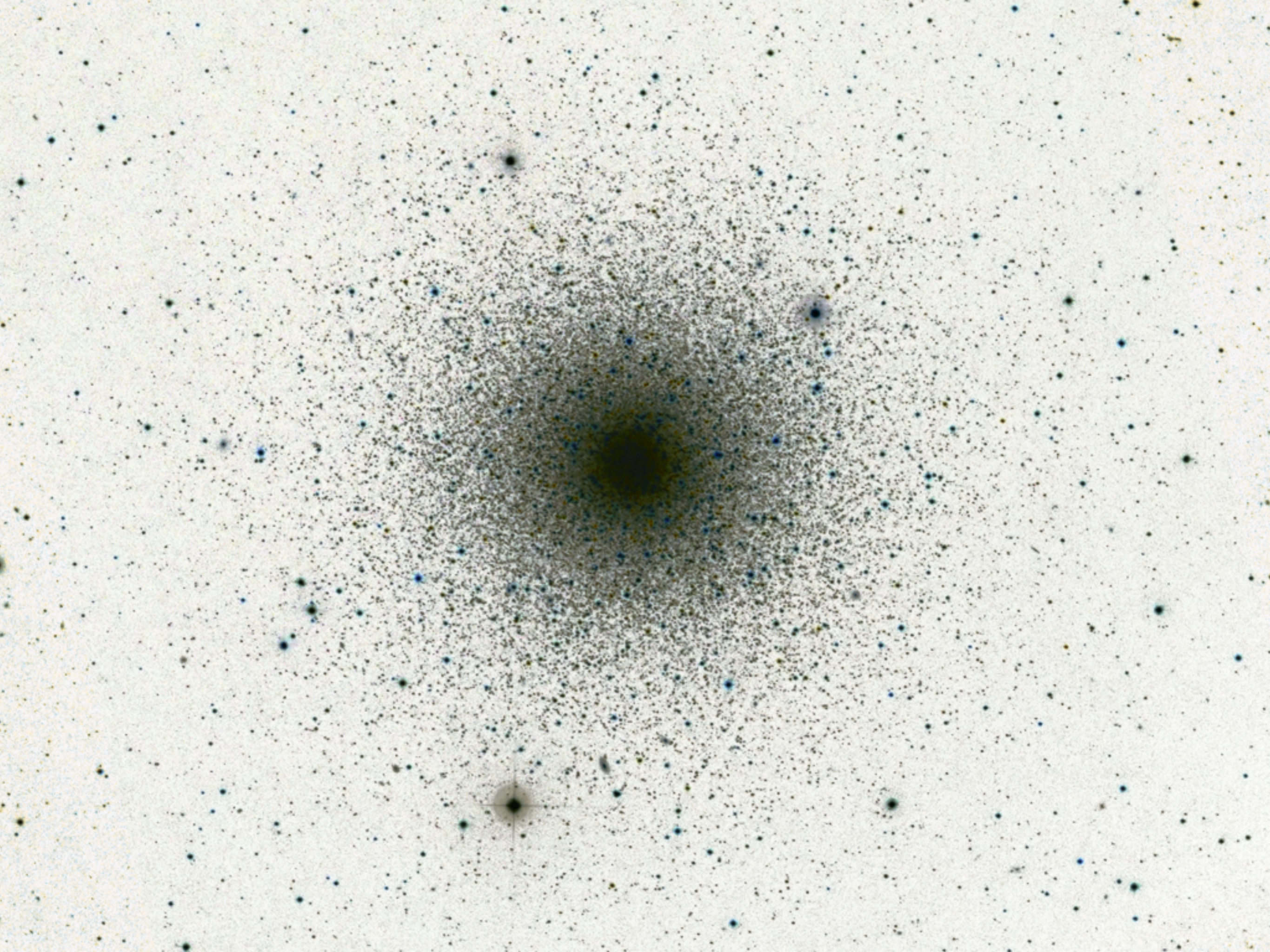
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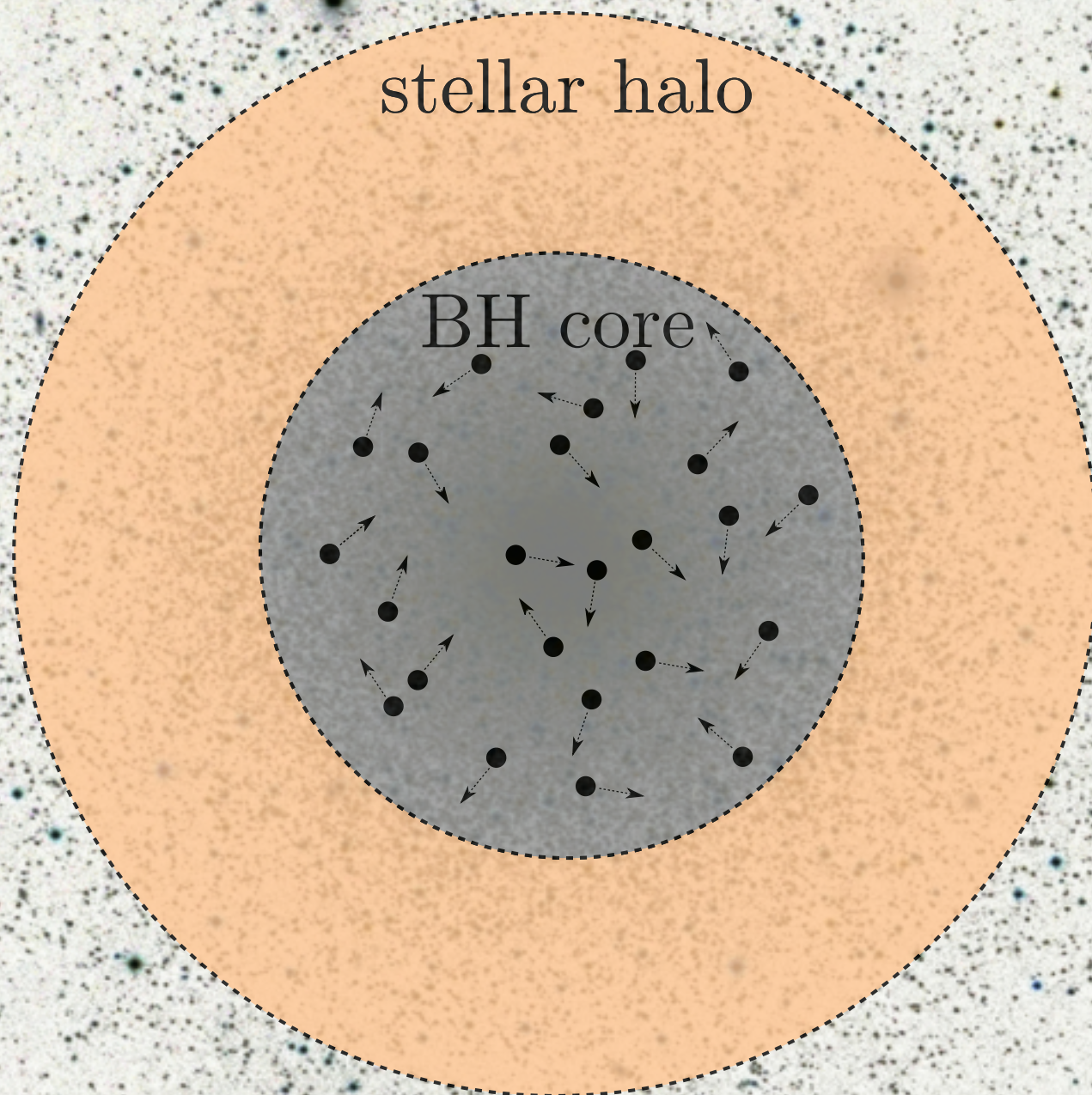


Post-Newtonian Dynamics in Dense Star Clusters: Formation, Masses, and Merger Rates of Highly-Eccentric Black Hole Binaries

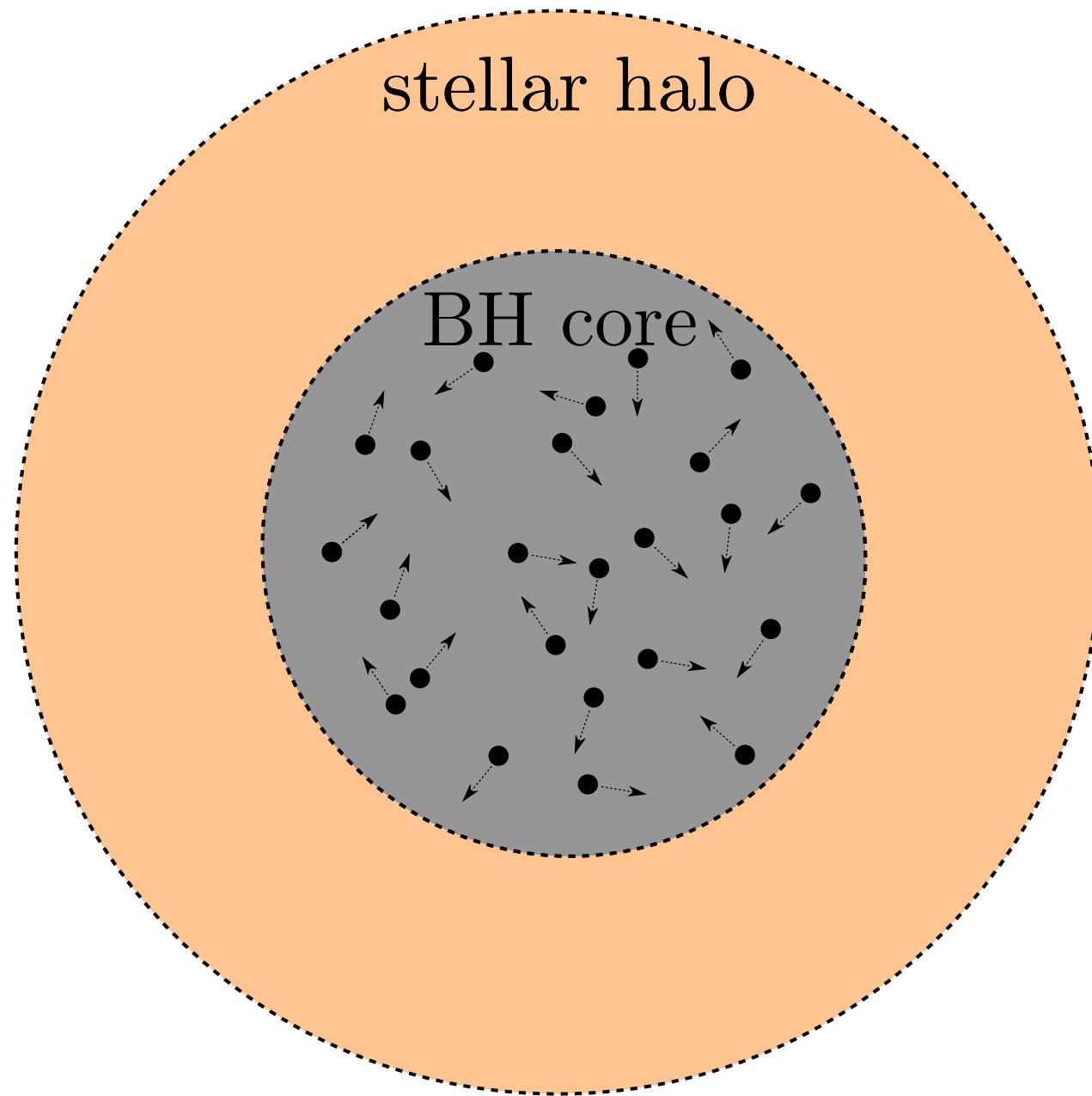
Carl L. Rodriguez,¹ Pau Amaro-Seoane,² Sourav Chatterjee,³ Kyle Kremer,⁴
Frederic A. Rasio,⁴ Johan Samsing,⁵ Claire S. Ye,⁴ and Michael Zevin⁴

Using state-of-the-art dynamical simulations of globular clusters, including radiation reaction during black hole encounters and a cosmological model of star cluster formation, we create a realistic population of dynamically-formed binary black hole mergers across cosmic space and time. We show that in the local universe, 10% of these binaries form as the result of gravitational-wave emission between unbound black holes during chaotic resonant encounters, with roughly half of those events having eccentricities detectable by current ground-based gravitational-wave detectors. The mergers





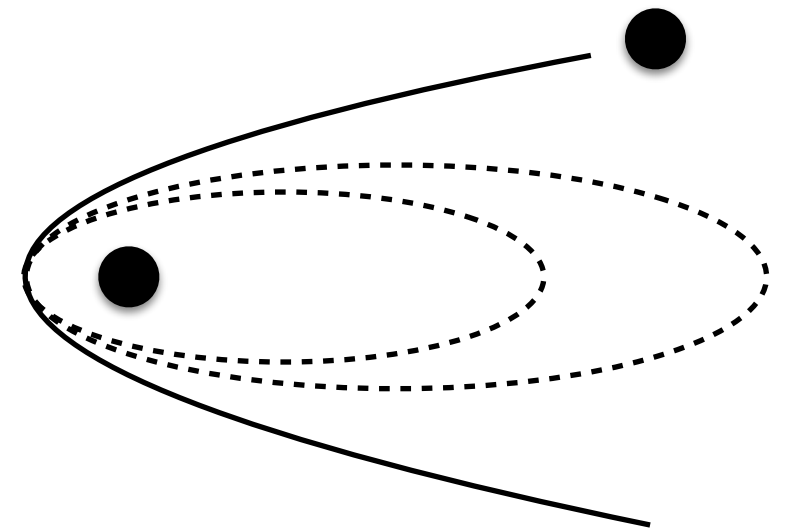
MODEL:



How do BBHs form and merge?

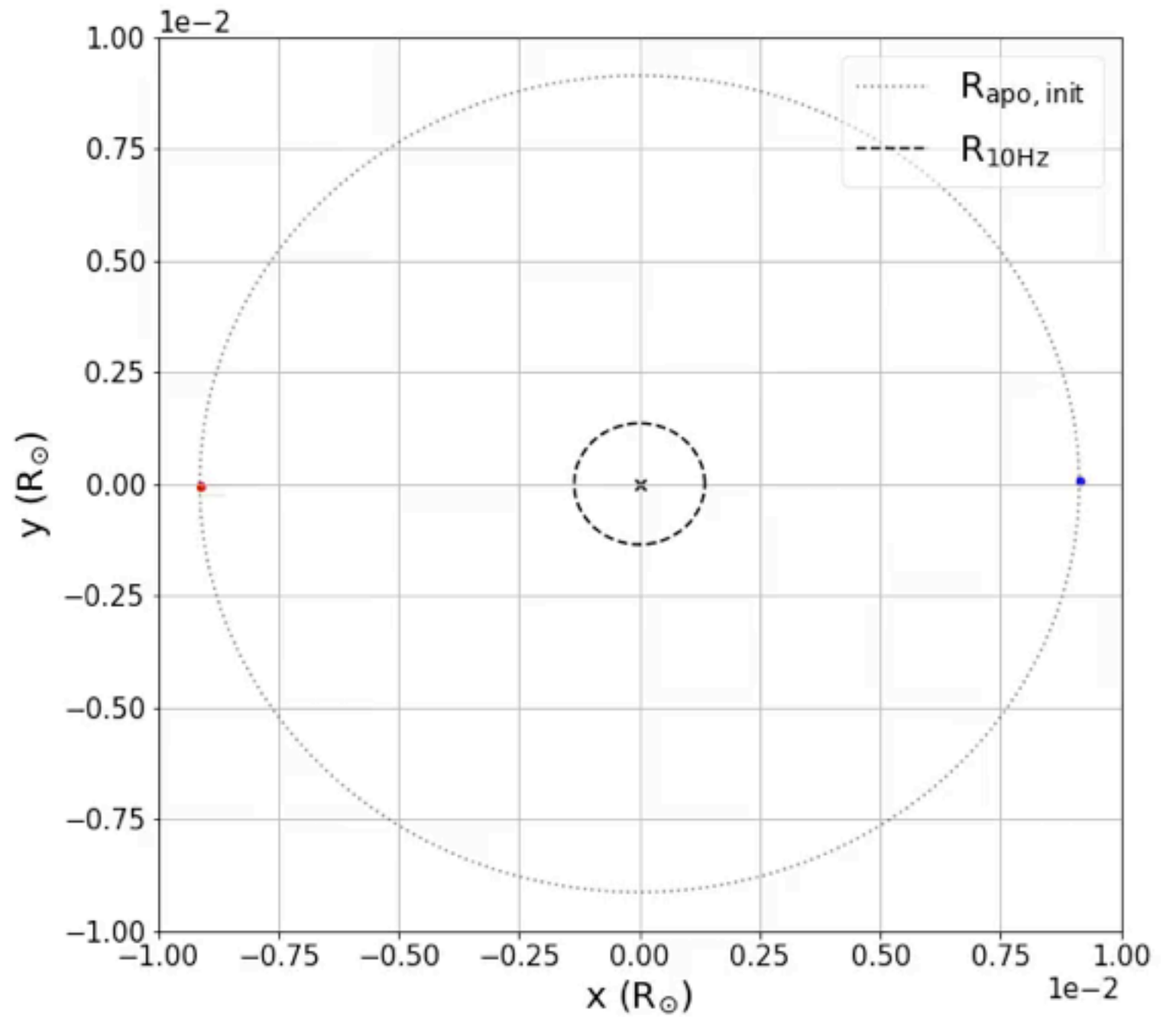
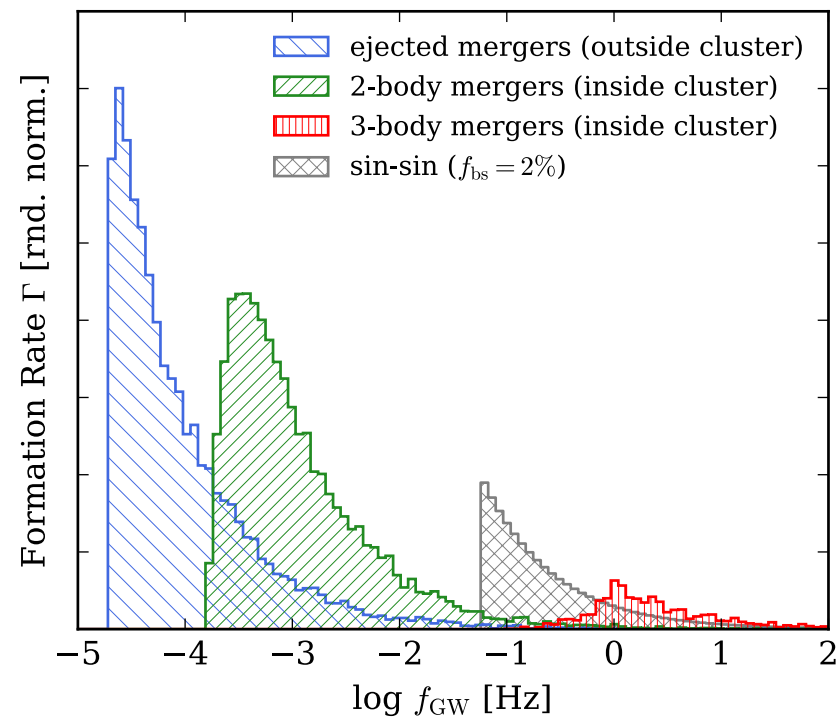
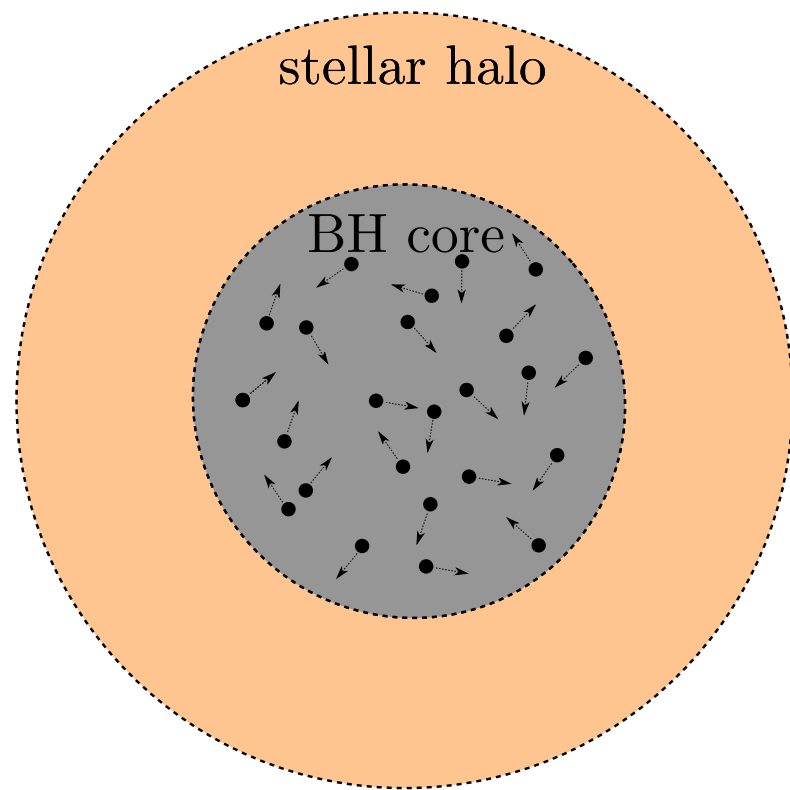
$$f \approx \frac{1}{\pi} \sqrt{\frac{2Gm}{r_f^3}}$$

$$r_f \approx \left(\frac{2Gm}{f^2 \pi^2} \right)^{1/3}$$



What is the peak freq. dist?
Correlate this with e!

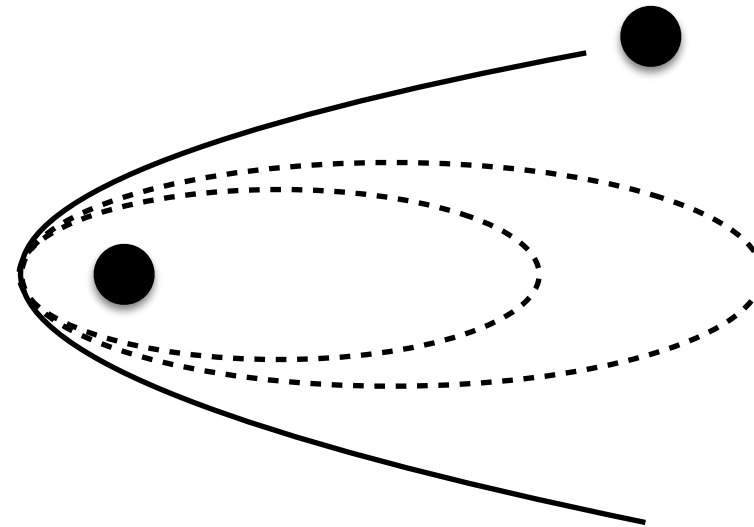
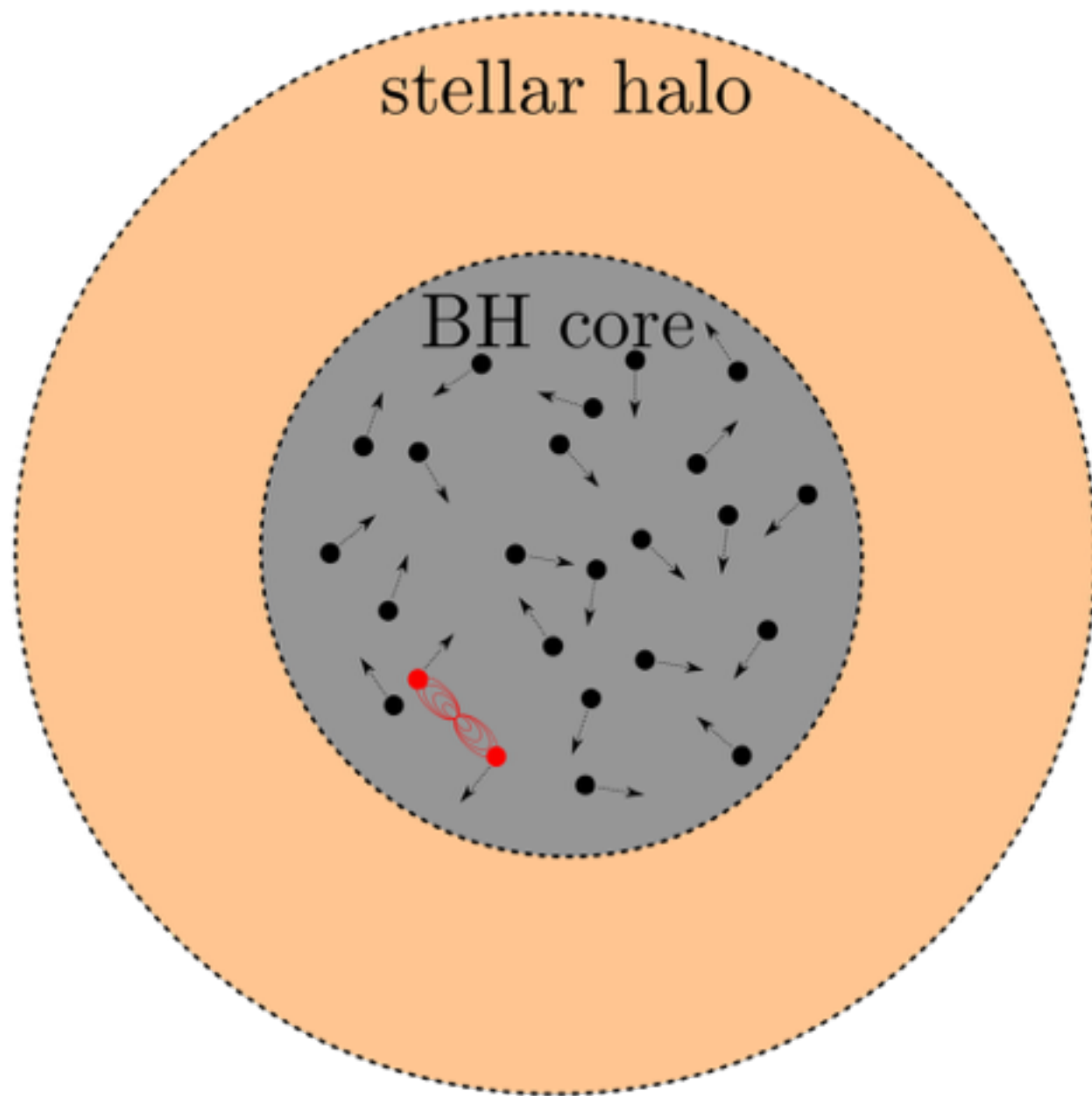
MODEL:



$$f \approx \frac{1}{\pi} \sqrt{\frac{2Gm}{r_f^3}}$$

Merger Type: Single-Single

Capture:

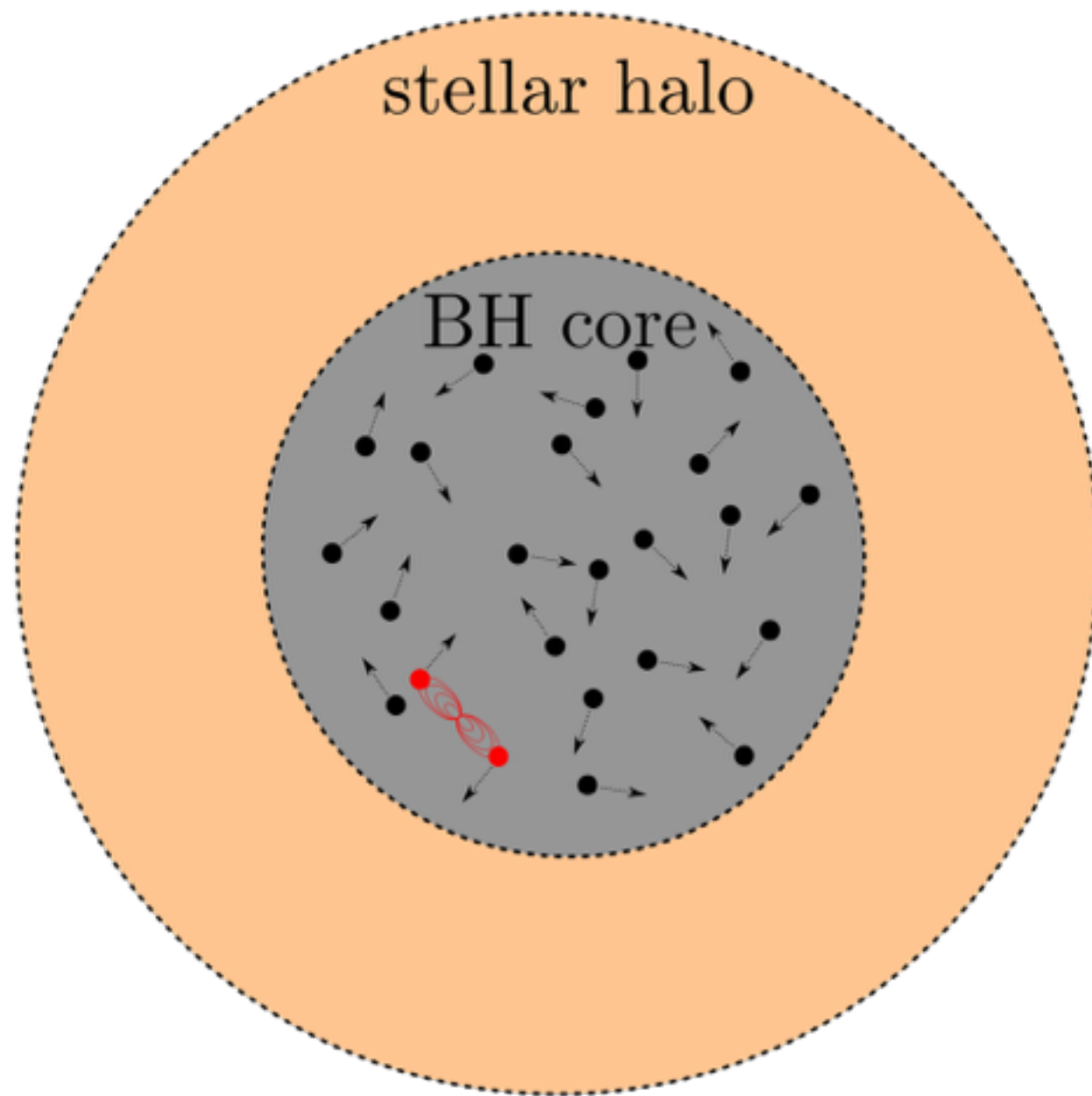


$$\Delta E_p \approx (85\pi/12)G^{7/2}c^{-5}m^{9/2}r_p^{-7/2}$$

$$E_{ss} \approx \mu v^2/2$$

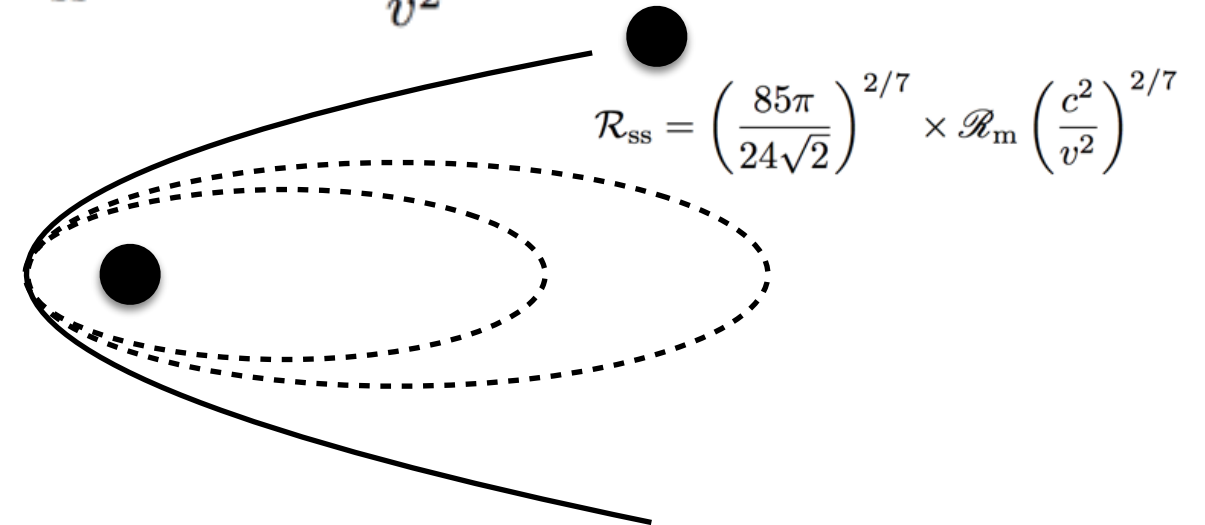
$$\mathcal{R}_{ss} = \left(\frac{85\pi}{24\sqrt{2}}\right)^{2/7} \times \mathcal{R}_m \left(\frac{c^2}{v^2}\right)^{2/7}$$

Merger Type: Single-Single



Cross section:

$$\sigma_{\text{ss}}^{<R} \approx 2\pi G \frac{2m R_{\text{ss}}}{v^2}$$



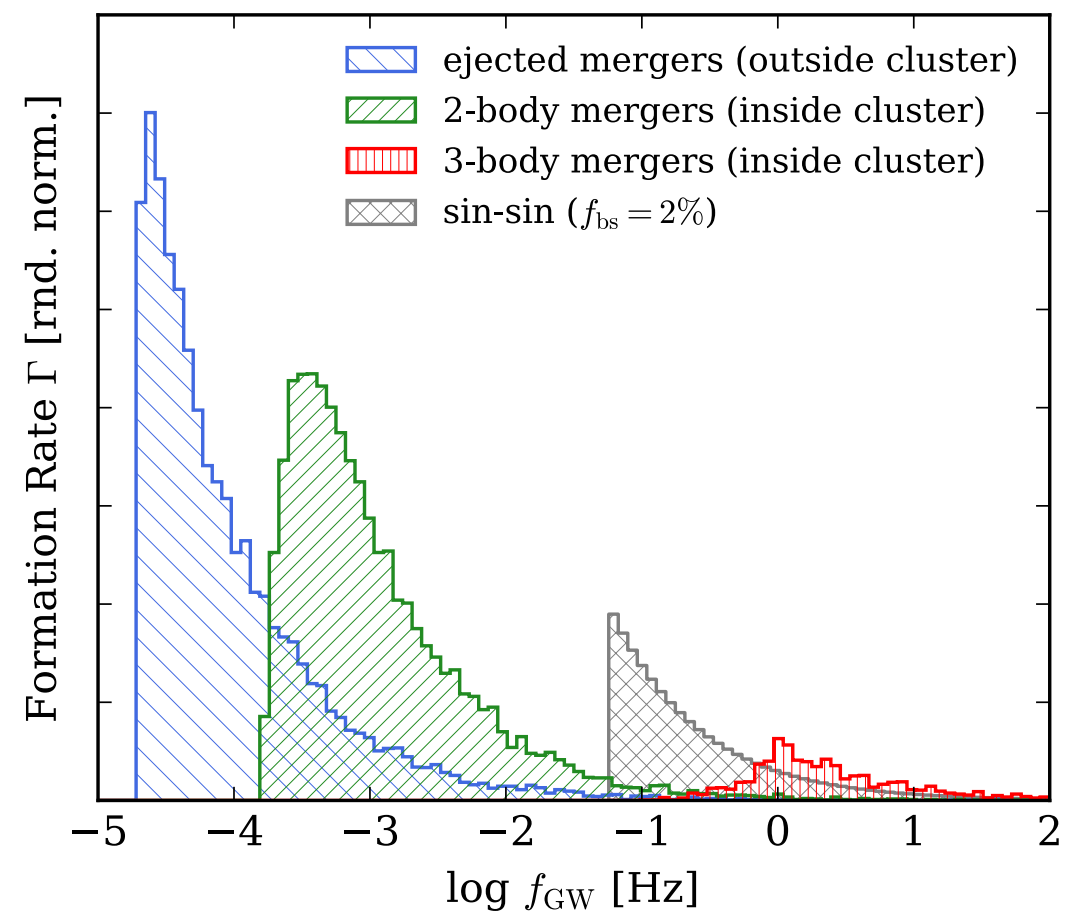
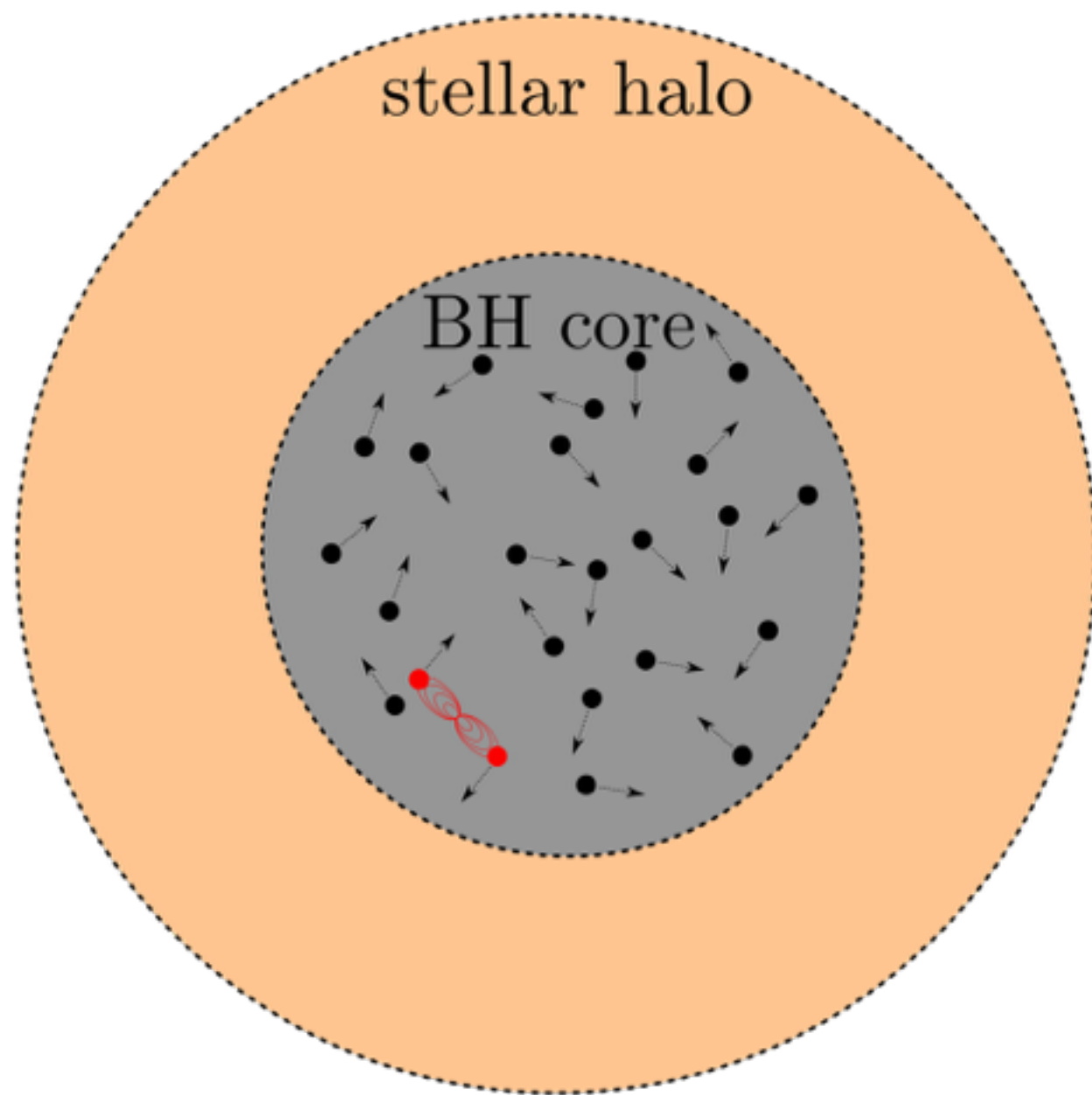
Rate:

$$d\Gamma_{\text{ss}}(r) = n(r) \sigma_{\text{ss}}^{<R}(r) v(r) \times n(r) 4\pi r^2 dr \times \frac{1}{2}$$

$$\Gamma_{\text{ss}} = \frac{8\pi^2 G}{m} \int_0^\infty \frac{R_{\text{ss}} \rho^2}{v} r^2 dr$$

$$\begin{aligned} \Gamma_{\text{ss}} &= n_0 \sigma_{\text{ss},0}^{<R} v_0 \frac{1}{2} N_s \times \frac{4}{3} \frac{\pi r_s^3 \rho_0}{M} \int_0^\infty \frac{\tilde{R}_{\text{ss}} \tilde{\rho}^2}{\tilde{v}} 3\tilde{r}^2 d\tilde{r} \\ &= n_0 \sigma_{\text{ss},0}^{<R} v_0 \frac{1}{2} N_s \times \xi_{\text{ss}}, \end{aligned}$$

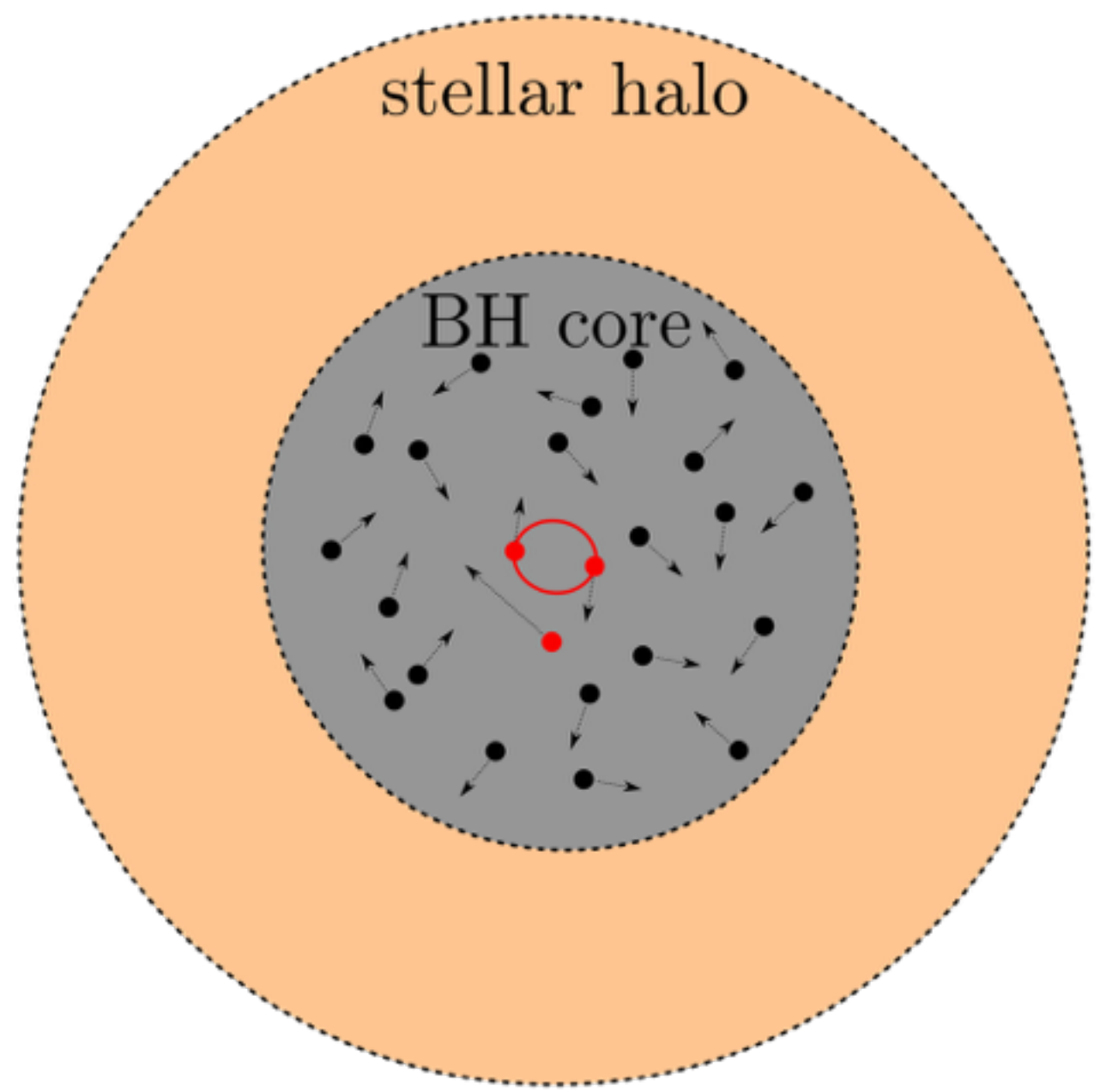
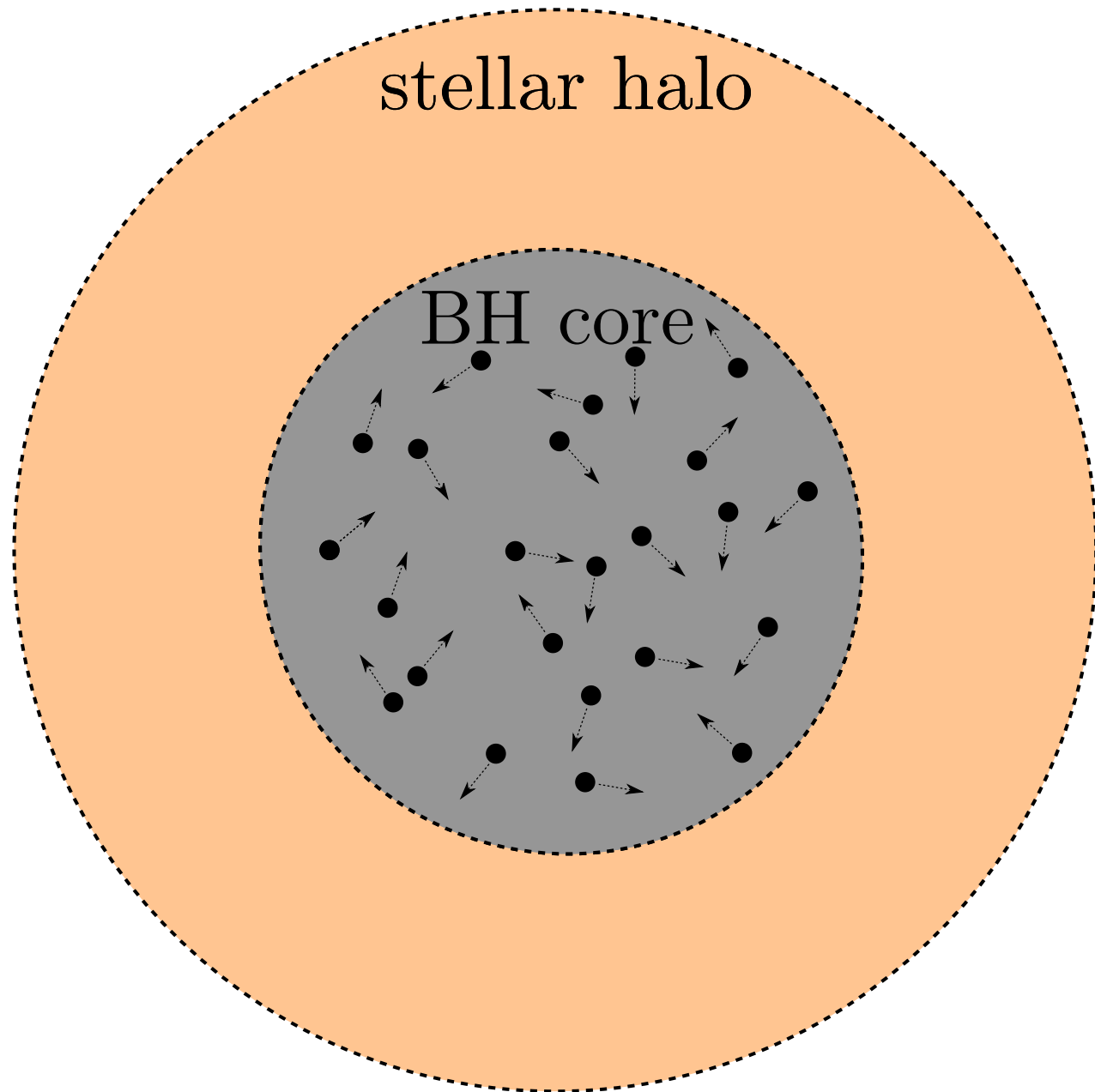
Merger Type: Single-Single



$$\mathcal{R}_{\text{ss}} = \left(\frac{85\pi}{24\sqrt{2}} \right)^{2/7} \times \mathcal{R}_{\text{m}} \left(\frac{c^2}{v^2} \right)^{2/7}$$

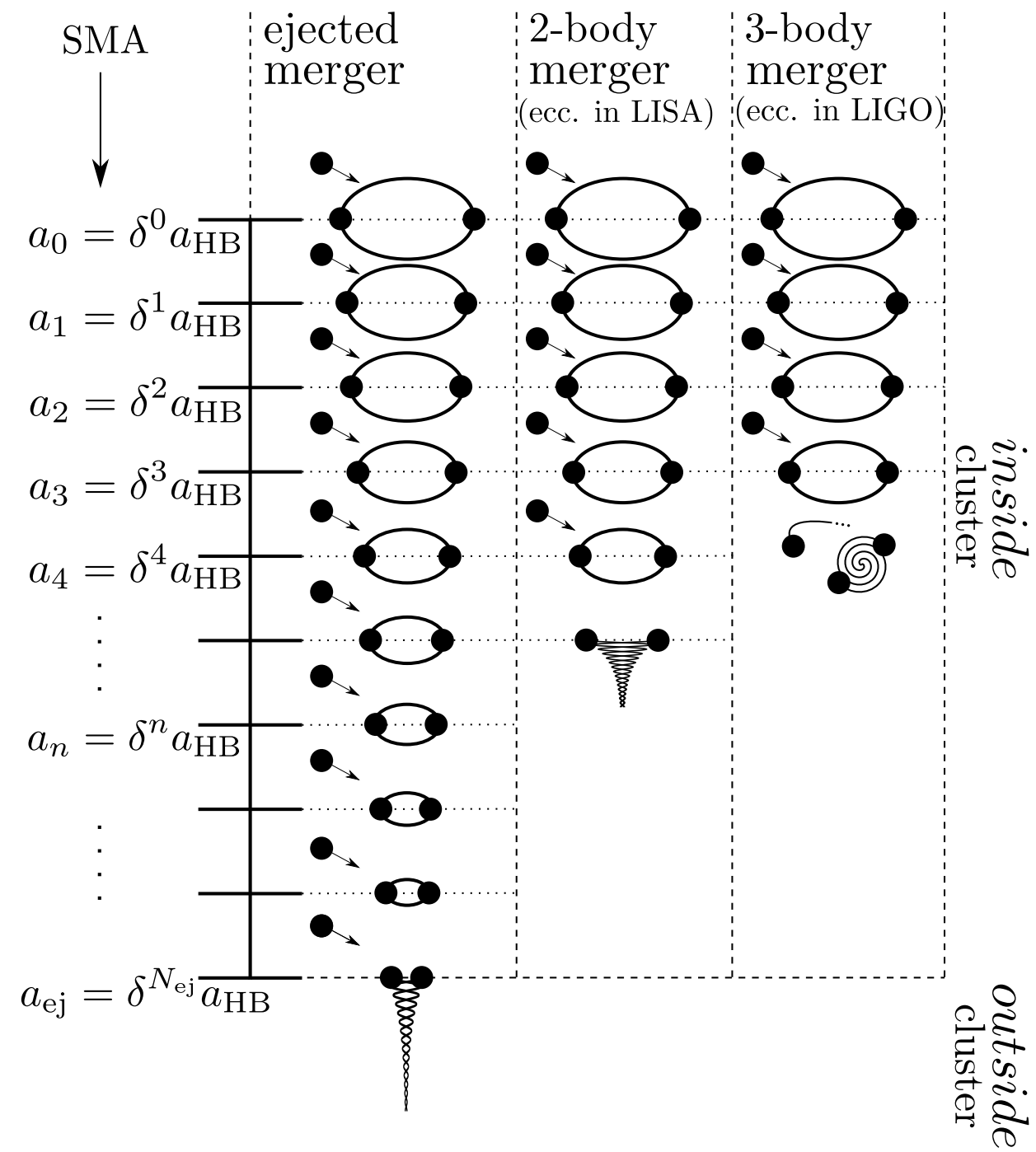
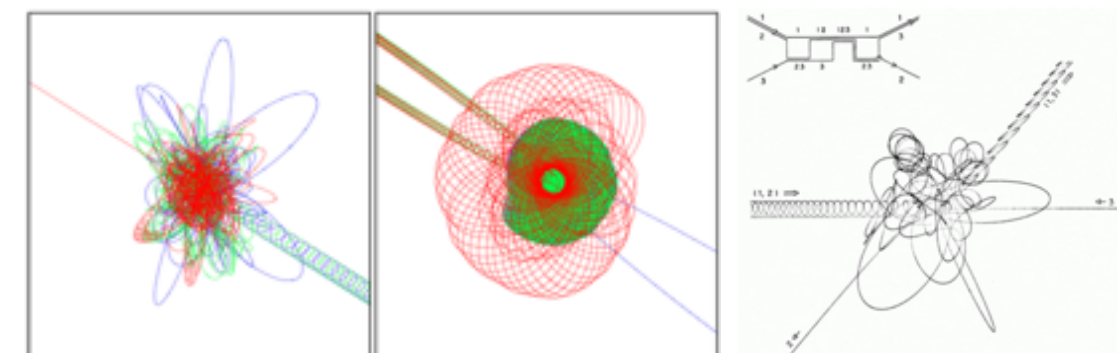
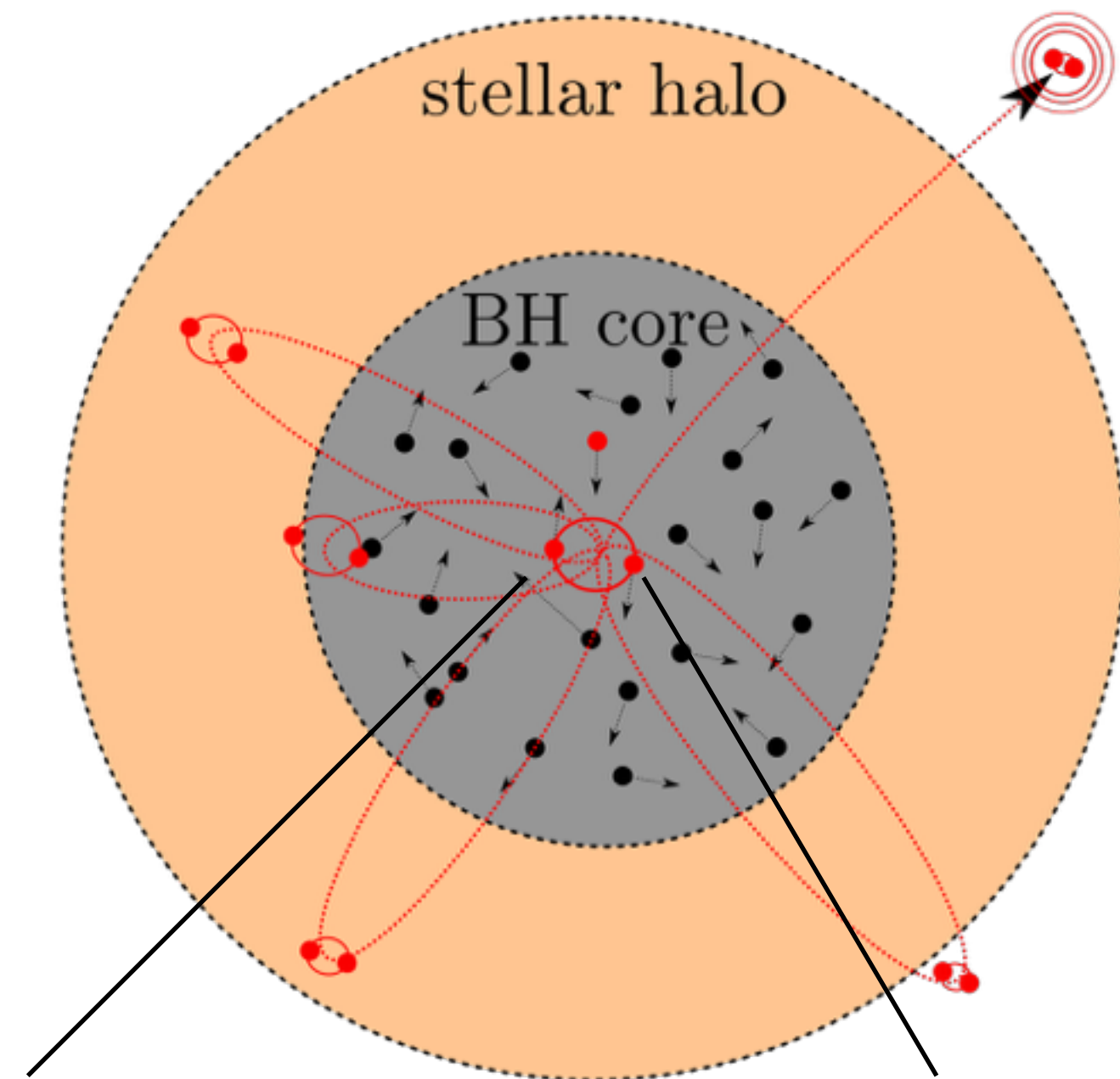
$$f \approx \frac{1}{\pi} \sqrt{\frac{2Gm}{r_f^3}}$$

Few-body BBH mergers: Formation of a BBH

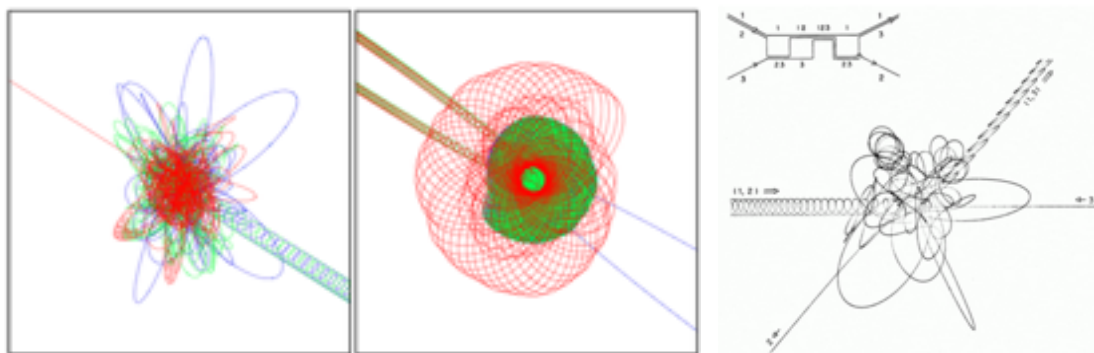
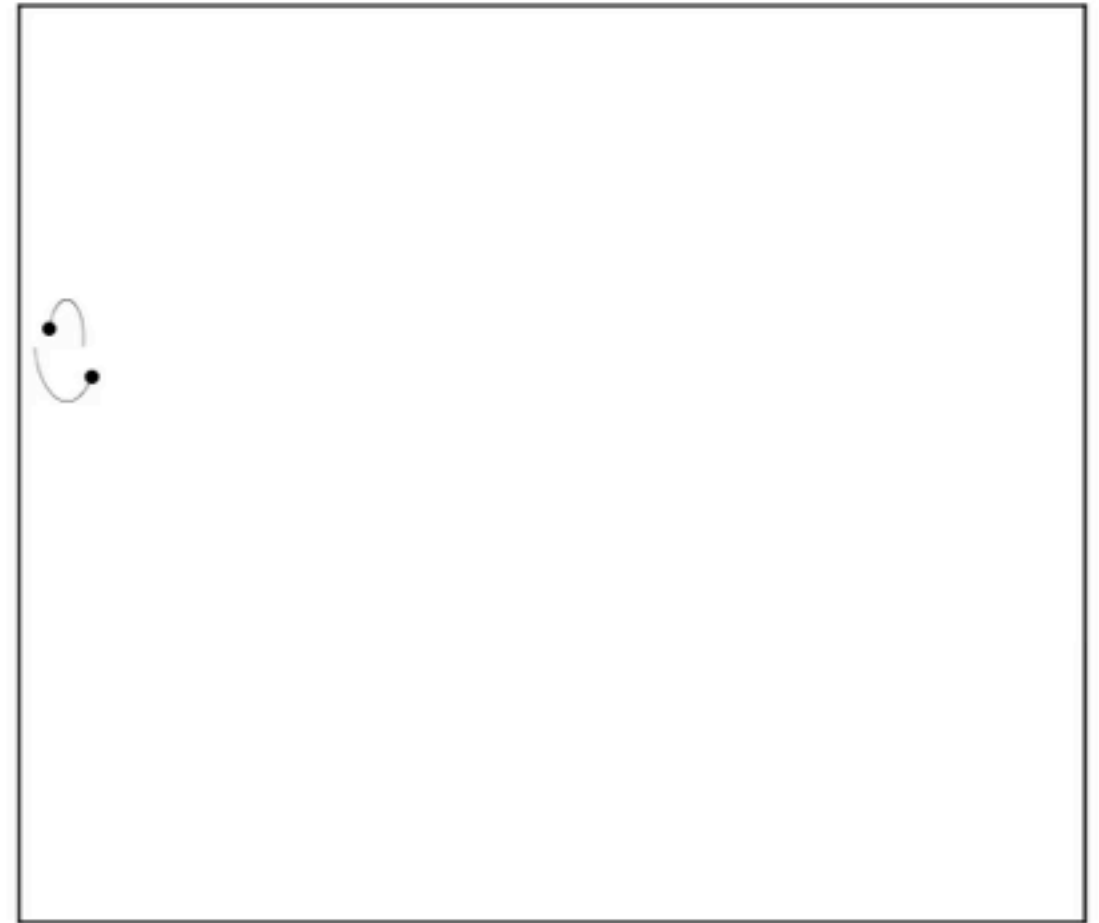
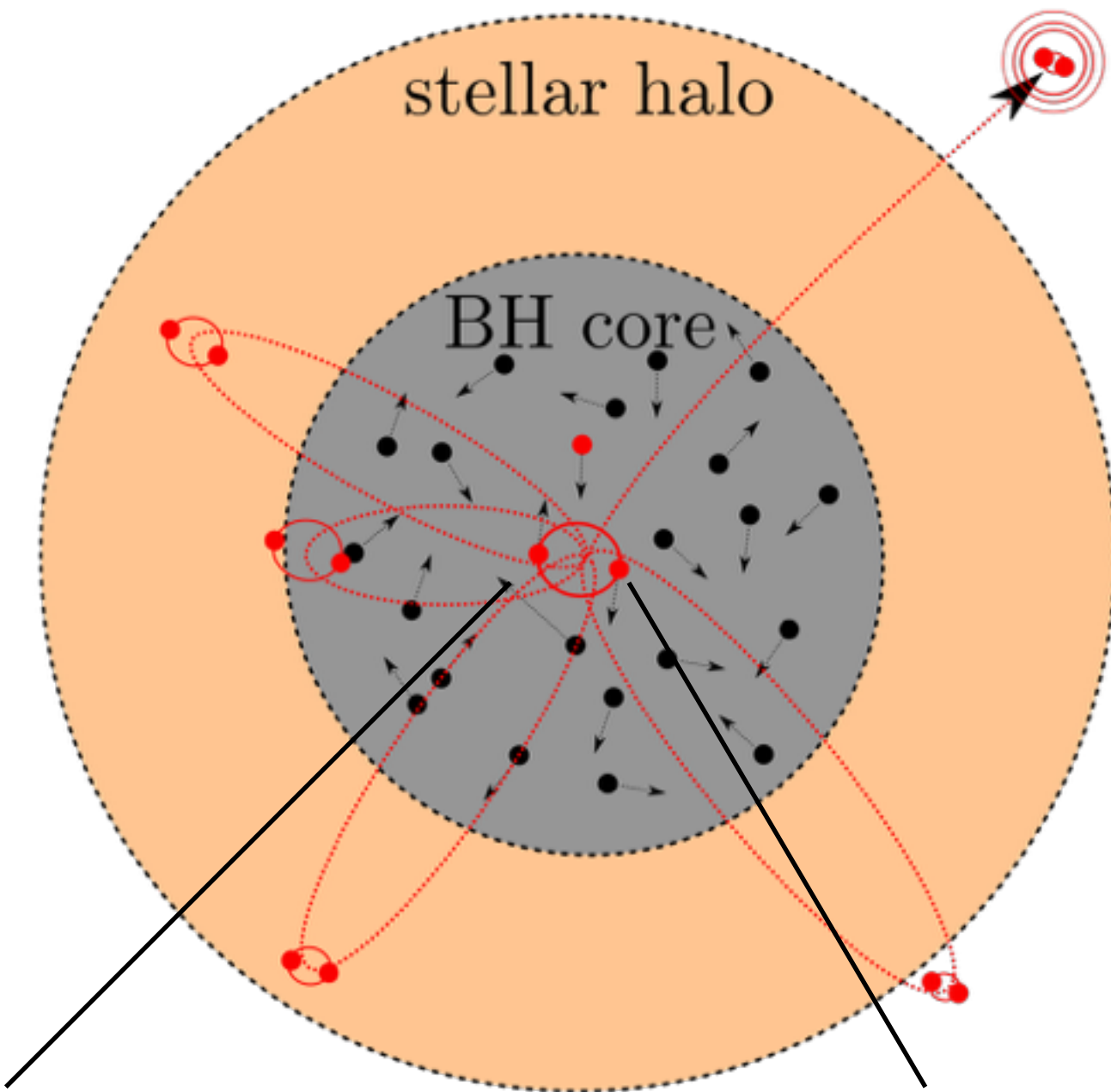


$$a_{\text{HB}} = \frac{3}{2} \frac{G m_{\text{BH}}}{v_{\text{dis}}^2}$$

Merger Type: Ejected Merger

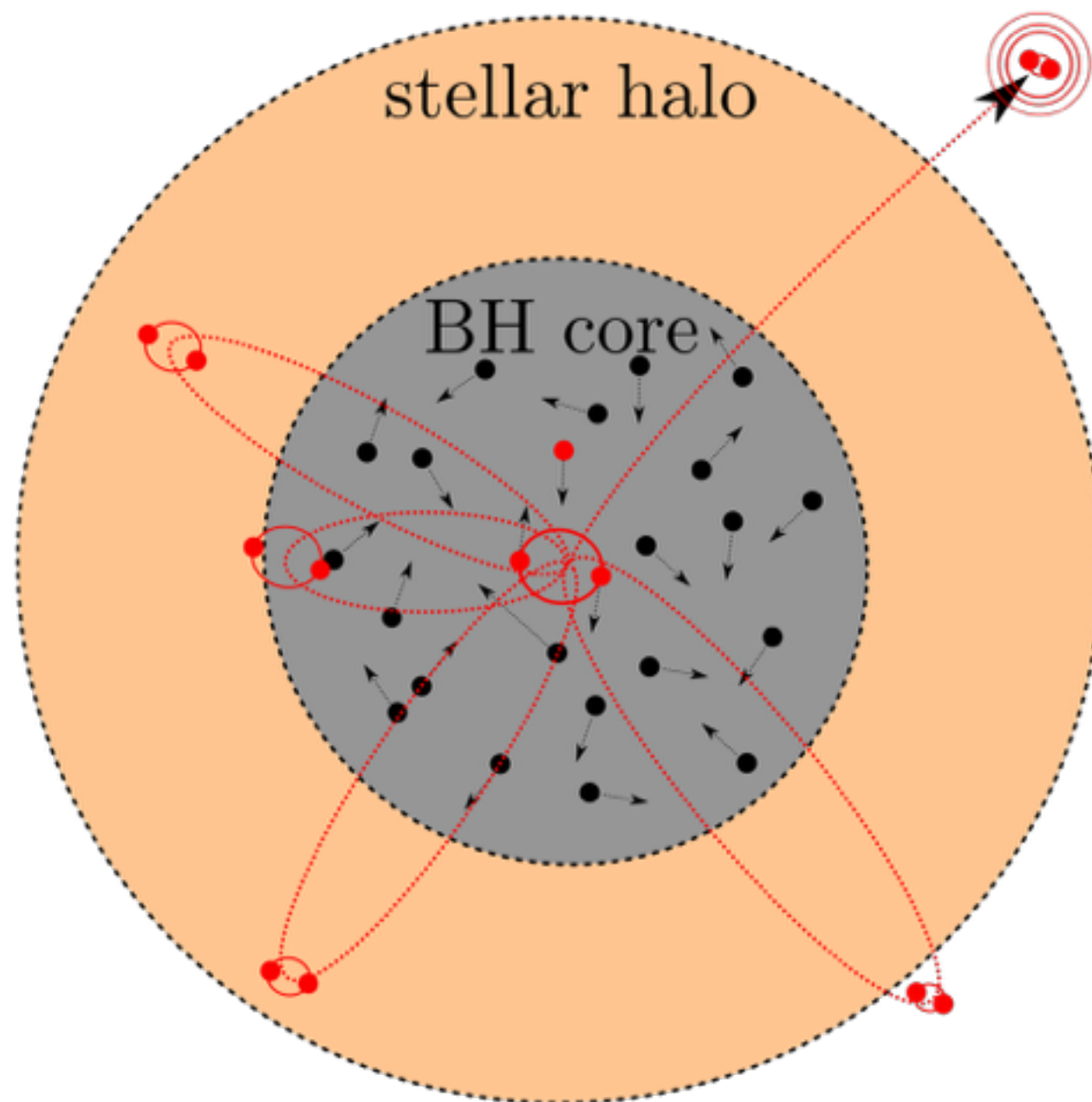


Merger Type: Ejected Merger



Merger Type: Ejected Merger

Classical way of forming BBH mergers
Suggested by: P. Zwart, S. McMillan



$$\Delta E_{\text{bs}} = (1/\delta - 1) \times E_{\text{B}}(a)$$

$$a_{\text{ej}} \approx \frac{1}{6} \left(\frac{1}{\delta} - 1 \right) \frac{Gm}{v_{\text{esc}}^2}$$

$$a_{\text{HB}} = \frac{3}{2} \frac{Gm_{\text{BH}}}{v_{\text{dis}}^2}$$

$$a(k) = a_{\text{HB}} \delta^k$$

$$da = -a(1 - \delta)dk$$

$$T_{\text{ej}} = \int_{a_{\text{ej}}}^{a_{\text{HB}}} \frac{1}{n_0 \sigma_{\text{bs}} v_0} \frac{da}{a(1 - \delta)},$$

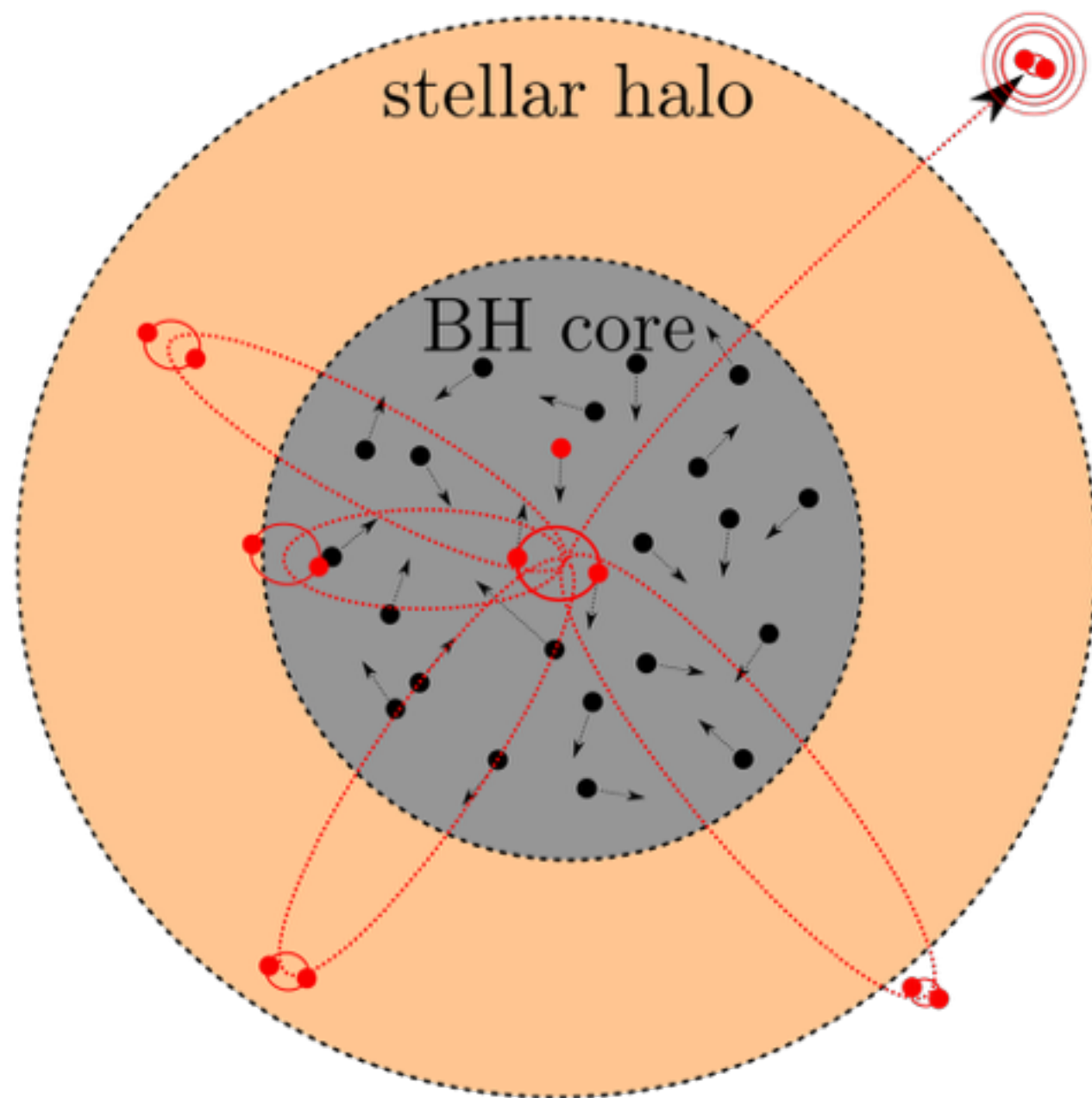
$$\approx \frac{(6\pi G)^{-1}}{(1 - \delta)} \frac{v_0}{n_0} \frac{m^{-1}}{a_{\text{ej}}},$$

Newtonian outcome.

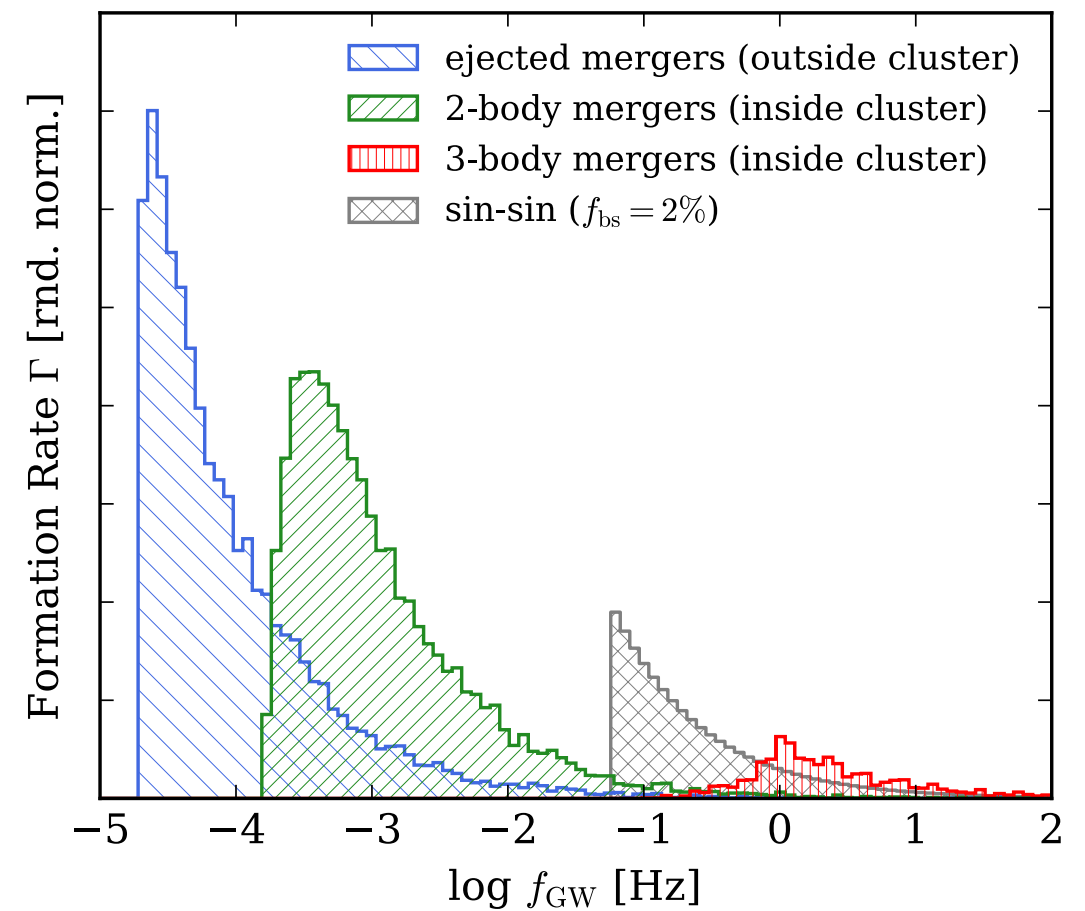
Leads to interesting rate,
but standard circular mergers.

$$N_{\text{bs}}(a_{\text{in}}, a_{\text{ej}}) = \int_{a_{\text{ej}}}^{a_{\text{in}}} \frac{1}{1 - \delta} \frac{1}{a} da = \frac{1}{1 - \delta} \ln \left(\frac{a_{\text{in}}}{a_{\text{ej}}} \right)$$

Merger Type: Ejected Merger

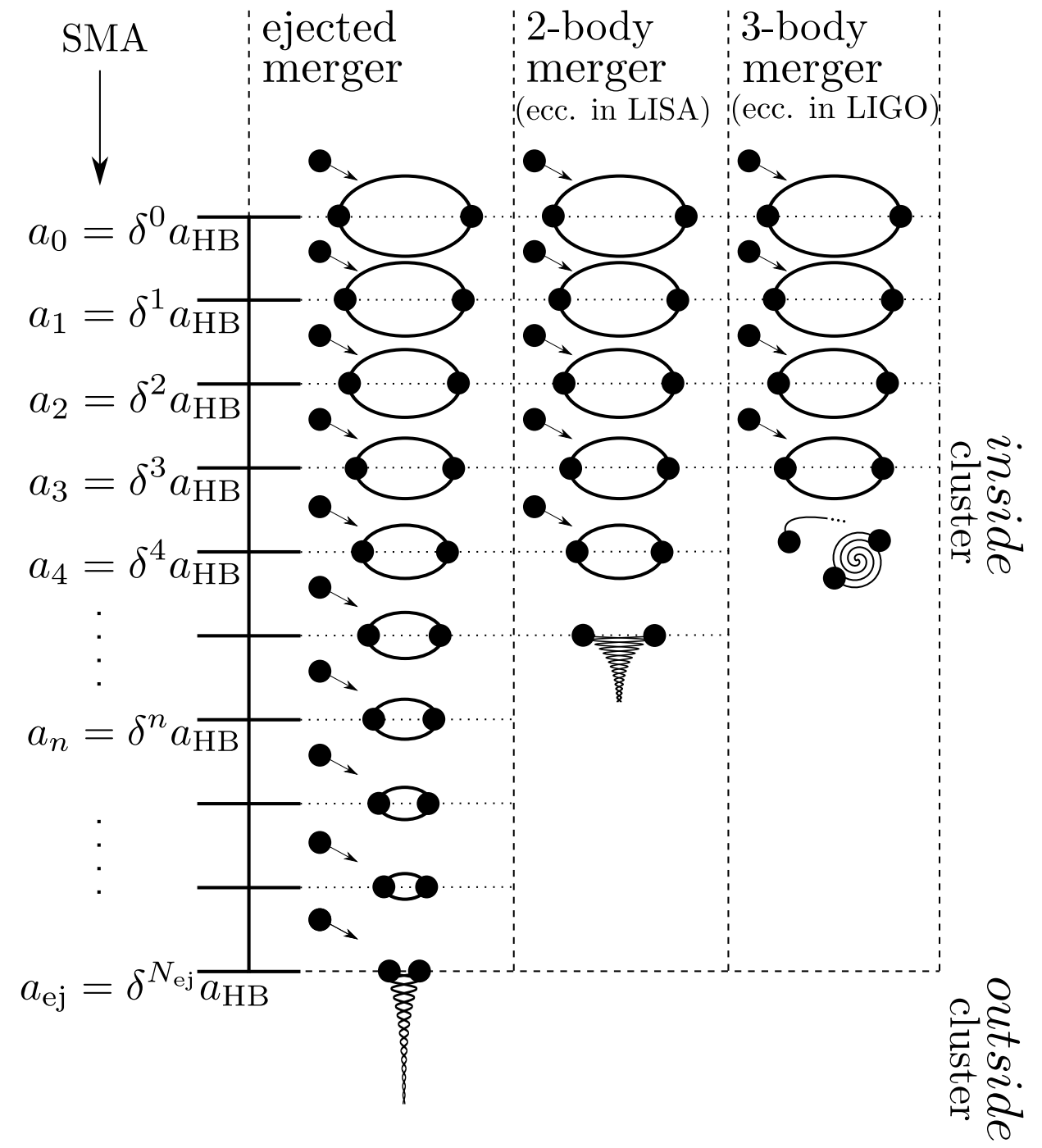
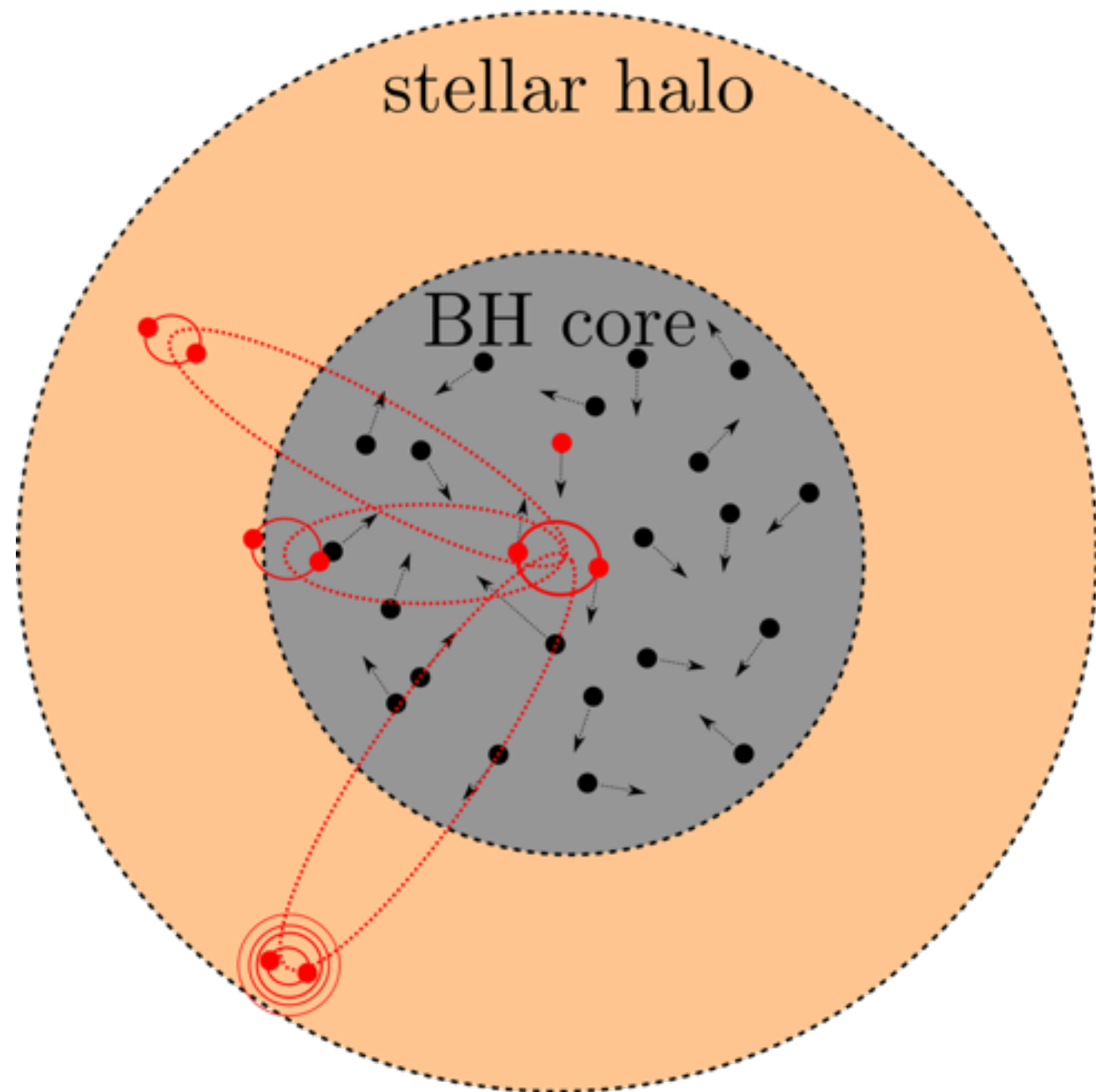


Circular GW sources

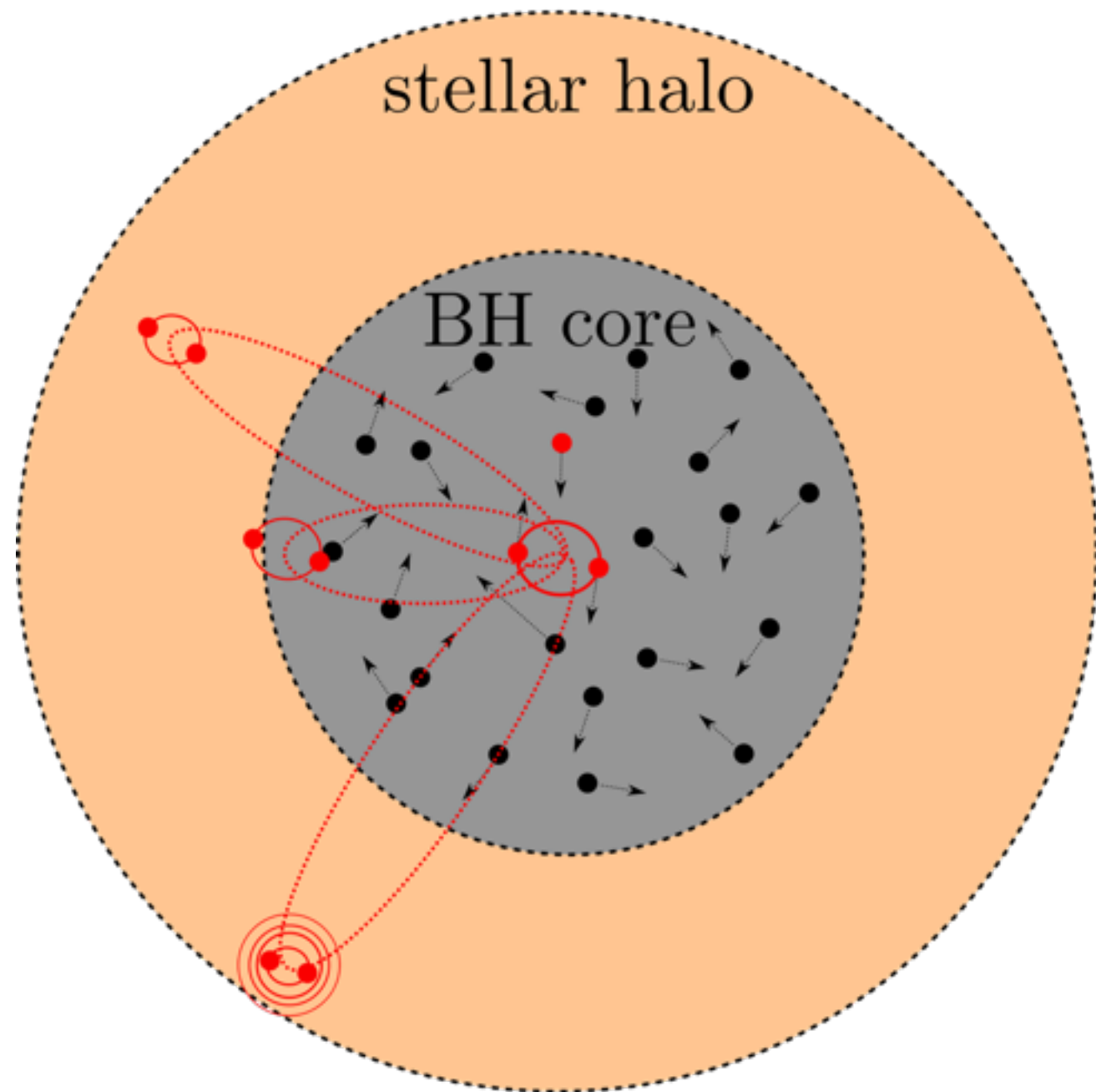


$$f_{r,0}^{\text{peak}}(\mathcal{T}) \approx 2 \cdot 10^{-5} \text{ Hz} \left(\frac{\mathcal{T}}{10^{10} \text{ yrs}} \right)^{-3/7} \left(\frac{a}{0.5 \text{ au}} \right)^{3/14} \left(\frac{m}{30 M_{\odot}} \right)^{-11/14}$$

Merger Type: 2-body Merger



Merger Type: 2-body Merger



$$P_{\text{IM}}(a_{\text{in}}, a_{\text{ej}}) \approx \frac{1}{1 - \delta} \int_{a_{\text{ej}}}^{a_{\text{in}}} \frac{P_{\text{IM}}(a)}{a} da \approx \frac{7}{10} \frac{P_{\text{IM}}(a_{\text{ej}})}{1 - \delta}$$

Time scales:

$$t_{\text{life}}(a, e) \approx t_{\text{life}}(a)(1 - e^2)^{7/2}$$

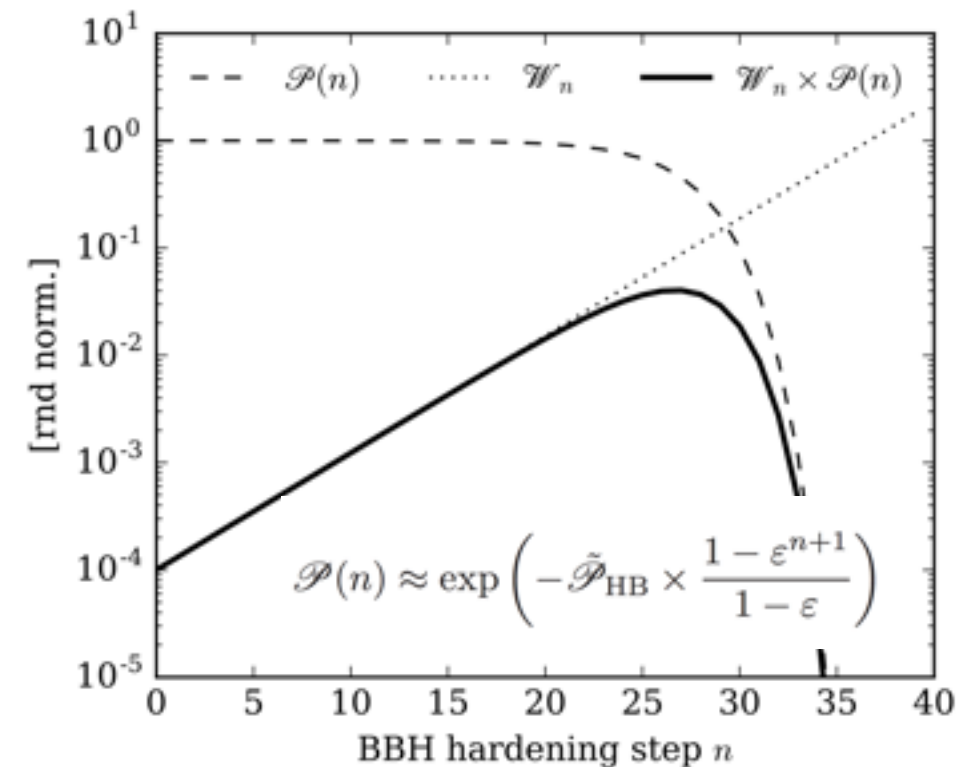
$$t_{\text{bs}}(a) \approx 1/\Gamma_{\text{bs}} \approx (n_s \sigma_{\text{bs}} v_{\text{disp}})^{-1}$$

Eccentricity:

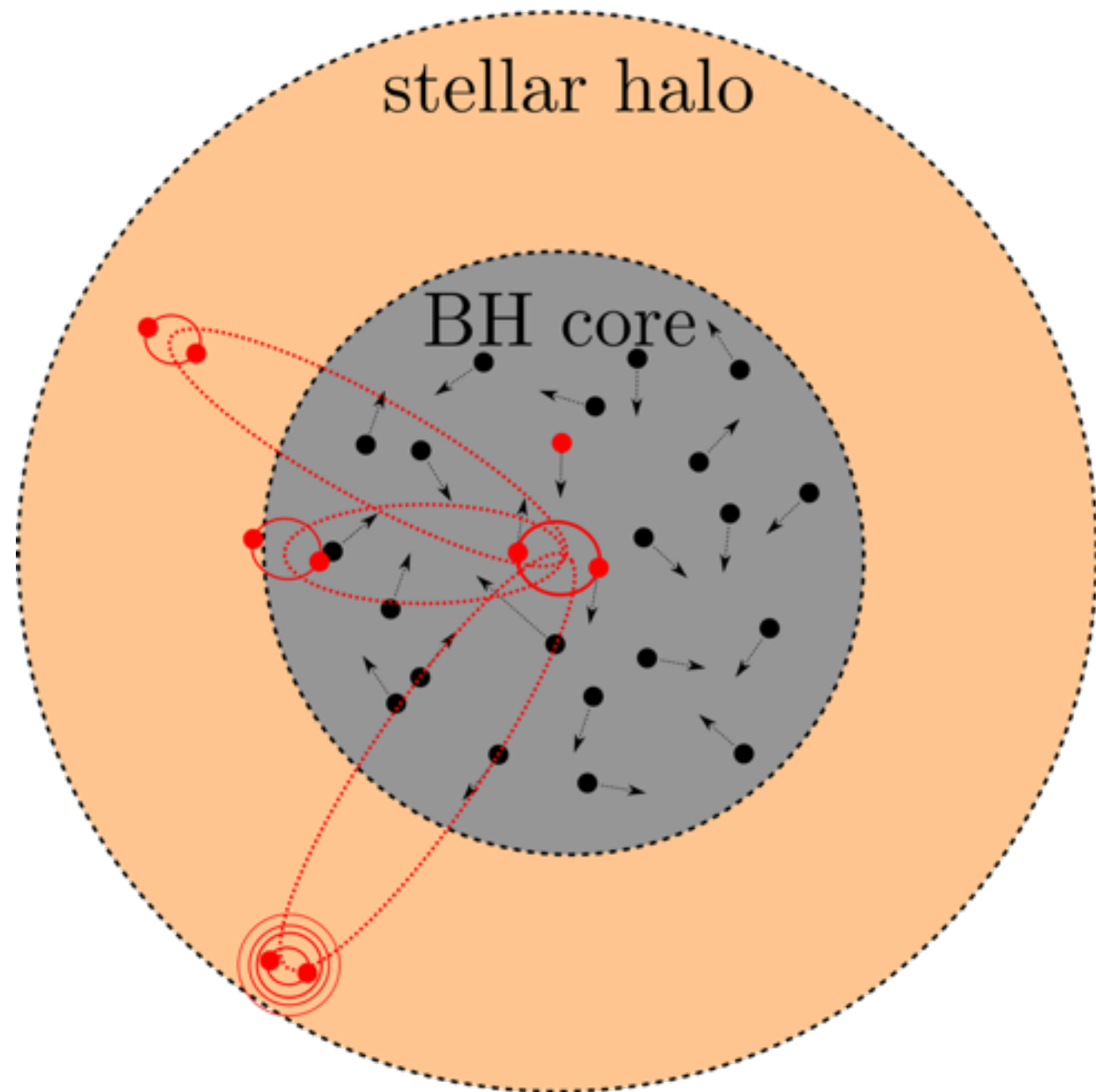
$$P(e) = 2e$$

Probability:

$$P_{\text{IM}}(a) \approx (t_{\text{bs}}(a)/t_{\text{life}}(a))^{2/7}$$



Merger Type: 2-body Merger



$$\frac{d\Gamma_{\text{TD}}}{d\log T} \propto \frac{dN_{\text{BBH}}}{dn} \propto \frac{dN_{\text{BBH}}}{d\log T} \\ \propto T^{-2/3} \exp\left(-\tilde{\mathcal{P}}_{\text{HB}} \times \frac{1 - \varepsilon(T/T_{\text{HB}})^{-20/21}}{1 - \varepsilon}\right)$$

Time scales:

$$t_{\text{life}}(a, e) \approx t_{\text{life}}(a)(1 - e^2)^{7/2}$$

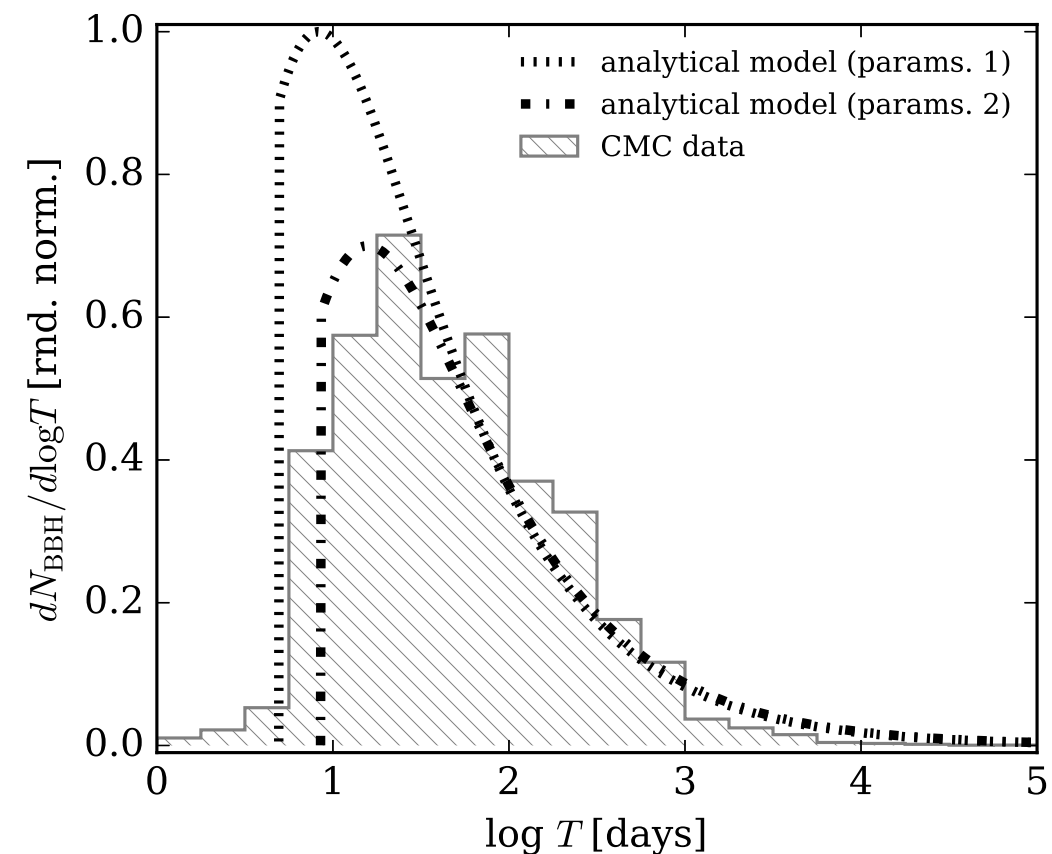
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Eccentricity:

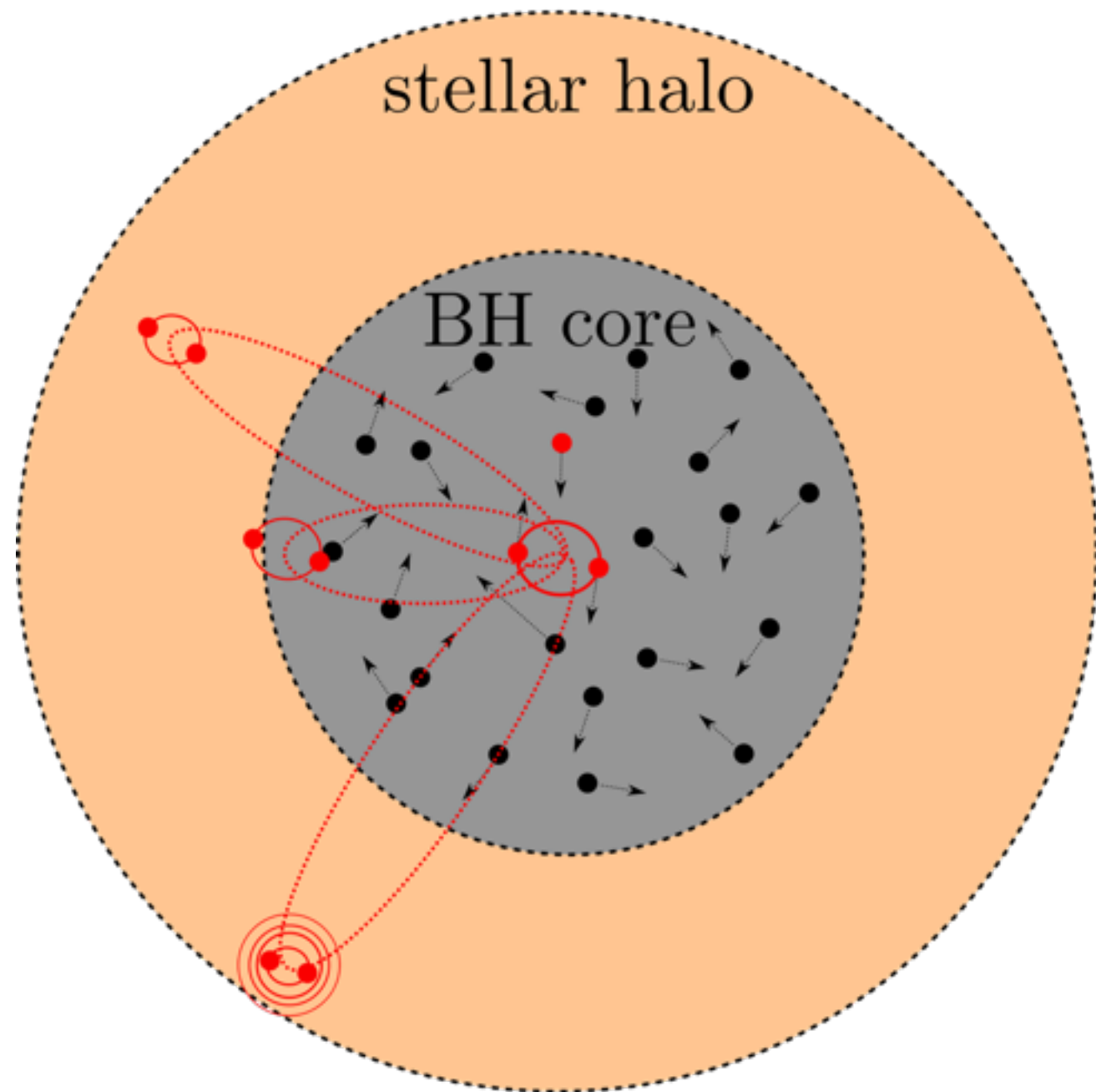
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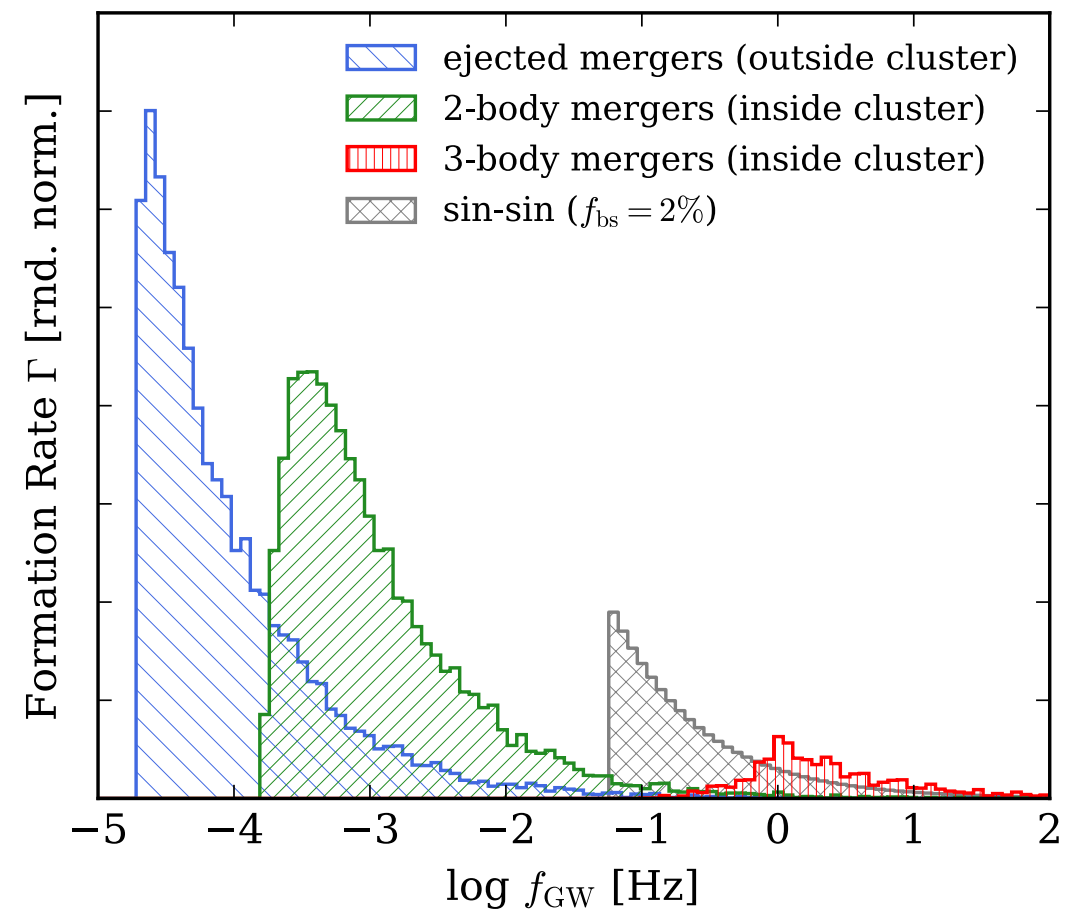
$$P_{\text{IM}}(a) \approx (t_{\text{bs}}(a)/t_{\text{life}}(a))^{2/7}$$



Merger Type: 2-body Merger

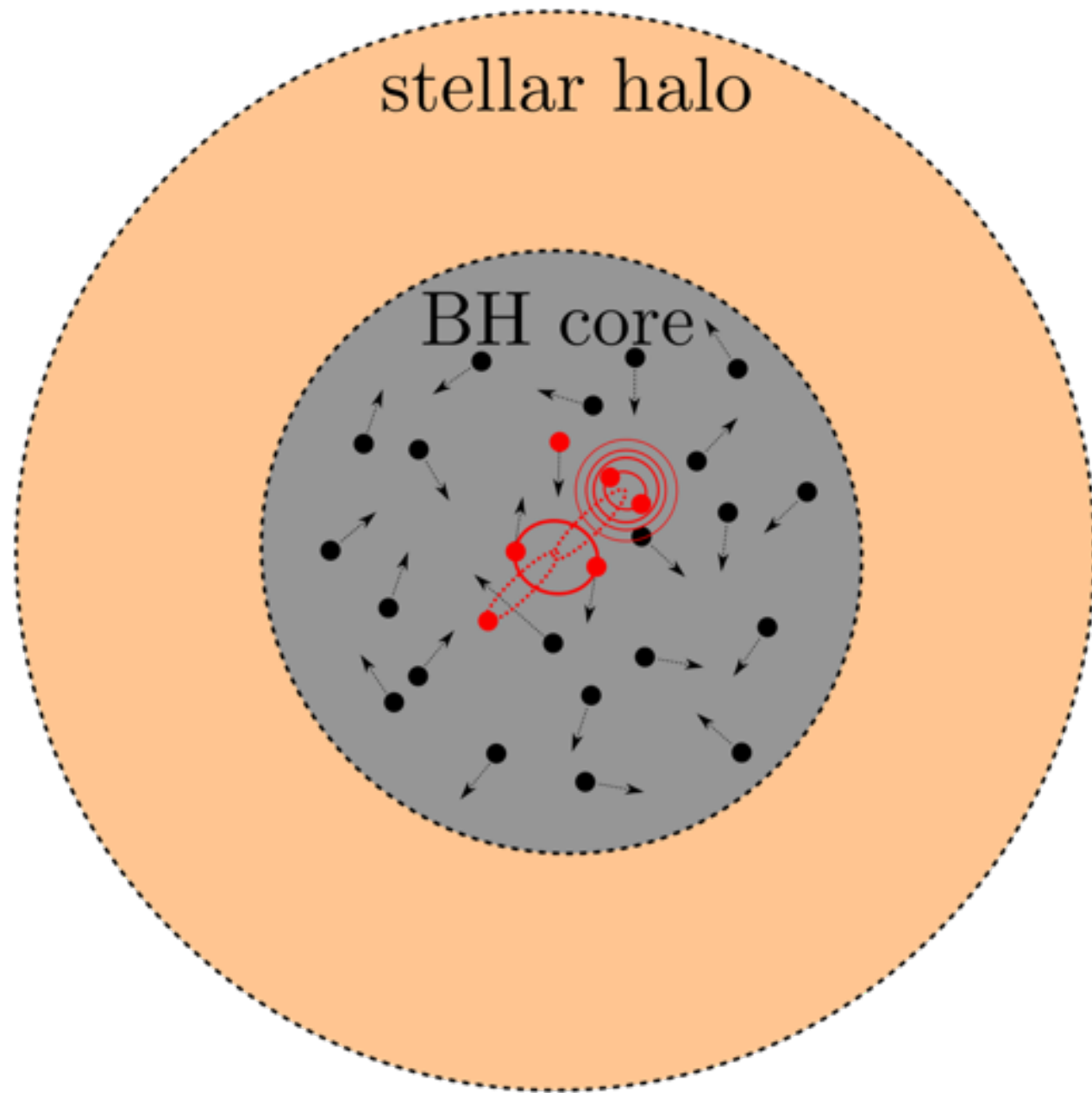


Eccentric LISA sources



$$f_{r,0}^{\text{peak}}(\mathcal{T}) \approx 2 \cdot 10^{-5} \text{ Hz} \left(\frac{\mathcal{T}}{10^{10} \text{ yrs}} \right)^{-3/7} \left(\frac{a}{0.5 \text{ au}} \right)^{3/14} \left(\frac{m}{30 M_{\odot}} \right)^{-11/14}$$

Merger Type: 3-body Merger

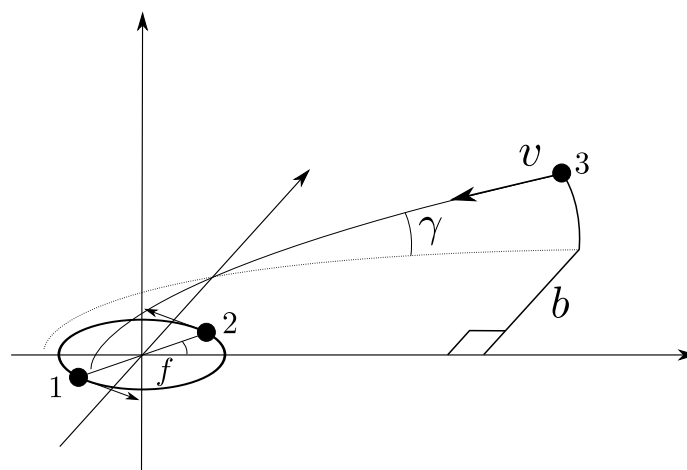
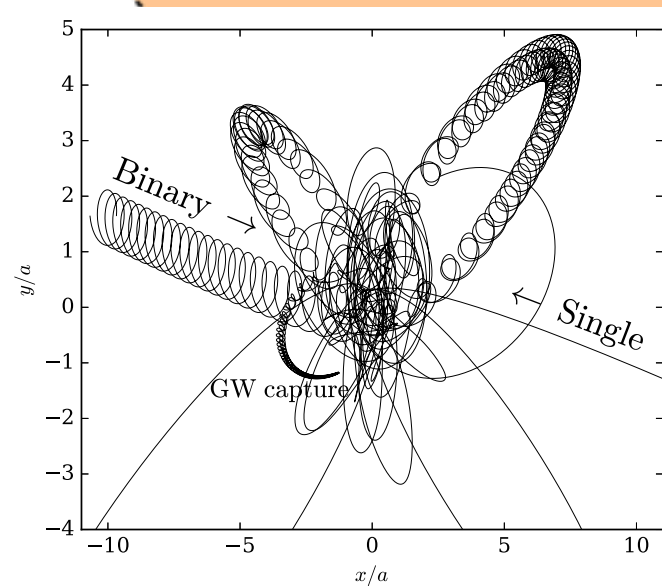
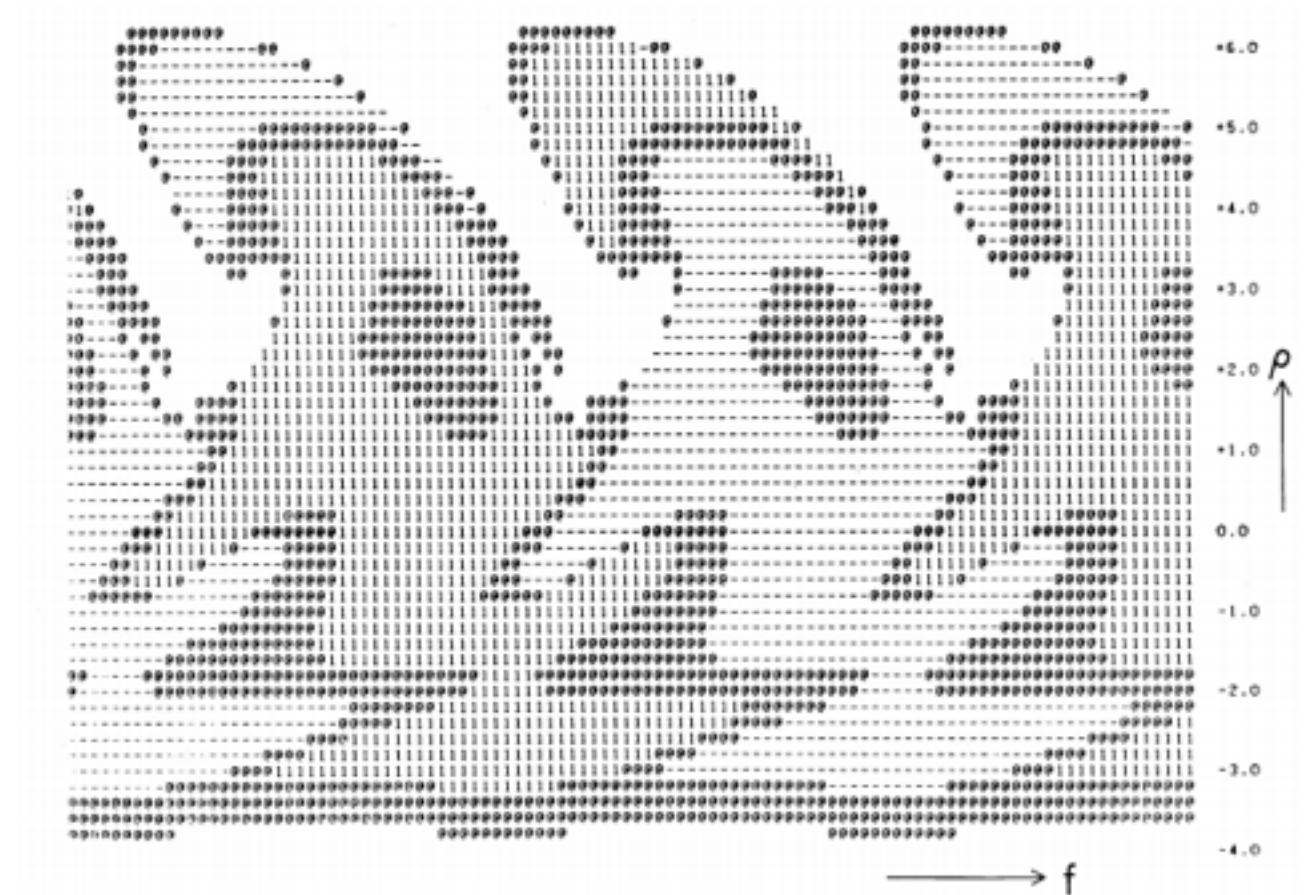
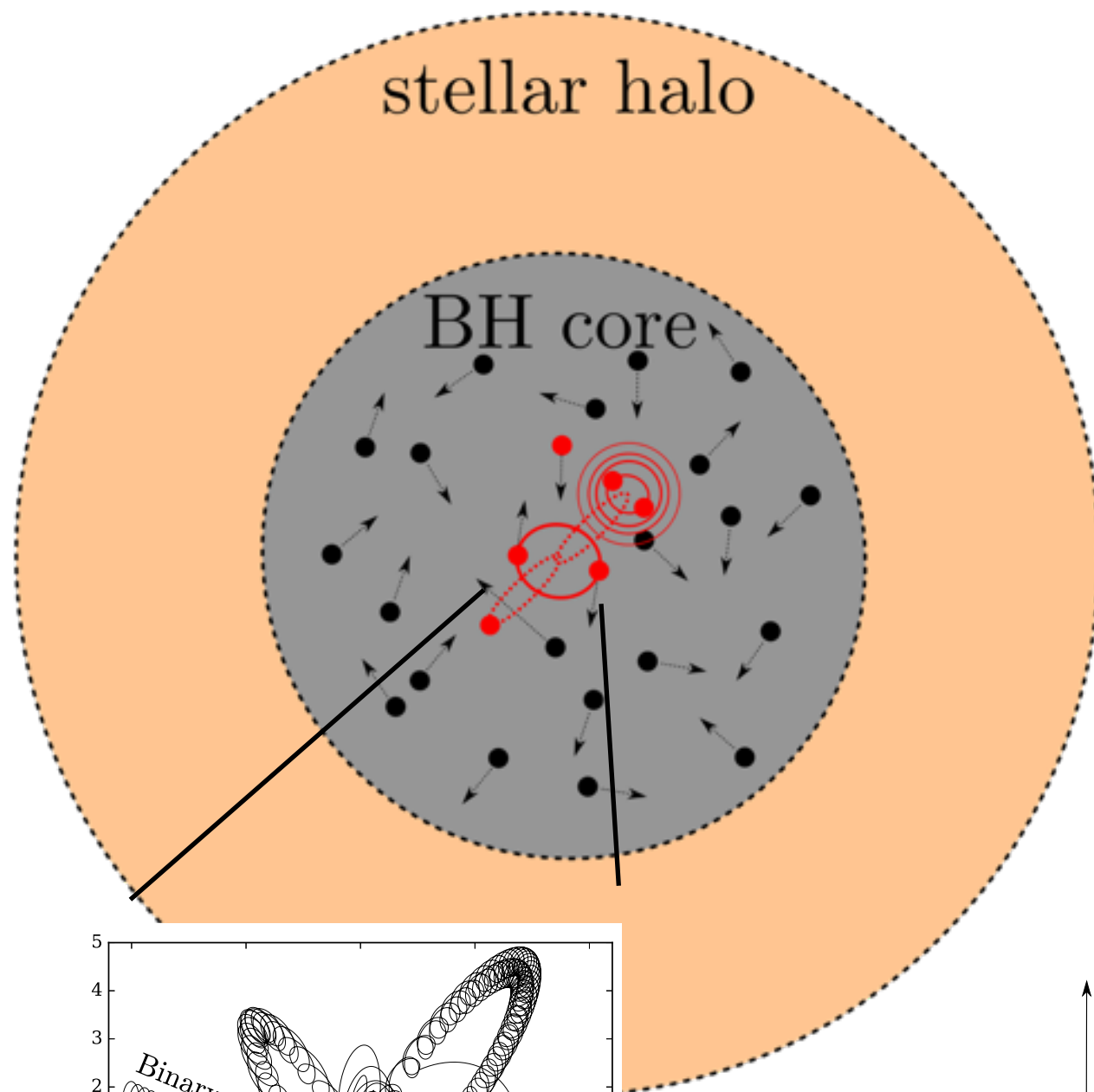


Post-Newtonian



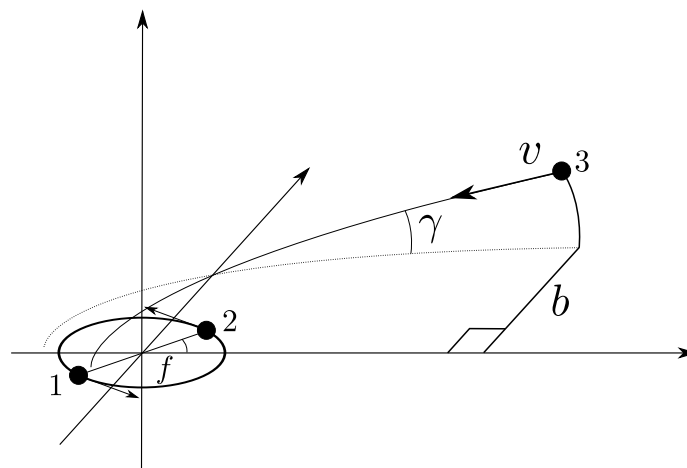
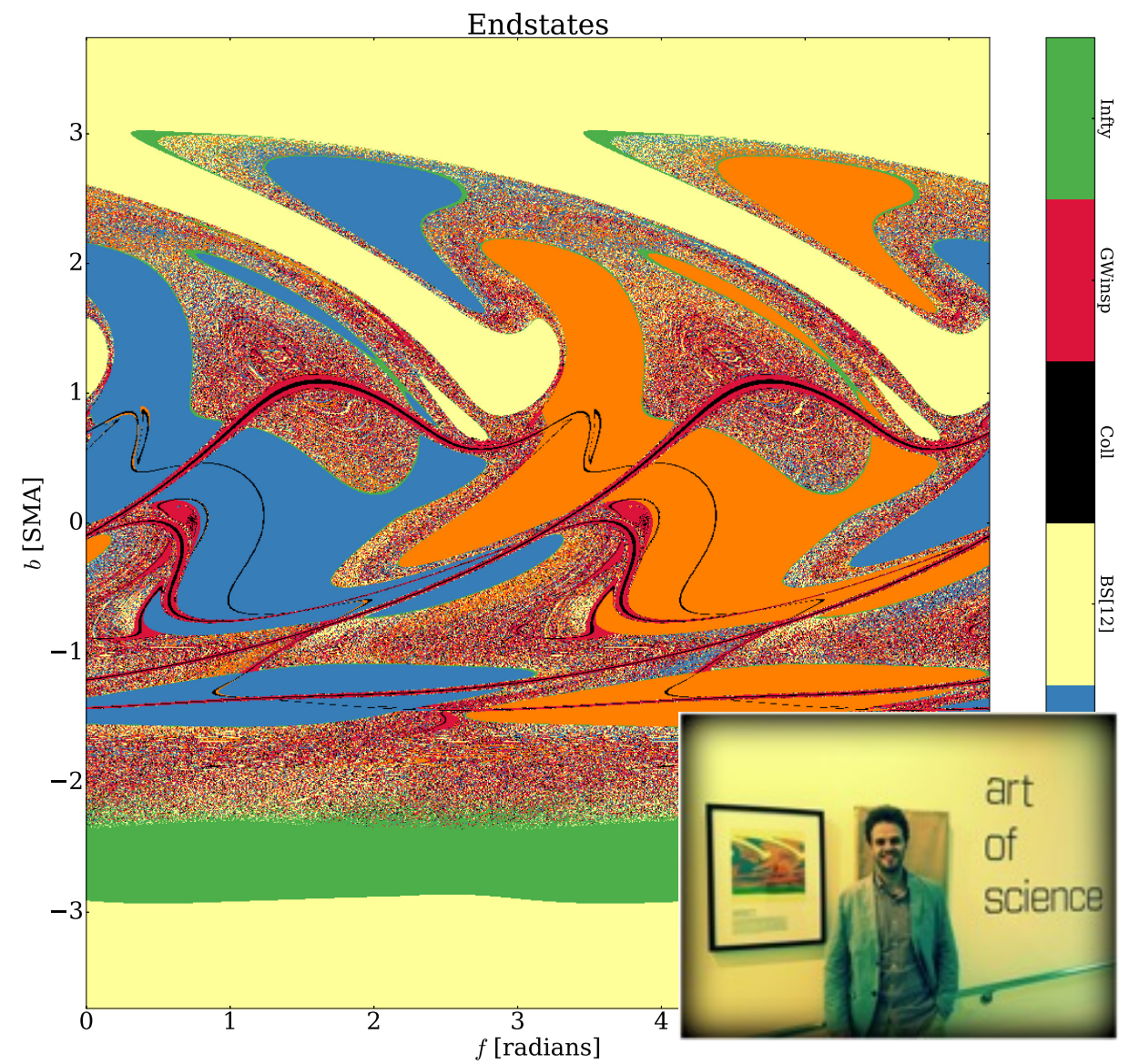
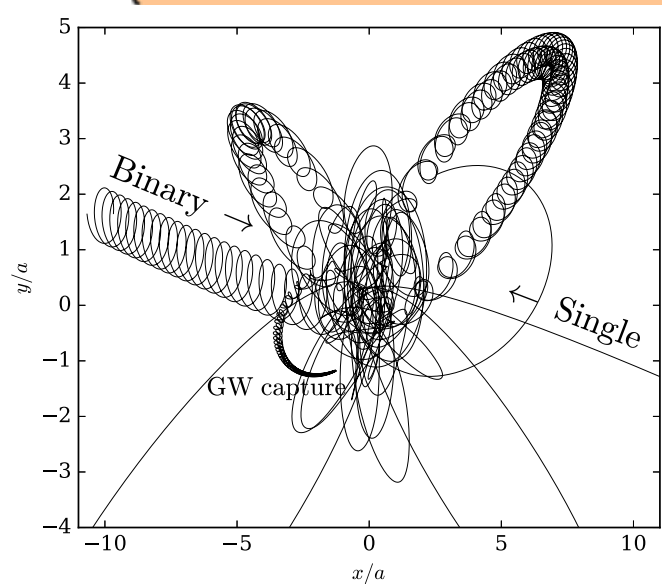
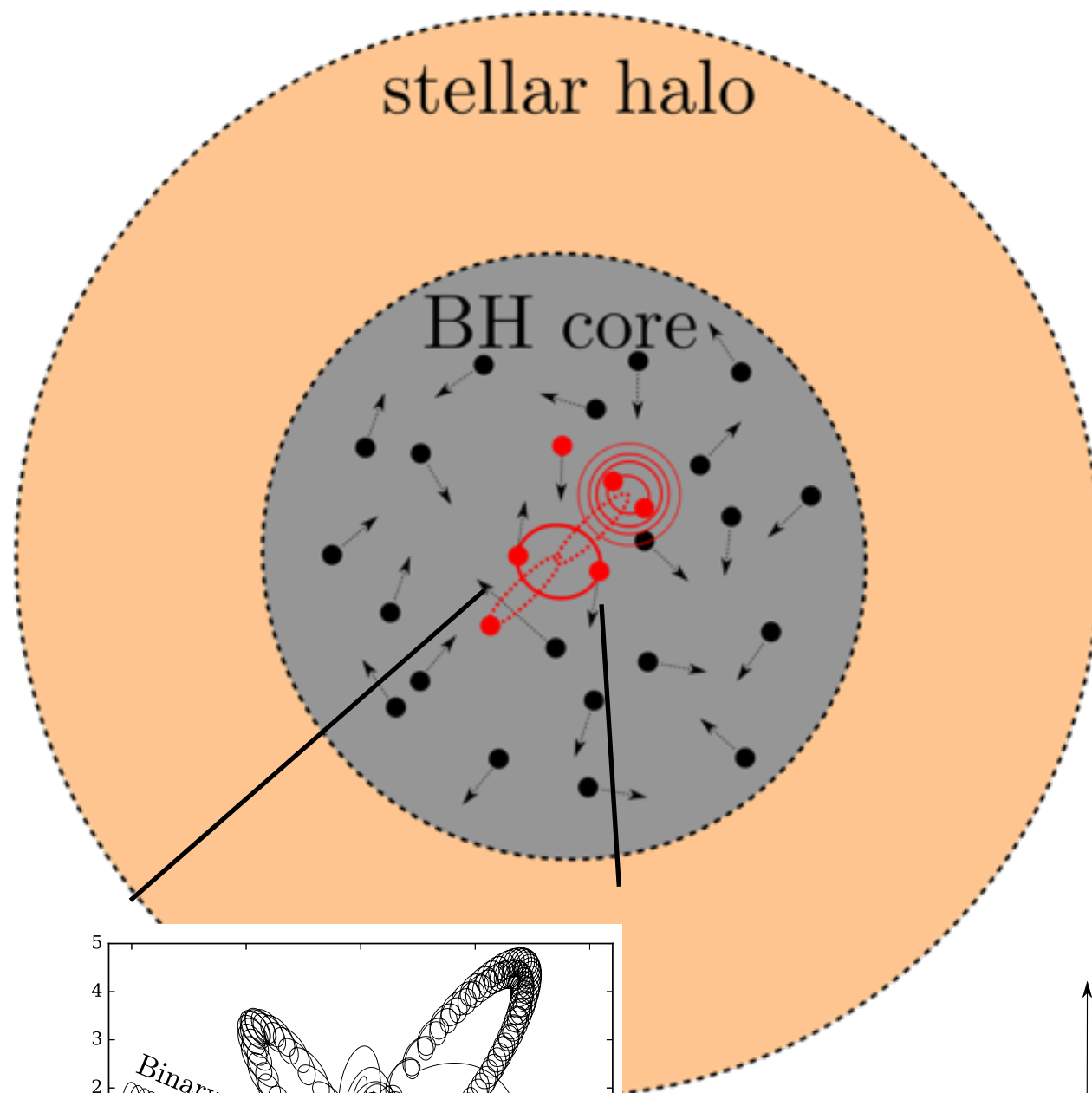
Merger Type: 3-body Merger

Topology

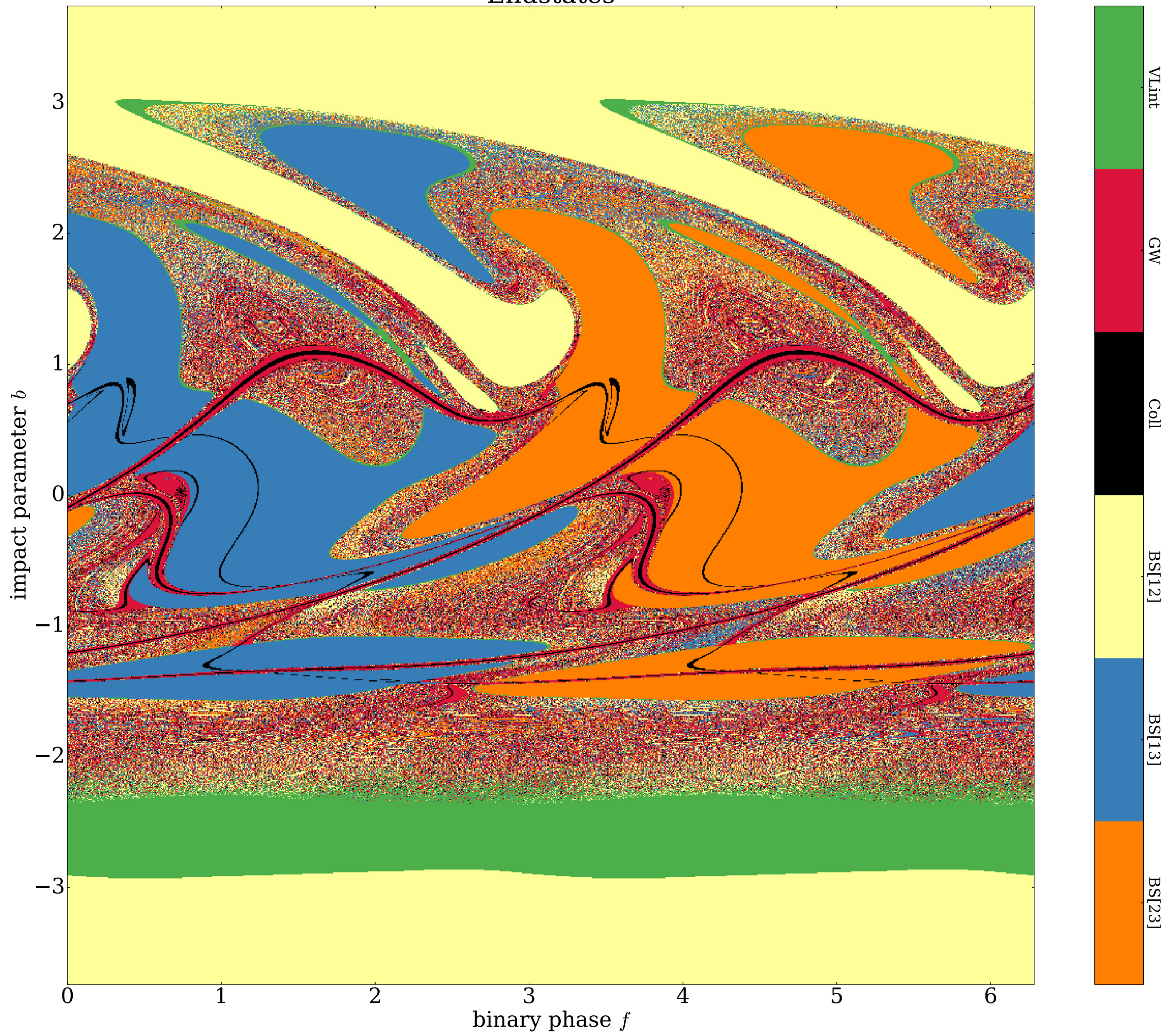


Merger Type: 3-body Merger

Topology

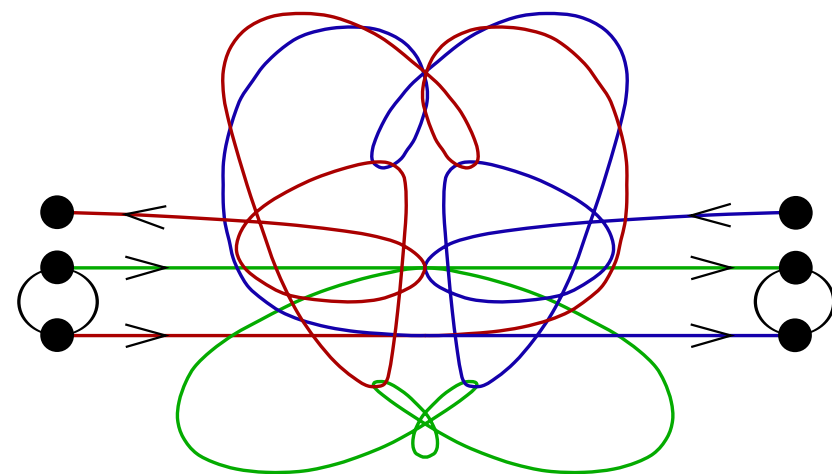
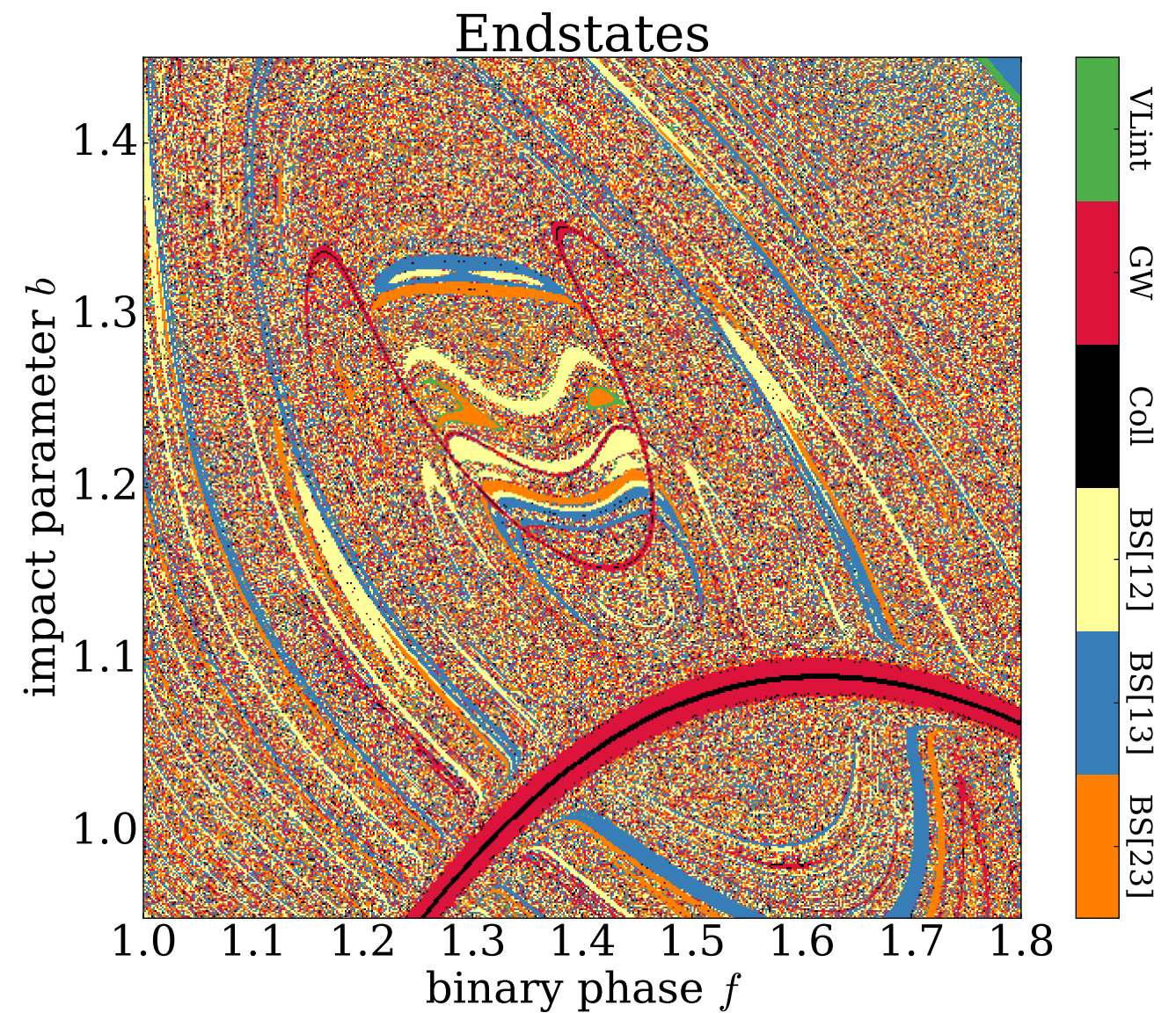
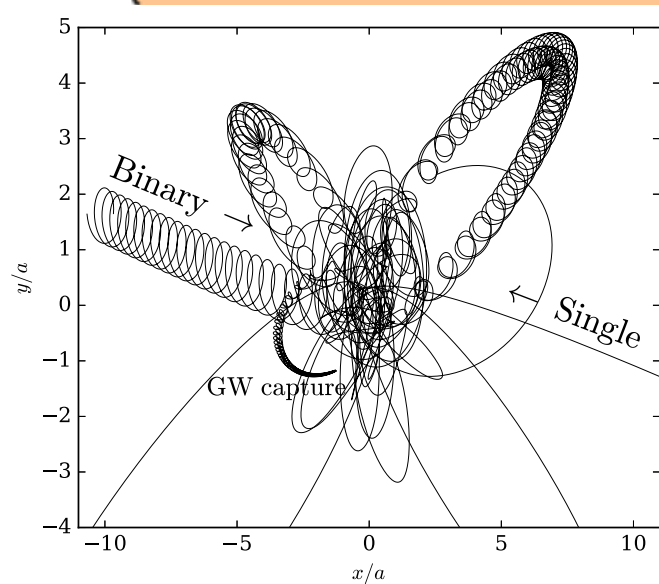
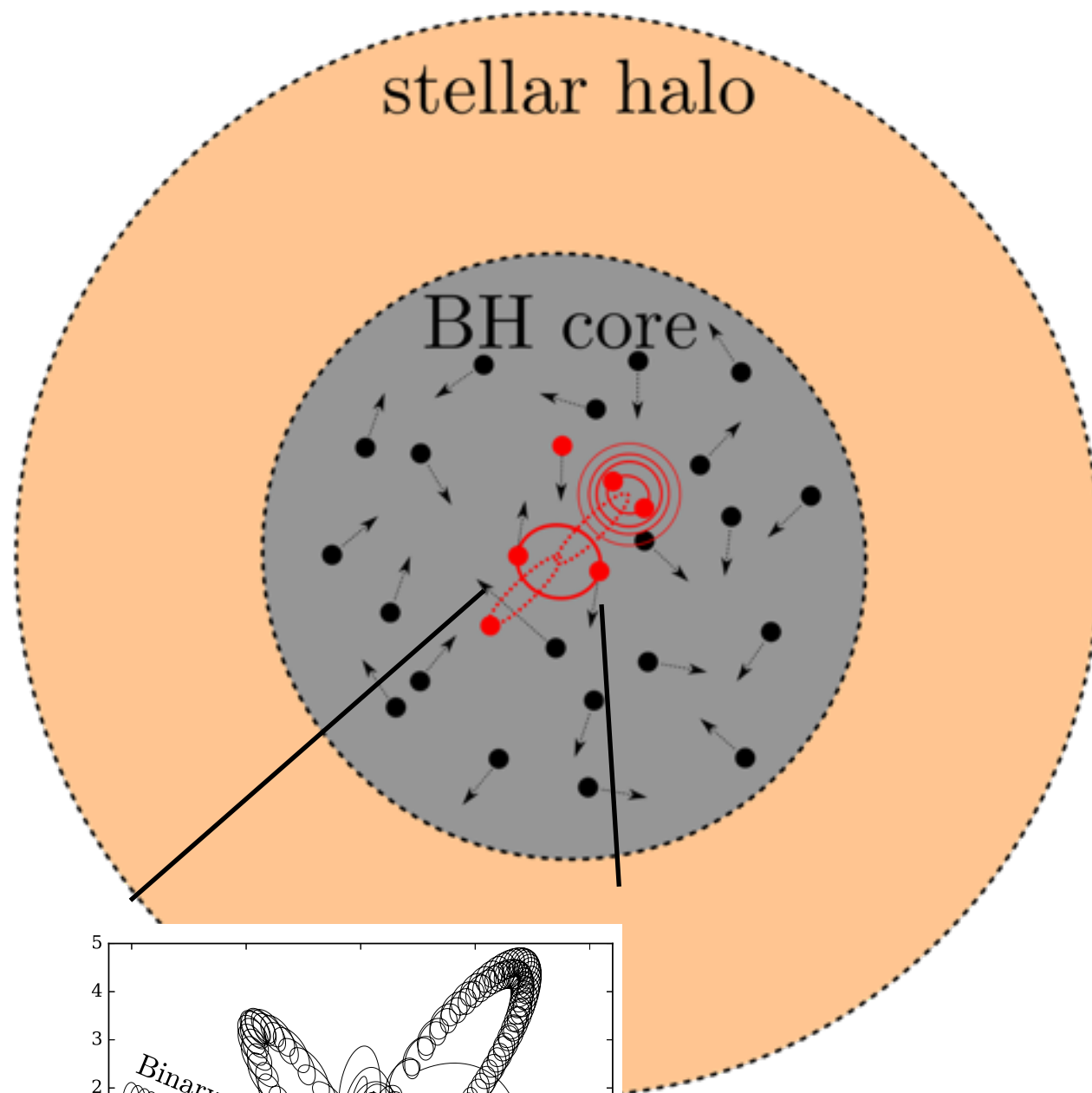


Endstates



Merger Type: 3-body Merger

Topology



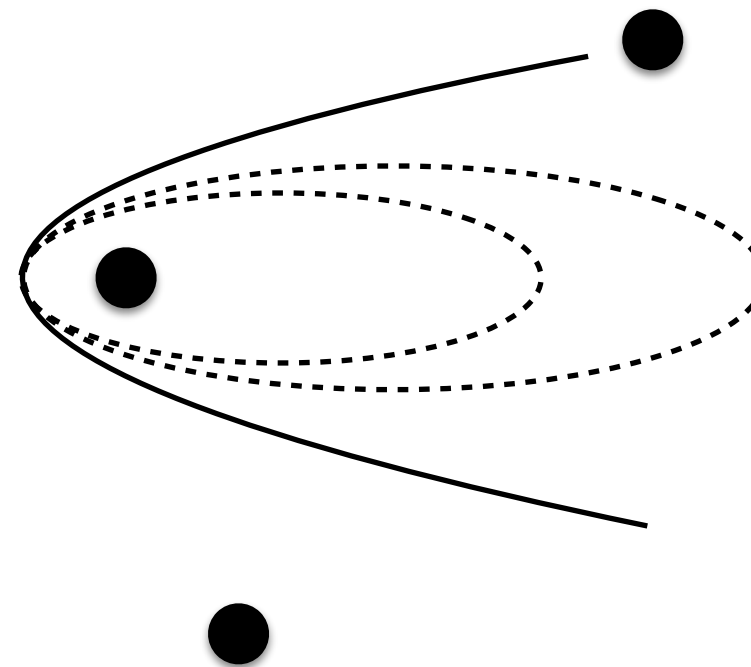
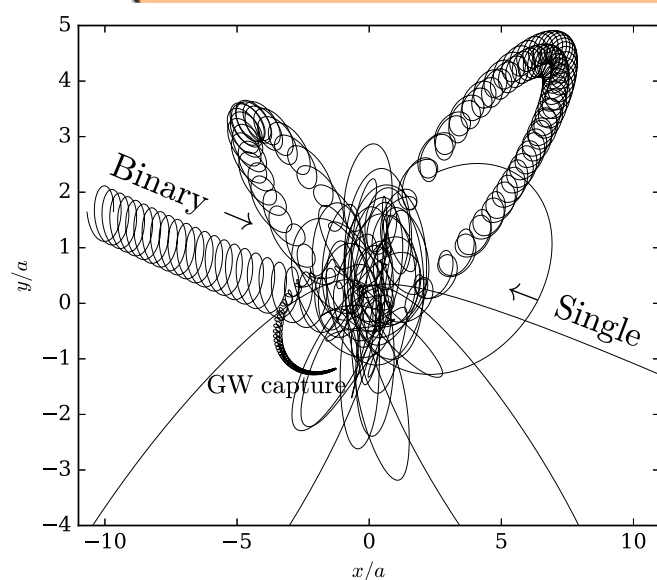
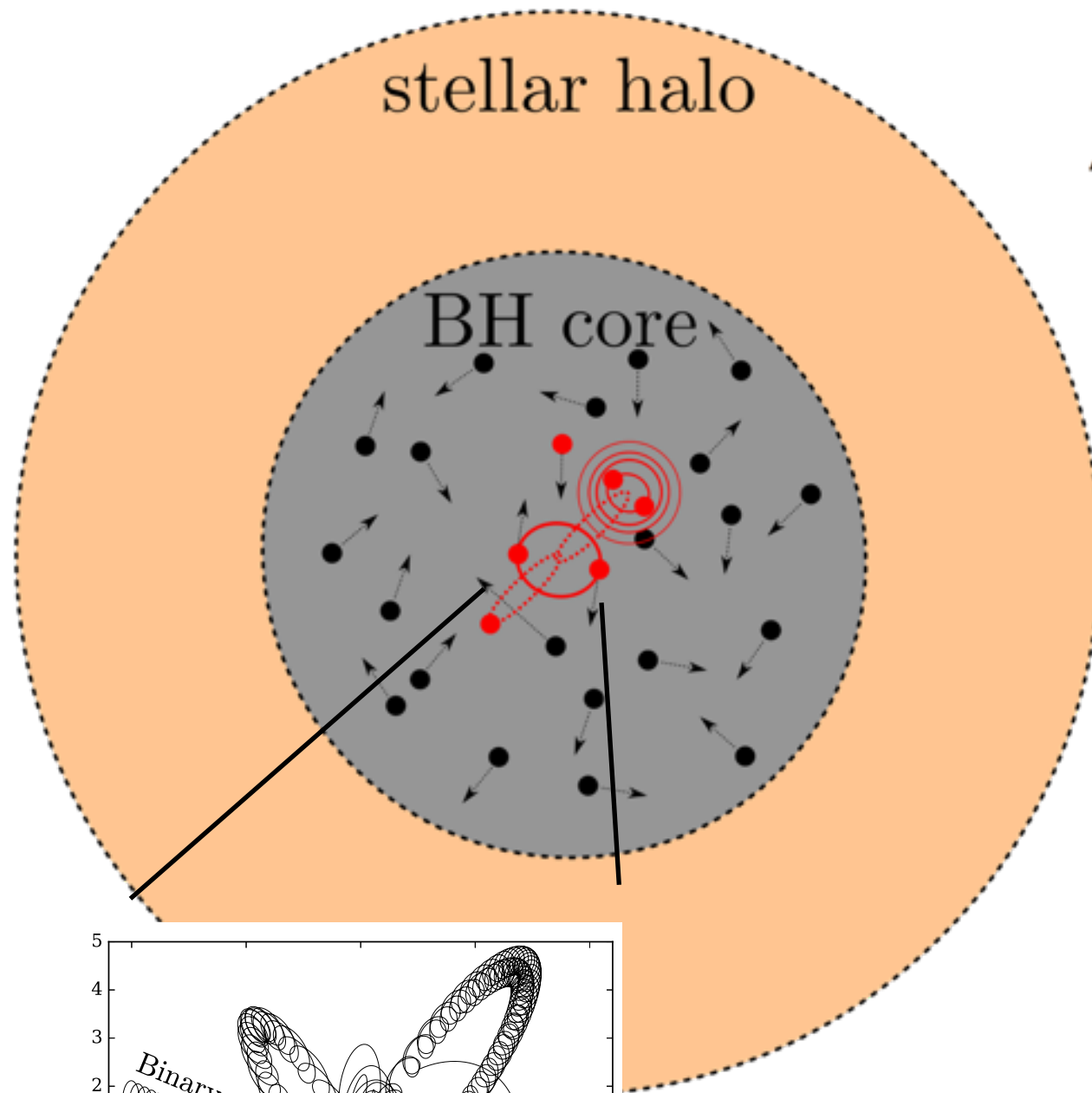
Merger Type: 3-body Merger

3-body GW capture:

$$E_B \approx Gm^2/(2a)$$

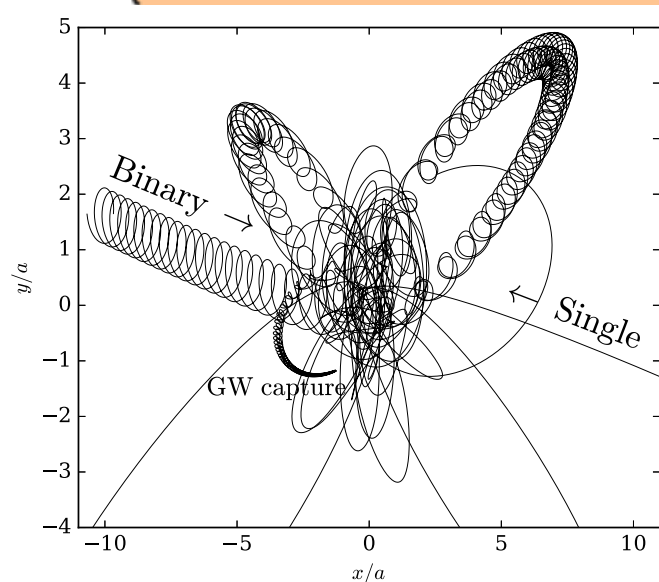
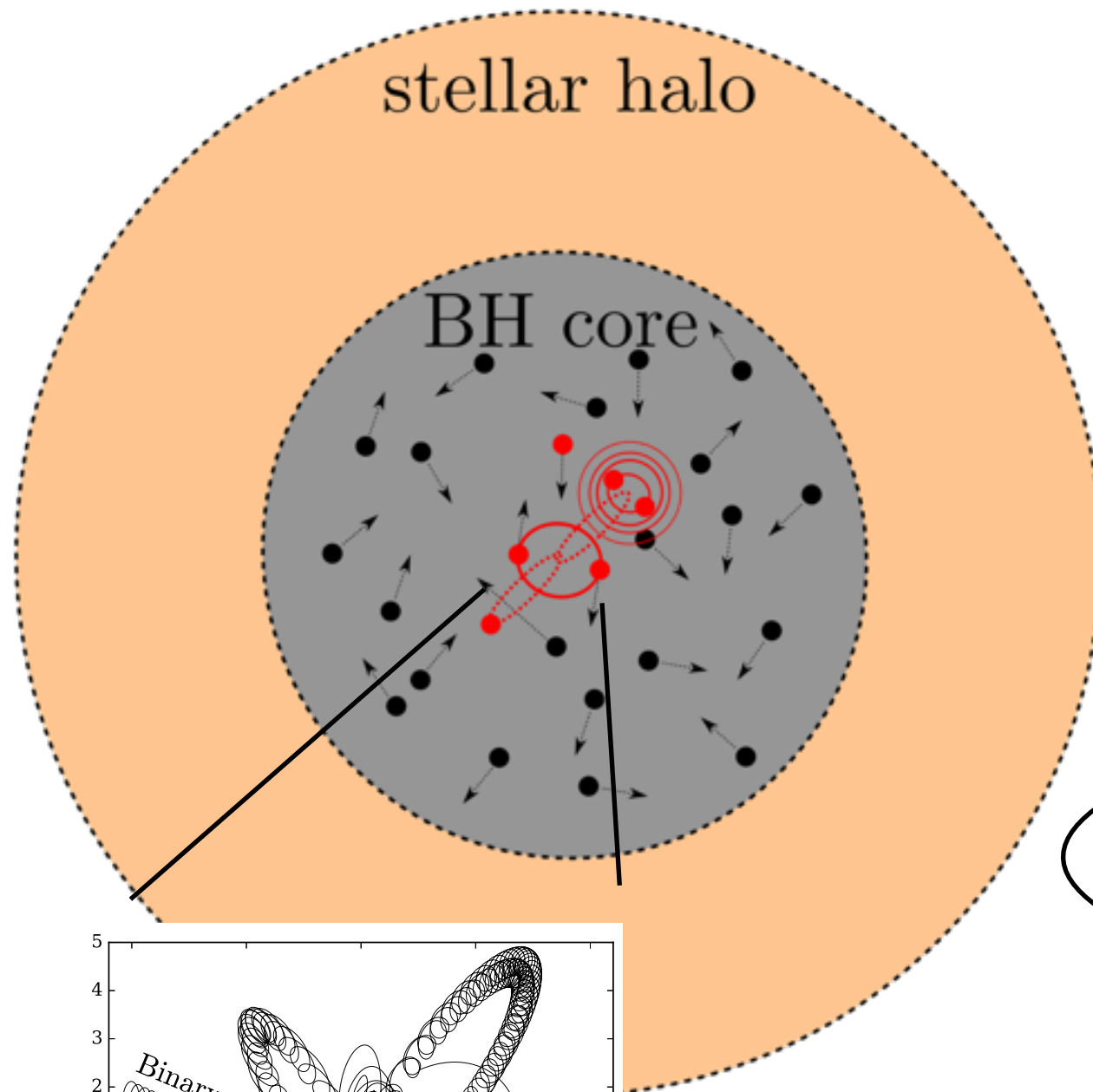
$$\Delta E_p \approx (85\pi/12)G^{7/2}c^{-5}m^{9/2}r_p^{-7/2}$$

$$\mathcal{R}_{bs} \approx \left(\frac{85\pi}{24\sqrt{2}} \right)^{2/7} \times \mathcal{R}_m \left(\frac{a}{\mathcal{R}_m} \right)^{2/7}$$



Merger Type: 3-body Merger

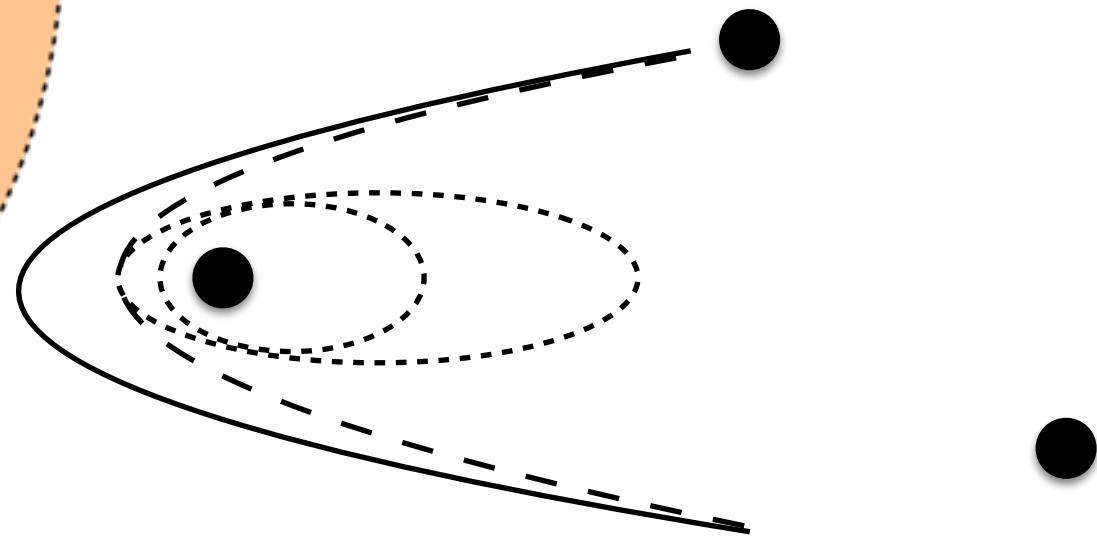
3-body Ecc. encounter:



$$r_f \approx \left(\frac{2Gm}{f^2 \pi^2} \right)^{1/3}$$

$$r_{\text{EM}} \approx r_f \times \frac{1}{2F(e_f)} \left(\frac{425}{304} \right)^{870/2299}$$

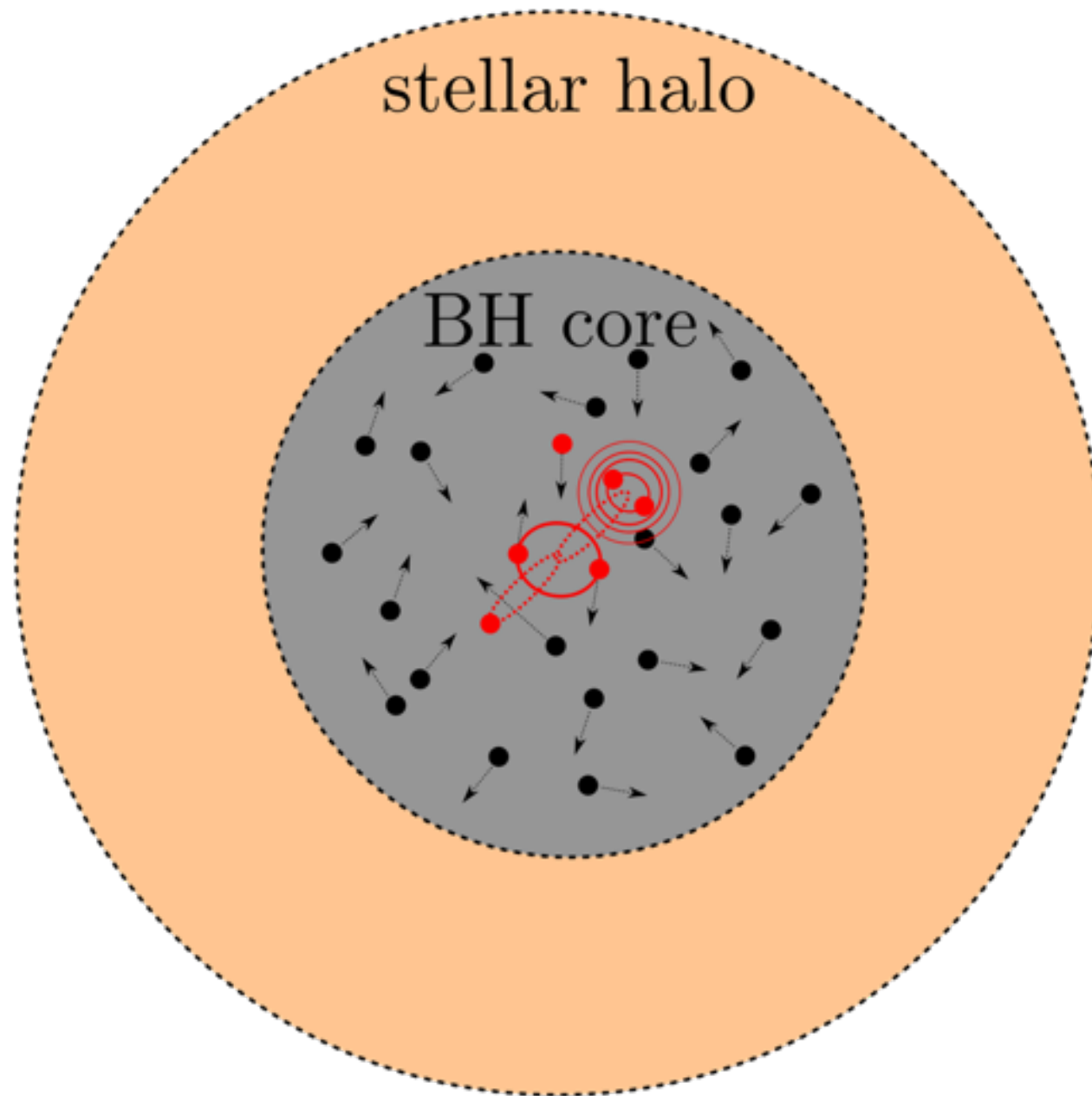
$$F(e) = \frac{e^{12/19}}{1+e} \left(1 + \frac{121}{304} e^2 \right)^{870/2299}$$



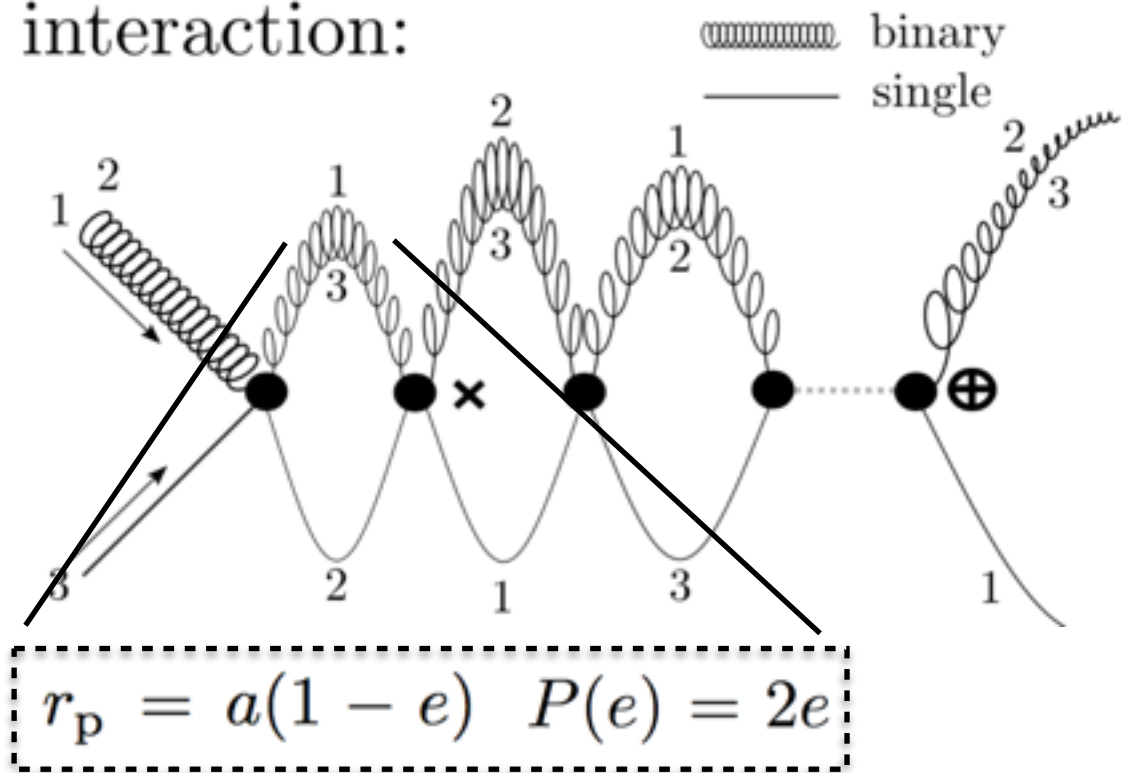
$$r_{\text{EM}} \approx \left(\frac{2Gm}{f^2 \pi^2} \right)^{1/3} \frac{1}{2} \frac{1+e_f}{e_f^{12/19}} \left[\frac{425}{304} \left(1 + \frac{121}{304} e_f^2 \right)^{-1} \right]^{870/2299}$$

Merger Type: 3-body Merger

3-body Ecc. Probability:



interaction:



$$P_{\text{EM}}(a) \approx \frac{2r_{\text{EM}}}{a} \times N_{\text{IMS}}$$

$$r_{\text{EM}} \approx \left(\frac{2Gm}{f^2\pi^2} \right)^{1/3} \frac{1}{2} \frac{1+e_f}{e_f^{12/19}} \left[\frac{425}{304} \left(1 + \frac{121}{304} e_f^2 \right)^{-1} \right]^{870/2299}$$

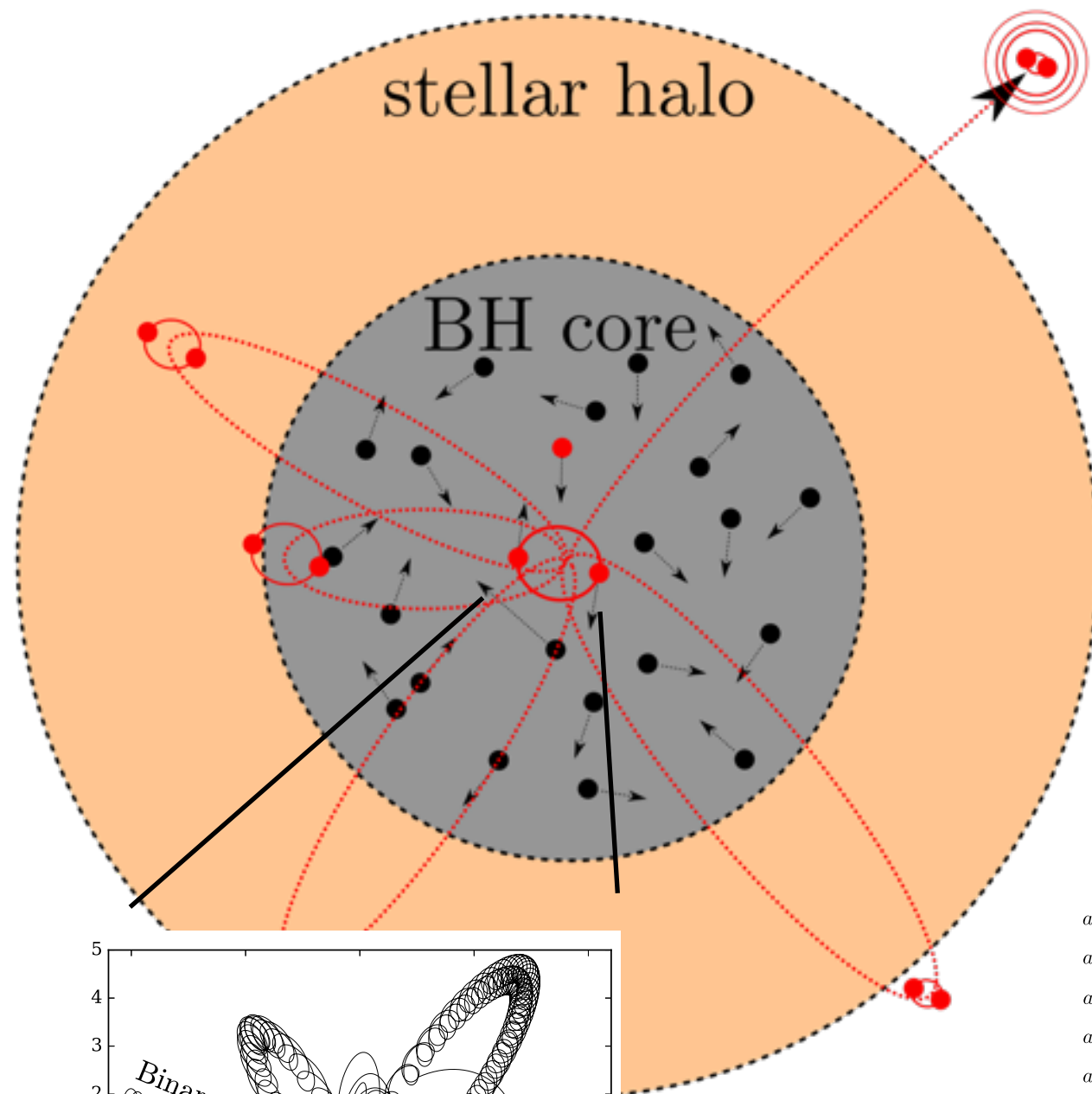
$$P_{\text{EM}}(a_{\text{in}}, a_{\text{ej}}) = \frac{1}{1 - \delta} \int_{a_{\text{ej}}}^{a_{\text{in}}} \frac{P_{\text{EM}}(a)}{a} da \approx \frac{P_{\text{EM}}(a_{\text{ej}})}{1 - \delta}$$

Merger Type: 3-body Merger

Eccentric Black Hole Mergers Forming in Globular Clusters

Authors: Johan Samsing

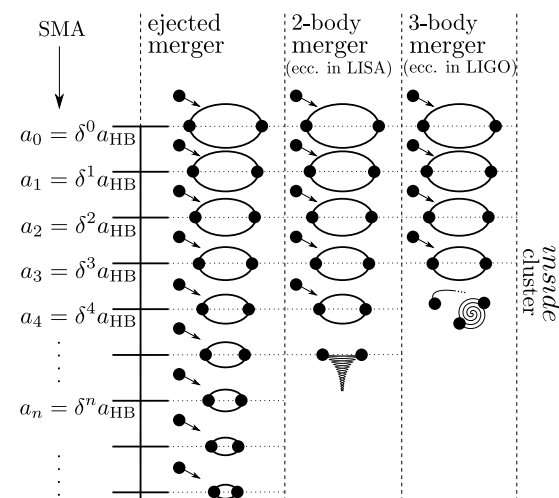
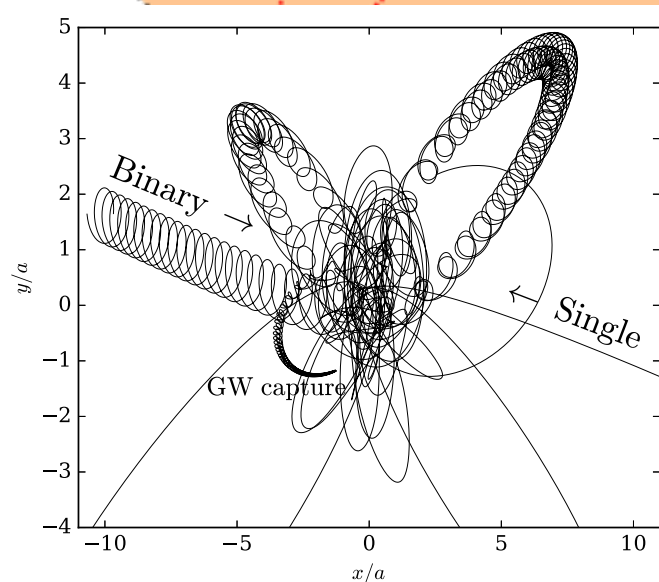
Newtonian codes underestimate the rate by more than a factor of 100!



number of tries

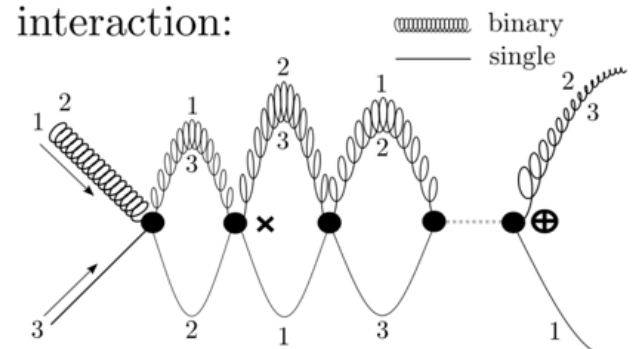
- GR: **1x1 = 1**

+GR: **20x20 = 400**



X

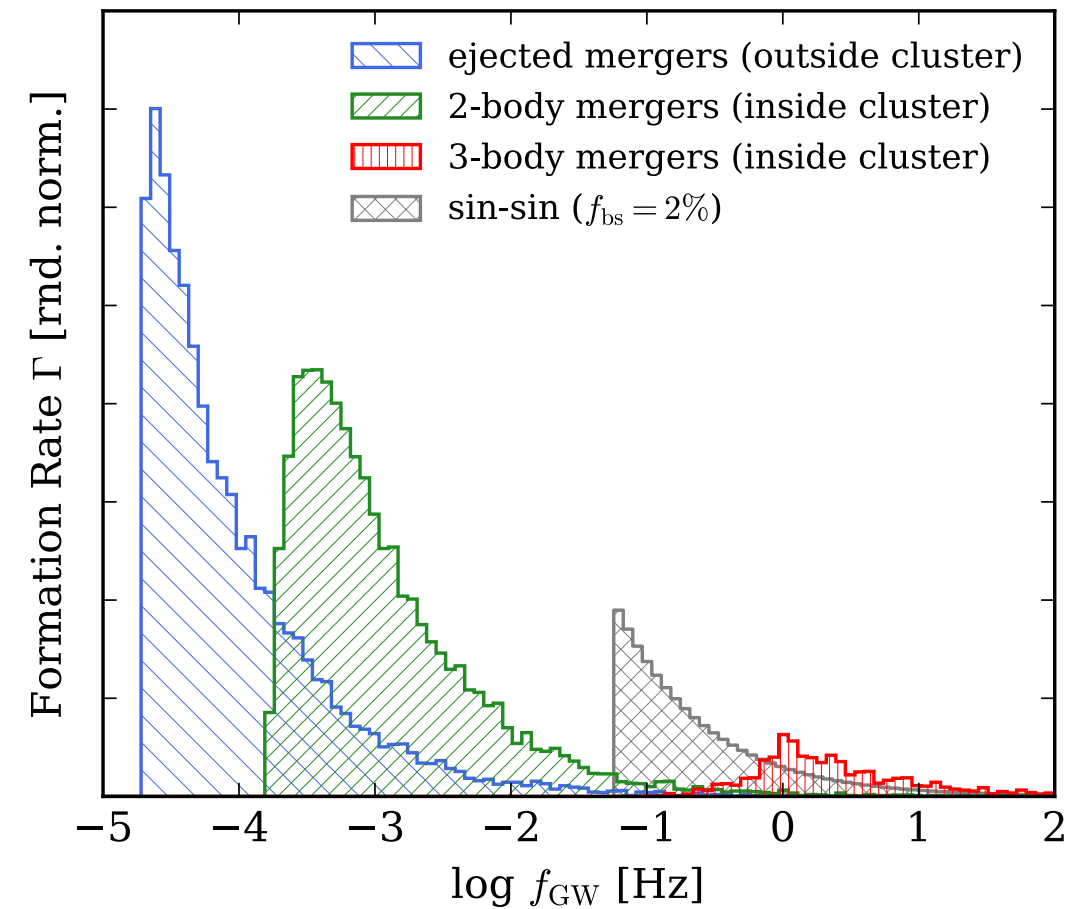
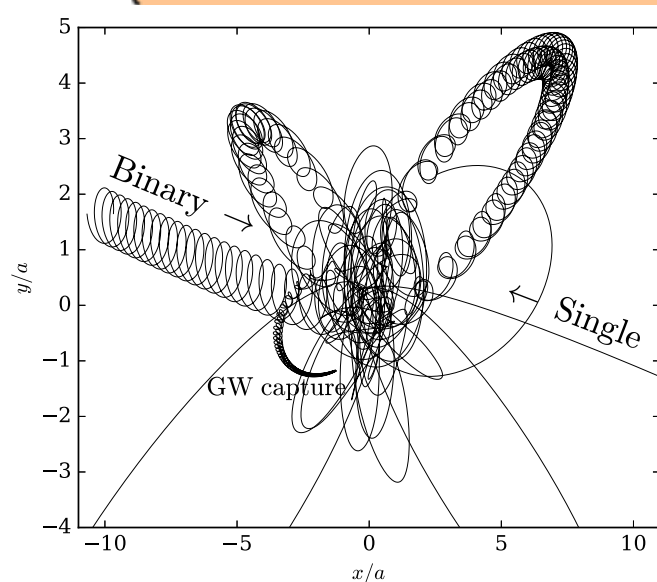
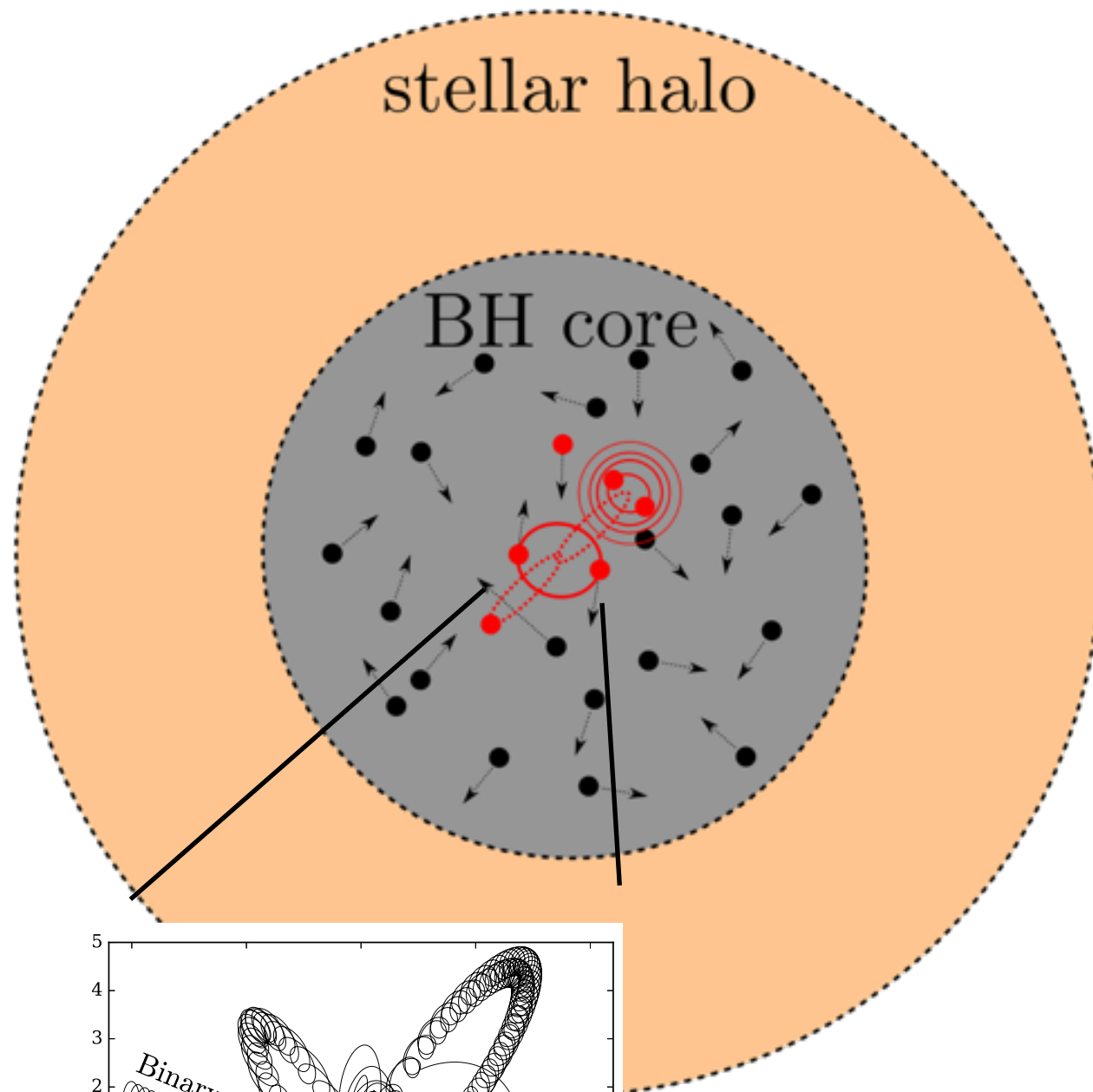
interaction:



$$N_{bs}(a_{in}, a_{ej}) = \int_{a_{ej}}^{a_{in}} \frac{1}{1-\delta} \frac{1}{a} da = \frac{1}{1-\delta} \ln \left(\frac{a_{in}}{a_{ej}} \right) \quad N_{IMS} \approx (\max(a_{IMS})/a)^{7/2}$$

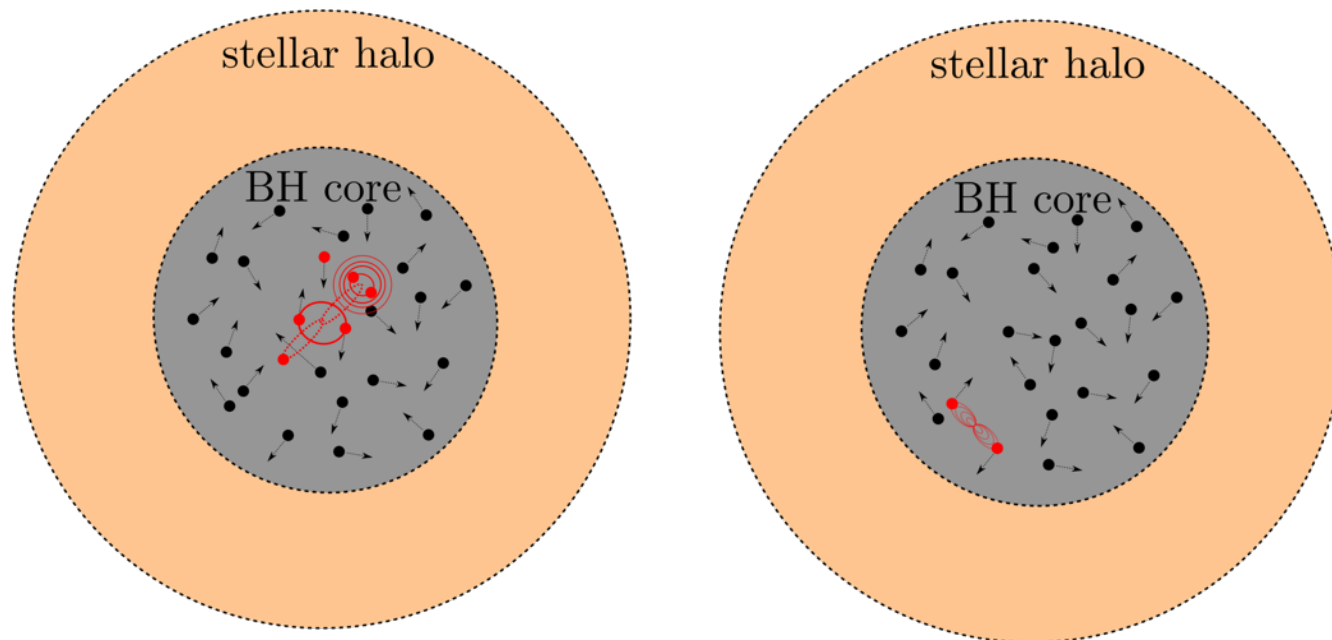
Merger Type: 3-body Merger

Eccentric LIGO sources

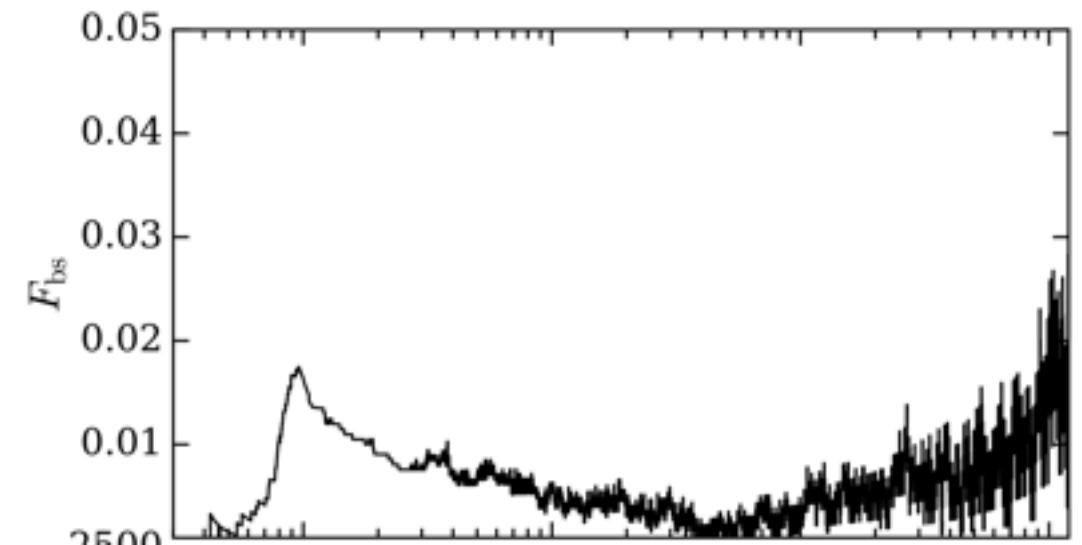


$$f_{r,0}^{\text{peak}}(\mathcal{T}) \approx 2 \cdot 10^{-5} \text{ Hz} \left(\frac{\mathcal{T}}{10^{10} \text{ yrs}} \right)^{-3/7} \left(\frac{a}{0.5 \text{ au}} \right)^{3/14} \left(\frac{m}{30 M_{\odot}} \right)^{-11/14}$$

Comparing binary-single and single-single:



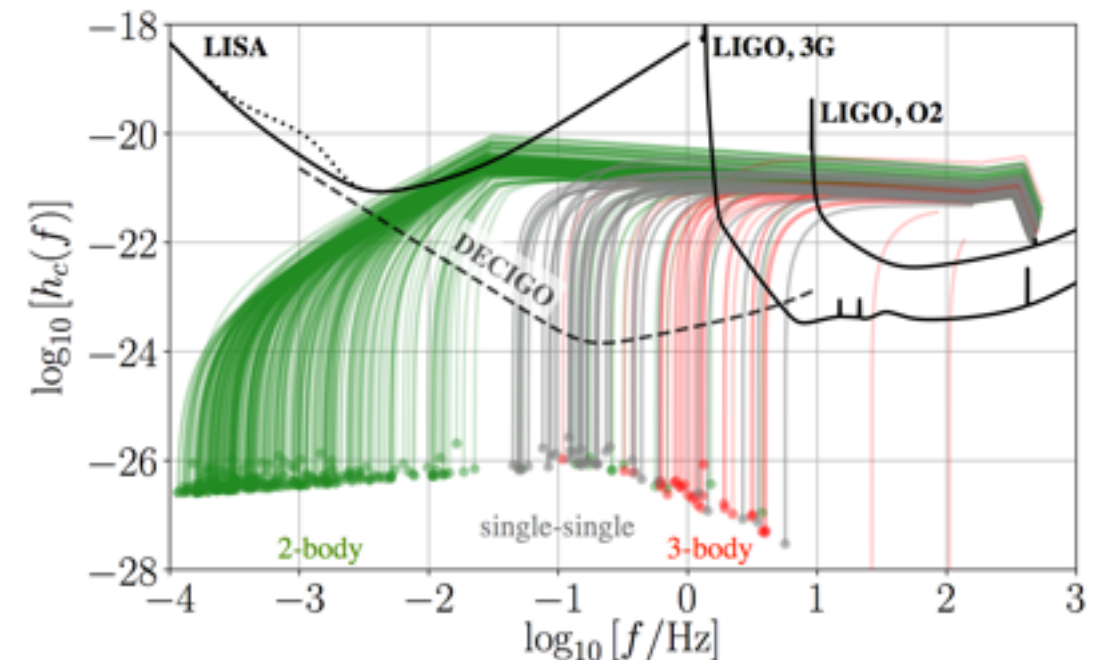
binary fraction



uniform sphere:

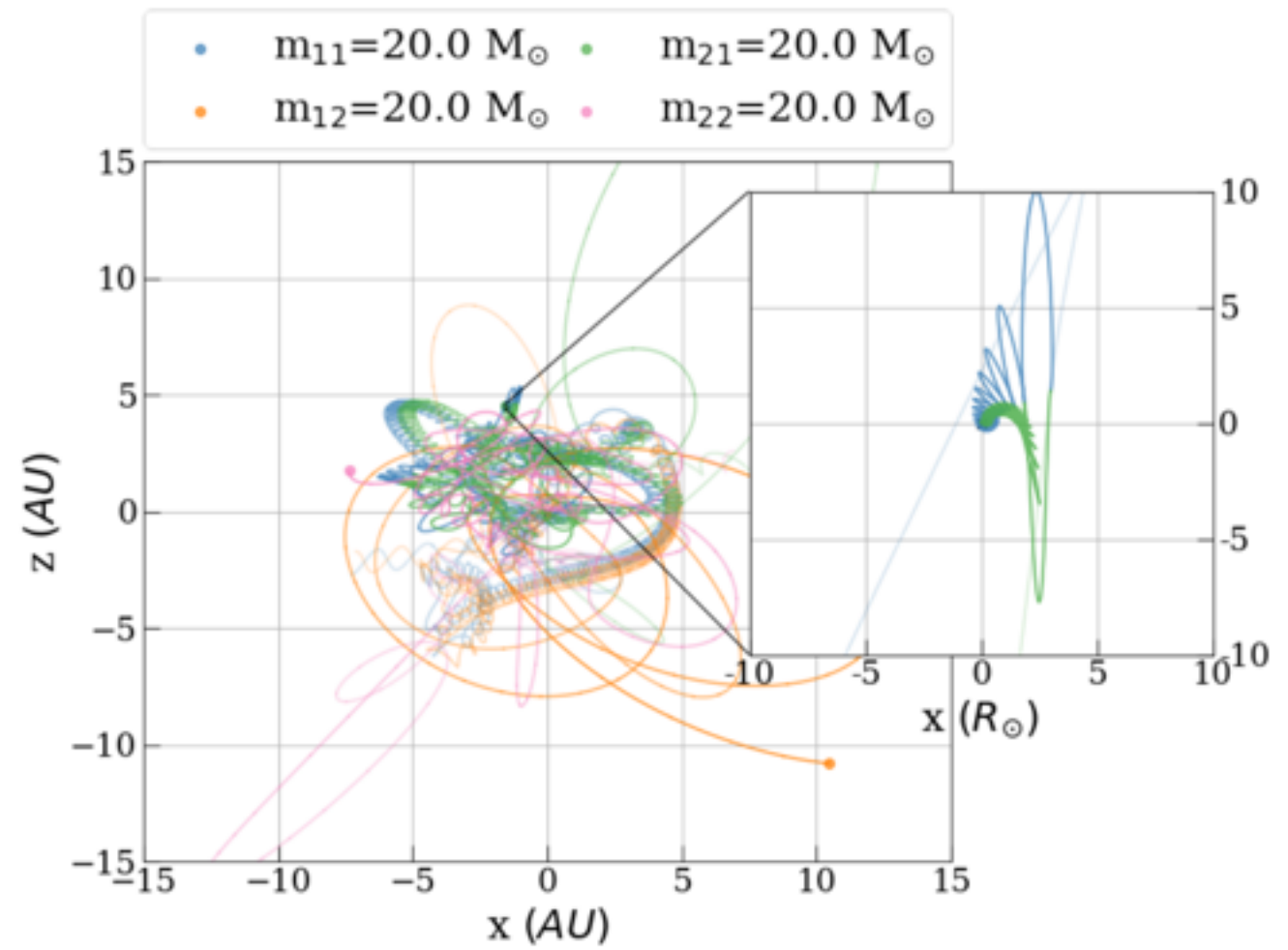
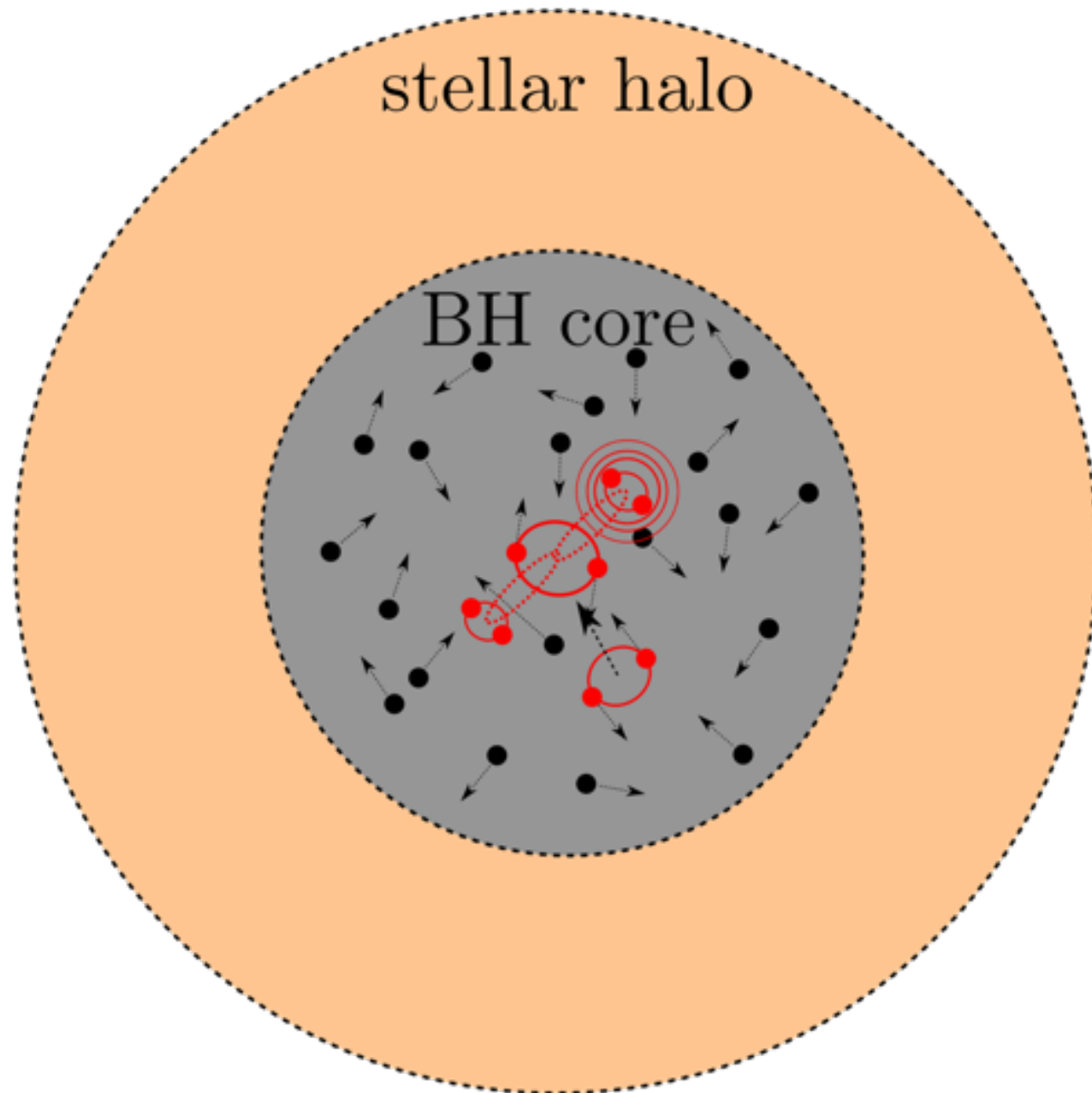
$$\frac{\Gamma_{bs}}{\Gamma_{ss}} \approx 6F_{bs}\mathcal{N} \times \left(\frac{\phi - 1}{12f_{ed}^2} \right)^{2/7} \frac{7}{5}$$

f distribution

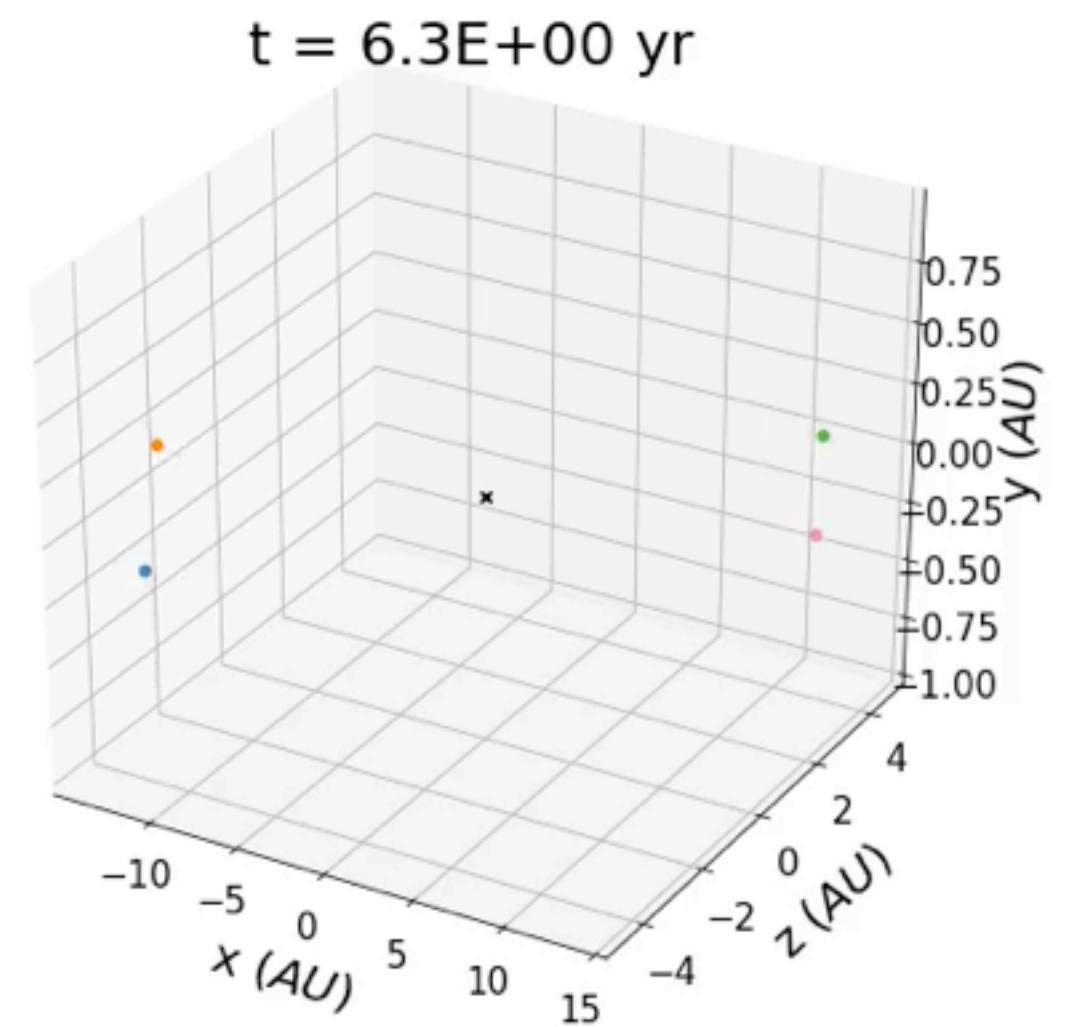
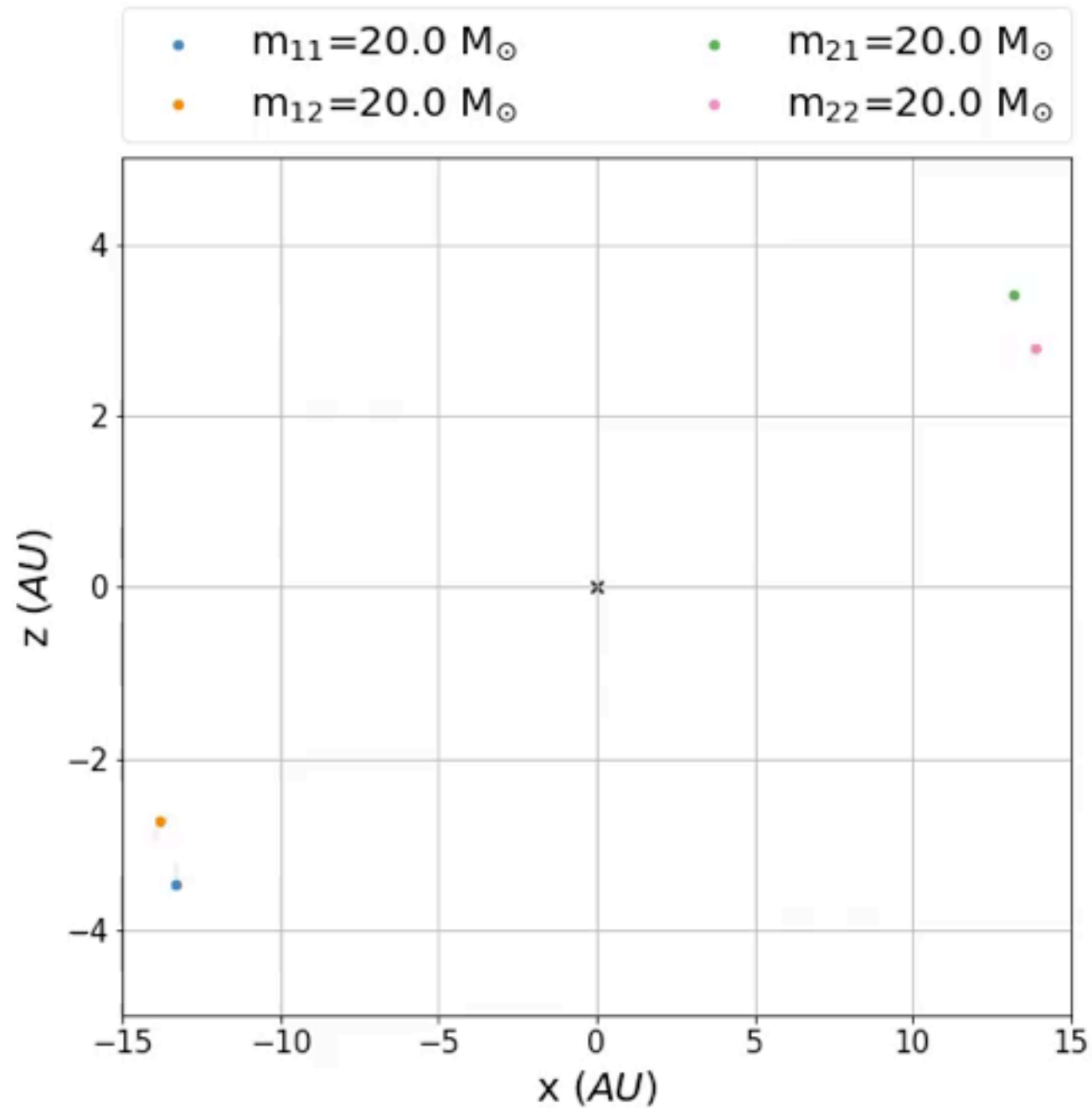
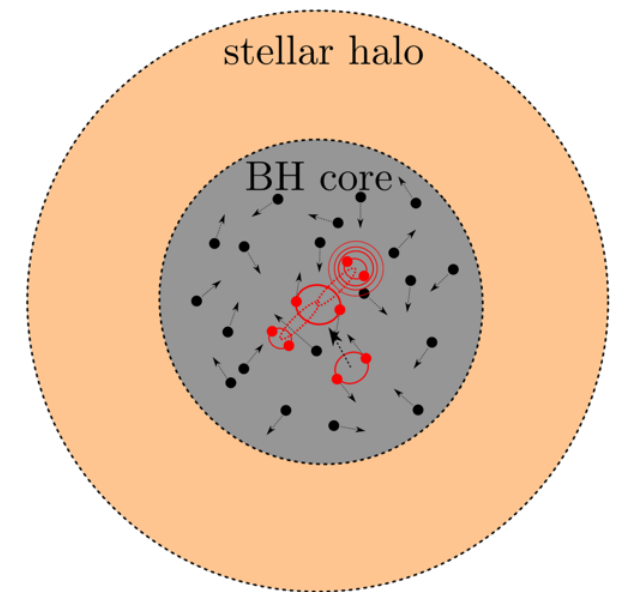


Can we here probe the BH core properties?

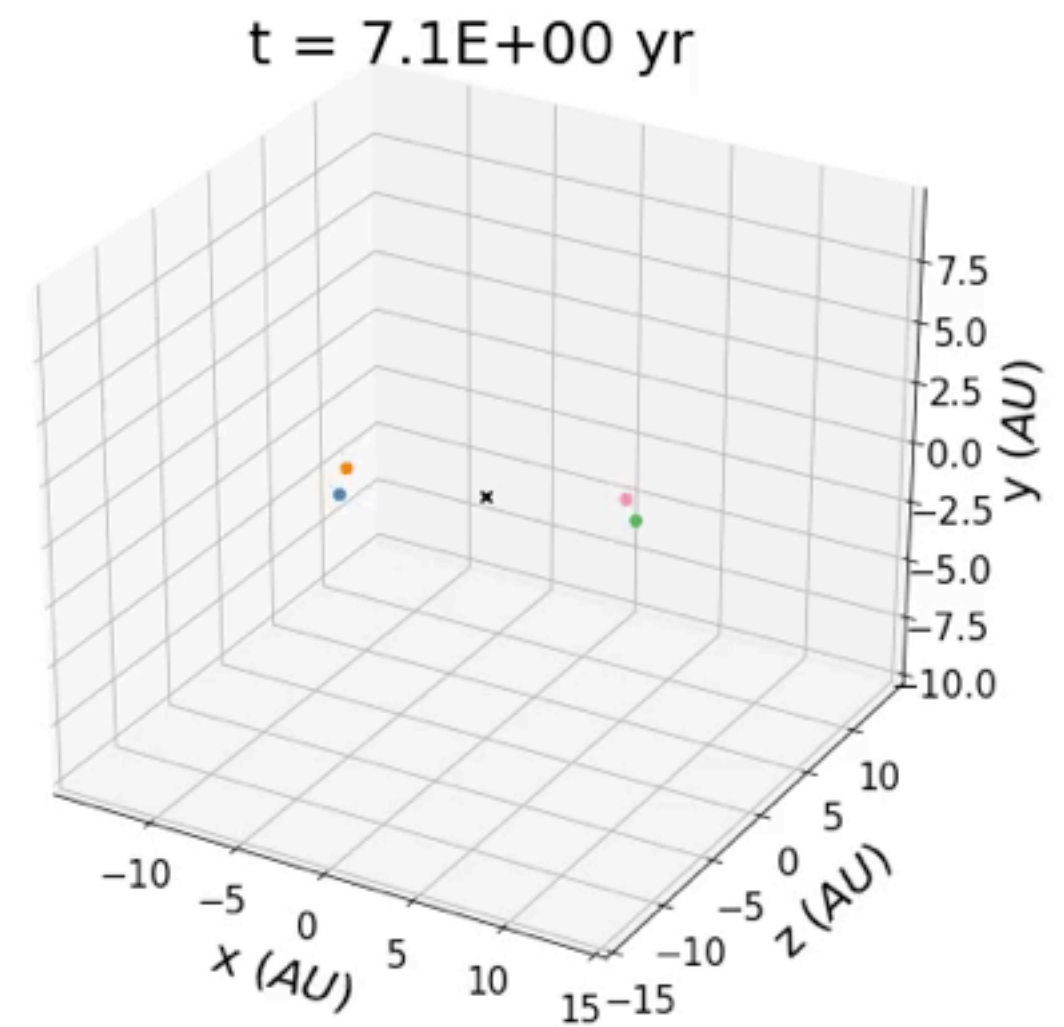
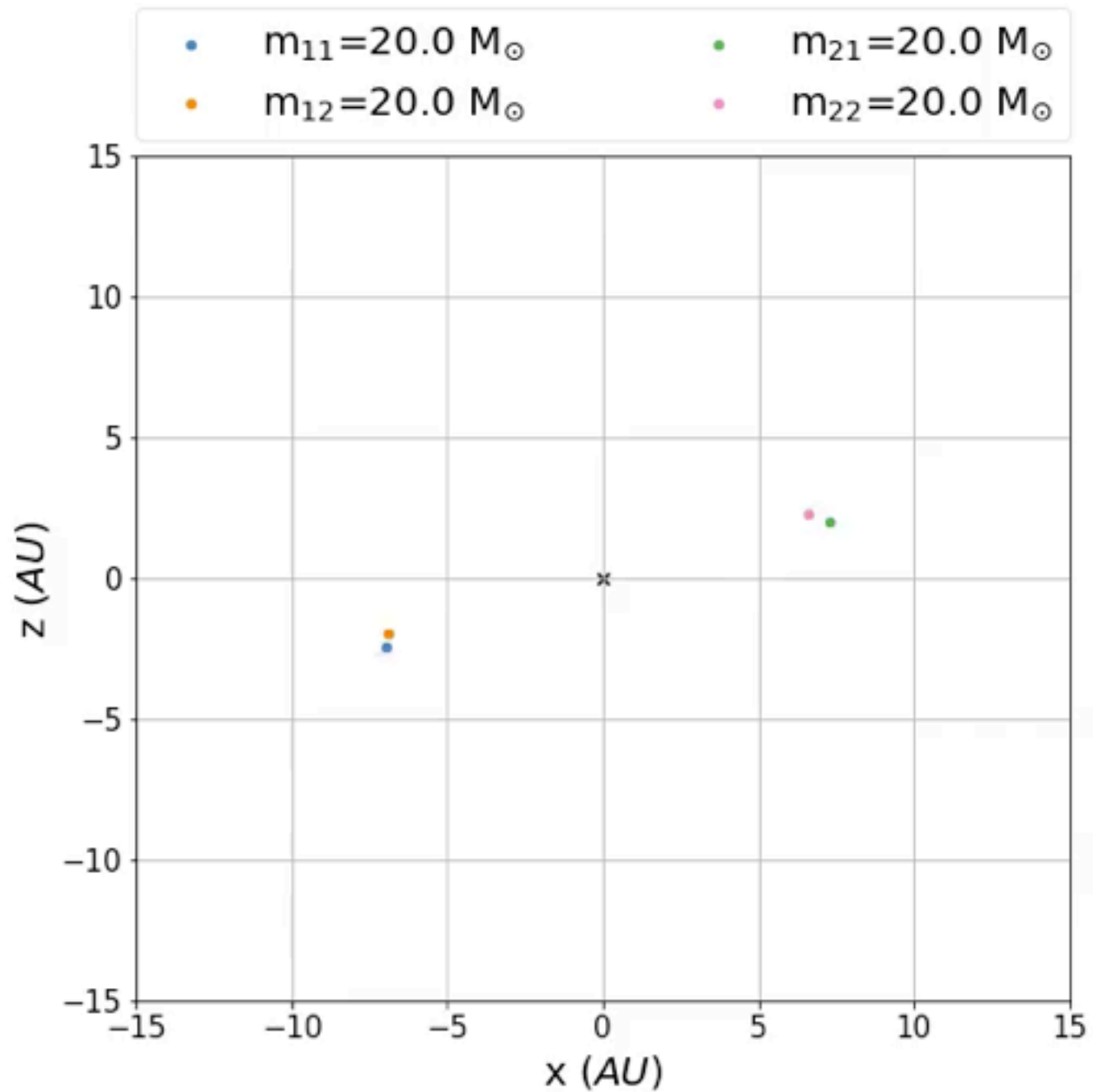
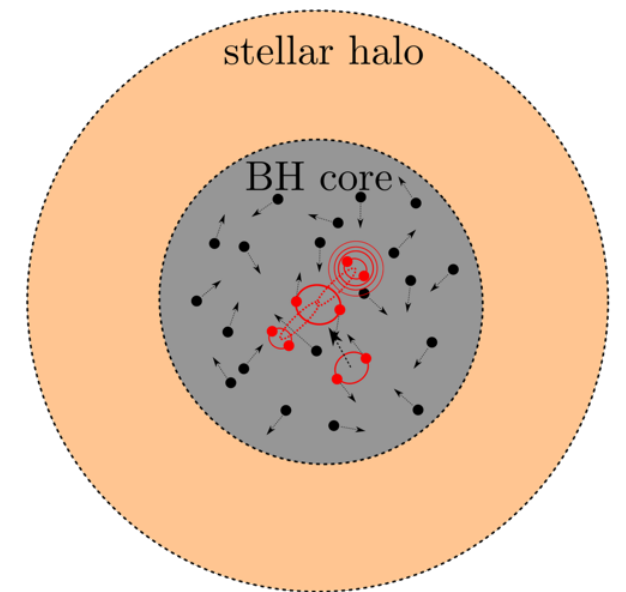
Merger Type: 4-body Merger

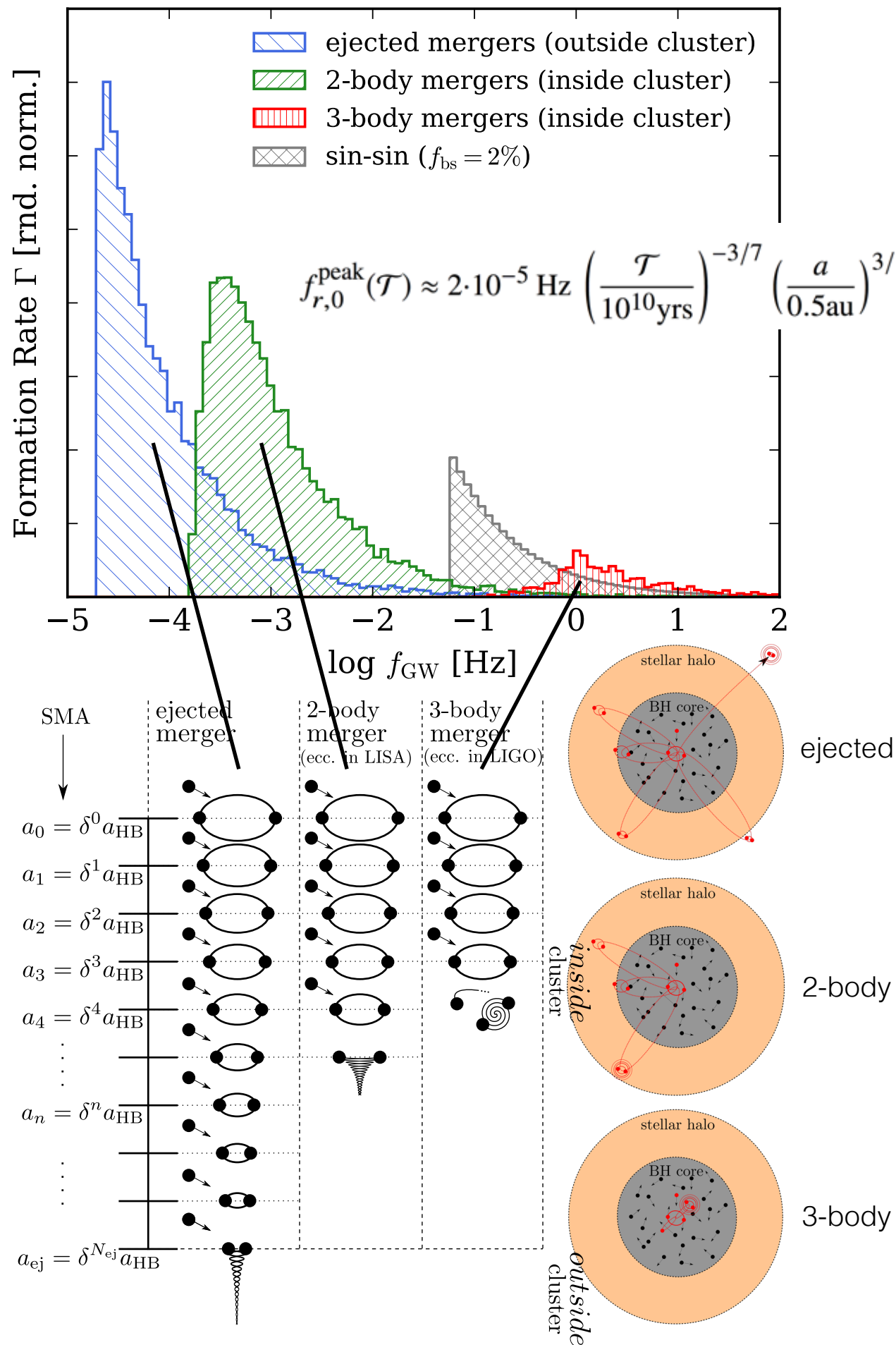


Merger Type: 4-body Merger



Merger Type: 4-body Merger





Peak Normalizations:

$$P_i \approx F_i \times \left(\frac{\tau_i(a_{\text{ej}})}{t_{\text{GW}}^{e=0}(a_{\text{ej}})} \right)^{2/7}$$

$$F_{\text{in}} \approx (7/10)/(1 - \delta) \approx 3$$

$$F_{\text{GW}} \approx (7/5)/(1 - \delta) \times N_{\text{MS}} \approx 120$$

$$a_{\text{ej}} \sim 0.5 \text{ AU} \quad M \sim 30 M_{\odot}$$

$$\tau_{\text{in}} \sim 10^7 \text{ years} \quad \tau_{\text{GW}} \sim 0.1 \text{ year}$$

$$P_{\text{in}} \approx 0.15 \quad P_{\text{GW}} \approx 0.03$$

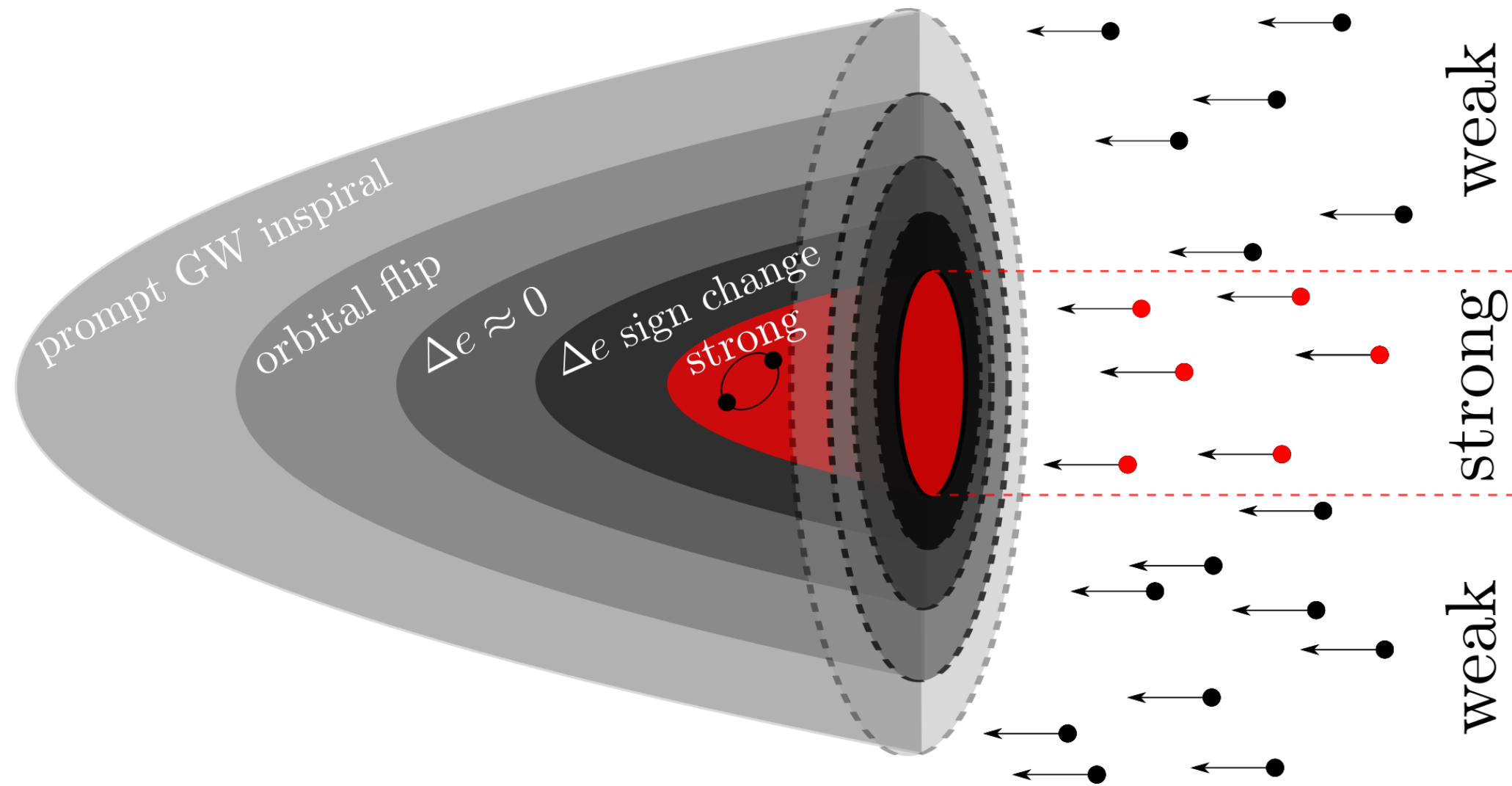
$$P(t_{\text{GW}}(a_{\text{ej}}) < T_{\text{H}}) \approx 0.35$$

$$0.82 \times 0.35 \approx 0.3$$

start to reach 10% high ecc. LIGO mergers

Merger Type: Secular-processes

- work done with Adrian Hamers (IAS)



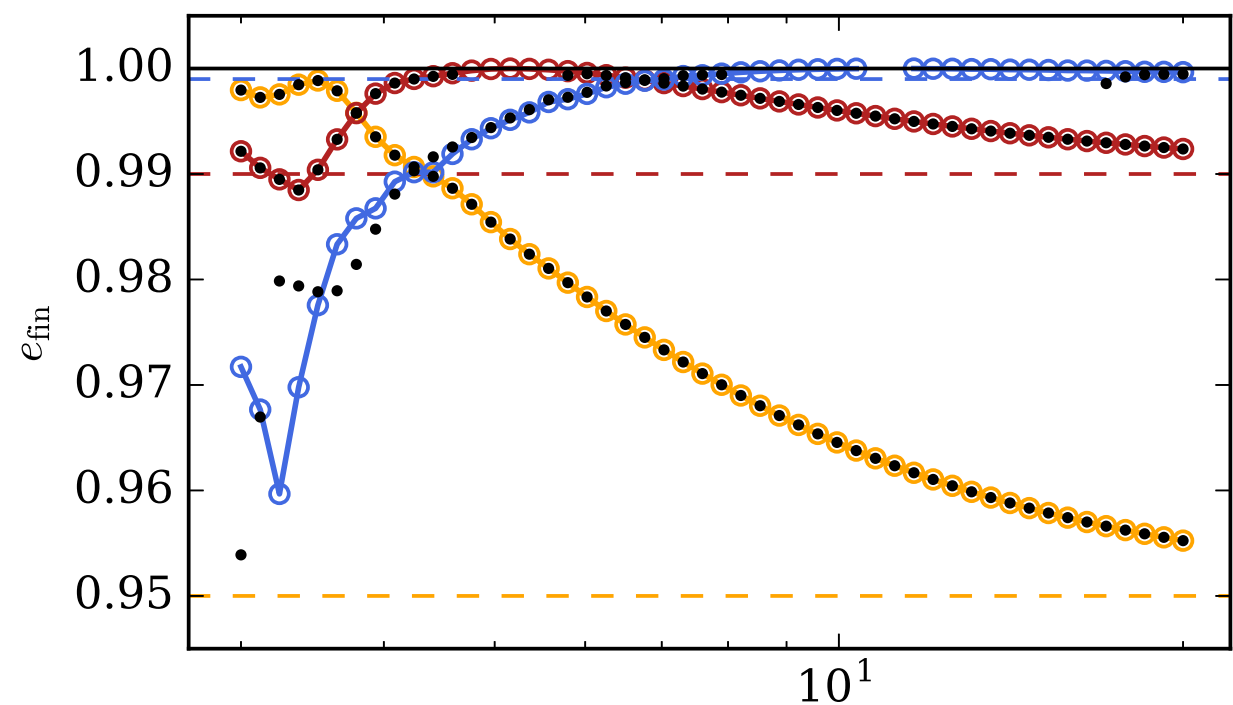
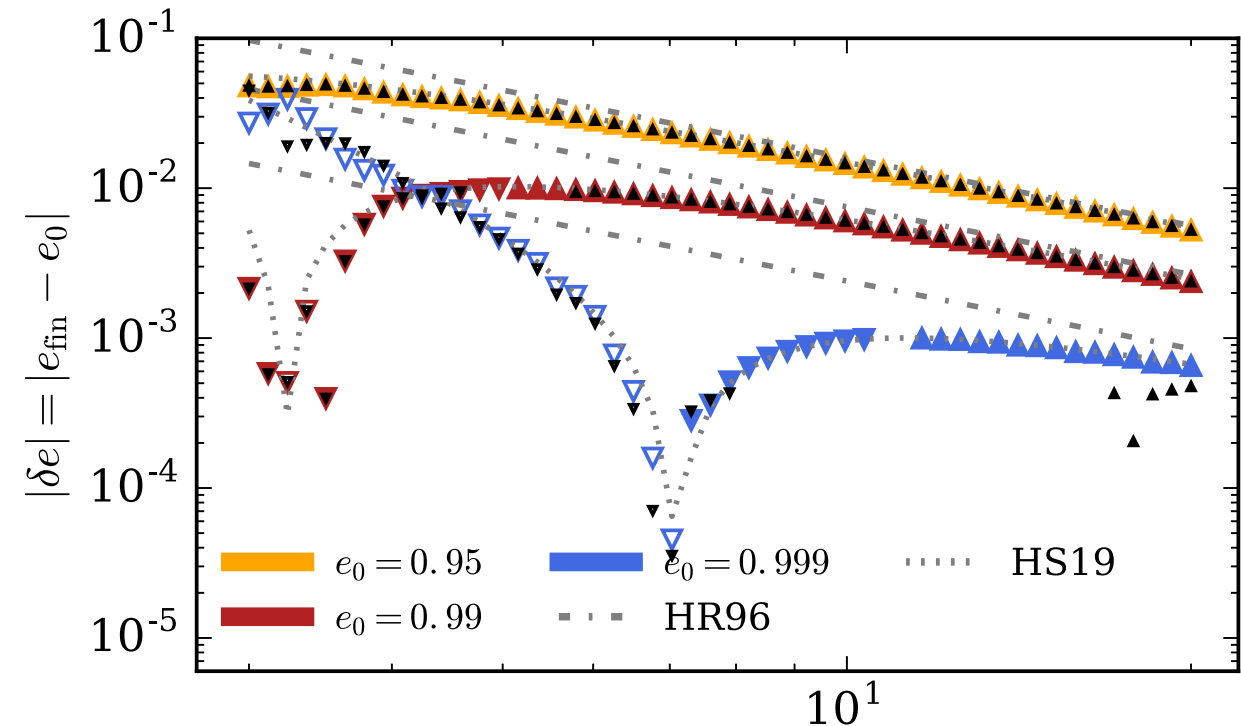
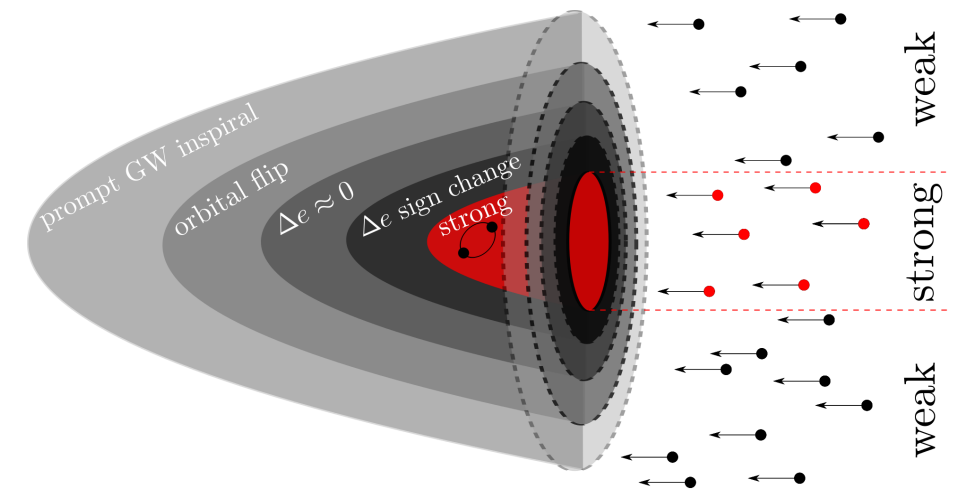
Merger Type: Secular-processes

1.order (Heggie, Rasio 96)

$$\delta e = -\frac{15\pi}{16} \left(\frac{2m_3^2 a^3}{M_{123} M_{12} r_p^3} \right)^{1/2} e \sqrt{1-e^2} \sin 2\Omega \sin^2 i$$

2.order (Hammers, Samsing 19)

$$\begin{aligned} \Delta e_{\text{SO}} = & \Delta e_{\text{FO}} + \epsilon^2 \frac{3}{512} \pi e_0 \left[-100 (1 - e_0^2) \sin 2\Omega \right. \\ & \left\{ (5 \cos i + 3 \cos 3i) \cos 2\omega + 6 \sin i \sin 2i \right\} \\ & + 4 \cos 2i \left\{ 3\pi (81e_0^2 - 56) - 200 (1 - e_0^2) \right. \\ & \left. \cos 2\Omega \sin 2\omega \right\} + 3\pi \left\{ 200e_0^2 \sin^4 i \cos 4\Omega \right. \\ & + 8 (16e_0^2 + 9) \sin^2 2i \cos 2\Omega \\ & \left. \left. + (39e_0^2 + 36) \cos 4i - 299e_0^2 + 124 \right\} \right], \end{aligned}$$



non-parabolic limit...

16 *Hamers & Samsing*

$$\begin{aligned}
 g_e^{(1)} = & \left[-\frac{15}{512} \pi \left(2541e_x^4 e_y + e_x^2 \left(36\pi \left(e_y^2 - 163e_z^2 + 35j_y^2 - 5j_z^2 \right) - 847j_x j_y \right) + e_x \left(2037e_y^3 - e_y \left(294e_z^2 + 637j_x^2 + 2592\pi j_x j_y + 3969j_y^2 + 4074j_z^2 - 420 \right) + 24e_z j_z (76\pi j_x - 77j_y) \right) \right. \\
 & \left. + 60\pi j_x^2 \left(3e_y^2 + 15e_z^2 - 3j_y^2 + j_z^2 \right) + 7j_x \left(j_y \left(163e_y^2 - 482e_z^2 - 62j_z^2 + 20 \right) - 264e_y e_z j_z + j_y^3 \right) \right. \\
 & \left. + 12\pi \left(3e_y^4 + e_y^2 \left(-91e_z^2 - 36j_y^2 + 3j_z^2 + 4 \right) + 72e_y e_z j_y j_z + 96e_z^4 + e_z^2 \left(81j_y^2 + 32j_z^2 - 12 \right) - 15j_y^4 + 15j_y^2 j_z^2 + 12j_z^4 - 4j_z^2 \right) - 49j_x^2 j_y \right], \\
 & \frac{15}{512} \pi \left(2541e_x^4 + 36\pi e_x^3 e_y + e_x^2 \left(2037e_y^3 - 2513e_z^2 - 1484j_x^2 + 1380\pi j_x j_y - 903j_y^2 - 7623j_z^2 + 420 \right) \right. \\
 & \left. + 4e_x \left(9\pi e_y^3 + e_y \left(3\pi \left(398e_z^2 - 35j_y^2 + 18j_z^2 + 4 \right) - 453\pi j_x^2 - 658j_x j_y \right) - 2e_z j_z (469j_x + 36\pi j_y) \right) - e_z^2 \left(707e_y^2 + 3213j_x^2 - 72\pi j_x j_y + 7 \left(j_y^2 - 480j_z^2 + 20 \right) \right) + 707e_y^2 j_x^2 \right. \\
 & \left. - 12\pi e_y^2 j_x j_y - 2037e_y^2 j_z^2 + 56e_y e_z j_z (47j_y - 12\pi j_x) + 1120e_z^4 - 49j_x^4 - 180\pi j_x^2 j_y + 7j_x^2 j_y^2 + 637j_x^2 j_z^2 + 140j_x^4 - 180\pi j_x j_y^3 + 120\pi j_x j_y j_z^2 + 144\pi j_x j_y + 903j_y^2 j_z^2 - 420j_z^4 \right), \\
 & \frac{15}{512} \pi \left(2892\pi e_x^2 e_z - e_y \left(e_z \left(-2219e_x^2 + 1561j_x^2 + 264\pi j_x j_y + 3059j_y^2 - 140 \right) + 2e_x j_z (2891j_x + 780\pi j_y) + 1120e_z^2 \right) + 3e_x^2 j_z (3143j_y - 492\pi j_x) + e_y^2 (2892\pi e_x e_z + 84\pi j_x j_z + 2471j_y j_z) \right. \\
 & \left. - 2e_x e_z \left(6\pi \left(224e_z^2 + 179j_y^2 - 28 \right) + 1206\pi j_x^2 - 679j_x j_y \right) + 707e_y^2 e_z + 3j_z \left(-7j_y \left(160e_z^2 + 51j_x^2 - 20 \right) + 4\pi j_x \left(96e_z^2 + 35j_x^2 - 12 \right) + 140\pi j_x j_y^2 - 301j_y^3 \right) \right); \\
 g_e^{(2)} = & \left[\frac{225}{32768} \pi \left(13041e_y^4 + 288e_x \pi e_y^3 + (10628e_z^2 + 36082e_z^2 + 2458j_x^2 - 23654j_y^2 - 32164j_z^2 + 640j_x j_y \pi - 672) e_y^2 \right. \right. \\
 & \left. \left. + 4 \left(72\pi e_x^2 + (920\pi j_x^2 + 1583j_y j_x + 8 \left(-527e_z^2 - 234j_y^2 + 9j_z^2 + 12 \right) \pi \right) e_x + 2e_z j_z (200\pi j_x + 11091j_y) \right) e_y^2 \right. \\
 & \left. + (20897e_x^4 + 2 \left(12032e_z^2 + 307j_x^2 - 32777j_y^2 - 33626j_z^2 - 13472j_x j_y \pi + 2018 \right) e_x^2 + 48e_z j_z (53j_x + 216j_y \pi e_x + 88096e_z^2 + 3825j_x^2 + 1921j_y^2 - 3136j_z^2 - 1770j_x^2 j_y^2 + 1568j_y^2 \right. \right. \\
 & \left. \left. + 4e_z^2 \left(8067j_x^2 + 400j_y \pi j_x - 19099j_y^2 - 2688j_z^2 - 3248 \right) + 3200j_x j_y^2 \pi - 960j_x j_y j_z^2 \pi + 1600j_x^2 j_y \pi - 1280j_x j_y \pi + 560 \right) e_y + 8e_z^2 e_z j_z (6472\pi j_x + 1167j_y) \right. \\
 & \left. - 8e_z j_z \left(-3883j_y^2 + 1080j_x \pi j_y^2 + (-17824e_z^2 - 3767j_x^2 + 2128) j_y + 40j_x \left(31e_z^2 + 15j_x^2 - 4 \right) \pi \right) - 4e_z^2 \left(7343j_x j_y + 216 \left(73e_z^2 - 19j_z^2 + j_z^2 \right) \pi \right) \right. \\
 & \left. + 4e_x \left(281j_y j_x^2 + 40 \left(115e_z^2 - 17j_y^2 - 21j_z^2 \right) \pi j_x^2 + j_y \left(-20602e_z^2 + 2237j_y^2 - 3802j_z^2 + 672 \right) j_x + 8 \left(864e_z^2 + 3 \left(243j_y^2 + 32j_z^2 - 36 \right) \pi \right) - 135j_y^4 - 12j_z^4 + j_z^2 \left(65j_z^2 + 308 \right) \right) \pi \right], \\
 & -\frac{225}{32768} \pi \left(20897e_x^4 + 432e_y \pi e_x^3 + (11626e_z^2 - 2884e_z^2 - 28358j_x^2 - 8294j_y^2 - 80388j_z^2 + 8896j_x j_y \pi + 4032) e_x^2 \right. \\
 & \left. + 4 \left(144\pi e_x^2 + (-3996\pi j_x^2 - 10513j_y j_x + 8 \left(1363e_z^2 - 170j_y^2 + 27j_z^2 + 24 \right) \pi \right) e_y + 2e_z j_z (2887j_x + 472j_y \pi) \right) e_x^2 \\
 & \left. + (13041e_y^4 + 2 \left(7126e_z^2 - 3209j_x^2 + 3531j_y^2 - 31626j_z^2 - 64j_x j_y \pi - 336 \right) e_y^2 + 16e_z j_z (8207j_y - 1256j_x \pi e_y + 57344e_z^2 + 4949j_x^2 + 2145j_y^2 - 448j_z^2) e_x \right. \\
 & \left. + (14558j_x^2 j_y^2 - 2464j_y^2 - 2028j_z^2 j_y^2 + 16588j_z^2 j_y^2 - 8064j_z^2 - 4e_z^2 \left(10699j_x^2 - 6272j_y \pi j_x - 2601j_y^2 - 16128j_z^2 + 2912 \right) - 1920j_x j_y^2 \pi - 2880j_x j_y j_z^2 \pi - 320j_x^2 j_y \pi + 1536j_x j_y \pi + 560 \right) e_x \right. \\
 & \left. + 4 \left(36\pi e_y^2 - (240\pi j_x^2 + 7679j_y j_x + 8 \left(81e_z^2 - 5j_y^2 + 9j_z^2 - 12 \right) \pi \right) e_y^2 + 6e_z j_z (613j_x + 312j_y \pi e_y^2) \right. \\
 & \left. + 4 \left((800\pi j_x^2 - 1795j_y j_x^2 + 40 \left(15e_z^2 + 30j_y^2 + 39j_z^2 - 8 \right) \pi j_x^2 - j_y \left(20602e_z^2 + 55j_y^2 + 3802j_z^2 - 3008 \right) j_x + 4 \left(706e_z^2 + 6 \left(135j_y^2 + 32j_z^2 - 36 \right) \pi \right) - 125j_y^4 - 24j_z^4 - 10j_y^2 \left(7j_z^2 + 12 \right) + 16 \right) \pi \right) e_y \right. \\
 & \left. + 4 \left(2e_z j_z \left(4219j_x^2 - 40j_y \pi j_x^2 + (14336e_z^2 + 2407j_y^2 - 1792) j_x + 88j_y \left(12e_z^2 + 5j_y^2 - 4 \right) \pi \right) \right), \\
 & -\frac{225}{2048} \pi \left(1024\pi e_x^2 + 32 \left(20\pi j_x^2 + 21j_y j_x + 4 \left(e_y^2 - 11j_y^2 - 2 \right) \pi \right) e_z^2 + 32e_y j_z (77j_x + 24j_y \pi e_z^2 + (75\pi j_x^2 + 518j_y j_x^2 - 10 \left(17e_z^2 + 17j_y^2 + 8 \right) \pi j_x^2 + j_y \left(4307e_z^2 + 373j_y^2 - 84 \right) e_z) \right. \\
 & \left. - \frac{225}{2048} \pi \left((j_x - 21e_z^2 + 2 \left(159j_y^2 + 8 \right) e_y^2 + 243j_y^2 - 176j_z^2 - 36 \right) \pi \right) e_z + 1387e_x^4 e_z + e_y j_z \left(849j_x^2 - 1302j_y \pi j_x^2 + (-2853e_z^2 + 769j_y^2 - 308) j_x + 4j_y \left(7e_z^2 + 55j_y^2 - 24 \right) \pi \right) + e_z^2 (1743e_y e_z + 4061j_y j_z - 806j_x j_z \pi) \right. \\
 & \left. + e_x \left(1365e_z^2 e_y^2 + j_z (28\pi j_x + 4021j_y \pi e_y^2) + e_z \left(672e_z^2 + 1769j_x^2 - 6357j_y^2 + 344j_x j_y \pi - 84 \right) e_y \right) + e_x \left(j_z \left(-1377j_y^2 + 880j_x \pi j_y^2 + (-6496e_z^2 - 1525j_x^2 + 812) j_y + 4j_x \left(448e_z^2 + 135j_y^2 - 36 \right) \pi \right) \right. \\
 & \left. - e_z^2 \left(2944\pi e_z^2 + (1534\pi j_x^2 + 3047j_y j_x - 2 \left(683e_z^2 + 1013j_y^2 + 184 \right) \pi \right) e_z + e_y j_z (4777j_x + 804j_y \pi) \right); \\
 h_e^{(0)} = & \left[\frac{3}{32} \pi \left(-1200\pi e_x^2 j_z + 15e_z^2 \left(1361e_y j_z + 80\pi e_z j_x + (369 + 384\pi^2) e_z j_y \right) - 6e_x \left(300\pi e_y^2 j_z + 15e_y e_z \left((64\pi^2 - 309) j_x + 160\pi j_y \right) + j_z \left(120\pi j_x^2 + (381 + 192\pi^2) j_x j_y + 20\pi j_y^2 \right) \right) \right. \\
 & \left. + 15e_z^2 e_z (440\pi j_x + 681j_y) + e_y j_z \left((1152\pi^2 - 3353) j_x^2 + 960\pi j_x j_y + 399j_y^2 \right) + e_z \left(720\pi j_x^2 - 313j_x^2 j_y + 1080\pi j_x j_y^2 - 917j_y^3 \right) \right], \\
 & \frac{3}{32} \pi \left(-4475e_z^2 j_z - 5e_z^2 (80\pi e_y j_z + 177e_z j_x - 480\pi e_z j_y) \right. \\
 & \left. + e_x \left(3635e_z^2 j_z + 10e_y e_z \left((65 + 192\pi^2) j_y - 200\pi j_x \right) + j_z \left(1043j_x^2 - 80\pi j_x j_y - (3001 + 384\pi^2) j_y^2 \right) \right) - 600\pi e_y^2 j_z - 5e_y^2 e_z \left((384\pi^2 - 387) j_x + 520\pi j_y \right) \right. \\
 & \left. + 2e_y j_z \left(-80\pi j_x^2 + (192\pi^2 - 371) j_x j_y + 340\pi j_y^2 \right) + e_z \left(29j_x^3 + 240\pi j_x^2 j_y + 95j_x j_y^2 + 360\pi j_y^3 \right) \right], \\
 & -\frac{3}{64} \pi \left(2400\pi e_x^2 e_z j_z + e_y \left(-620e_x e_z j_z + 45 \left(11 + 16\pi^2 \right) e_z^2 j_x + j_z^2 \left(48\pi^2 j_x + 7j_x - 240\pi j_y \right) \right) + 15e_x e_z^2 \left(80\pi j_x + (133 - 48\pi^2) j_y \right) + e_x j_z^2 \left((77 - 48\pi^2) j_y - 680\pi j_x \right) + 1200\pi e_y^2 e_z j_z - 12e_z j_x j_z (20\pi j_x + 69j_y) \right); \\
 \end{aligned}$$

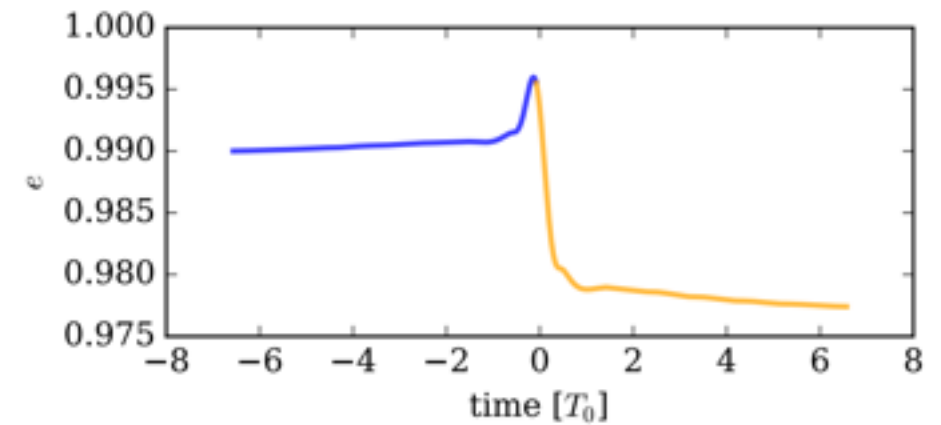
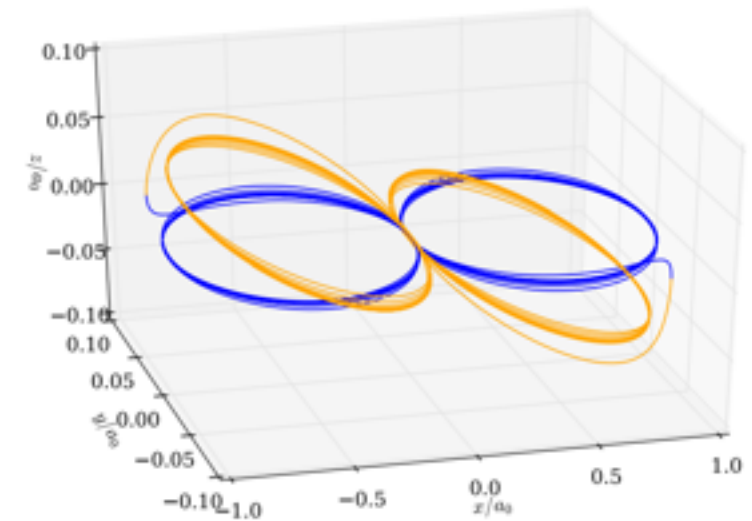
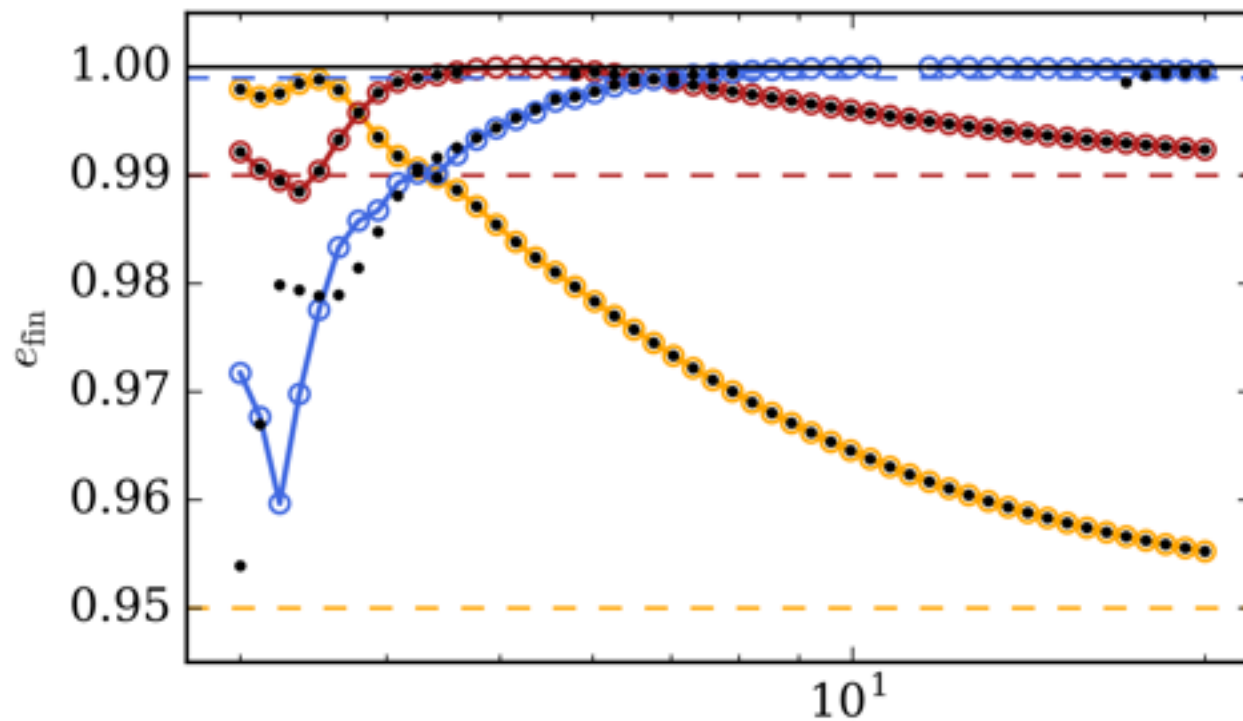
The functions associated with the vector angular-momentum changes are given by

$$\begin{aligned}
 f_j^{(0)} = & \left[-\frac{3}{2} \pi (5e_y e_z - j_y j_z) \lambda \frac{3}{2} \pi (5e_x e_z - j_x j_z) \lambda, 0 \right]; \\
 f_j^{(1)} = & \left[-\frac{75}{16} \pi (-7e_x e_y e_z + e_x j_y j_z + e_y j_x j_z + e_z j_x j_y) \lambda, \frac{15}{32} \pi \left(e_z \left(-73e_x^2 - 3e_y^2 + 15j_x^2 + 5j_y^2 - 4 \right) + 10j_z (3e_x j_x + e_y j_y) + 32e_z^2 \right), \frac{15}{32} \pi \left(e_y \left(3e_x^2 - 32e_z^2 - 5j_x^2 - 15j_y^2 + 4 \right) - 10e_x j_x j_y + 3e_y^3 \right) \right]; \\
 g_j^{(0)} = & \left[\frac{3}{16} \pi \left(75e_x^2 j_y + 60\pi e_x e_y j_y + j_x \left(5j_x j_y - 6\pi \left(10e_y^2 + j_z^2 \right) \right) - 50e_y e_z j_z + 10e_z^2 (5j_y - 9\pi j_x) \right), \right. \\
 & \left. -\frac{3}{16} \pi \left(15e_z^2 (5j_x + 4\pi j_y) - 10e_x (6\pi e_y j_x + 5e_y j_y + 15e_z j_z) + 50e_z^2 j_x + 90\pi e_z^2 j_y + 5j_x^3 - 10j_x j_z^2 + 6\pi j_y j_z^2 \right), \right. \\
 & \left. -\frac{15}{8} \pi (5e_x e_y j_z + 5e_x e_z j_y + 5e_y e_z j_x + j_x j_y j_z) \right]; \\
 h_j^{(1)} = & \left[\frac{15}{512} \pi \left(-2541e_x^4 j_y + e_y \left(7j_x \left(121e_x^2 + 682e_z^2 - 63j_y^2 + 62j_z^2 - 20 \right) - 24 \left(116\pi e_x^2 j_y - 322e_x e_z j_z + \pi j_y \left(41e_z^2 - 5j_y^2 - 5j_z^2 + 8 \right) + 49j_x^3 \right) - 96\pi e_x^2 e_z j_z \right. \right. \\
 & \left. \left. + 3e_y^2 (1080\pi e_x j_x + 343e_x j_y - 32\pi e_z j_z) + e_x \left(7j_y \left(42e_z^2 + 91j_x^2 - 258j_z^2 - 60 \right) + 360\pi j_x \left(11e_z^2 + j_z^2 \right) + 120\pi j_x j_y^2 + 903j_y^3 \right) + e_y^2 (456\pi j_y - 707j_x) \right) \right. \\
 & \left. - 64e_z j_z \left(3\pi \left(8e_z^2 - 1 \right) + 15\pi j_x^2 - 7j_x j_y \right) \right), \\
 & \frac{15}{256} \pi \left(e_x^2 (847j_x + 1446\pi j_y) - e_x^2 (3e_y (558\pi j_x + 511j_y) + 6748e_z j_z) + e_x \left(7j_x \left(196e_z^2 - 359e_z^2 - 64j_y^2 + 91j_z^2 + 40 \right) - 6\pi j_y \left(29e_z^2 - 406e_z^2 + 35j_y^2 - 50j_z^2 - 28 \right) - 343j_x^3 - 270\pi j_x^2 j_y \right) \right. \\
 & \left. - 54\pi e_y^2 j_x - 1372e_y^2 e_z j_z + e_y \left(2e_z^2 (12\pi j_x + 511j_y) + 210\pi j_x^2 + 217j_x^2 j_y + 6\pi j_x \left(25j_y^2 - 30j_z^2 - 12 \right) - 217j_y j_z^2 \right) + 8e_z j_z \left(280e_z^2 + 49j_x^2 - 60\pi j_x j_y + 56j_y^2 - 35 \right) \right), \\
 & \frac{15}{512} \pi \left(36\pi e_x^2 j_z + e_x^2 (3549e_y j_z - 492\pi e_z j_x + 2219e_z j_y) + 2e_x \left(18\pi e_z^2 j_z + e_y e_z (60\pi j_y - 21j_x) + j_z \left(-6\pi \left(32e_z^2 + 25j_y^2 - 4 \right) + 30\pi j_x^2 + 49j_x j_y \right) \right) \right. \\
 & \left. + e_z \left(-7j_y \left(337e_z^2 + 23j_x^2 - 20 \right) - 36\pi j_x \left(17e_z^2 + 5j_y^2 - 4 \right) - 180\pi j_x j_y^2 + 7j_y^3 \right) + e_y j_z \left(2037e_z^2 - 1071j_x^2 + 360\pi j_x j_y - 469j_y^2 + 420 \right) - 3360e_y e_z^2 j_z - 32e_z^2 (36\pi j_x + 35j_y) \right); \\
 \end{aligned}$$

2 *Hamers & Samsing*

Terms of order	Number of terms in Δe	
	$E = 1$	$E > 1$
ϵ_{SA} (all)	16	60
ϵ_{SA}	2	8
$\epsilon_{\text{SA}} \epsilon_{\text{oct}}$	14	52
ϵ_{SA}^2 (all)	193	55,895
ϵ_{SA}^2	17	1,871
$\epsilon_{\text{SA}}^2 \epsilon_{\text{oct}}$	60	16,035
$\epsilon_{\text{SA}}^2 \epsilon_{\text{oct}}^2$	116	37,989
ϵ_{SA}^3 (all)	1,146	2,931,541
ϵ_{SA}^3	54	38,366
$\epsilon_{\text{SA}}^3 \epsilon_{\text{oct}}$	175	289,496
$\epsilon_{\text{SA}}^3 \epsilon_{\text{oct}}^2$	311	856,072
$\epsilon_{\text{SA}}^3 \epsilon_{\text{oct}}^3$	606	1,747,607

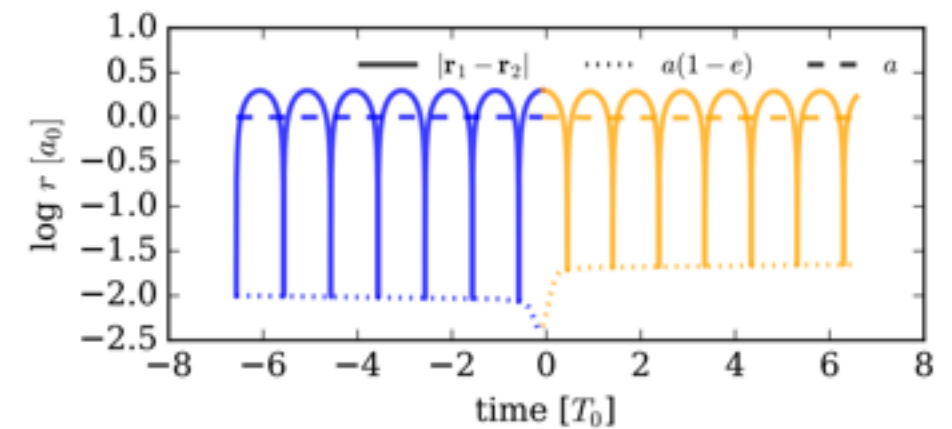
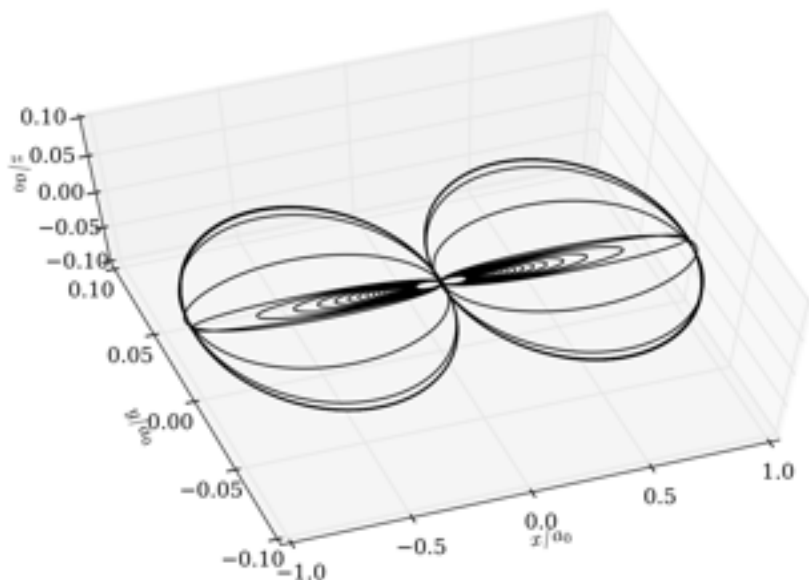
Merger Type: Secular-processes



PN effects?

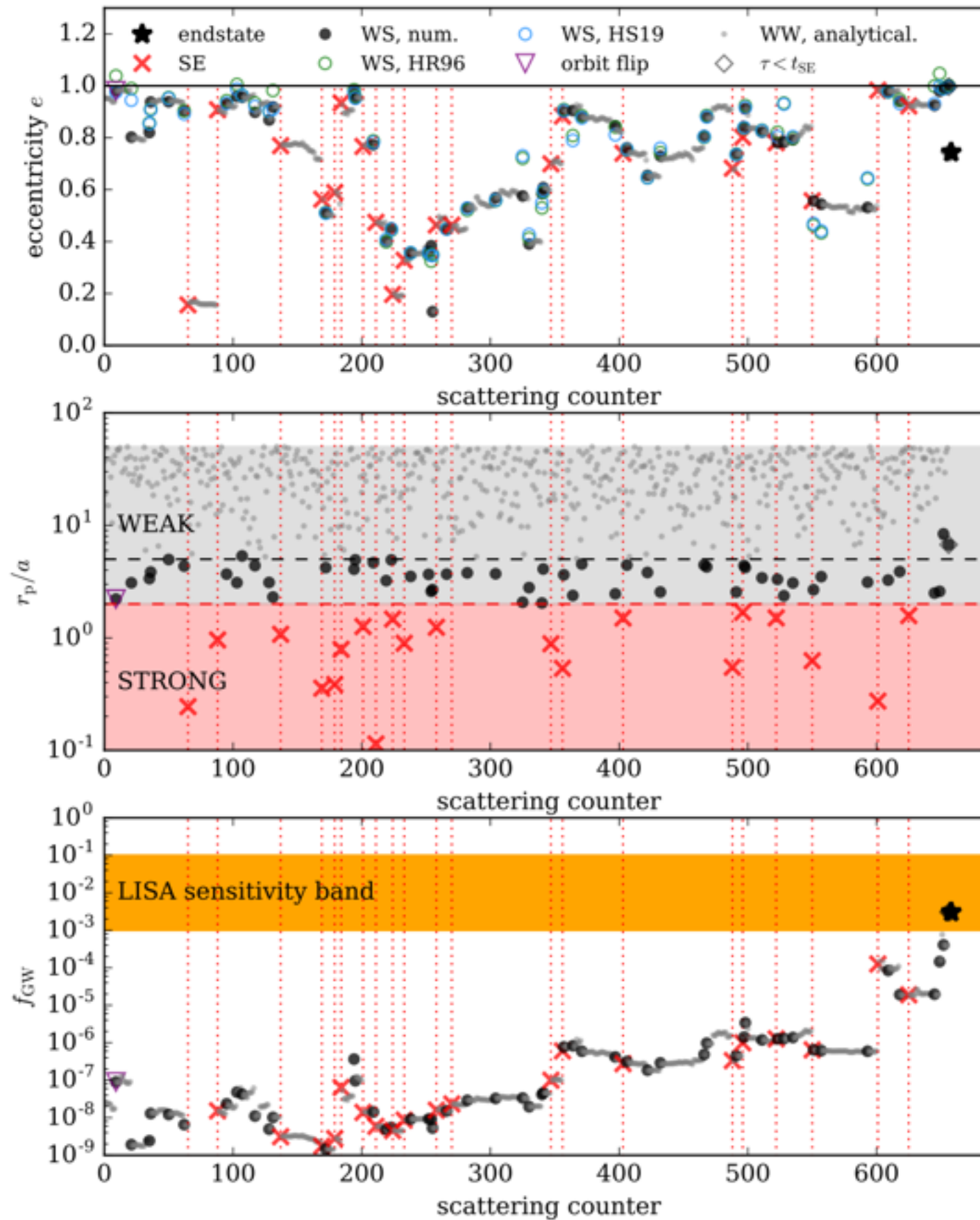
2.5

1,2

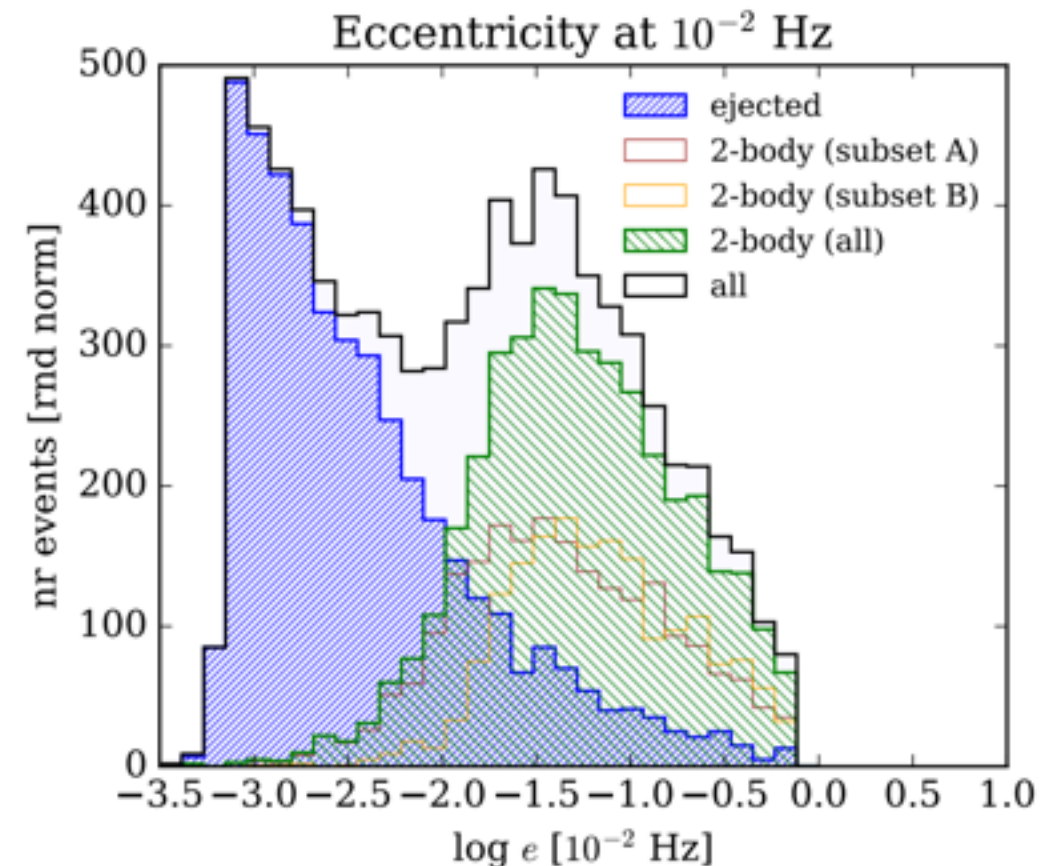
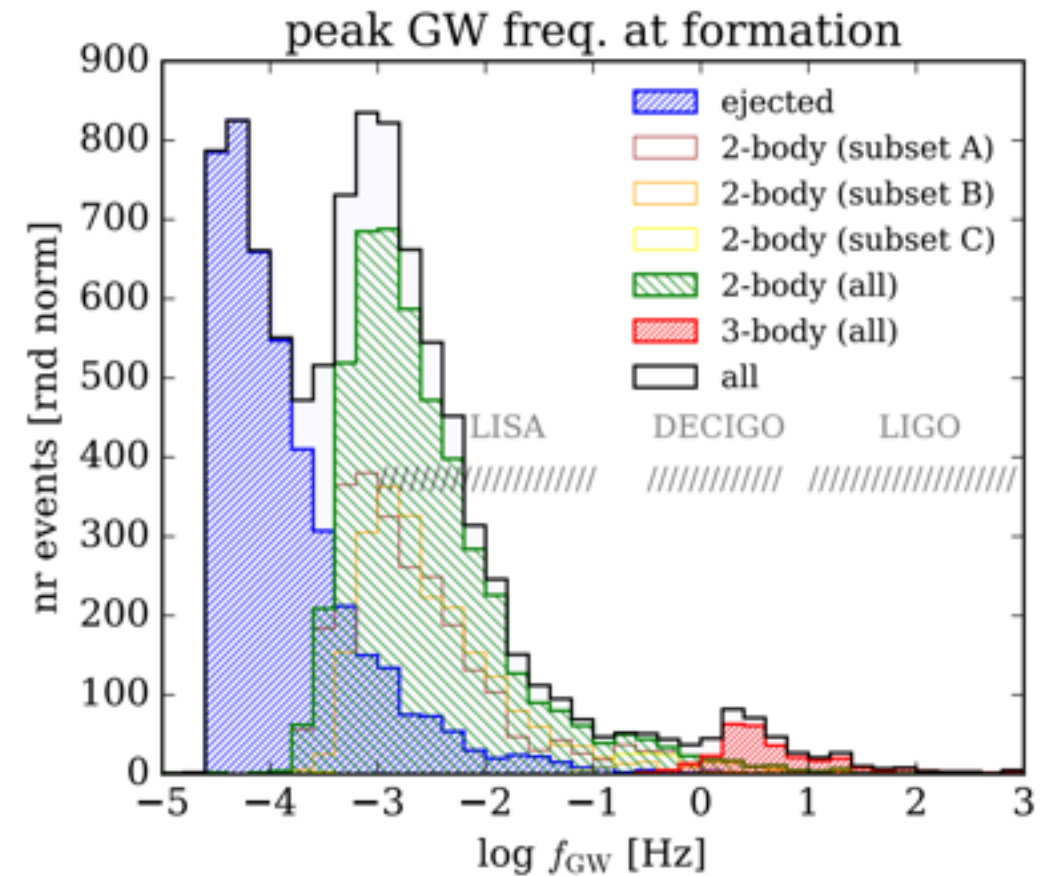
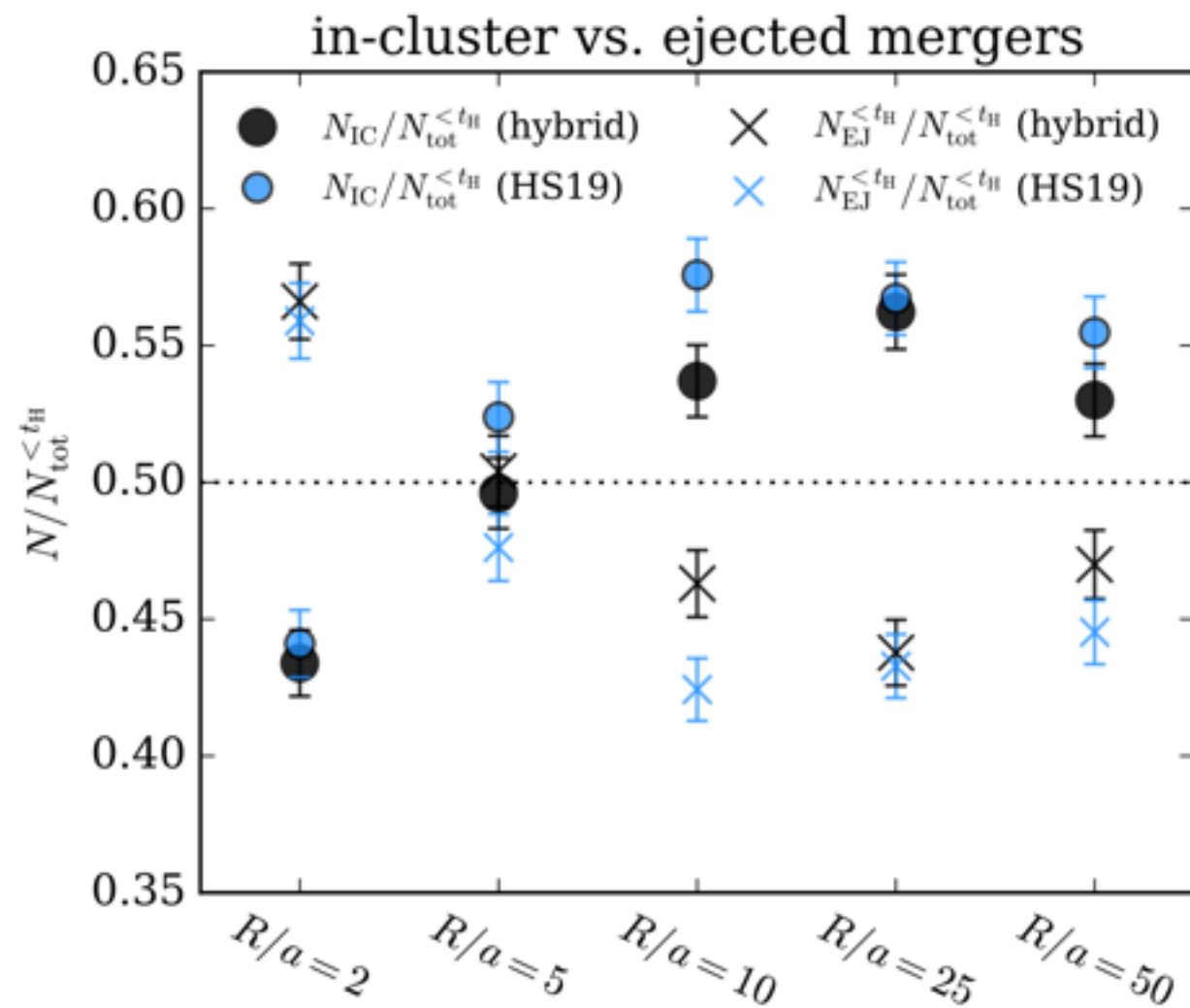


$$\begin{aligned} \frac{r_p}{a_0} &\gtrsim \frac{a_0^{2/3}}{\mathcal{R}_m^{2/3}} (1 - e_0^2)^{2/3} \\ &\gtrsim 10^3 \times \left(\frac{a_0}{0.5 \text{ AU}} \right)^{2/3} \left(\frac{m}{20 M_\odot} \right)^{-2/3} \left(1 - (e_0/0.99)^2 \right)^{2/3} \end{aligned}$$

Merger Type: Secular-processes

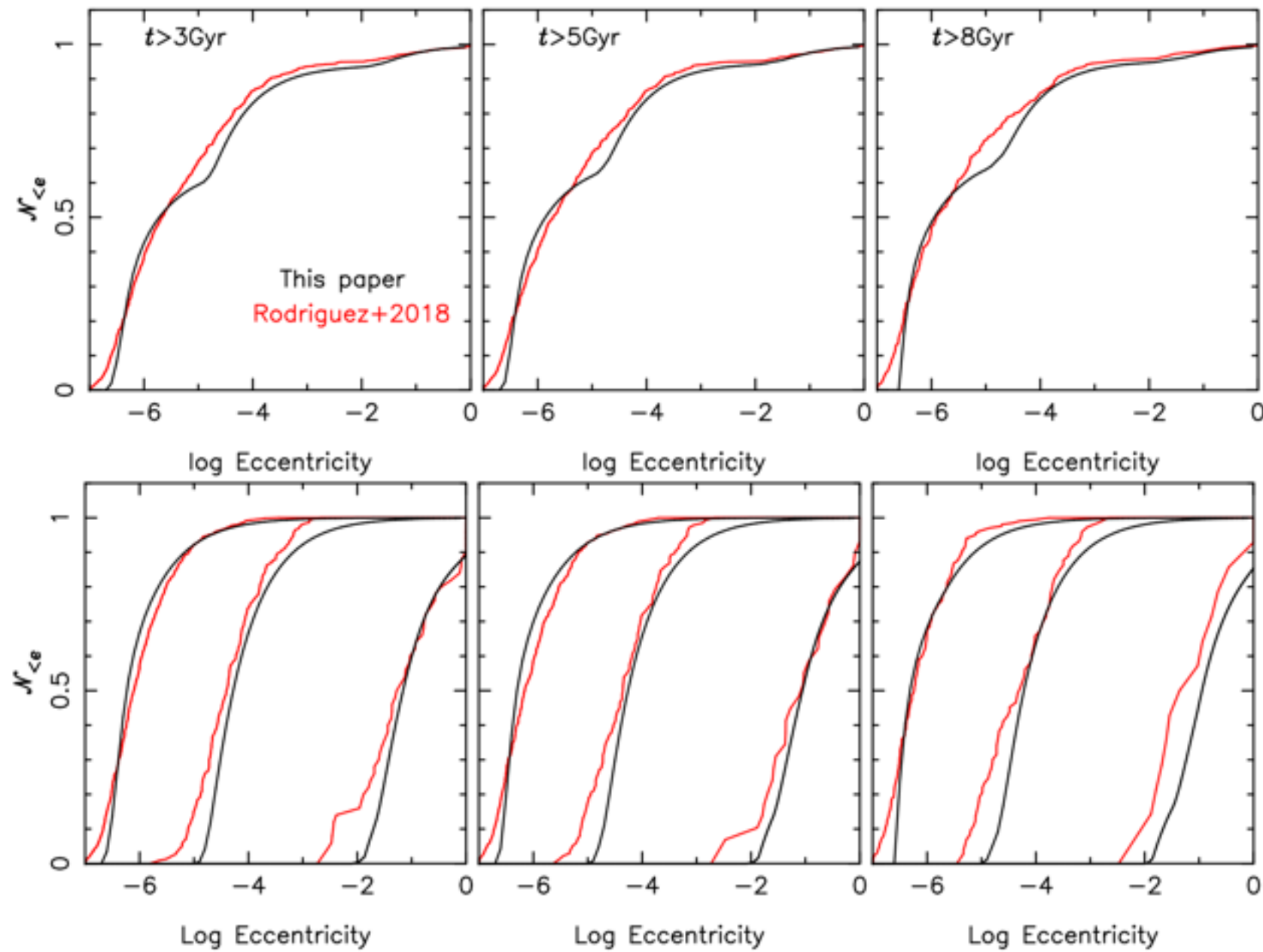


Results from our MC code

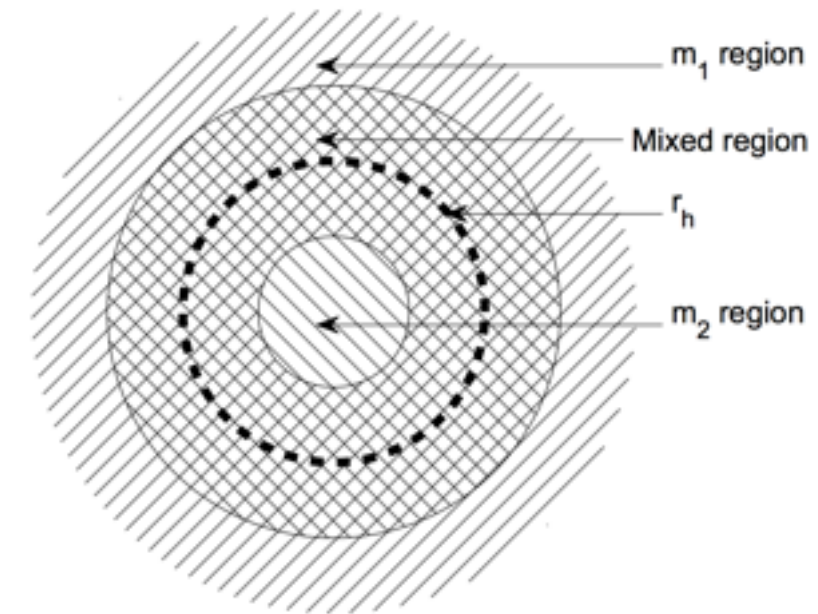


more BBHs are driven to merger!

Promising..



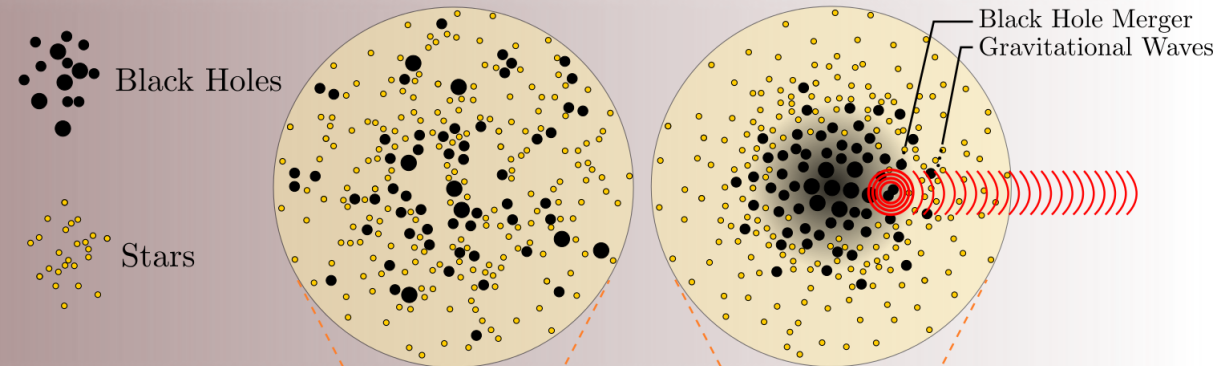
Antonini et al. 2019



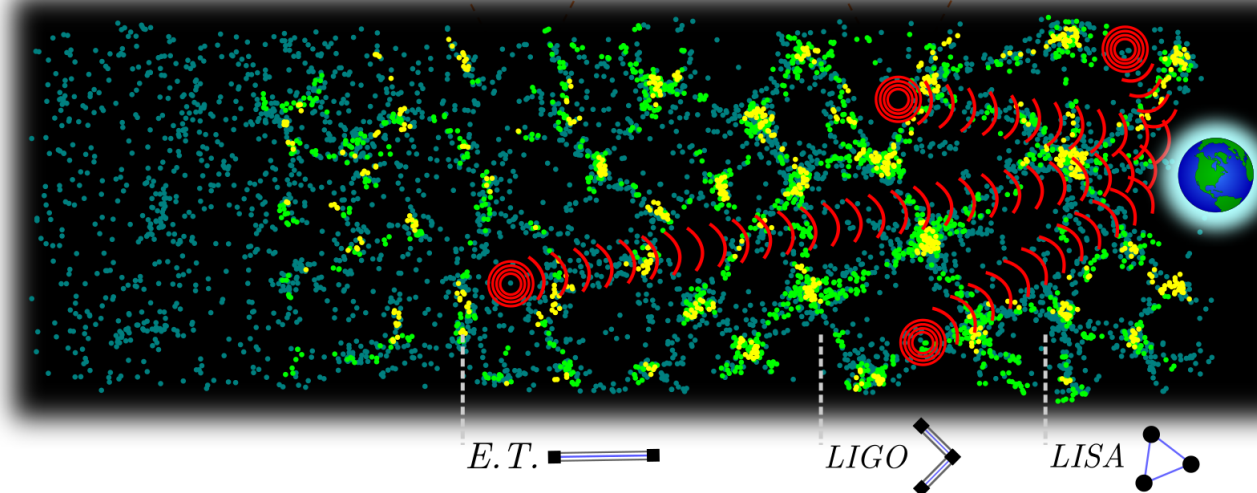
Future

Overview

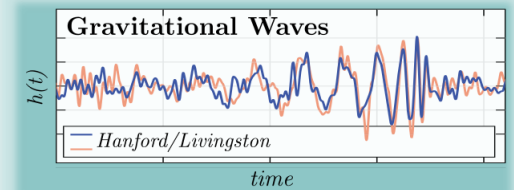
Cluster Evolution →



Cosmological Evolution →



Observables



- Masses of the two BHs
- Spins of the two BHs
- Orbital Eccentricity
- Distance
- Perturbations, etc.

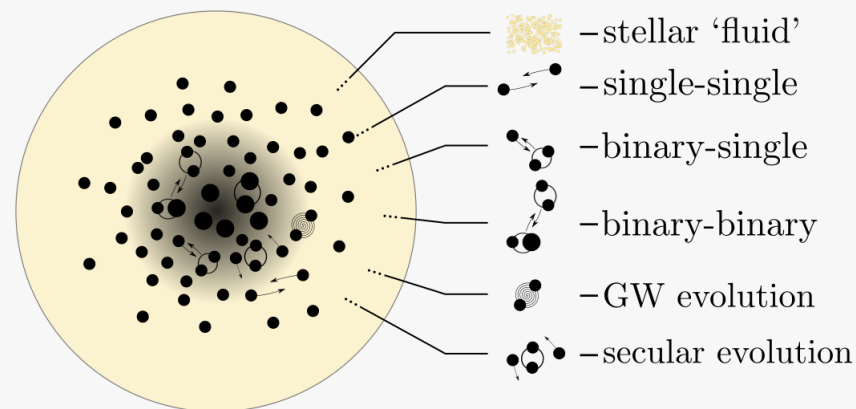
Implications

- Black Hole Formation
- Astrophysical Channels
- Relativistic Dynamics
- Tests of General Relativity
- Cosmological Evolution

Theoretical Modeling

Cluster Model

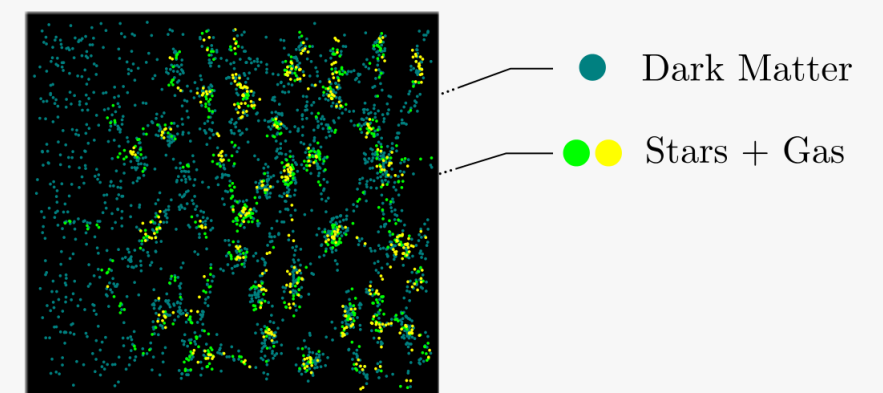
Interacting Components



Evolve using Boltzmann eqs.

Cosmology Model

Interacting Components



Evolve using N-body Codes



THANK YOU