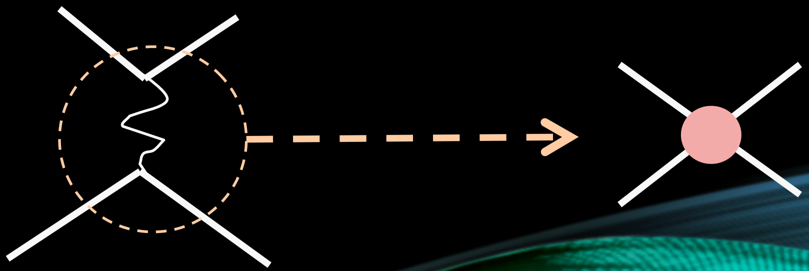
INTRODUCTION TO EFFECTIVE FIELD THEORY

Nordic Winter School 2019



C.P. Burgess

OUTLINE

- Motivation and Overview
  - Decoupling and quantifying theoretical error
  - Effective field theories & a toy model
- Particle physics
  - Known unknowns and unknown unknowns
  - Technical naturalness?
- Gravity and cosmology
  - Time-dependence and EFTs
- Relevance to present puzzles
  - A brief cold shower
  - Inflation, Black holes, Dark matter and Dark energy

MOTIVATION & OVERVIEW

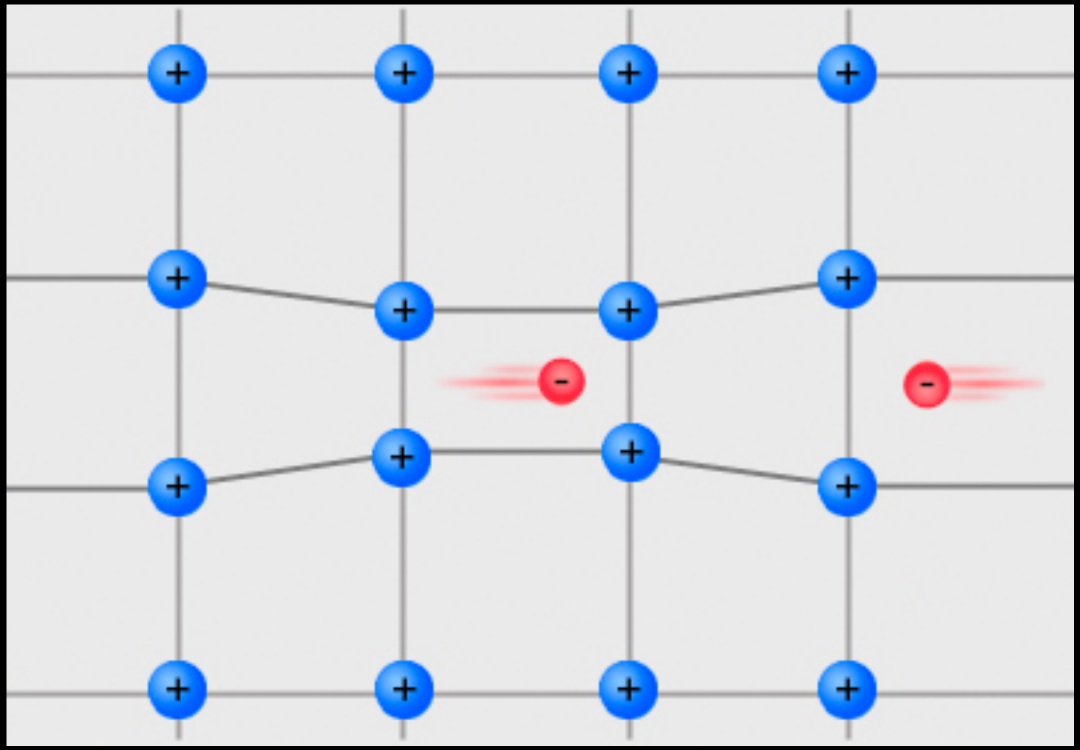
DECOUPLING

- ***Decoupling***: most details of small-distance physics are not needed when understanding long-distance physics (this is why science is possible)

- It turns out that QFT very generally shares this property: how to see it explicitly? Can it be exploited to simplify calculations?

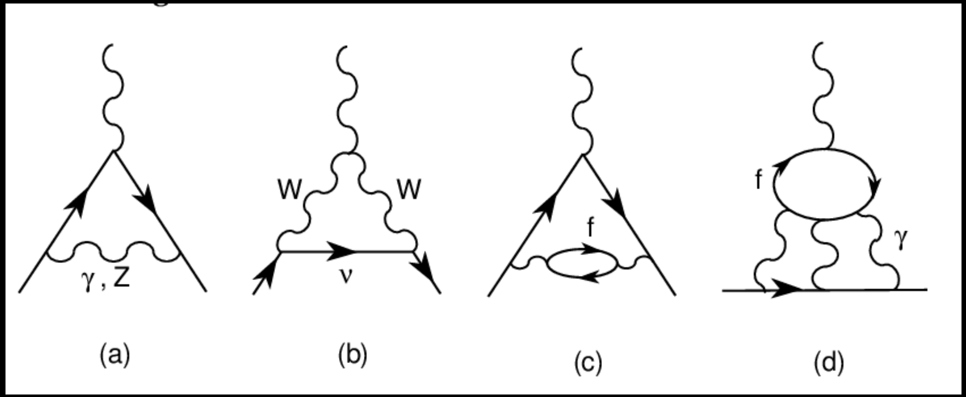
ACCURACY OF CALCULATIONS

- Some predictions work too well
- eg superconductivity relies on electrons pairing into Cooper pairs:
  - neglect Coulomb interaction
  - keep interactions via lattice ions
  - predictions work at 10% level



RENORMALIZATION

- ***Renormalization***: Why does it make sense to subtract infinities and then compare with experiment to 10 decimal places?



$$a_\mu = 1159652188.4(4.3) \quad 10^{-12} \text{ (exp)}$$

$$a_\mu = 1159652140(27.1) \quad 10^{-12} \text{ (th)}$$

$$\mathscr{L}_{\text{int}} = ieA_\mu \overline{\psi}\gamma^\mu \psi$$

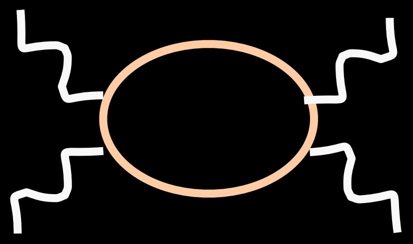
$$e^3 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{(p+q+m)^2 (p+q)^2}$$

$$\sim e \left(\frac{e}{4\pi}\right)^2 \left[\int \frac{\mathrm{d}^4 p}{p^4} + q \int \frac{\mathrm{d}^4 p}{p^5} + \cdots \right]$$

RENORMALIZATION

- ***Renormalizable***: don't need new parameters beyond "e" and "m" for other observables

$$\mathscr{L}_{\text{int}} = ieA_\mu \overline{\psi}\gamma^\mu\psi$$



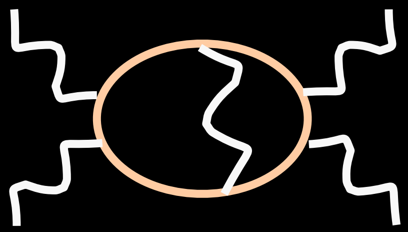
$$e^4 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{(p+q+m)^4} \sim \left(\frac{e^2}{4\pi}\right)^2 q^4 \int \frac{\mathrm{d}^4 p}{p^8} + \cdots$$

$$\sim \left(\frac{e^2}{4\pi}\right)^2 \frac{q^4}{m^4} + \cdots$$

RENORMALIZATION

- ***Renormalizable***: higher orders do not make things worse

$$\mathscr{L}_{\text{int}} = ieA_\mu \bar{\psi}\gamma^\mu\psi$$



$$e^6 \int \left(\frac{\mathrm{d}^4 p}{(2\pi)^4}\right)^2 \frac{1}{(p+q+m)^6 (p+q)^2} \sim e^2 \left(\frac{e}{4\pi}\right)^4 q^4 \int \frac{\mathrm{d}^4 p}{p^8} + \cdots$$

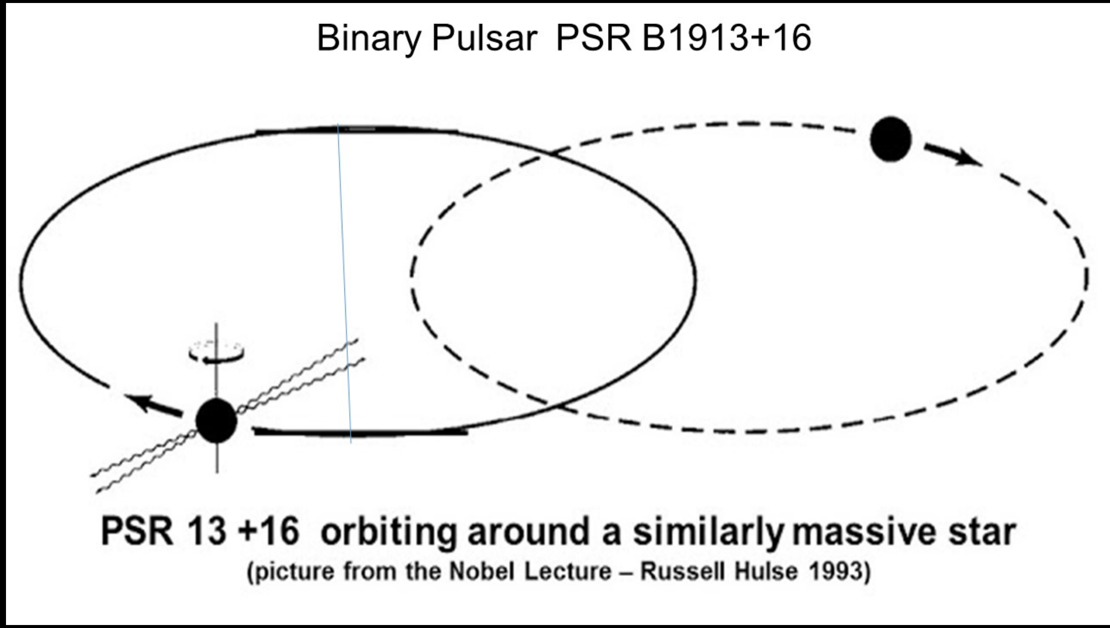
$$\sim e^2 \left(\frac{e}{4\pi}\right)^4 \frac{q^4}{m^4} + \cdots$$

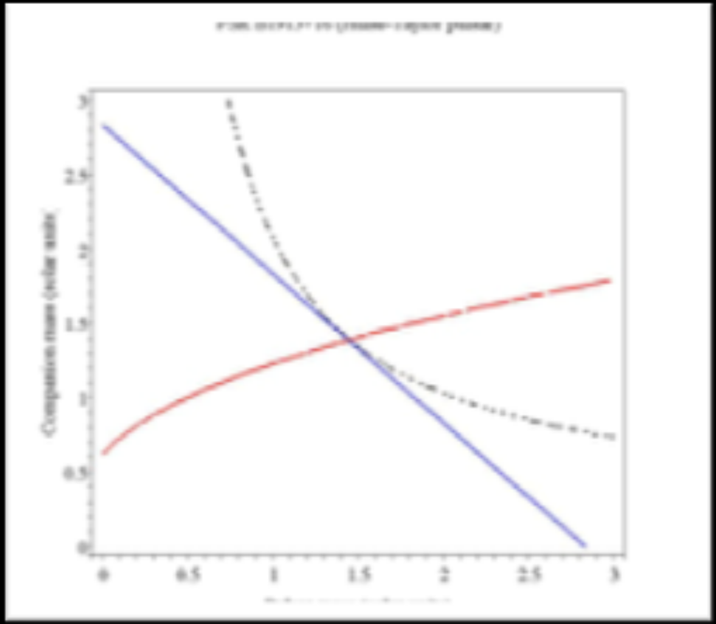
RENORMALIZATION

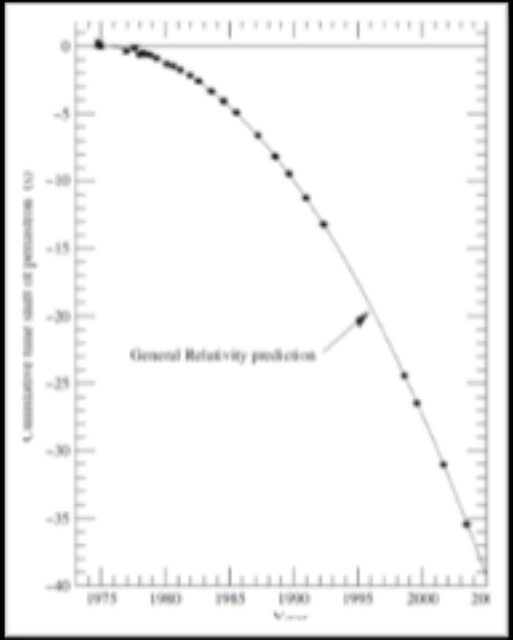
- Do nonrenormalizable theories (eg GR) make sense? If not why is comparison with observations meaningful?

$$\dot{P} = -2.408(10)\ \ 10^{-12} \text{ (exp)}$$

$$\dot{P} = -2.40243(5)\ \ 10^{-12} \text{ (th)}$$





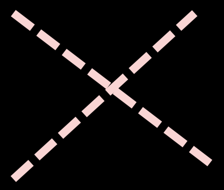


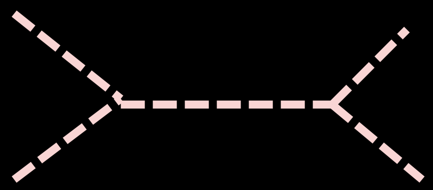
RENORMALIZATION

- Nonrenormalizable theories have couplings with dimension inverse power of mass

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_p}$$

$$\mathscr{L}_{GR} = (\partial h)^2 + \frac{1}{M_p} h(\partial h)^2 + \frac{1}{M_p^2} h^2(\partial h)^2 + \cdots$$





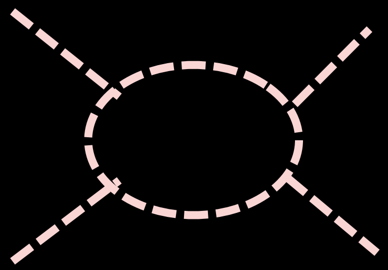
$$\mathscr{A}_{\text{tree}} \simeq \frac{q^2}{M_p^2} \left(= 8\pi i G \, \frac{s^3}{tu} \right)$$

De Witt

RENORMALIZATION

- Higher orders diverge worse and worse
- New types of divergences

$$\mathscr{L} = (\partial h)^2 + \frac{1}{M_p} h(\partial h)^2 + \frac{1}{M_p^2} h^2(\partial h)^2 + \cdots$$



$$\mathscr{A}_{\text{loop}} \simeq \frac{q^2}{M_p^4} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{(p+q)^6}{(p+q)^8}$$

$$\simeq \left(\frac{q}{4\pi M_p^2} \right)^2 \left[\int \frac{\mathrm{d}^4 p}{p^2} + q^2 \int \frac{\mathrm{d}^4 p}{p^4} + \cdots \right]$$

TOY MODEL

- Effective field theories (EFTs) are the formalism for addressing these questions.
  - Designed to exploit hierarchies of scale m/M as efficiently as possible.

- Toy model: concrete example that illustrates the construction
  - $\lambda \ll 1$ is semiclassical limit

$$\mathscr{L} = -(\partial\phi)^*(\partial\phi) - V(\phi^*\phi)$$

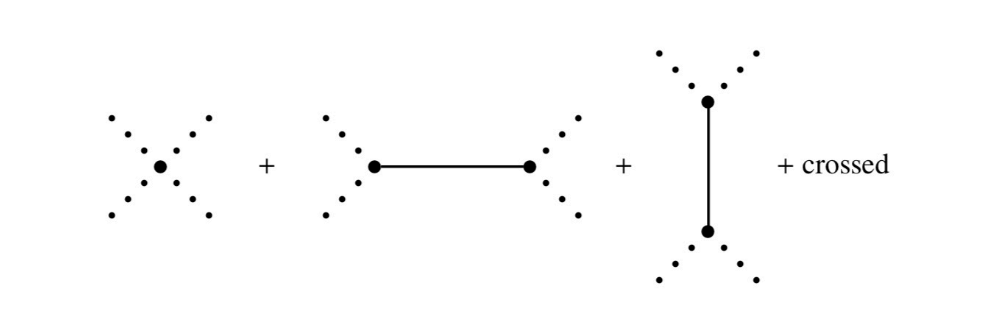
$$V(\phi^*\phi) = \frac{\lambda}{4}(\phi^*\phi - v^2)^2$$

$$\phi = v + \frac{1}{\sqrt{2}}\left(\hat{\phi}_R + i\hat{\phi}_I\right)$$

$$m_R^2 = \lambda v^2 \qquad m_I^2 = 0$$

TOY MODEL

- Can calculate eg tree level scattering of massless particles



$$\mathscr{L} = -(\partial\phi)^*(\partial\phi) - V(\phi^*\phi)$$

$$V(\phi^*\phi) = \frac{\lambda}{4}(\phi^*\phi - v^2)^2$$

$$\phi^*\phi - v^2 = \sqrt{2}\, v\phi_R + \frac{1}{2}(\phi_R^2 + \phi_I^2)$$

$$\mathscr{A} = -\frac{3i\lambda}{2} + \frac{i(\lambda v)^2}{2m_R^2}\left[\frac{1}{1+2p\cdot q/m_R^2} + \frac{1}{1-2q\cdot q'/m_R^2} + \frac{1}{1-2p\cdot q'/m_R^2}\right] + \mathcal{O}\left(\frac{\lambda}{4\pi}\right)^2$$

TOY MODEL

- Low-energy limit: $E << m_R$

$$\mathscr{A} = -\frac{3i\lambda}{2} + \frac{i(\lambda v)^2}{2m_R^2}\left[\frac{1}{1+2p\cdot q/m_R^2} + \frac{1}{1-2q\cdot q'/m_R^2} + \frac{1}{1-2p\cdot q'/m_R^2}\right]$$

$$\simeq 2i\lambda\left[\frac{(p\cdot q)^2 + (p\cdot q')^2 + (q\cdot q')^2}{m_R^4}\right] + \mathcal{O}\left[\lambda\left(\frac{q}{m_R}\right)^6\right] + \mathcal{O}\left(\frac{\lambda}{4\pi}\right)^2$$

- Amplitude suppressed by $(E/m_R)^4$ in low-energy limit. What if E/m_R smaller than coupling? Is dominant low-energy contribution at one loop?

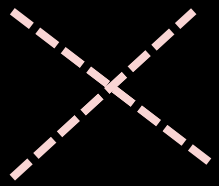
TOY MODEL

- This low-energy amplitude

$$\mathscr{A} \simeq 2i\lambda \left[\frac{(p \cdot q)^2 + (p \cdot q')^2 + (q \cdot q')^2}{m_R^4} \right] + \mathcal{O}\left(m_R^{-6}\right)$$

is what would have been obtained at lowest order from the following lagrangian

$$\mathscr{L} = -\frac{1}{2}\partial_\mu \hat{\varphi}\, \partial^\mu \hat{\varphi} + \frac{\lambda}{4m_R^4}(\partial_\mu \hat{\varphi}\, \partial^\mu \hat{\varphi})(\partial_\nu \hat{\varphi}\, \partial^\nu \hat{\varphi})$$



TOY MODEL

- Turns out 2 to 2 amplitude is proportional to *$(E/m_R)^4$* at each loop. N to N' scattering is proportional to *$(E/m_R)^{N+N'}$*. Easy way to see why?

- Obvious once recognized at low energies *all* massless scattering (also N to N') is governed by `effective lagrangian'

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\hat{\varphi}\,\partial^\mu\hat{\varphi} + G_{\rm eff}\,(\partial_\mu\hat{\varphi}\,\partial^\mu\hat{\varphi})(\partial_\nu\hat{\varphi}\,\partial^\nu\hat{\varphi})$$

$$G_{\rm eff} = \frac{\lambda}{4m_R^4}\left[1 + \mathscr{O}(\lambda)\right]$$

- All other interactions are suppressed by more powers of *E/m_R*. (Suppression of low-energy scattering better than a tree level result.)

TOY MODEL

Two steps to see why:

- Expand in powers of E/M as early as possible.
  - Do so by *integrating out* heavy particle to obtain effective lagrangian (applicable to all low-energy observables), rather than observable by observable.
- Make symmetries manifest in this effective lagrangian

SYMMETRIES

- Toy model has U(1) symmetry

$$\mathscr{L} = -(\partial\phi)^*(\partial\phi) - V(\phi^*\phi)$$

$$V(\phi^*\phi) = \frac{\lambda}{4}(\phi^*\phi - v^2)^2$$

$$\phi = v + \frac{1}{\sqrt{2}}\left(\hat{\phi}_R + i\hat{\phi}_I\right)$$

$$m_R^2 = \lambda v^2 \qquad m_I^2 = 0$$

$$\phi \to e^{i\theta}\phi$$

$$\begin{pmatrix} \phi_R \\ \phi_I \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_R \\ \phi_I \end{pmatrix}$$

How to realize symmetry using only light field?

SYMMETRIES

- Redefine variables

$$\phi = (v + \chi)\, e^{i\xi/v}$$

Symmetry as realized on low-energy field is not linear

- Symmetry transformation

$$\phi \to e^{i\theta}\phi \quad \text{implies} \quad \xi \to \xi + \theta v \quad \chi \to \chi$$

- Toy model lagrangian becomes

$$\mathscr{L} = (\partial\chi)^2 + \left(1 + \frac{\chi}{v}\right)^2 (\partial\xi)^2 - V(\chi)$$

EFFECTIVE FIELD THEORIES

- To integrate out heavy particle: split fields into high- and low-energy parts (won't be unique way to do this)

$$\hat{\phi} \leftrightarrow \{\hat{H}, \hat{L}\} \quad \text{where} \quad E(L) < \Lambda \quad \text{and} \quad E(H) > \Lambda$$

- Low-energy observables can be computed from

$$\langle \hat{L}(x_1) \cdots \hat{L}(x_n) \rangle = \int \mathscr{D}\hat{L}\mathscr{D}\hat{H}\, \hat{L}(x_1) \cdots \hat{L}(x_n)\, \exp\left(iS[\hat{L}, \hat{H}] \right)$$

- Want to efficiently identify effects of heavy physics in powers of $1/M$.

EFFECTIVE FIELD THEORIES

- Want:

$$\langle \hat{L}(x_1)\cdots\hat{L}(x_n)\rangle = \int \mathscr{D}\hat{L}\mathscr{D}\hat{H}\ \hat{L}(x_1)\cdots\hat{L}(x_n)\ \exp\left(iS[\hat{L},\hat{H}]\right)$$

- Define Wilson action:

$$\exp\left(iS_W[\hat{L}]\right) = \int \mathscr{D}\hat{H}\ \exp\left(iS[\hat{L},\hat{H}]\right)$$

- Then

$$\langle \hat{L}(x_1)\cdots\hat{L}(x_n)\rangle = \int \mathscr{D}\hat{L}\ \hat{L}(x_1)\cdots\hat{L}(x_n)\ \exp\left(iS_W[\hat{L}]\right)$$

- Any Λ dependence of S_W cancels in $< L(x_1)\ldots L(x_n) >$.

EFFECTIVE FIELD THEORIES

- The Wilson action defined by:

$$\exp\left(S_W[\hat{L}]\right) = \int \mathscr{D}\hat{H}\ \exp\left(iS[\hat{L},\hat{H}]\right)$$

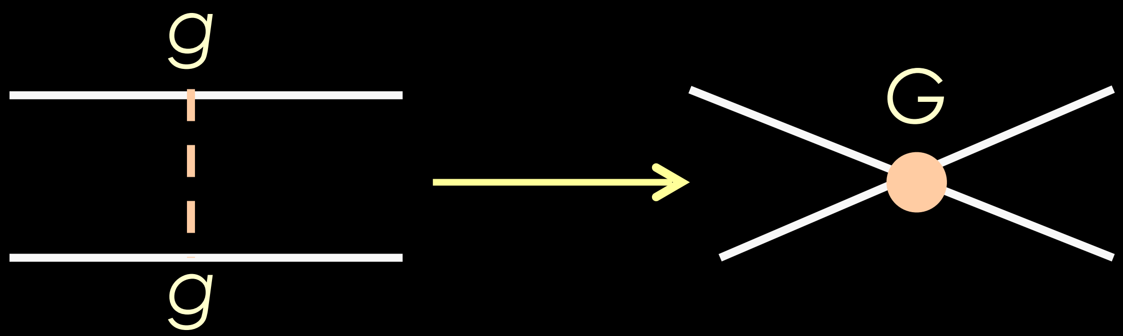
behaves "as if" it is the classical action for the low-energy theory.

- S_W defined this way would be nonlocal, BUT becomes local once expanded in powers of 1/M (consequence of uncertainty principle):

$$G(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2+M^2} = \left[\frac{1}{M^2} + \frac{\Box}{M^4} + \cdots\right]\ \delta^4(x-y)$$

EFFECTIVE FIELD THEORIES

- Expanding propagator in *1/M* gives local result:

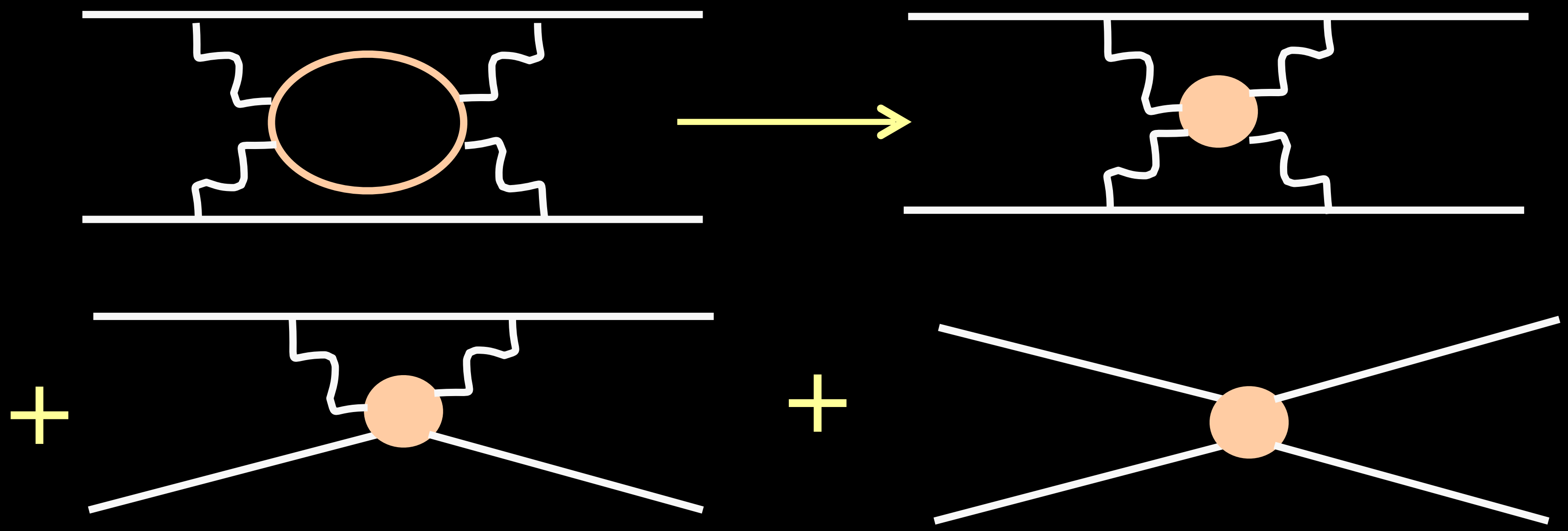


$$G = \frac{g^2}{M^2}$$

- New coupling is generically nonrenormalizable, but underlying theory could be renormalizable so must be predictive
- Predictivity comes from compulsory low-energy approximation

EFFECTIVE FIELD THEORIES

- Not quite so simple as contracting a line once loops included (eg integrating out muons in QED)



EFFECTIVE FIELD THEORIES

- Circling back: renormalization. Recall definition of S_W depends on cutoff

$$\exp\left(iS_W[\hat{L},\Lambda]\right) = \int_\Lambda \mathscr{D}\hat{H}\ \exp\left(iS[\hat{L},\hat{H}]\right)$$

- But cutoff also enters into its use:

$$\langle \hat{L}(x_1)\cdots\hat{L}(x_n)\rangle = \int^\Lambda \mathscr{D}\hat{L}\ \hat{L}(x_1)\cdots\hat{L}(x_n)\ \exp\left(iS_W[\hat{L},\Lambda]\right)$$

- Any Λ dependence of S_W cancels in $< L(x_1)\ldots L(x_n) >$: *nothing depends on the cutoff!*

INTRODUCTION TO EFT (LECTURE 2)

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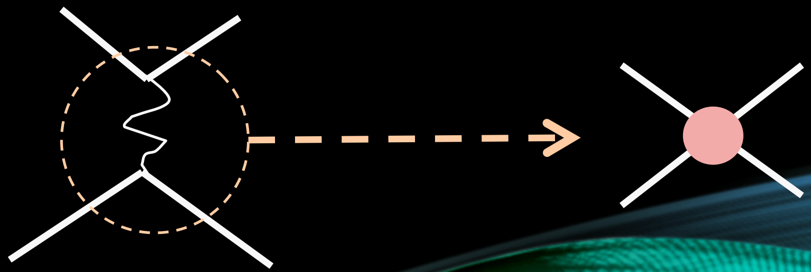

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TOY MODEL RECAP

- Scattering of massless states is suppressed at each order in the loop expansion by powers of E/m_R

- This can be understood by building an EFT for the light (Goldstone) particle alone.

- Low energy field nonlinearly realizes symmetry

$$\mathscr{L} = -(\partial\phi)^*(\partial\phi) - V(\phi^*\phi)$$

$$V(\phi^*\phi) = \frac{\lambda}{4}(\phi^*\phi - v^2)^2$$

$$\phi = (v + \chi)\, e^{i\xi/v}$$

$$\phi \to e^{i\theta}\phi \quad \text{implies} \quad \xi \to \xi + \theta v \quad \chi \to \chi$$

TOY MODEL EFT

- Wilson action for the Toy Model must be invariant under shift symmetry, so built from only derivatives of the light field

$$S_W[\xi + \theta] = S_W[\xi] \Rightarrow S_W = S_W[\partial\xi]$$

$$\mathscr{L}_W = -\frac{1}{2}\partial_\mu\xi\,\partial^\mu\xi + G_{\rm eff}\,(\partial_\mu\xi\,\partial^\mu\xi)(\partial_\nu\xi\,\partial^\nu\xi) + \cdots$$

- Could get G by integrating out heavy field, but better simply to "match": choose it to reproduce a result of the full theory (eg a scattering amplitude).

$$G_{\rm eff} = \frac{\lambda}{4m_R^4}$$

TOY MODEL EFT

- To drive home how S_W describes *all* low energy physics (and that EFT need not be restricted to expansions about vacuum configuration) consider time-dependent solution to full theory's field equation

$$\Box \phi = \frac{\lambda}{2}(\phi^*\phi - v^2)\phi$$

- Consider the time-dependent solution:

$$\phi = \varrho_0 \, e^{i\omega t} \qquad \text{where} \qquad \varrho_0 = \sqrt{v^2 + \frac{2\omega^2}{\lambda}}$$

with energy

$$\varepsilon = \dot{\phi}^*\dot{\phi} + \frac{\lambda}{4}(\phi^*\phi - v^2)^2 = \omega^2\left(v^2 + \frac{3\omega^2}{\lambda}\right)$$

Energy gain because field climbs the potential to balance centrifugal force

TOY MODEL EFT

- How does the EFT know about the radial field climbing the potential given there is no radial field in the EFT?

$$\mathscr{L}_W = -\frac{1}{2}\partial_\mu \xi\, \partial^\mu \xi + G_{\mathrm{eff}}\,(\partial_\mu \xi\, \partial^\mu \xi)(\partial_\nu \xi\, \partial^\nu \xi) + \cdots$$

- This gives the field equation

$$\partial_\mu \left\{ \partial^\mu \xi \left[1 - 4G_{\mathrm{eff}}\,(\partial \xi)^2 + \cdots\right] \right\} = 0$$

with solution

$$\frac{\xi}{\sqrt{2}v} = \omega t$$

TOY MODEL EFT

- Compute the energy of this solution within the EFT:

$$\varepsilon = \mathscr{H} = \Pi\dot{\xi} - \mathscr{L} = \frac{1}{2}\dot{\xi}^2 + 3G_{\text{eff}}\,\dot{\xi}^4$$

which with the matched value for G becomes:

$$G_{\text{eff}} = \frac{\lambda}{4m_R^4} \qquad \varepsilon = v^2\omega^2 + \frac{3\omega^4}{\lambda}$$

in agreement with the full theory

REDUNDANT INTERACTIONS

- What about other terms with same dimension (or less) and so same (or lower) power of $1/m_R$ in its coefficient?

$$\text{e.g.} \quad \mathscr{L} = G_1 (\partial_\mu \xi) \Box \partial^\mu \xi \quad \text{or} \quad G_2 \partial_\mu \partial_\nu \xi \, \partial^\mu \partial^\nu \xi$$

- These differ only by a total derivative (so are not independent)
- The first can be removed to this order in 1/M with field redefinition

$$\delta\xi = G_1 \Box \xi$$

REDUNDANT INTERACTIONS

- Field redefinitions can be used to remove any term in the effective action that vanishes when evaluated at the solution to the lowest order field equations - for the Toy Model: $\Box \xi = 0$

if $S[\xi] = S_0[\xi] + \epsilon S_1[\xi] + \cdots$ then when $\delta\xi = \epsilon \, F[\xi]$

$$\delta S[\xi] = \epsilon \int \mathrm{d}^4 x \, \frac{\delta S_0}{\delta \xi(x)} \, F(\xi) + \cdots$$

which can be used to remove any term in S_1 that vanishes using the e.o.m. of S_0.

DIMENSIONAL REGULARIZATION

- Dimensional regularization does not introduce a new scale (apart from logs)

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D} \left[\frac{p^{2A}}{(p^2 + q^2)^B} \right] = \frac{1}{(4\pi)^{D/2}} \left[\frac{\Gamma(A + D/2)\Gamma(B - A - D/2)}{\Gamma(B)\Gamma(D/2)} \right] (q^2)^{A-B+D/2}$$

  - Divergences arise as poles as D goes to 4
  - Convenient because it preserves symmetries (eg gauge invariance) broken by cutoffs, and simplifies dimensional reasoning

EFTS IN DIM REG

- Dimensionally regularize both the full theory and EFT

$$\mathscr{L}_{\text{full}}(\chi, \xi) \qquad\qquad \mathscr{L}_{\text{EFT}}(\xi)$$

- Renormalize in any convenient way (eg minimal subtraction)
- Match the couplings in the EFT by demanding they give same observables as for the full theory
- Any error introduced by keeping very high energy modes of light field is absorbed into the effective couplings.

POWER COUNTING

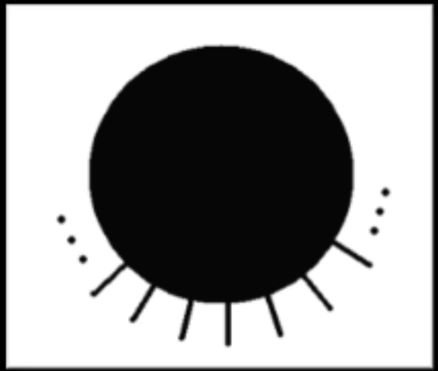
- The real power with EFTs comes beyond leading order in E/M.
- Need an algorithm to systematically identify which interactions and which Feynman graphs must be included to any specific order.
- When all the low-energy scales are similar in size this algorithm amounts to dimensional analysis of Feynman graphs
  - much simplest to do this using dimensional regularization.

POWER COUNTING

- Consider lagrangian of form

$$\mathscr{L} = f^4 \sum_n c_n \mathcal{O}_n\left(\frac{\partial}{M}, \frac{\xi}{v}\right)$$

- Consider (amputated) Feynman graph with
  - E external lines,
  - I internal lines
  - V_n vertices involving d_n derivatives and f_n fields



- All such diagrams satisfy:

$$L = 1 + I - \sum_n V_n \qquad 2I + E = \sum_n f_n V_n$$

POWER COUNTING

- Internal lines bring:

$$\text{Line Factor} = \left[\frac{M^2 v^2}{f^4}\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{1}{p^2}\right]^I$$

- Vertices bring

$$\text{Vertex Factor} = \prod_n\left[\frac{f^4}{v^{f_n}}\left(\frac{p}{M}\right)^{d_n}(2\pi)^4\,\delta^4(p)\right]^{V_n}$$

- Number of independent integrals

$$I - \sum_n V_n + 1 = L$$

POWER COUNTING

$$\text{Line Factor} = \left[\frac{M^2 v^2}{f^4}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{1}{p^2}\right]^I \quad \text{Vertex Factor} = \prod_n \left[\frac{f^4}{v^{f_n}}\left(\frac{p}{M}\right)^{d_n}(2\pi)^4\,\delta^4(p)\right]^{V_n}$$

- Net power of f⁴: $$-I + \sum V_n = 1 - L$$
- Net power off 1/v: $$-2I + \sum f_n V_n = E$$
- In dimensional reg: p becomes q so net power of q, M is $$M^{4L}\left(\frac{q}{M}\right)^N \qquad \begin{aligned} N &= 4L - 2I + \sum d_n V_n \\ &= 2 + 2L + \sum (d_n - 2) V_n \end{aligned}$$

POWER COUNTING

- Combining terms

$$\mathscr{A}_E(q) \sim \frac{q^2 f^4}{M^2}\left(\frac{1}{v}\right)^E \left(\frac{Mq}{4\pi f^2}\right)^{2L} \prod_n \left[c_n \left(\frac{q}{M}\right)^{d_n-2}\right]^{V_n}$$

- For Toy Model $\quad f^2 = Mv \qquad M = m_R$

$$\mathscr{A}_E(q) \sim q^2 v^2 \left(\frac{1}{v}\right)^E \left(\frac{q}{4\pi v}\right)^{2L} \prod_n \left[c_n \left(\frac{q}{m_R}\right)^{d_n-2}\right]^{V_n}$$

POWER COUNTING

$$\mathscr{A}_E(q) \sim q^2 v^2 \left(\frac{1}{v}\right)^E \left(\frac{q}{4\pi v}\right)^{2L} \prod_n \left[c_n \left(\frac{q}{m_R}\right)^{d_n-2}\right]^{V_n}$$

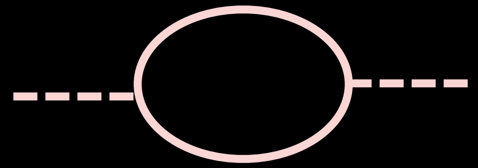
- Notice always positive powers of q since d_n is 2 or larger (actually 4 or larger in the toy model)

- Interactions with no derivatives (ie scalar potential) are potentially dangerous at low energies

POWER COUNTING FOR GRAVITY

- Aside: a similar expression can be derived for gravity:

$$\mathscr{L} = -\frac{M_p^2}{2} R + c_1 R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} + \frac{c_2}{m^2} R_{\mu\nu\lambda\rho} R^{\lambda\rho\alpha\beta} R_{\alpha\beta}{}^{\mu\nu} + \cdots$$



- Coefficient of curvature cubed term is set by mass of particle integrated out, and smallest m wins in the denominator

$$\mathscr{A}_E(q) \sim q^2 M_p^2 \left(\frac{1}{M_p}\right)^E \left(\frac{q}{4\pi M_p}\right)^{2L} \prod_{d_n \geq 4} \left[\left(\frac{q}{M_p}\right)^2 \left(\frac{q}{m}\right)^{d_n - 4}\right]^{V_n}$$

POWER COUNTING FOR GRAVITY

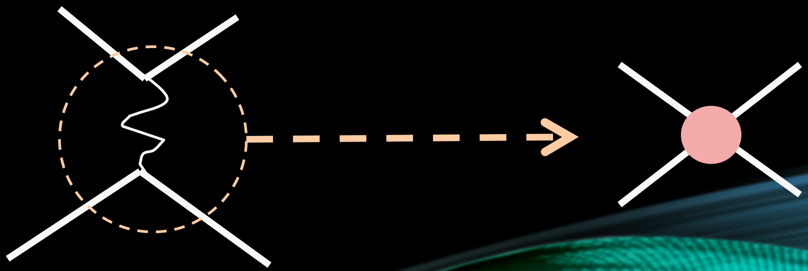
$$\mathscr{L} = -\frac{M_p^2}{2}R + c_1 R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} + \frac{c_2}{m^2}R_{\mu\nu\lambda\rho}R^{\lambda\rho\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} + \cdots$$

$$\mathscr{A}_E(q) \sim q^2 M_p^2 \left(\frac{1}{M_p}\right)^E \left(\frac{q}{4\pi M_p}\right)^{2L} \prod_{d_n \geq 4}\left[\left(\frac{q}{M_p}\right)^2 \left(\frac{q}{m}\right)^{d_n-4}\right]^{V_n}$$

- Dominant contribution: L=0 and V_n = 0 for d_n > 2 (ie classical GR)
- Next-to-leading contributions: L = 1 and V_n =0 for d_n >2 (ie 1-loop GR); or L = 0 and V_n = 1 for d_n = 4 term (tree level with one insertion of R^2 term)
- Size of quantum corrections: $\left(\frac{q}{4\pi M_p}\right)^2$

INTRODUCTION TO EFT (LECTURES 3 & 4)

Nordic Winter School 2019



C.P. Burgess

UNITARITY BOUNDS

- For toy model power counting:

$$\mathscr{A}_E(q) \sim q^2 v^2 \left(\frac{1}{v}\right)^E \left(\frac{q}{4\pi v}\right)^{2L} \prod_n \left[c_n \left(\frac{q}{m_R}\right)^{d_n - 2}\right]^{V_n}$$

- Could gauge the U(1) symmetry to get a gauge boson with mass:

$$M_A^2 = 2g^2 v^2$$

- Massive gauge boson is in the low energy theory if $g^2 \ll \lambda$

UNITARITY BOUNDS

- For toy model power counting:

$$\mathscr{A}_E(q) \sim q^2 v^2 \left(\frac{1}{v}\right)^E \left(\frac{q}{4\pi v}\right)^{2L} \prod_n \left[c_n \left(\frac{q}{m_R}\right)^{d_n - 2} \right]^{V_n}$$

- Theory breaks down when

$$q \sim 4\pi v \sim \frac{4\pi M_A}{g}$$

- Often quoted as a “unitarity bound”: when low-energy cross section exceeds unitarity limit

ZERO DERIVATIVE INTERACTIONS

- For toy model power counting:

$$\mathscr{A}_E(q) \sim q^2 v^2 \left(\frac{1}{v}\right)^E \left(\frac{q}{4\pi v}\right)^{2L} \prod_n \left[c_n \left(\frac{q}{m_R}\right)^{d_n - 2}\right]^{V_n}$$

- What about mass term for light particle

$$V = m^2\phi^2 \Rightarrow c_n = m^2 v^2 / f^4 = m^2/m_R^2$$

- Mass insertions have $d_n = 0$ and come with factors

$$(m^2/m_R^2)(m_R^2/q^2) = m^2/q^2$$

ZERO DERIVATIVE INTERACTIONS

- For relativistic particles q >> m so perturbing in m/q is OK.

- For q ~ m the kinematics becomes non-relativistic and so path integral becomes dominated by Schrodinger action, which scales t and x differently

- Leads to different form of low energy theory (eg NRQED or NRQCD or HQET) and it is the relevant interaction that signals the instability towards this transition at low energies.

RELEVANT INTERACTIONS

- Normally a relevant interaction signals a transition to a new scaling regime

$$x^\mu \to \tilde{x}^\mu = s x^\mu \qquad \phi \to \tilde{\phi} = \phi/s$$

$$S = \int \mathrm{d}^4 x \left[(\partial\phi)^2 + m^2\phi^2 \right] = \int \mathrm{d}^4 \tilde{x} \left[(\tilde{\partial}\tilde{\phi})^2 + \frac{m^2}{s^2}\tilde{\phi}^2 \right]$$

So mass term becomes more important as s tends to zero: although can perturb in the mass for relativistic problems, once q ~ m nonrelativistic scaling takes over

EFTS IN PARTICLE PHYSICS

Known unknowns

RAYLEIGH SCATTERING

- As a first practical example of EFT methods consider photons scattering from a neutral body much smaller than wavelength.

$$a \ll \lambda$$

- Lowest dimension interaction between photon and neutral field is

$$\mathscr{L} = c\Psi^*\Psi\,\nabla\cdot\mathbf{E} + \frac{g}{2}\Psi^*\Psi\,\mathbf{E}^2 + \cdots$$

where first term is redundant and g has dimensions $(\text{length})^3$ and is called the object's polarizability. Amplitude and cross section for photon scattering from neutral body then is

$$\mathscr{A} = igkk'\epsilon\cdot\epsilon' \qquad \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{g^2k^4}{32\pi^2}(1+\cos^2\theta) \qquad \sigma = \frac{g^2k^4}{6\pi}$$

QED (BELOW ELECTRON MASS)

- Another illustrative example of EFTs at work is QED

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}(\gamma^\mu D_\mu + m)\psi - eA_\mu J^\mu$$

- Most general possible low-energy interactions of these kinds of fields

- Integrate out the electron

$$\mathscr{L}_{\rm eff} = -eA_\mu J^\mu - \frac{1}{4}Z\,F_{\mu\nu}F^{\mu\nu} + \frac{b_1}{m^4}\left[(F_{\mu\nu}F^{\mu\nu})^2 + \frac{7}{4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right] + \cdots$$

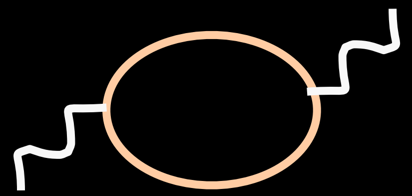
QED (BELOW ELECTRON MASS)

- Integrate out the electron

$$\mathscr{L}_{\text{eff}} = -eA_\mu J^\mu - \frac{1}{4} Z F_{\mu\nu}F^{\mu\nu} + \frac{b_1}{m^4}\left[(F_{\mu\nu}F^{\mu\nu})^2 + \frac{7}{4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right] + \cdots$$

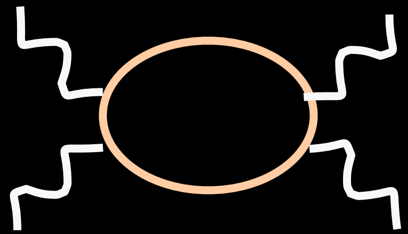
- Compute Z using vacuum polarization graph

$$Z = 1 - \frac{\alpha}{3\pi}\left[\frac{1}{\epsilon} - \gamma_k + \ln\left(\frac{m^2}{\mu^2}\right)\right] \qquad (D = 4 - 2\epsilon)$$



- Compute b_1 using box graph

$$b_1 = \frac{\alpha^2}{90}$$



QED (BELOW ELECTRON MASS)

- The four-photon term provides the simplest way to compute low-energy photon-photon scattering cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \simeq \frac{139}{4\pi^2}\left(\frac{\alpha^2}{90}\right)^2\left(\frac{E_{\mathrm{cm}}^6}{m^8}\right)\left(3+\cos^2\theta\right)^2$$

- There are also redundant operators

$$\partial_\mu F^{\mu\nu}\partial^\lambda F_{\lambda\nu} \propto e^2 J^\nu J_\nu \qquad F_{\mu\nu}\Box F^{\mu\nu} \propto eF^{\mu\nu}\partial_\mu J_\nu \simeq -e^2 J^\mu J_\mu$$

- In this case redundant interactions generate current-current interactions, whose presence can be ignored only at places where there are no currents

QED (BELOW ELECTRON MASS)

- The Maxwell term is also interesting, must rescale $A_\mu = Z^{-1/2} A'_\mu$

$$\mathscr{L}_{\text{eff}} = -\, e_{\text{phys}} A'_\mu J^\mu - \frac{1}{4} F'_{\mu\nu} F^{\mu\nu\prime} + \cdots$$

- Upshot: low-energy influence of electron is suppressed by m only after appropriate redefinition of e. (Precise statement of decoupling.)

$$e = Z^{1/2} e_{\text{phys}} = e_{\text{phys}} \left[1 - \frac{\alpha}{6\pi} \left(\frac{1}{\epsilon} - \gamma_k + \ln \left(\frac{m^2}{\mu^2} \right) \right) \right]$$

- Corrections to macroscopic classical E+M given by powers of E/m rather than α. This is at root of why Rutherford scattering is same in classical and quantum calculation.

QED (ABOVE ELECTRON MASS)

- For E bigger than m, the renormalization Z gives information about large logs

$$Z = 1 - \frac{\alpha}{3\pi}\left[\frac{1}{\epsilon} - \gamma_k + \ln\left(\frac{m^2}{\mu^2}\right)\right]$$

- In modified minimal subtraction remove just first two terms

$$A_\mu = Z_{\overline{MS}}^{-1/2} A'_\mu \qquad Z_{\overline{MS}} = 1 - \frac{\alpha}{3\pi}\left(\frac{1}{\epsilon} - \gamma_k\right)$$

- Corresponding charge is not itself physical

$$\alpha_{\overline{MS}} = \left(\frac{Z_{\overline{MS}}}{Z}\right)\alpha_{\text{phys}} = \alpha_{\text{phys}}\left[1 + \frac{\alpha}{3\pi}\ln\left(\frac{m^2}{\mu^2}\right)\right]$$

QED (ABOVE ELECTRON MASS)

- Since physical charge cannot depend on mu, must have

$$\mu^2 \frac{\partial \alpha_{\overline{MS}}}{\partial \mu^2} = -\frac{\alpha_{\overline{MS}}^2}{3\pi}$$

- Because MSbar is mass-independent its RG evolution is easy to solve

$$\frac{1}{\alpha_{\overline{MS}}(\mu)} = \frac{1}{\alpha_{\overline{MS}}(\mu_0)} - \frac{1}{3\pi} \ln\left(\frac{\mu^2}{\mu_0^2}\right) \qquad \text{with} \qquad \alpha_{\overline{MS}}(\mu = m) = \alpha_{\text{phys}}$$

This is RG improved in that it holds even when both terms on RHS are similar size

QED (ABOVE ELECTRON MASS)

- Why do we care? Consider E >> m limit of scattering

$$\sigma(E, m_e, \alpha_{\text{phys}}) = \frac{1}{E^2} F\left(\frac{m_e}{E}, \alpha_{\text{phys}}, f, \theta_k\right)$$

where there is a sum over soft photons up to energies

$$E_\gamma = fE \quad \text{with} \quad 1 > f \gg m/E$$

- Cannot Taylor expand F due to log(m/E) singularities, but these are not present when using MSbar couplings. Identify log(E/m) by setting mu=E in

$$\sigma(E, m_e, \alpha_{\text{phys}}) = \frac{1}{E^2}\left[F_0\left(\frac{E}{\mu}, \alpha_{\overline{MS}}(\mu), f, \theta_k\right) + \mathcal{O}(m/E)\right]$$

QED (INCLUDING MUONS)

- Next consider QED at energies above the muon mass

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \overline{\psi}(\gamma^\mu D_\mu + m)\psi - \overline{\chi}(\gamma^\mu D_\mu + M)\chi$$

- Most general possible low-energy interactions of these kinds of fields

- Integrate out the muon gives

$$\mathscr{L}_{\text{eff}} = -\frac{1}{4}Z\,F_{\mu\nu}F^{\mu\nu} - Z_e\overline{\psi}(\gamma^\mu D_\mu + Z_m m)\psi + \frac{b_1}{M^4}\left[(F_{\mu\nu}F^{\mu\nu})^2 + \frac{7}{4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right] + \cdots$$

QED (INCLUDING MUONS)

- Integrating out muon gives $F_{mn}{}^4$ term with coefficient M^{-4}. Integrating out the electron gives m^{-4}. At lower energies smallest mass wins.

$$\mathscr{L}_{\text{eff}} \supset b_1 \left(\frac{1}{M^4} + \frac{1}{m^4} \right) \left[(F_{\mu\nu}F^{\mu\nu})^2 + \frac{7}{4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 \right] + \cdots$$

Barring selection rules should expect smallest mass to dominate in denominators, but largest mass wins in numerators. From that point of view the large size of the Planck mass makes sense

$$\mathscr{L} \supset -\frac{1}{2}(m^2 + M^2 + M_p^2)R + \cdots$$

while the cosmological constant is a puzzle…

QED (INCLUDING MUONS)

- In minimal subtraction both muons and electrons contribute to the running of the EM coupling

$$\alpha_{\overline{MS}} = \left(\frac{Z_{\overline{MS}}}{Z}\right)\alpha_{\text{phys}} = \alpha_{\text{phys}}\left[1 + \frac{\alpha}{3\pi}\ln\left(\frac{m^2}{\mu^2}\right) + \frac{\alpha}{3\pi}\ln\left(\frac{m^2}{M^2}\right)\right]$$

and so

$$\mu^2\,\frac{\partial\alpha_{\overline{MS}}}{\partial\mu^2} = -\,\frac{2\alpha_{\overline{MS}}^2}{3\pi}$$

$$\frac{1}{\alpha_{\overline{MS}}(\mu)} = \frac{1}{\alpha_{\overline{MS}}(\mu_0)} - \frac{2}{3\pi}\ln\left(\frac{\mu^2}{\mu_0^2}\right)$$

with $\alpha_{\overline{MS}}(\mu = \sqrt{mM}) = \alpha_{\text{phys}}$

QED (INCLUDING MUONS)

- Minimal subtraction makes it seem as if muons play a role in running also at energies below the muon mass.

$$\alpha_{\overline{MS}} = \left(\frac{Z_{\overline{MS}}}{Z}\right)\alpha_{\text{phys}} = \alpha_{\text{phys}}\left[1 + \frac{\alpha}{3\pi}\ln\left(\frac{m^2}{\mu^2}\right) + \frac{\alpha}{3\pi}\ln\left(\frac{m^2}{M^2}\right)\right]$$

  better is to make its decoupling manifest: decoupling subtraction.

- Can have decoupling and the convenience of MSbar running by using MSbar for EFT with electrons and muons above muon mass; MSbar for EFT with electrons only between m and M.
- Match the coupling constant across the thresholds as particle is integrated out.

QED (INCLUDING MUONS)

- Decoupling subtraction:

If m < mu < M:

$$\mu^2 \frac{\partial \alpha_{\overline{MS}}}{\partial \mu^2} = -\frac{\alpha_{\overline{MS}}^2}{3\pi}$$

$$\alpha_{\overline{MS}}(\mu = m) = \alpha_{\text{phys}}$$

$$\frac{1}{\alpha_{\overline{MS}}(\mu)} = \frac{1}{\alpha_{\text{phys}}} - \frac{1}{3\pi} \ln\left(\frac{\mu^2}{\mu_0^2}\right)$$

If mu > M

$$\mu^2 \frac{\partial \alpha_{\overline{MS}}}{\partial \mu^2} = -\frac{2\alpha_{\overline{MS}}^2}{3\pi}$$

$$\frac{1}{\alpha_{\overline{MS}}(\mu)} = \frac{1}{\alpha_{\text{phys}}} - \frac{1}{3\pi} \ln\left(\frac{M^2}{m^2}\right) - \frac{2}{3\pi} \ln\left(\frac{\mu^2}{m^2}\right)$$

WEAK INTERACTIONS

- The weak interactions were a starting point for understanding EFTs. Integrating out the W boson leads to the Fermi lagrangian

$$\mathscr{L}_{\rm sm} \supset g W_\mu \overline{\psi} \gamma^\mu \gamma_L \psi + \text{c.c.}$$

$$\mathscr{L}_F = \sqrt{2} G_F \, \overline{\psi} \gamma_\mu \gamma_L \psi \; \overline{\psi} \gamma^\mu \gamma_L \psi$$

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 M_W^2}$$

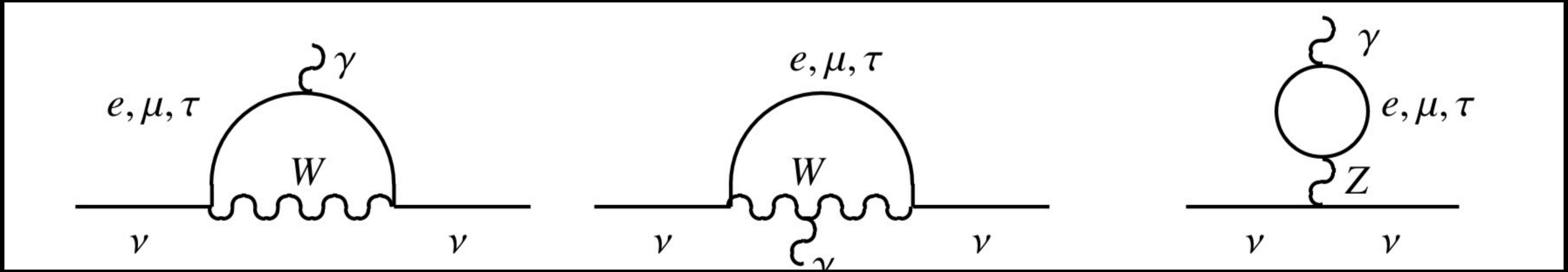
- Do not also expect weak interactions to get corrections proportional to smaller masses as lighter fields are integrated out.
  - Selection rules (parity, flavour transformations, etc) always require at least one W propagator, so effective interactions need not always be dominated by the lightest particle integrated out.

WEAK INTERACTIONS

- The possibility of having lighter masses can lead to surprises, however. eg:

$$\sigma(\nu\nu \to \gamma\gamma) \sim G_F^4 E^6$$

$$\sigma(\nu\nu \to \gamma\gamma\gamma) \sim \left(\frac{\alpha}{4\pi}\right)^3 \frac{G_F^2 E^{10}}{m_e^8}$$



WEAK INTERACTIONS

- Integrating out the W boson gives

$$\mathscr{L}^{\text{eff}}_{\nu 1,2\gamma} = C^{(1)}_{ab}\, M^{ab}_{\mu\nu}\, F^{\mu\nu} + C^{(2)}_{ab}\, M^{ab}_{\mu\nu}\, F^{\mu\lambda}F_{\lambda}{}^{\nu} + \mathscr{L}_F$$

$$M^{ab}_{\mu\nu} := i\overline{\nu}^a\gamma_\mu\gamma_L\partial_\nu\nu^b - i\partial_\nu\overline{\nu}^a\gamma_\mu\gamma_L\nu^b$$

$$C^{(2)}_{ab}(\mu) = \frac{2\sqrt{2}\,\alpha\, G_F}{\pi M_W^2}\left[1 + \frac{4}{3}\ln\left(\frac{M_W^2}{\mu^2}\right)\right]\delta_{ab}$$

- Symmetric derivative on neutrino leads to redundant operators
- Chirality requires odd number of gamma matrices

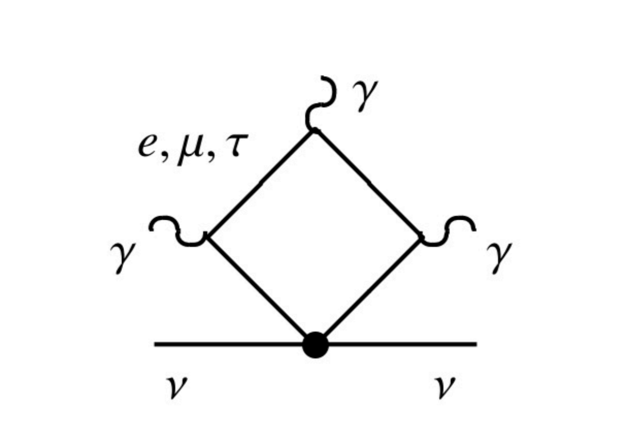
WEAK INTERACTIONS

- Evolving down to lower scales focus on graph involving only one factor of M_W^{-2}

$$\mathscr{L}^{\text{eff}}_{\nu 3\gamma} = \frac{e\, v_{ab}\, \alpha}{90\pi\, m_e^4} \left(\frac{G_F}{\sqrt{2}}\right)\left[5\,(N^{ab}_{\mu\nu}\, F^{\mu\nu})(F_{\lambda\rho}\, F^{\lambda\rho}) - 14\,(N^{ab}_{\mu\nu}\, F^{\nu\lambda}\, F_{\lambda\rho}\, F^{\rho\mu})\right]$$

$$v_{ab} := v_{ab\,ee}(\mu = m_e) = U^*_{ea}\, U_{eb} + \delta_{ab}\left(-\frac{1}{2} + 2s_w^2\right)$$

$$N^{ab}_{\alpha\beta} = \partial_\alpha\left(\overline{\nu}^a \gamma_\beta \gamma_L \nu^b\right) - (\alpha \leftrightarrow \beta)$$



- Redundant for 1,2 photons since involves derivative of neutrino current

EFTS IN PARTICLE PHYSICS

Unknown unknowns

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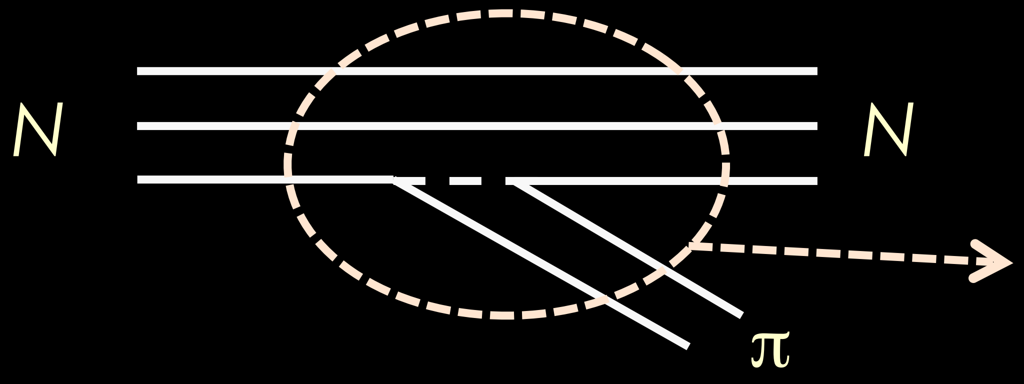
UNKOWN UV THEORY

- Low-energy degrees of freedom can be qualitatively different from high-energy ones
  - e.g. pions, or atoms, or planets can be ‘elementary’ fields at low energies while their constituents are ‘elementary’ at high energies



$$L_{eff} = \frac{1}{F}\partial_\mu \pi \left(N\gamma^\mu N \right)$$

if $E < 4\pi F \sim 1\ \mathrm{GeV}$

TECHNICAL NATURALNESS

- The SM is most general renormalizable theory built from given particle content and gauge symmetry
  - smells like a low-energy EFT

- But SM also contains relevant interactions (those that get larger at low energies) like

$$\mathscr{L}_{\mathrm{SM}} \supset -\zeta + w^2 H^\dagger H$$

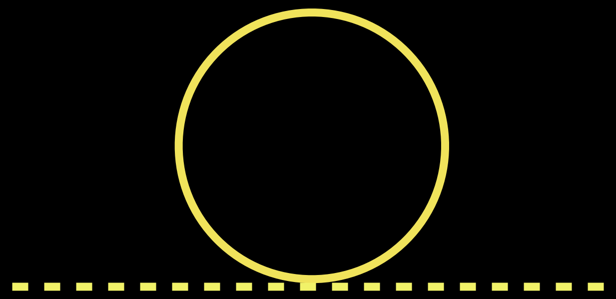
Is this a problem?

TECHNICAL NATURALNESS

- Imagine embedding the SM into some UV theory, for simplicity take it simply to be a singlet scalar, S, of mass M
- Compute the Higgs mass both in the EFT (the SM) below M and in the UV theory above M

$$\mathscr{L} = \mathscr{L}_{\mathrm{SM}} - \frac{1}{2}(\partial S)^2 - \frac{1}{2}M^2 S^2 - \frac{1}{2}g^2 S^2 H^\dagger H + \cdots$$

$$\begin{aligned} m_H^2 &= 2w_{\mathrm{he}}^2(\mu) + (SM\ loops) - \frac{g^2 M^2}{8\pi^2}\ln\left(\frac{M^2}{\mu^2}\right) \\ &= 2w_{\mathrm{le}}^2(\mu) + (SM\ loops) \end{aligned}$$



TECHNICAL NATURALNESS

- The effective constant w_{le} is order the weak scale always, while w_{he} is order M, everywhere except precisely at mu = M
- The same holds for higher thresholds: must adjust initial UV coupling with high precision to arrive at low energies with the SM value.
- This is not how hierarchies of scale usually work: normally if a parameter is small, its small size can be understood at any scale one chooses to ask: eg why are atoms larger than nuclei?

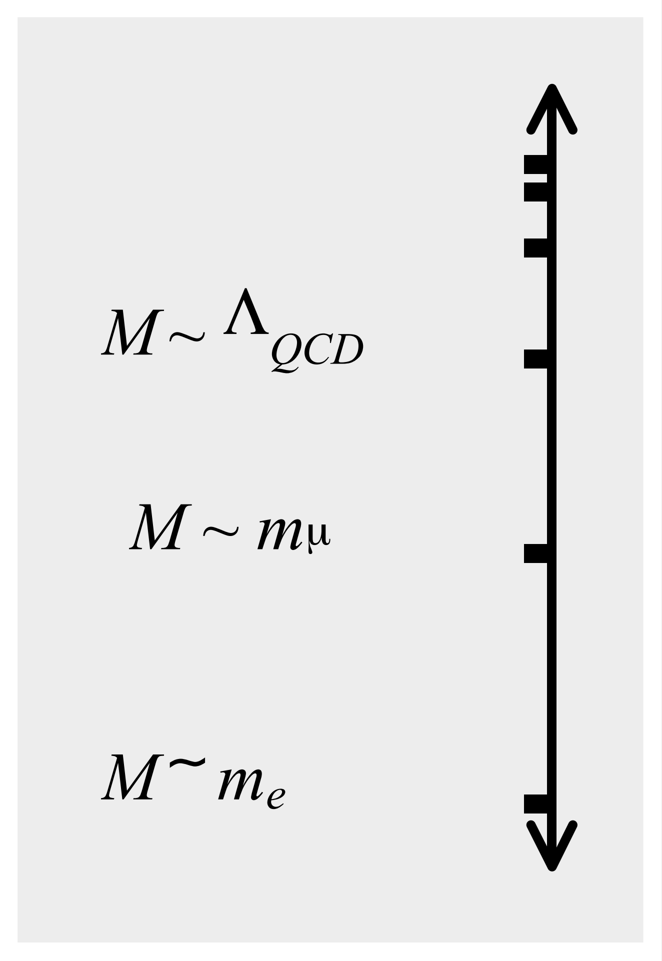
TECHNICAL NATURALNESS

- "Technically natural" understanding of why a parameter is small:
  - why is it small in the UV theory?
  - why does it stay small as one integrates out scales between the UV and measurement scale?

$$\hat{\alpha}\hat{m}_e \ll \Lambda_{QCD}$$

$$\delta m_e \sim \alpha \ln\left(\frac{m_\mu}{m_e}\right)$$

$$\alpha m_e \ll m_p$$



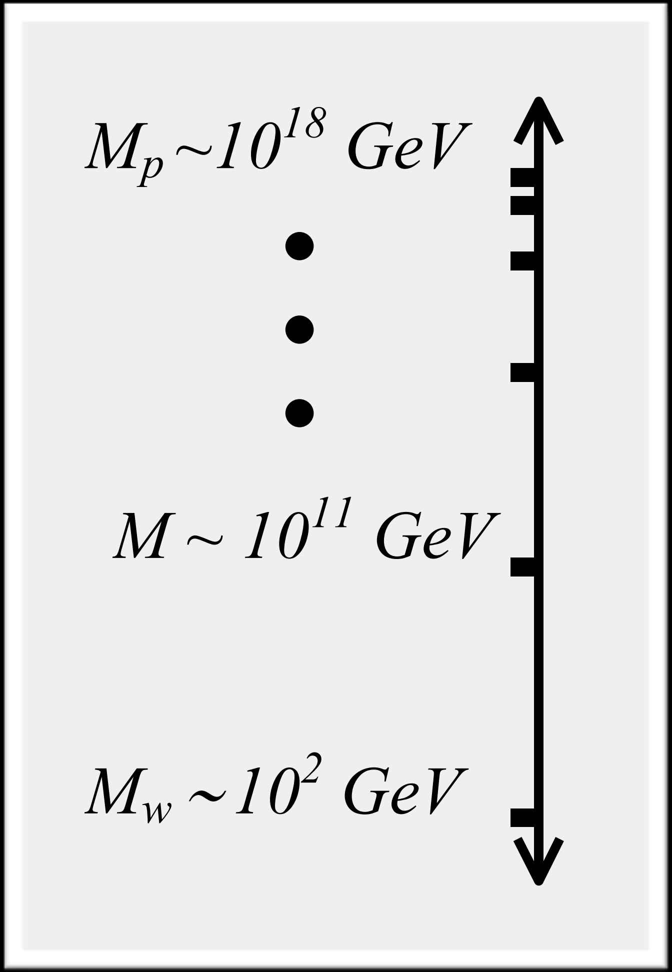
TECHNICAL NATURALNESS

- The small size of the Higgs mass relative to Planck scale is not automatically technically natural:
  - add fermions
  - make Higgs composite
  - deny M_p is a scale

$$w \sim \hat{M}$$

$$w \sim M$$

$$w \sim m_H$$



TECHNICAL NATURALNESS

- Threat to the naturalness argument: cosmological constant is small, and the scales where it is unnatural are well understood.

$$\zeta \sim M_W^4$$

$$\zeta \sim m_e^4$$

$$\zeta \sim (0.01eV)^4$$