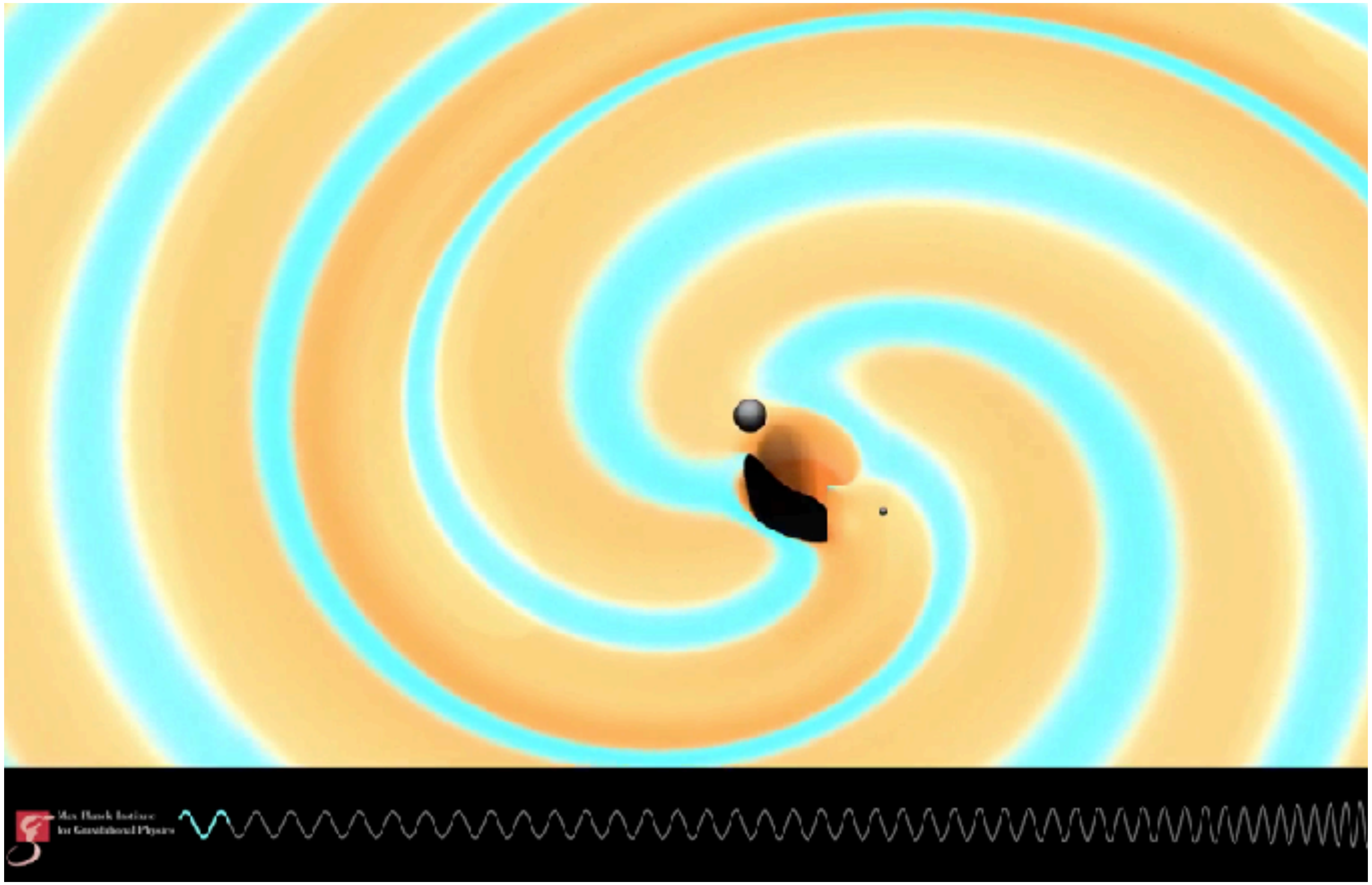
The basic physics of GWs cont.:

Generation of GWs

binary systems as GW sources



credit: SXS

Recall from last lecture: properties of GWs

- Waves are transverse, two polarizations

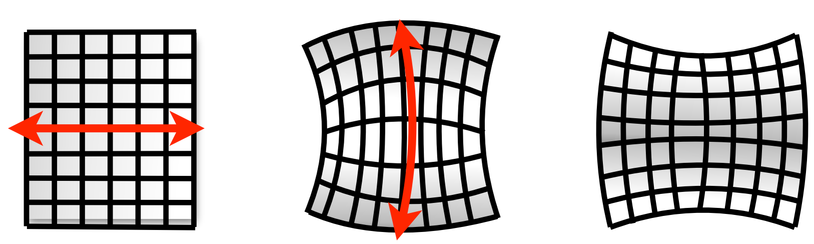
$$h^{\mathrm{TT}}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R_{txtx} = -R_{tyty} = -\frac{1}{2}\ddot{h}_+$$

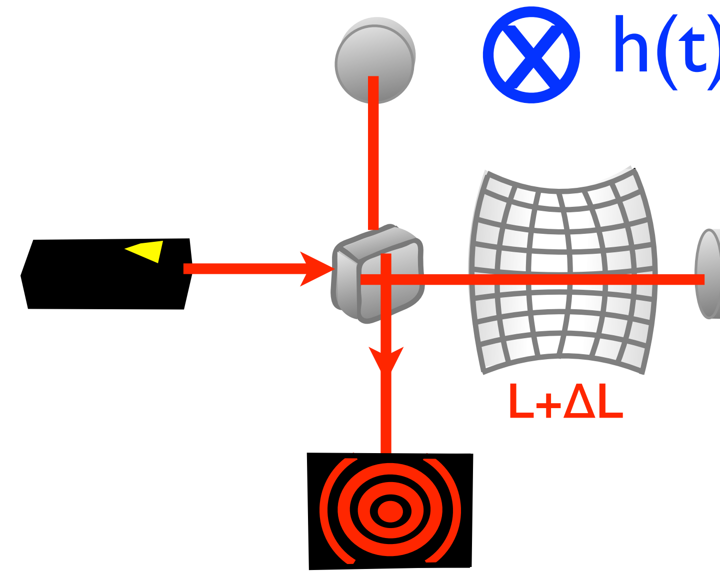
$$R_{txty} = R_{tytx} = -\frac{1}{2}\ddot{h}_\times$$

- Effect of GWs:

plus-polarization h_+ cross-polarization h_x



$$\frac{\Delta \mathrm{L}}{\mathrm{L}} \sim h(t)$$

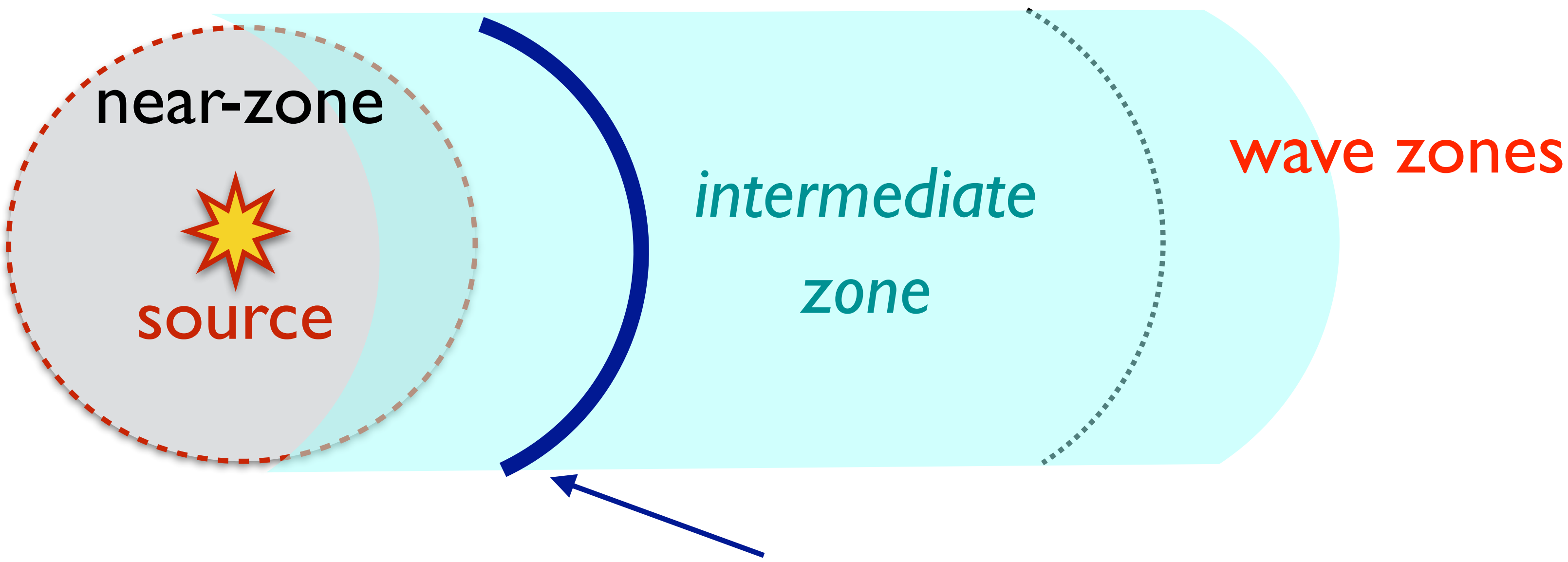


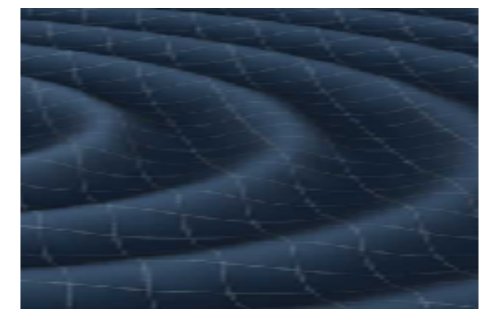
- Energy carried by GWs: $T^{\mathrm{GW}}_{ab} = -\frac{1}{32\pi}\left\langle \frac{\partial h^{ij}}{\partial x^a}\frac{\partial h_{ij}}{\partial x^b}\right\rangle$ ← average

(TT gauge metric)

Production of GWs

Information flow from source to wave zone





Moderately close to source: *source fields are **weak***

Starting point for approx. calculations: Einstein equations in harmonic coordinates

$$\Box \bar{h}^{\alpha\beta} = -16\pi\, \tau^{\alpha\beta} \qquad \partial_\alpha \tau^{\alpha\beta} = 0$$

Includes: T^{ab} of source & gravitational field energy-momentum & $O(h^2)$ contributions

[Landau-Lifshitz formulation of GR]

Example of approximate theoretical treatment

Iterative approximation scheme to solve:

For weak-field, slow motion sources

$$\Box \bar{h}^{\alpha\beta} = -16\pi\, \tau^{\alpha\beta} \quad \& \quad \partial_\alpha \tau^{\alpha\beta} = 0$$

- Formal Green function solution to wave equation:

$$\bar{h}^{\alpha\beta}(t, \mathbf{x}) = 4 \int d^3x' \, \frac{\tau^{\alpha\beta}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

- Analyze propagation of information from moderately far zone (weak source fields) through intermediate zone to wave zone
- Different approximations in each zone
- Matched asymptotic expansions to obtain composite solution

Leading order result in the wave zone

- Further approximate the solution in the far field:

$$\bar{h}^{\alpha\beta}(t,\mathbf{x}) = 4\int d^3x' \, \frac{\tau^{\alpha\beta}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

Distance source to field point

Far field: $|\mathbf{x}-\mathbf{x}'| \approx r$

Slow-motion: $t-|\mathbf{x}-\mathbf{x}'| \approx t-r$

Retarded time

Calculations lead to:

$$h_{ij}^{\mathrm{TT}} \approx \frac{2}{r}\frac{G}{c^4}\frac{\partial^2}{\partial t^2} I_{ij}^{\mathrm{TT}}(t-r/c)$$

tiny, ~10^{-49} in cgs units

TT projection of the quadrupole moment

$$I^{ij}(t) = \int x^i x^j \tau^{00}(t,\mathbf{x})\, d^3\mathbf{x}$$

Beyond quadrupole radiation

- Relativistic sources generate GWs not only from time-varying quadrupole
- Two families of moments (like electric and magnetic):
    - Moments of mass distribution (mass moments)
    - Moments of angular-momentum distribution (current moments)
    - Moments in asymptotic radiation $\neq$ source moments at higher order

Order of magnitude estimates

- Source parameters:

Mass M, size L, rate of quadrupolar oscillations ω, distance r

Internal kinetic energy of quadrupolar dynamics

$$E^Q_{\rm kin} \sim M L^2 \omega^2$$

- GW amplitude:

$$h^{\rm GW}_{ij} \sim \frac{2}{r}\,\frac{G}{c^4}\,\frac{\partial^2}{\partial t^2} I_{ij} \sim \frac{G}{c^4}\,\frac{\omega^2 M L^2}{r} \sim \frac{G}{c^2}\,\frac{E^Q_{\rm kin}/c^2}{r}$$

$$\sim 10^{-21}\left(\frac{E^Q_{\rm kin}}{M_\odot c^2}\right)\left(\frac{100{\rm Mpc}}{r}\right)$$

100Mpc=300 million light years

Order of magnitude estimates

- Source parameters:

Mass M, size L, rate of quadrupolar oscillations ω, distance r

Internal kinetic energy of quadrupolar dynamics

$$E_{\mathrm{kin}}^{Q} \sim M L^2 \omega^2$$

- GW energy flux:

$$F_{\mathrm{GW}} \sim \frac{c^3}{G} \left(\frac{d h_{ij}^{\mathrm{GW}}}{dt} \right)^2$$

← $\sim h_{GW}^2 \omega^2$

$$\sim 10^{42} \left(\frac{f}{100\mathrm{Hz}} \right)^2 \left[10^{-21} \left(\frac{E_{\mathrm{kin}}^{Q}}{M_{\odot} c^2} \right) \left(\frac{100\mathrm{Mpc}}{r} \right) \right]^2 \mathrm{erg\,cm^{-2}\,s^{-1}}$$

c.f. flux from Sirius (brightest star in night sky): $F_{\mathrm{Sirius}} \sim 10^{-4}$ erg cm^2 s^{-1}

GW luminosity

$$P_{\mathrm{GW}} = \frac{1}{5}\,\frac{G}{c^5}\,\left\langle \frac{d^3Q_{ij}}{dt^3}\,\frac{d^3Q_{ij}}{dt^3} \right\rangle$$

energy/time

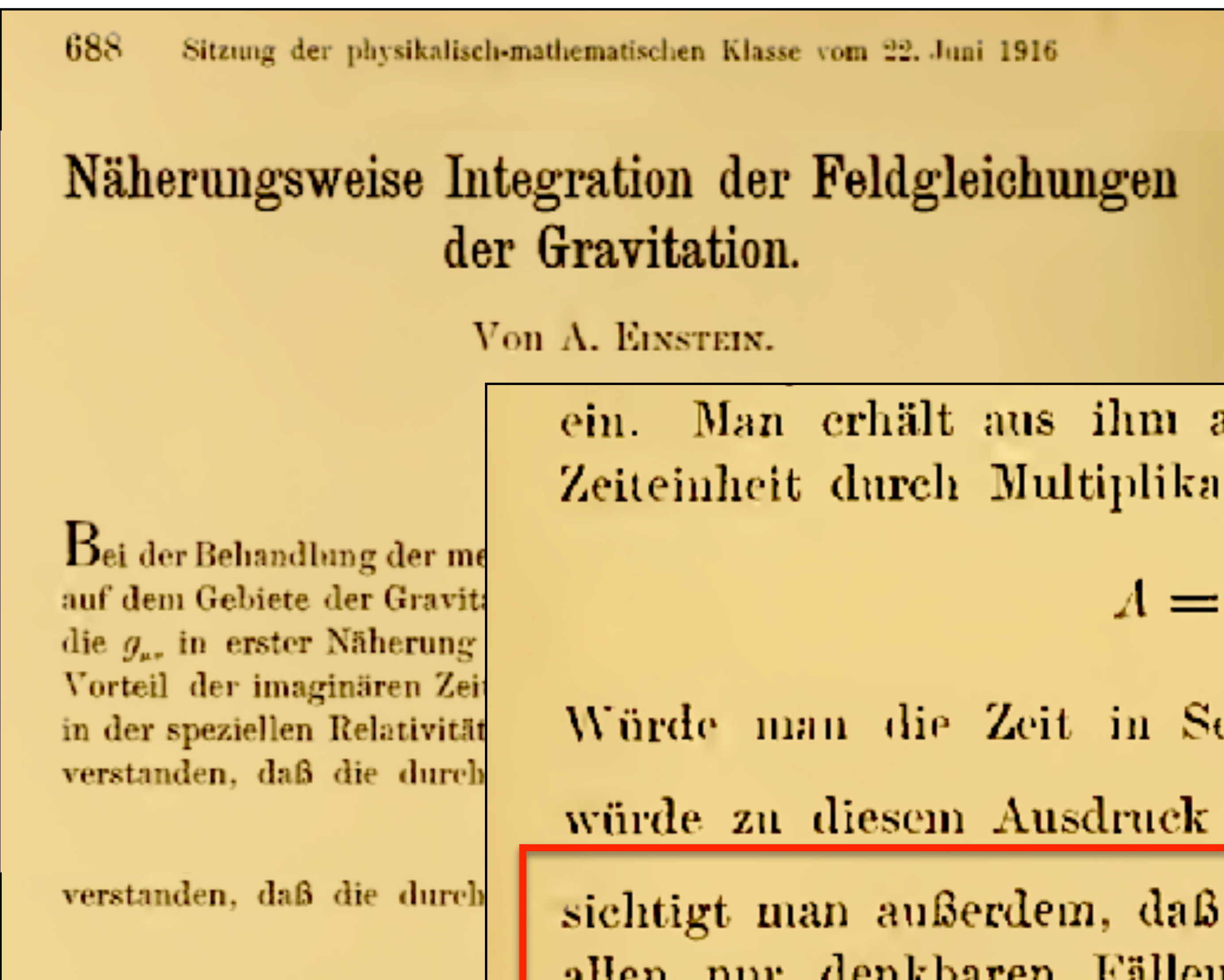
Quadrupole formula

$$\frac{G}{c^5} \sim 3 \times 10^{-60} \mathrm{cm}^3 \mathrm{s}^{-2} \mathrm{g}^{-1}$$

symmetric, trace-free quadrupole moment:

$$Q_{ij} = \int d^3x \left(x^i x^j - \frac{1}{3}\delta^{ij}|\mathbf{x}|^2 \right) \tau^{00}(t, \mathbf{x})$$

= mass density for a Newtonian source

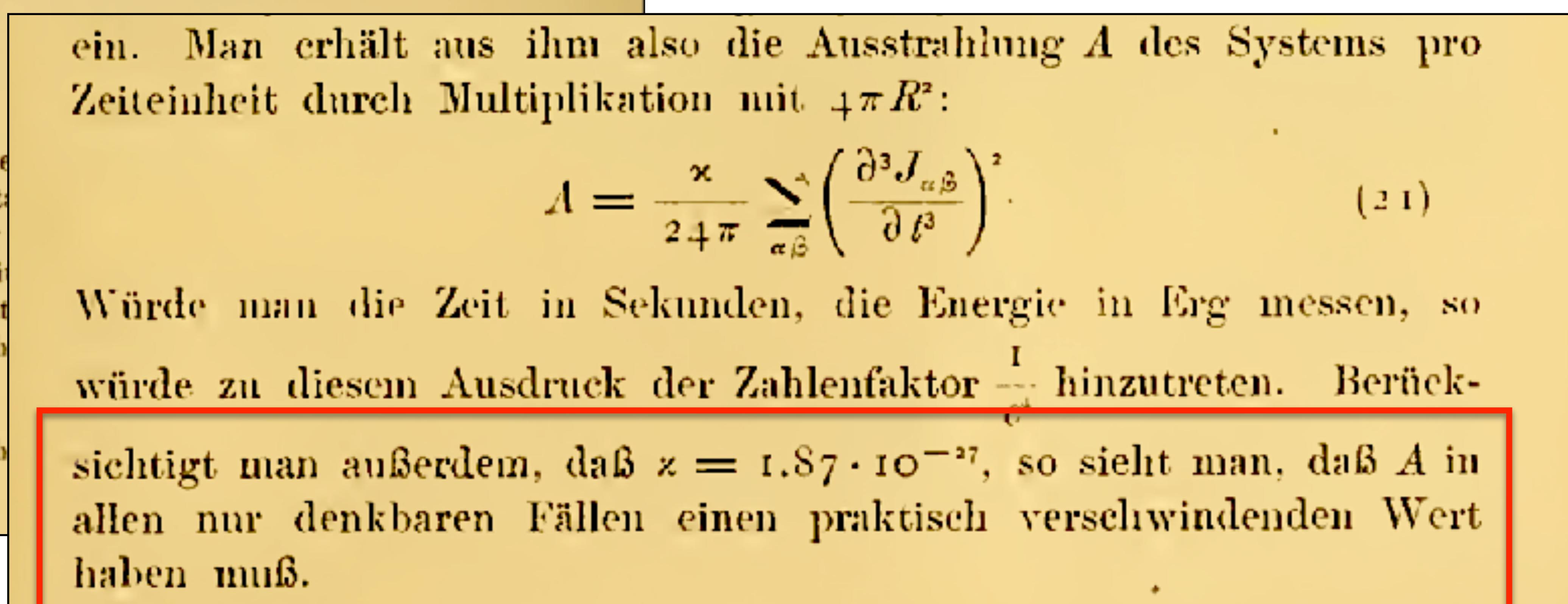

688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. Einstein.

Bei der Behandlung der me
auf dem Gebiete der Gravit
die $g_{\mu\nu}$ in erster Näherung
Vorteil der imaginären Zei
in der speziellen Relativität
verstanden, daß die durch

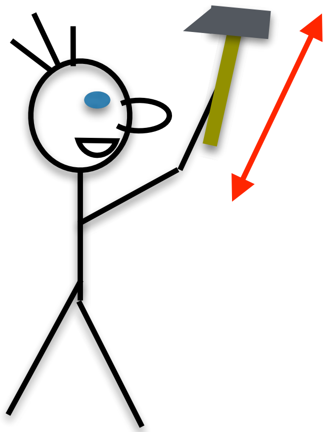
verstanden, daß die durch


ein. Man erhält aus ihm also die Ausstrahlung A des Systems pro Zeiteinheit durch Multiplikation mit $4\pi R^2$:

$$A = \frac{\varkappa}{24\pi} \sum_{\alpha\beta} \left(\frac{\partial^3 J_{\alpha\beta}}{\partial t^3} \right)^2. \qquad (21)$$

Würde man die Zeit in Sekunden, die Energie in Erg messen, so würde zu diesem Ausdruck der Zahlenfaktor $\frac{1}{c^4}$ hinzutreten. Berücksichtigt man außerdem, daß $\varkappa = 1.87 \cdot 10^{-27}$, so sieht man, daß A in allen nur denkbaren Fällen einen praktisch verschwindenden Wert haben muß.

Order of magnitude estimates



Person hammers for 10 sec, 2x per sec: **10s/(0.5s)=20 times = *N***

L~ 50 cm, M ~ 2kg, P~0.5s v~L/P

energy expended: $E_{\rm kin} \sim N \times (1/2)\, M\, L^2/P^2 \sim 10^8 {\rm erg}$

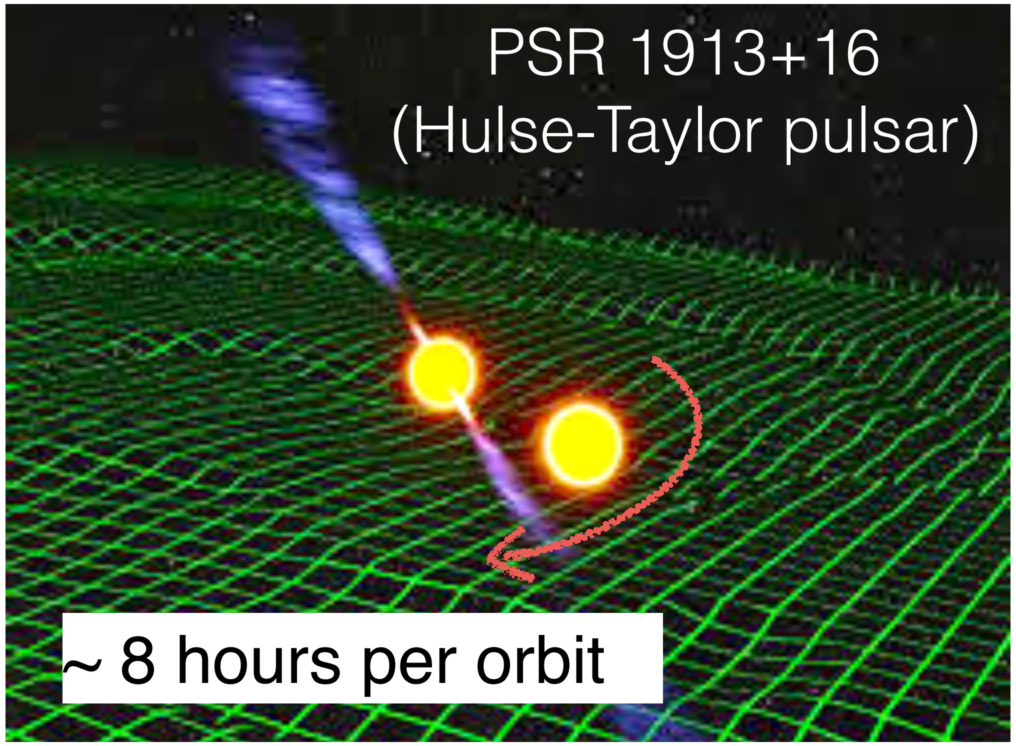
quadrupole moment ***Q~ M L²***

- GW power: $P_{\rm GW} \sim \frac{G}{c^5}\frac{1}{5}(\dddot{Q})^2 \sim \frac{G}{c^5}\frac{1}{5}(\frac{ML^2}{P^3})^2 \sim 10^{-45} {\rm erg\, s^{-1}}$
- Total energy in GWs: $E_{GW} \sim P_{GW} \times 10\ s \sim 10^{-44}\ erg$
- Gravitons generated: frequency $\omega \sim 2\pi / P \sim 10\ s^{-1}$, energy $E \sim \hbar\omega \sim 10^{-26}$ erg

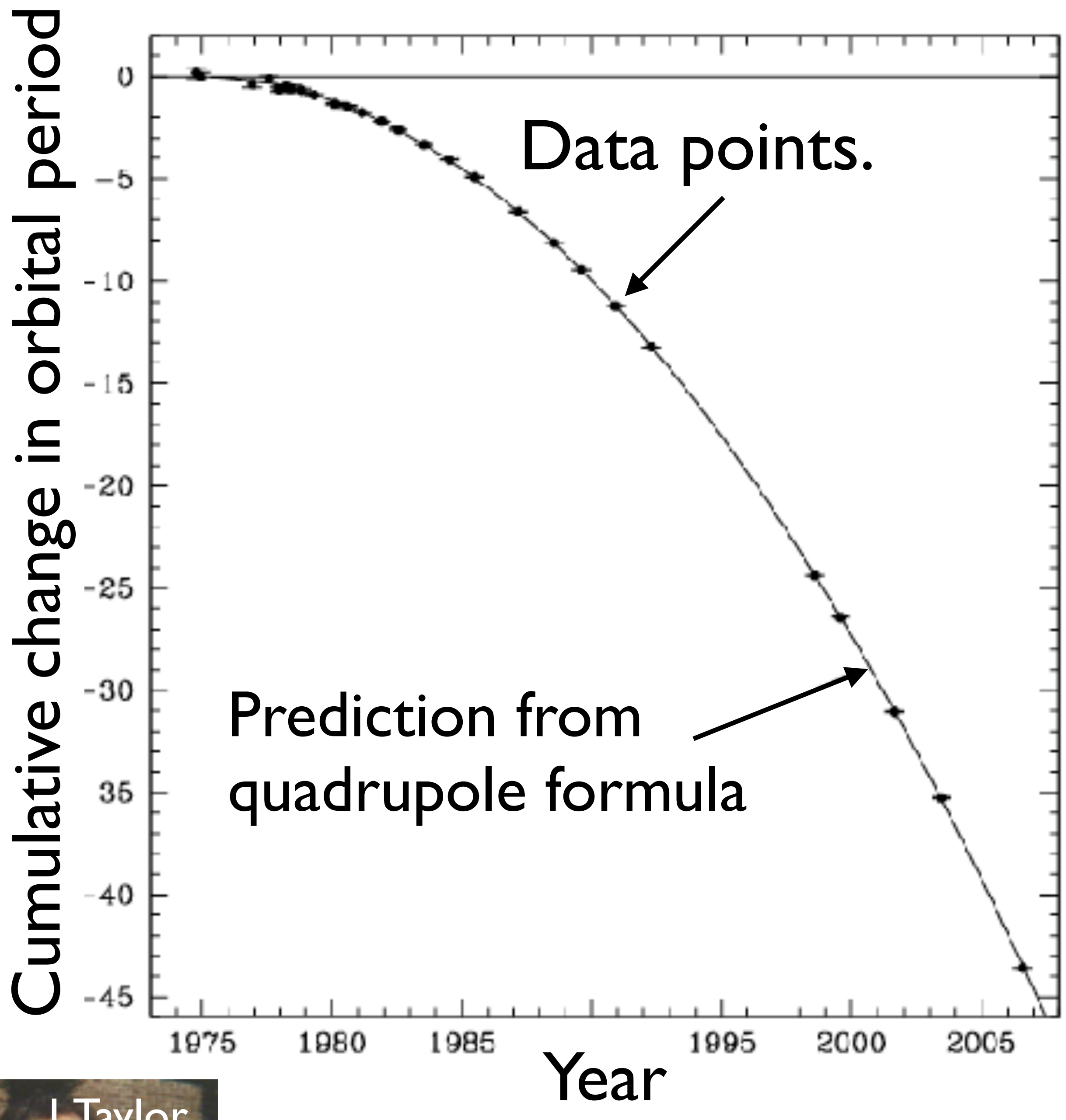
\# of gravitons ~ $E_{GW} / E \sim 10^{-18}$

Effect of GWs on their source

- GWs carry away energy and angular momentum
- Orbital decay measured in binary pulsars

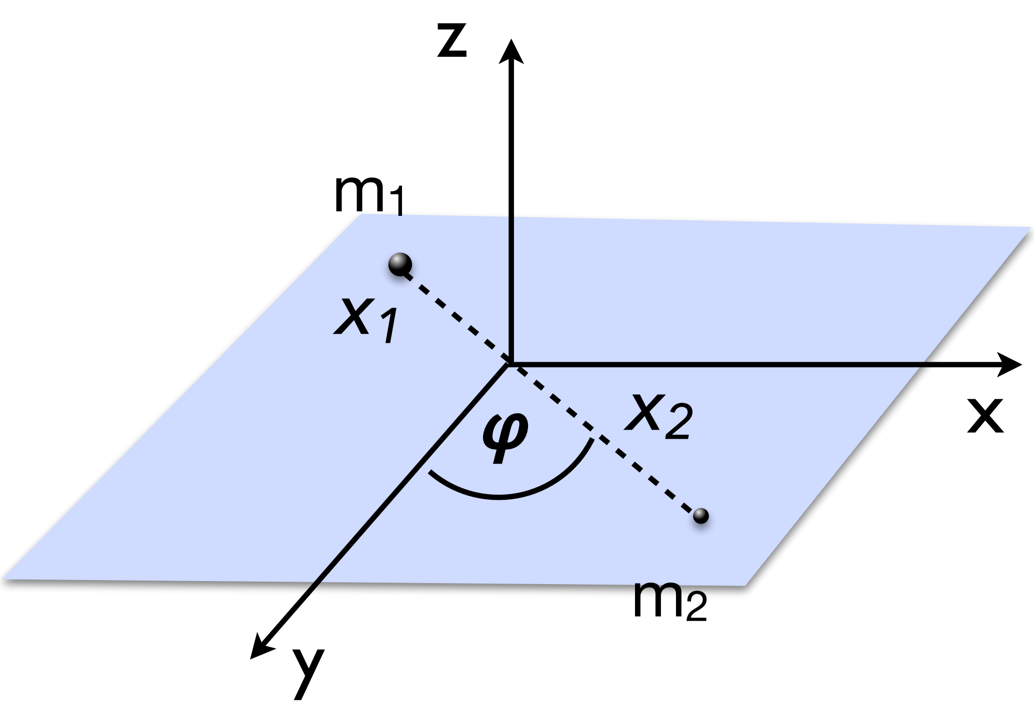


[credit: MPA]



[Taylor & Weisberg 2010]

Newtonian binary system



- Center-of-mass coordinates

relative displacement

$$\mathbf{x_1} = \frac{m_2}{M}\mathbf{x} \qquad \mathbf{x_2} = -\frac{m_1}{M}\mathbf{x}$$

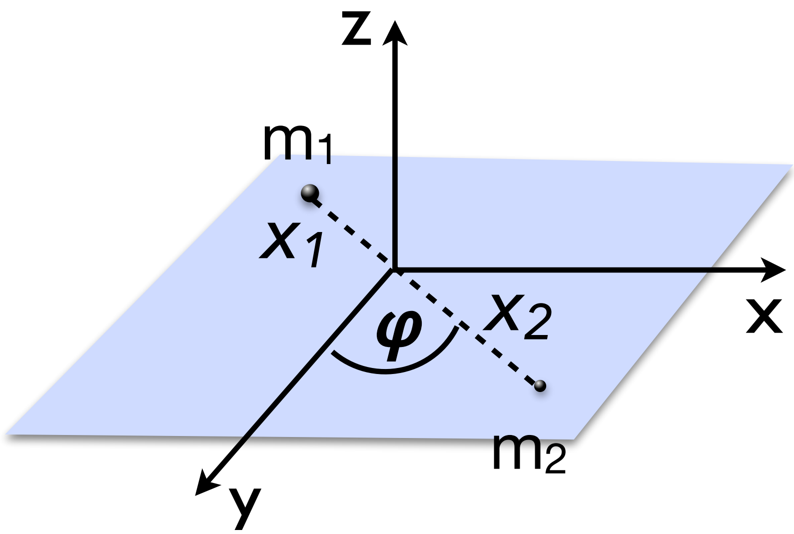
total mass $M = m_1 + m_2$

reduced mass $\mu = m_1 m_2 / M$

- Plane polar coordinates $x^i = (r\cos\varphi, r\sin\varphi, 0)$

Will assume that the objects are point masses

Newtonian binary system



- Plane polar coordinates

$$x^i = (r\cos\varphi, r\sin\varphi, 0)$$

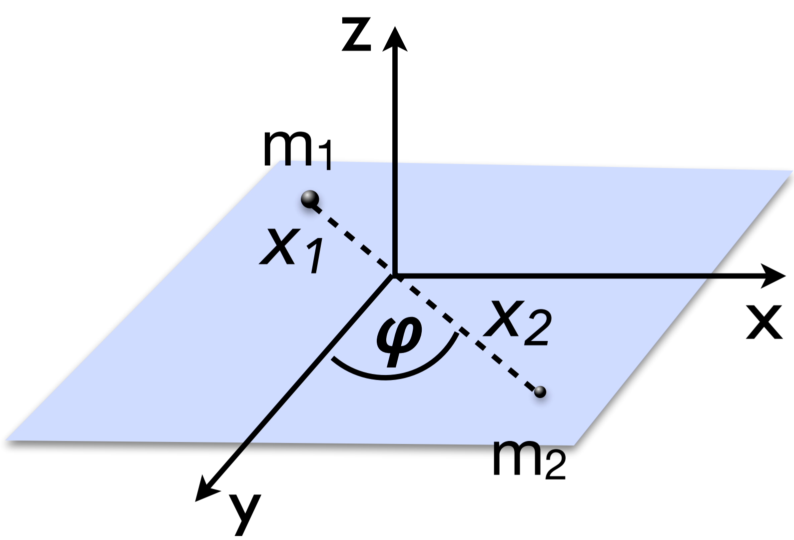
relative separation

- Orbital dynamics described by the Lagrangian

$$S = \int dt \left[\frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{\mu M}{r}\right]$$

- Equations of motion: $\mu\ddot{r} - \mu r\dot{\varphi}^2 + \dfrac{\mu M}{r^2} = 0$ $\qquad \mu r^2\ddot{\varphi} + 2\mu r\dot{r}\dot{\varphi} = 0$

Newtonian binary system



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- Equations of motion: $\mu\ddot{r} - \mu r\dot{\varphi}^2 + \frac{\mu M}{r^2} = 0$ $\qquad \mu r^2\ddot{\varphi} + 2\mu r\dot{r}\dot{\varphi} = 0$

- Specialize to circular orbits $\dot{\varphi}^2 \equiv \Omega^2 = \frac{M}{r^3} \longrightarrow$

$$r = M^{1/3}\Omega^{-2/3}$$

GW energy loss from a Newtonian circular-orbit binary system

- quadrupole moment:

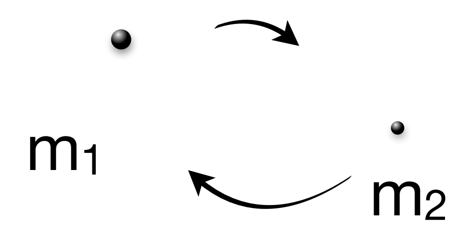
$$Q^{ij} = \int_{\text{source}} d^3x\, \rho \left(x^i x^j - \frac{1}{3}\delta^{ij}|\mathbf{x}|^2 \right)$$

$$= \sum_{A=1,2} m_A \left(x_A^i x_A^j - \frac{1}{3}\delta^{ij}|\mathbf{x}_A|^2 \right)$$

$$= \left(x^i x^j - \frac{1}{3}\delta^{ij}|\mathbf{x}|^2 \right) \left[m_1 \frac{m_2^2}{M^2} + m_2 \frac{m_1^2}{M^2} \right]$$

$$= \mu r^2 \left(n^i n^j - \frac{1}{3}\delta^{ij} \right)$$

Where $n^i = \dfrac{x^i}{r}$ is a radial unit vector



center of mass coordinates:

$$\mathbf{x_1} = \frac{m_2}{M}\mathbf{x}$$

$$\mathbf{x_2} = -\frac{m_1}{M}\mathbf{x}$$

relative displacement

$\mu = m_1 m_2 / M$

$M = m_1 + m_2$

GW energy loss from a Newtonian circular-orbit binary system

- Introduce another unit vector $\phi^i = \frac{v^i}{r\Omega} = (-\sin\varphi, \cos\varphi, 0)$
- Properties: $\dot{n}^i = \Omega\phi^i \qquad \dot{\phi}^i = -\Omega n^i$ $\qquad n^i = \frac{x^i}{r} = (\cos\varphi, \sin\varphi, 0)$
- Quadrupole: $Q^{ij} = \mu r^2 \left(n^i n^j - \frac{1}{3}\delta^{ij} \right)$
- Time derivatives:

$$\dddot{Q}_{ij} = \mu r^2 \frac{d^2}{dt^2}\left(\Omega\phi^i n^j + \Omega n^i \phi^j\right)$$

$$= \mu r^2 \Omega \frac{d}{dt}\left(-\Omega n^i n^j + \Omega\phi^i\phi^j + \Omega\phi^i\phi^j - \Omega n^i n^j\right)$$

$$= 2\Omega^3 \mu r^2 \left(-n^i\phi^j - n^j\phi^i - n^i\phi^j - n^j\phi^i\right)$$

$$= -4\Omega^3 \mu r^2 \left(n^i\phi^j + n^j\phi^i\right)$$

GW energy loss from a Newtonian circular-orbit binary system

- Further properties of the unit vectors: $n^i n_i = 1 \qquad \phi^i \phi_i = 1 \qquad \phi^i n_i = 0$

- had: $\dddot{Q}_{ij} = -4\Omega^3 \mu r^2 \left(n^i \phi^j + n^j \phi^i\right)$

- GW power: $$P_{\rm GW} = \frac{1}{5}\langle \frac{d^3 Q_{ij}}{dt^3}\frac{d^3 Q_{ij}}{dt^3}\rangle = \frac{16}{5}\Omega^6 \mu^2 r^4 \left(2+0\right)$$

$$= \frac{32}{5}\mu^2 M^{4/3}\Omega^{10/3}$$

$r = M^{1/3}\Omega^{-2/3}$

for Newtonian circular orbits

Energy balance: $P_{\rm GW}$ = - (average energy loss rate from the binary)

Back-reaction on the orbit

- Orbital energy:

$$E_{\rm orbit} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{\mu M}{r} = \frac{1}{2}\mu(M\Omega)^{2/3} - \mu(M\Omega)^{2/3} = -\frac{1}{2}\mu(M\Omega)^{2/3}$$

$$E_{\rm orbit} = -\frac{1}{2}\mu(2\pi)^{2/3}M^{2/3}T^{-2/3}$$

$$T = \frac{2\pi}{\Omega}$$ Orbital period

- Energy balance:

$$\frac{dE_{\rm orbit}}{dt} = -P_{\rm GW}$$

$$\frac{dE_{\rm orbit}}{dt} = \frac{1}{3}\mu(2\pi)^{2/3}M^{2/3}T^{-5/3}\frac{dT}{dt}$$

$$P_{\rm GW} = \frac{32}{5}\mu^2 M^{4/3}(2\pi)^{10/3}T^{-10/3}$$

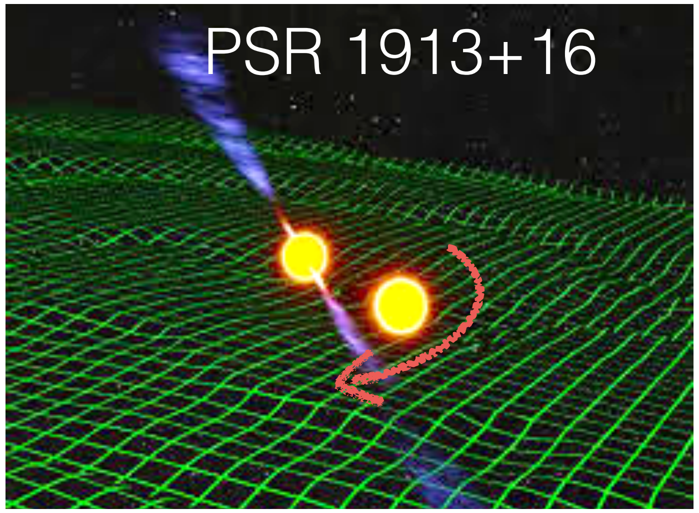
$$\frac{dT}{dt} = -\frac{96}{5}(2\pi)^{8/3}M^{2/3}\mu T^{-5/3}$$

Rate of change in orbital period due to GW losses

Orbital decay of the Hulse-Taylor binary pulsar

Measurements:

Pulsar mass	m_1=1.4414 M⊙
Companion mass	m_2=1.3867 M⊙
Orbital period	T=0.322997448930 days
Orbital decay rate	dT/dt=-75.9μs/yr=**-2.4 x 10^{-12}**
Orbital eccentricity	e=0.6171338



Our results for dT/dt assumed a circular-orbit binary. Extra factor for eccentric orbits:

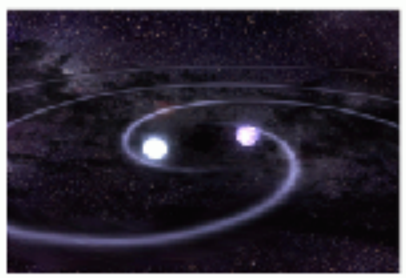
$$\frac{dT}{dt} = -\frac{96}{5}(2\pi)^{8/3}M^{2/3}\mu T^{-5/3} \times \left[\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}}\right]$$

$$\approx -2.4 \times 10^{-12}$$

Approximately how long until merger?

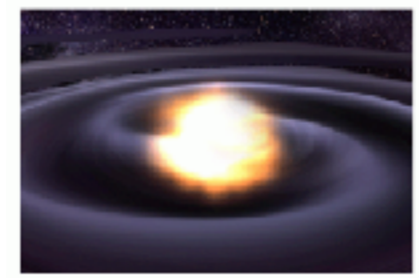
Write as: $$\frac{dT}{dt} = -\alpha T^{-5/3} \qquad \alpha = \frac{96}{5}(2\pi)^{8/3} M^{2/3} \mu$$ (circular orbits)

T=T_{now}



t=0

T~0



t=t_{merge}

$$\frac{1}{\alpha}\int_{\mathrm{T_{now}}}^{0} dT\, T^{5/3} = \int_0^{t_{\rm merge}} dt$$

$$t_{\rm merge} = \frac{3}{8\alpha} T_{\rm now}^{8/3}$$

$\approx 10^8$ yrs for the Hulse-Taylor binary (when including eccentricity)

GW strain amplitude from a Newtonian binary

Had: $$h_{ij} = \frac{2}{D}\ddot{Q}_{ij}$$

Distance to source

Already computed $$\ddot{Q}_{ij} = 2\mu r^2\Omega^2\left(\phi^i\phi^j - n^i n^j\right)$$

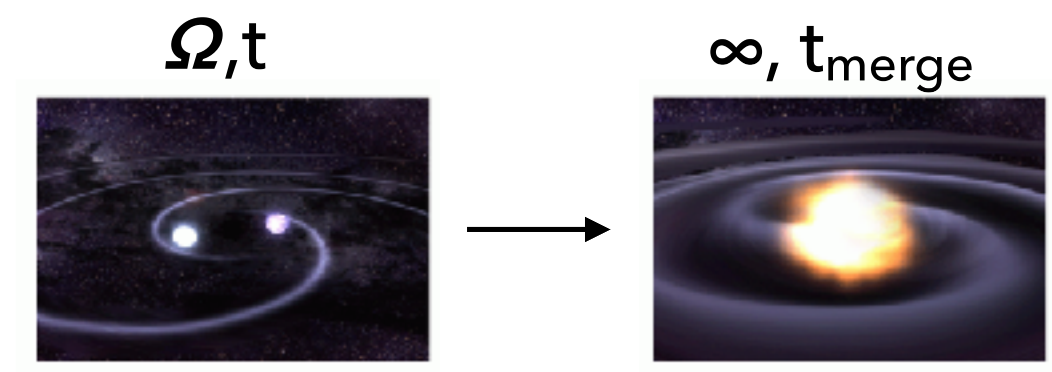
$$= -2\mu M^{2/3}\Omega^{2/3}\begin{pmatrix}\cos(2\varphi) & \sin(2\varphi) & 0\\ \sin(2\varphi) & -\cos(2\varphi) & 0\\ 0 & 0 & 0\end{pmatrix}$$

GW strain amplitude from a Newtonian binary

GW phase evolution: start from result for orbital decay

$$\frac{dT}{dt} = -\frac{96}{5}(2\pi)^{8/3}M^{2/3}\mu T^{-5/3} \longrightarrow \frac{d\Omega}{dt} = \frac{96}{5}\mu M^{2/3}\Omega^{11/3}$$

Integrate: $$\Omega = \frac{5^{3/8}}{8\mu^{3/8}M^{1/4}\left(t_{\text{merge}} - t\right)^{3/8}}$$

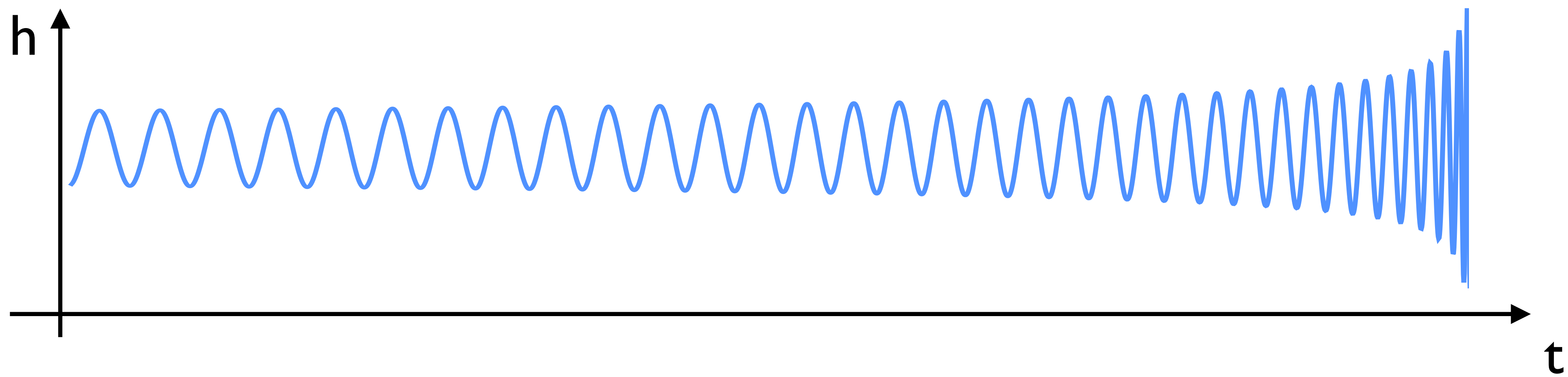


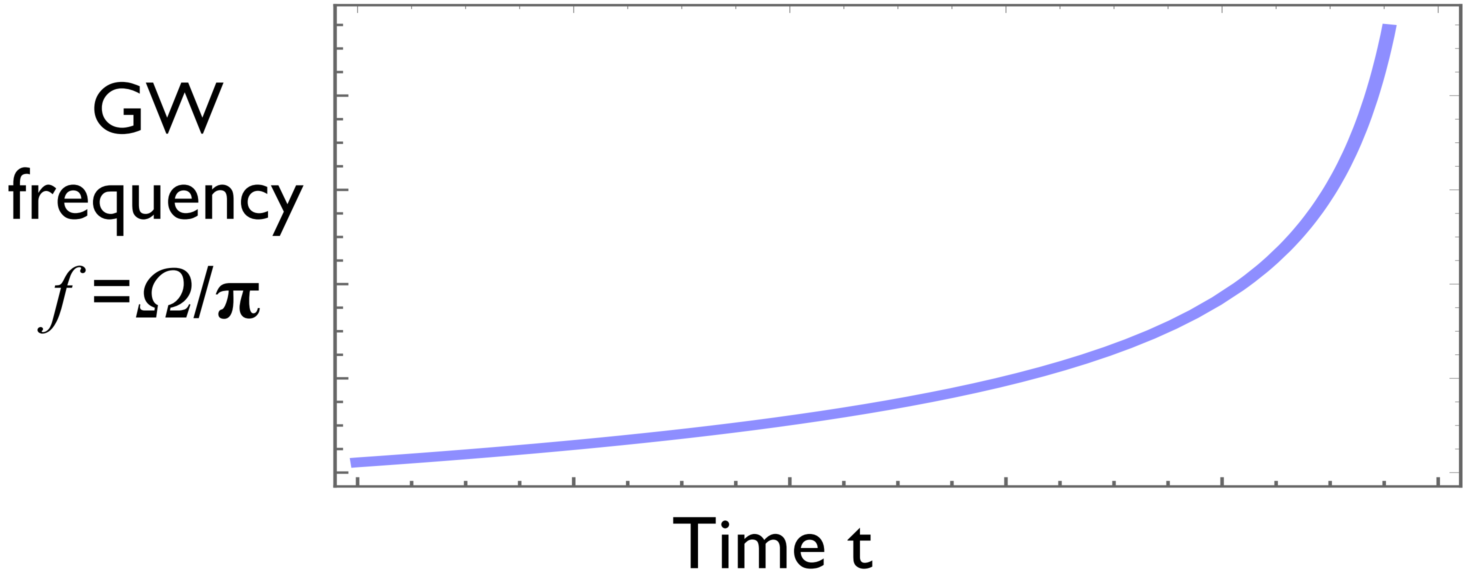
Integrate once more: $$\varphi = \varphi_{\text{merge}} - \frac{\left(t_{\text{merge}} - t\right)^{5/8}}{5^{5/8}\mu^{3/8}M^{1/4}}$$

GW strain amplitude from a Newtonian binary

Finally: $$h \sim -\frac{\mathcal{M}}{2D}\left(\frac{t_{\mathrm{merge}}-t}{5\mathcal{M}}\right)^{-1/4}\cos\left[2\varphi_{\mathrm{merge}}-2\left(\frac{t_{\mathrm{merge}}-t}{5\mathcal{M}}\right)^{5/8}\right]$$

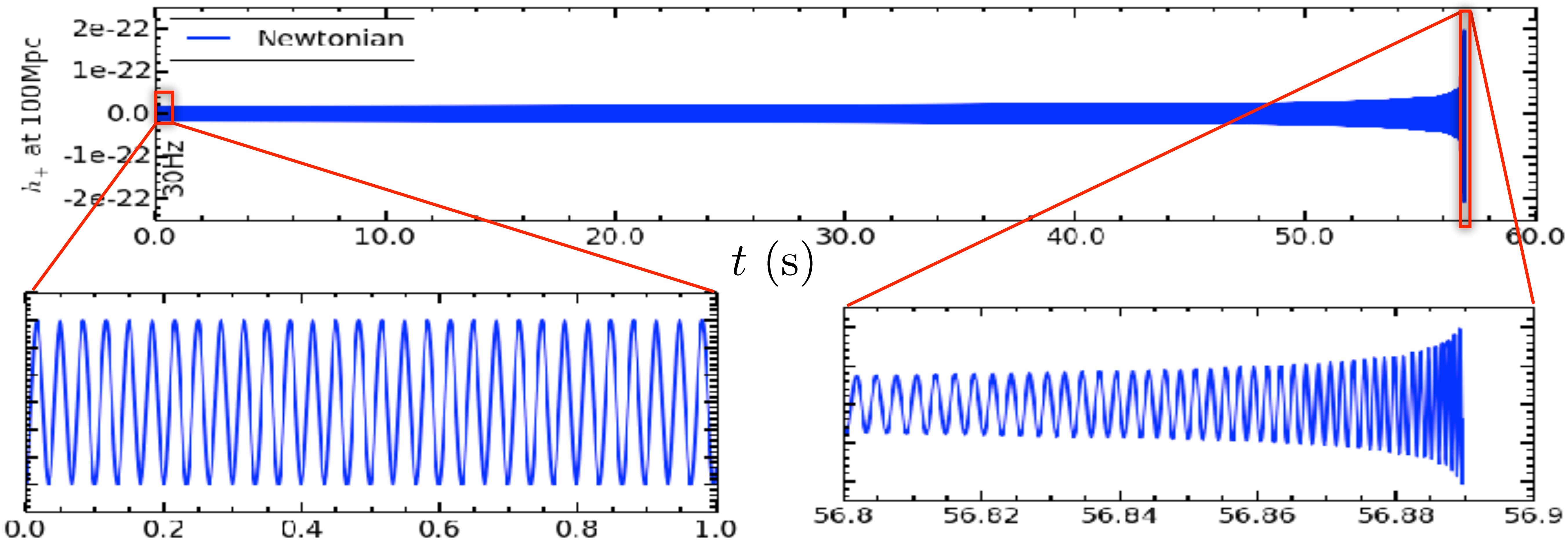
$$\mathcal{M} = \mu^{3/5}M^{2/5}$$ Chirp mass





$$f = \frac{5^{3/8}}{8\pi\mathcal{M}^{5/8}(t_{\mathrm{merge}}-t)^{3/8}}$$

Corrections to Newtonian waveform

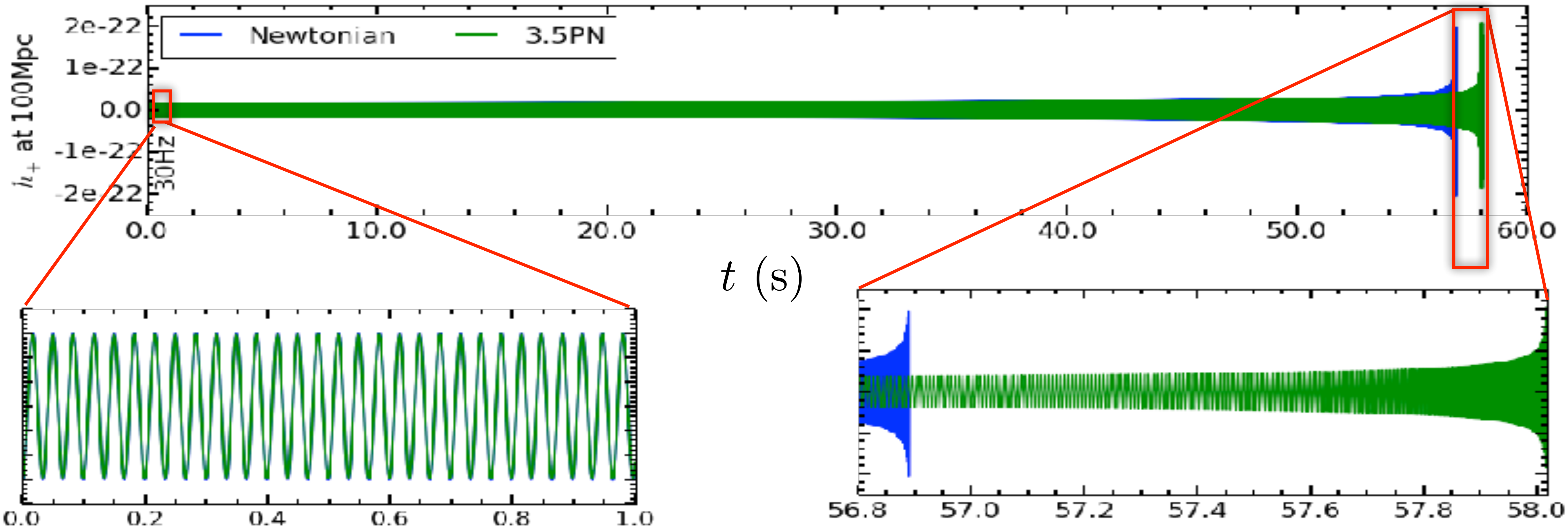


credit: Ben Lackey

$$\mathrm{Phase}(t) = 0\mathrm{PN}(t; \mathcal{M})$$

PN: post-Newtonian, relativistic corrections to Newtonian dynamics & waveforms

Post-Newtonian waveform (nonspinning binary)



credit: Ben Lackey

$$\text{Phase}(t) = 0\text{PN}(t;\mathcal{M})\,[1 + \underbrace{1\text{PN}(t;\boldsymbol{\nu})}_{(v/c)^2} + \cdots + \underbrace{3.5\text{PN}(t;\boldsymbol{\nu})}_{(v/c)^7}]$$

Depends not only on chirp mass but also on symmetric mass ratio:

$$\boldsymbol{\nu} = \frac{m_1 m_2}{(m_1+m_2)^2} = \frac{\mu}{M}$$