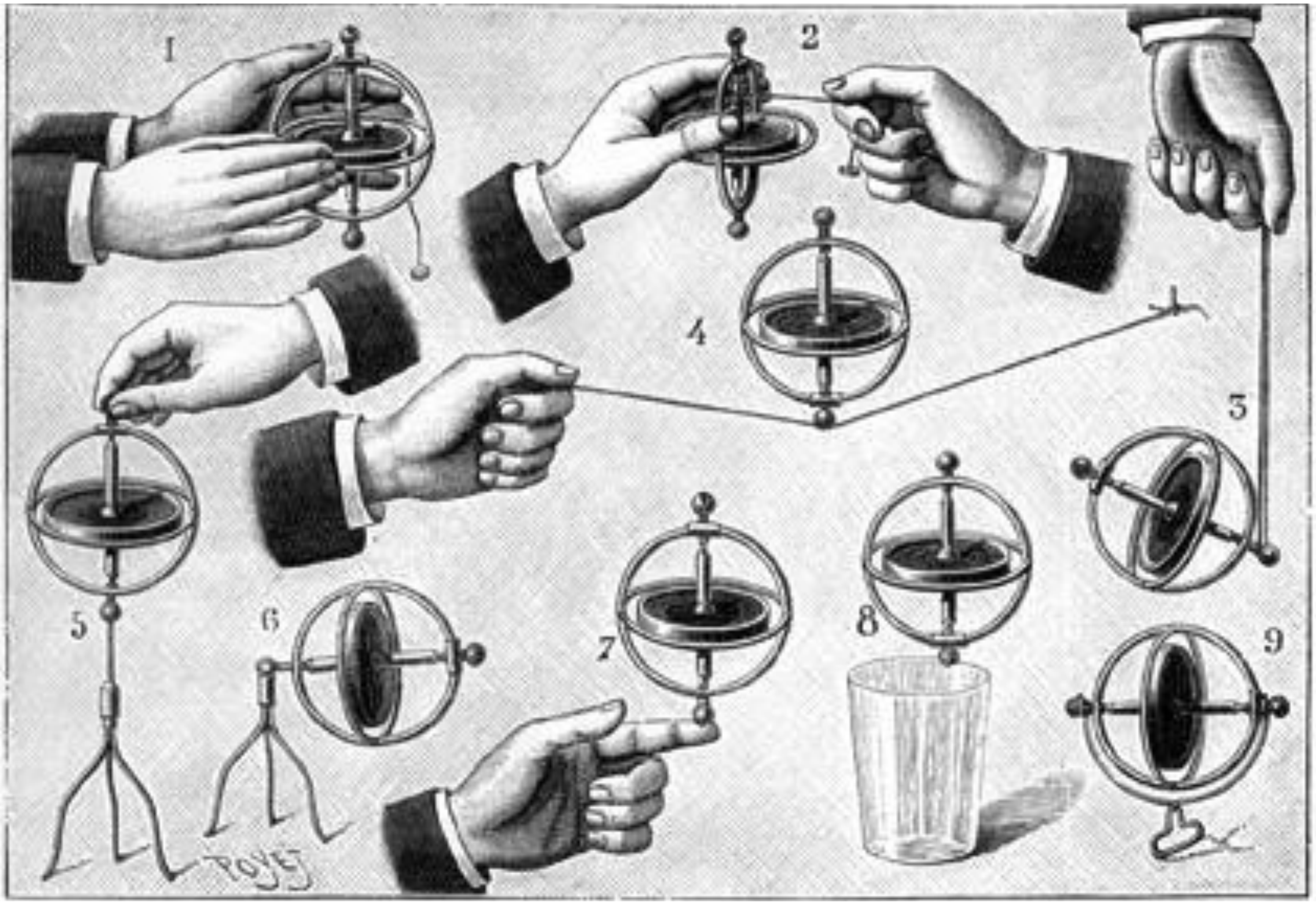
Collective Neutrino Flavor Oscillations



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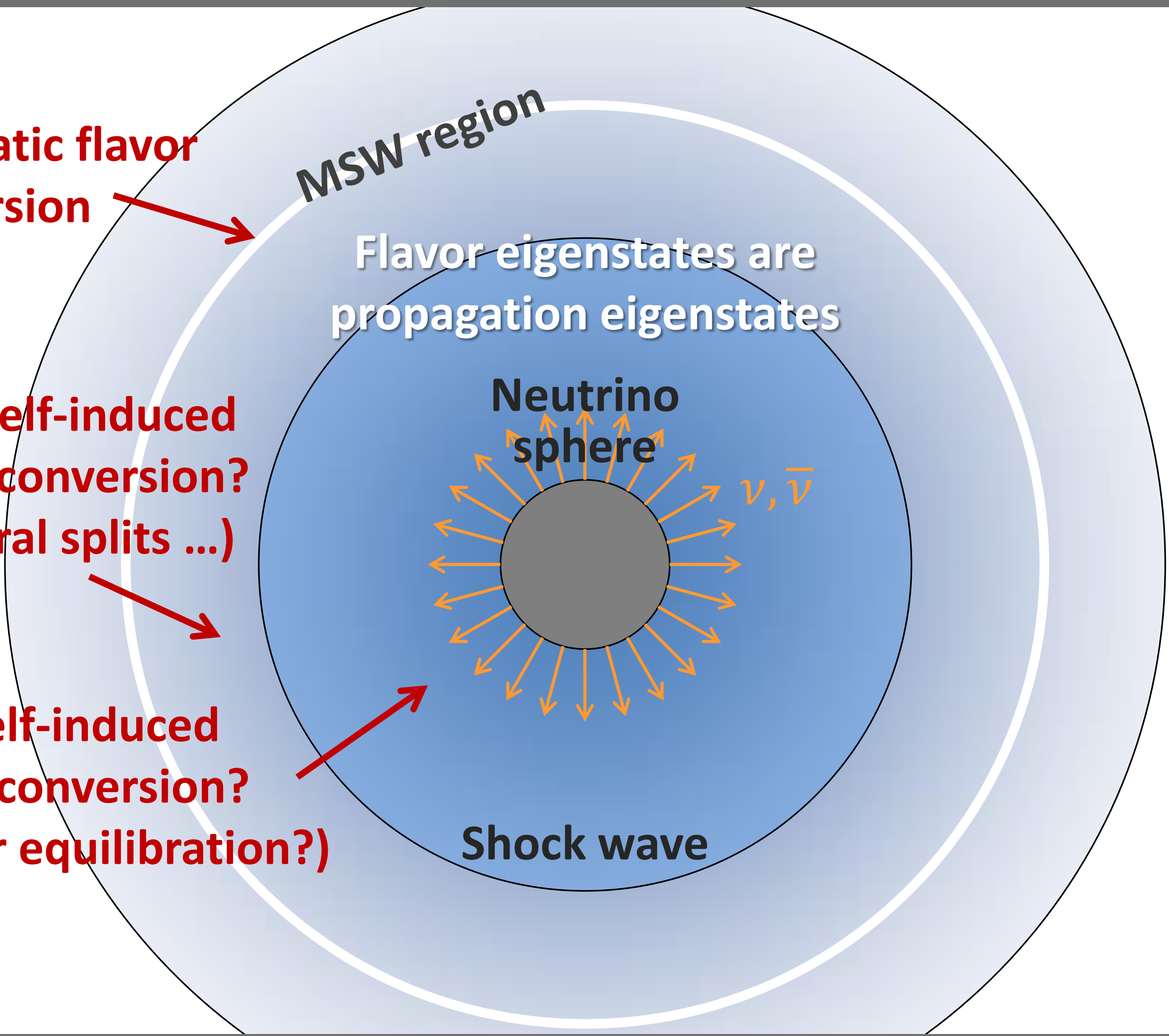
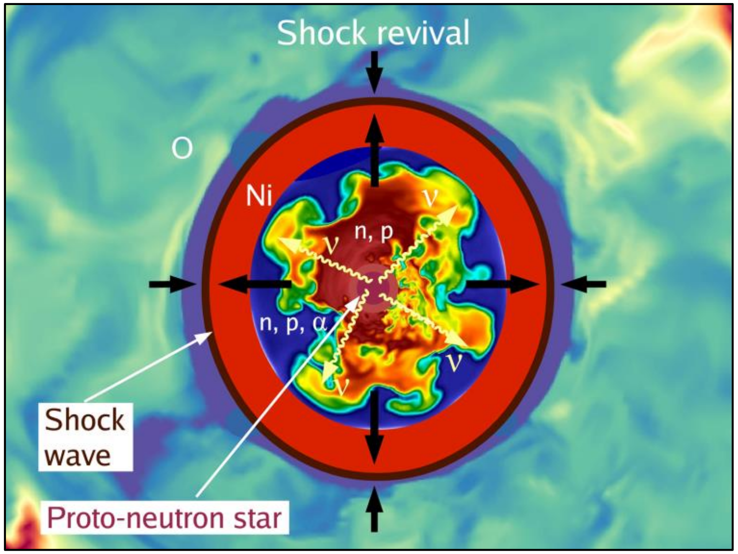
Flavor Conversion in Core-Collapse Supernovae

- Adiabatic flavor conversion
- Slow self-induced flavor conversion? (Spectral splits …)
- Fast self-induced flavor conversion? (Flavor equilibration?)



Why worry about detailed neutrino transport?



- Explosion mechanism:
  Shock-wave revival by nu energy deposition
- Nucleosynthesis in neutrino irradiated outflows in SNe and NS-mergers depends on flavor (beta reactions!)
- Signal interpretation of DSNB and next nearby SN
- **Collective flavor conversion: interesting theoretical problem in its own right**

Qualitative Scenarios

Neutrino source with nontrivial flavor structure
(depending on energy and angle distribution)

After leaving the source region:

- Flavor non-equilibrium survives in simple calculable ways (spectral swaps, other signatures, or no nu-nu effect at all)?
- Equilibration within constraints of conservation laws caused by instabilities and/or random matter effects?
- No simple answer, need case by case numerical simulation? How to implement in practice?

Kinetic Equation for Neutrino Transport

Flavor-dependent phase-space densities (occupation number matrices)

$$\varrho = \begin{pmatrix} f_{\nu_e} & f_{\langle\nu_e|\nu_\mu\rangle} & f_{\langle\nu_e|\nu_\tau\rangle} \\ f_{\langle\nu_\mu|\nu_e\rangle} & f_{\nu_\mu} & f_{\langle\nu_\mu|\nu_\tau\rangle} \\ f_{\langle\nu_\tau|\nu_e\rangle} & f_{\langle\nu_\tau|\nu_\mu\rangle} & f_{\nu_\tau} \end{pmatrix}$$

Diagonal: Usual occupation numbers
Off-diag: Flavor coherence information

and similar for $\bar{\nu}$

Transport equation

$$\underbrace{\left(\partial_t + \vec{v}\cdot\vec{\nabla}_x\right.}_{\text{Streaming}} \underbrace{\left.- \vec{F}\cdot\vec{\nabla}_p\right)}_{\substack{\text{Gravitational forces} \\ \text{(redshift, deflection)}}} \varrho(t,\vec{x},\vec{p}) = \underbrace{-\mathrm{i}\,[\mathcal{H}(t,\vec{x},\vec{p}),\varrho(t,\vec{x},\vec{p})]}_{\substack{\text{Flavor oscillations} \\ (\text{vacuum, matter, } \nu\nu)}} + \underbrace{\mathcal{C}[\varrho(t,\vec{x},\vec{p})]}_{\text{Collisions}}$$

β

- $e^- + p \rightleftharpoons n + \nu_e$
- $e^+ + n \rightleftharpoons p + \bar{\nu}_e$
- $e^- + A \rightleftharpoons \nu_e + A^*$
- $\nu + n, p \rightleftharpoons \nu + n, p$
- $\nu + A \rightleftharpoons \nu + A$
- $\nu + e^\pm \rightleftharpoons \nu + e^\pm$
- $N + N \rightleftharpoons N + N + \nu + \bar{\nu}$
- $e^+ + e^- \rightleftharpoons \nu + \bar{\nu}$
- $\nu_x + \nu_e, \bar{\nu}_e \rightleftharpoons \nu_x + \nu_e, \bar{\nu}_e$
  ($\nu_x = \nu_\mu, \bar{\nu}_\mu, \nu_\tau$, or $\bar{\nu}_\tau$)
- $\nu_e + \bar{\nu}_e \rightleftharpoons \nu_{\mu,\tau} + \bar{\nu}_{\mu,\tau}$

Typical approximations in numerical simulations:

- **Reducing 6+1 dimensions**
  (Angular moments, ray-by-ray, …)
- **No gravitational deflection**
- **No flavor conversion (large matter effect!)**
- **No muons**
- **3-species transport: $\nu_e, \bar{\nu}_e, \nu_x$**

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Streaming — Gravitational forces (redshift, deflection) — Flavor oscillations (vacuum, matter, $\nu\nu$) — Collisions

Flavor evolution governed by "Hamiltonian matrix" (here for 2 flavors)

$$\mathcal{H} = \underbrace{\frac{\Delta m^2}{4E}\begin{pmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{pmatrix}}_{\text{Vacuum oscillations}} + \underbrace{\sqrt{2}G_\mathrm{F}\begin{pmatrix} n_e & 0 \\ 0 & 0\end{pmatrix}}_{\text{MSW effect}} + \underbrace{\sqrt{2}G_\mathrm{F}\int\frac{d^3\vec{p}}{(2\pi)^3}(\varrho+\overline{\varrho})}_{\text{Nu-nu interactions, nus feed back on each other}}$$

Vacuum oscillations — MSW effect — Nu-nu interactions, nus feed back on each other

- **Flavor evolution is caused by off-diagonal $\mathcal{H}$ elements (vacuum or nu-nu term)**
- **For $\Delta m^2 = 0$, nu-nu term can still cause run-away modes!**

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5000 citations

Neutrino oscillations in matter

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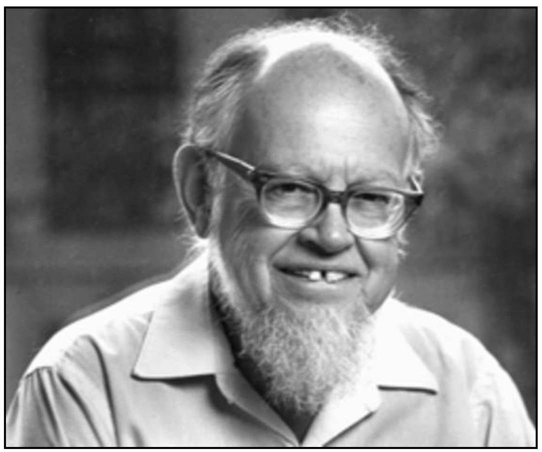
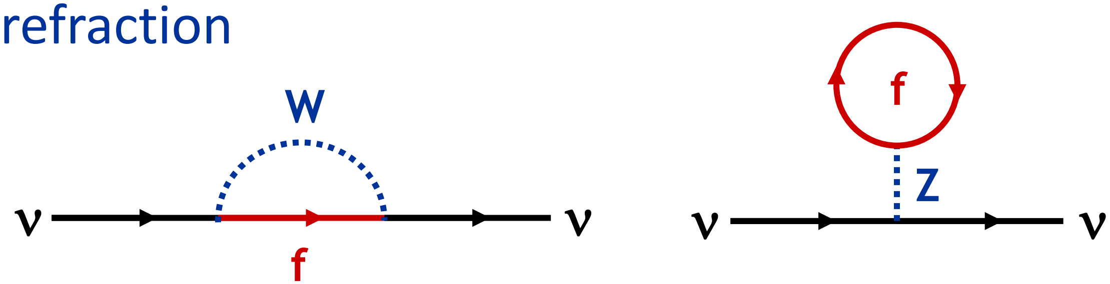
(Received 6 October 1977; revised manuscript received 5 December 1977)

The effect of coherent forward scattering must be taken into account when considering the oscillations of neutrinos traveling through matter. In particular, for the case of massless neutrinos for which vacuum oscillations cannot occur, oscillations can occur in matter if the neutral current has an off-diagonal piece connecting different neutrino types. Applications discussed are solar neutrinos and a proposed experiment involving transmission of neutrinos through 1000 km of rock.



Lincoln Wolfenstein

Neutrinos in a medium suffer flavor-dependent refraction



$$V_{\rm weak} = \sqrt{2} G_{\rm F} \times \begin{cases} N_{\rm e} - N_{\rm n}/2 & \text{for } \nu_{\rm e} \\ -N_{\rm n}/2 & \text{for } \nu_{\mu} \end{cases}$$

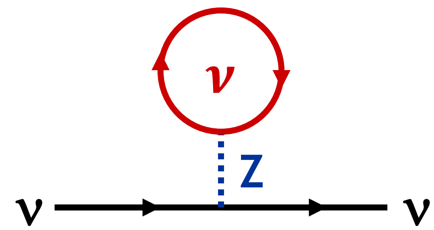
Typical density of Earth: 5 g/cm^3

$$\Delta V_{\rm weak} \approx 2 \times 10^{-13}\ {\rm eV} = 0.2\ {\rm peV}$$

Flavor-Off-Diagonal Refractive Index

2-flavor neutrino evolution as an effective 2-level problem

$$\mathrm{i}\,\frac{\partial}{\partial t}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = \mathcal{H}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix}$$



Effective mixing Hamiltonian

$$\mathcal{H} = \frac{\mathcal{M}^2}{2E} + \sqrt{2}G_\mathrm{F}\begin{pmatrix} N_e - \frac{N_n}{2} & 0 \\ 0 & -\frac{N_n}{2}\end{pmatrix} + \sqrt{2}G_\mathrm{F}\begin{pmatrix} N_{\nu_e} & N_{\langle\nu_e|\nu_\mu\rangle} \\ N_{\langle\nu_\mu|\nu_e\rangle} & N_{\nu_\mu}\end{pmatrix}$$

Mass term in flavor basis: causes vacuum oscillations

Wolfenstein's weak potential, causes MSW "resonant" conversion together with vacuum term

Flavor-off-diagonal potential, caused by flavor oscillations. (J.Pantaleone, PLB 287:128,1992)

Flavor oscillations feed back on the Hamiltonian: Nonlinear effects!

Self-Induced Flavor Conversion

Flavor conversion (vacuum or MSW) for a neutrino of given momentum p

- Requires lepton flavor violation by masses and mixing

$$\nu_e(p) \to \nu_\mu(p)$$

$$\boxed{\frac{\Delta m^2_{\mathrm{atm}}}{2E} = 10^{-10}\,\mathrm{eV} = 0.5\ \mathrm{km}^{-1}}$$

Pair-wise flavor exchange by ν–ν refraction (forward scattering)

- No net flavor change of pair
- Requires dense neutrino medium (collective effect of interacting neutrinos)
- **Can even occur without masses/mixing (and then does not depend on $\Delta m^2/2E$)**
- Familiar as neutrino pair process $\mathcal{O}(G_\mathrm{F}^2)$ Here as coherent refractive effect $\mathcal{O}(G_\mathrm{F})$

$$\nu_e(p) + \bar\nu_e(k) \to \nu_\mu(p) + \bar\nu_\mu(k)$$

$$\nu_e(p) + \nu_\mu(k) \to \nu_\mu(p) + \nu_e(k)$$

$$\boxed{\sqrt{2}G_\mathrm{F}n_\nu = 10^{-5}\,\mathrm{eV} = 0.5\ \mathrm{cm}^{-1}}$$

$$E = 12.5\ \mathrm{MeV}$$
$$R = 80\ \mathrm{km}$$
$$L_\nu = 40 \times 10^{51}\,\mathrm{erg/s}$$

Brief history of collective flavor oscillations

1992–2005	Synchronised oscillations by nu-nu interaction (pioneers)
2006–2009	Self-induced conversion, bipolar oscillations, flavor pendulum, (theory of) spectral swaps/splits, multiple splits, 3-flavor effects, … (goldrush period)
2009–2012	Multi-angle matter effect, halo effect, linearised stability analysis, spurious instabilities, … (reality begins to strike)
2013–2015	Spontaneous spatial symmetry breaking
2015	Non-stationary modes (“pulsating modes”)
2016–now	Full space-time dependence acknowledged, dispersion relation in the linearised regime, normal-mode analysis
2016–now	Fast flavor conversion, supported by nontrivial angle distribution, triggering, initial/boundary conditions, nonlinear regime, …
2011–now	Many-body Hamiltonian, exact solutions, quantum vs. classical, …

Different Ways to Describe Flavor Oscillations

Two-flavor oscillations described by a Schrödinger equation for "flavor spinor"

$$i\partial_t\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = \mathcal{H}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix}$$

- Flavor amplitudes for single-neutrino wave function
- or field operators (flavor oscillation ~ Bogoliubov transformation)

in vacuum with $\mathcal{H} = \frac{\Delta m^2}{2E}\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}$

Equivalent commutator equation in terms of "density matrix" ϱ

$$i\partial_t\varrho = [\mathcal{H},\varrho]$$

$$\varrho = \begin{pmatrix}\langle\nu_e|\nu_e\rangle & \langle\nu_e|\nu_\mu\rangle\\ \langle\nu_\mu|\nu_e\rangle & \langle\nu_\mu|\nu_\mu\rangle\end{pmatrix}$$

- Single-particle density matrix
- or field bilinears
- or expectation values of field bilinears (occupation numbers) ("matrix of densities")

Expand Hermitean 2×2 matrices in Pauli matrices

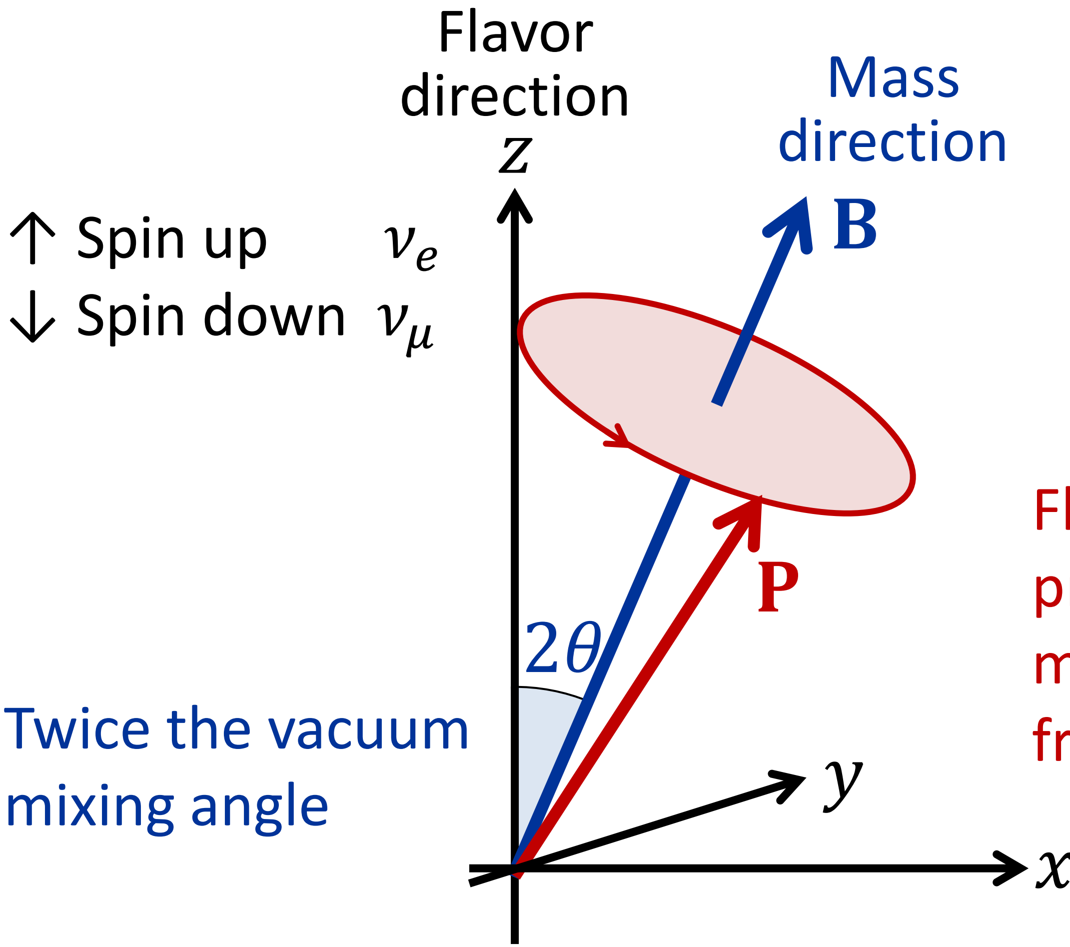
$\rho = \mathrm{Tr}(\rho) + \frac{1}{2}\mathbf{P}\cdot\boldsymbol{\sigma}$ and $\mathcal{H} = \frac{\Delta m^2}{2E}\mathbf{B}\cdot\boldsymbol{\sigma}$ with $\mathbf{B} = (\sin 2\theta, 0, \cos 2\theta)$

Equivalent spin-precession form

$\dot{\mathbf{P}} = \omega\mathbf{B}\times\mathbf{P}$ with $\omega = \frac{\Delta m^2}{2E}$

$\mathbf{P}$ is "polarization vector" or "Bloch vector" (real numbers or Hermitean field bilinears)

Flavor Oscillation as Spin Precession



Flavor polarization vector precesses around the mass direction with frequency $\omega = \Delta m^2/2E$

Adding Matter

Schrödinger equation including matter

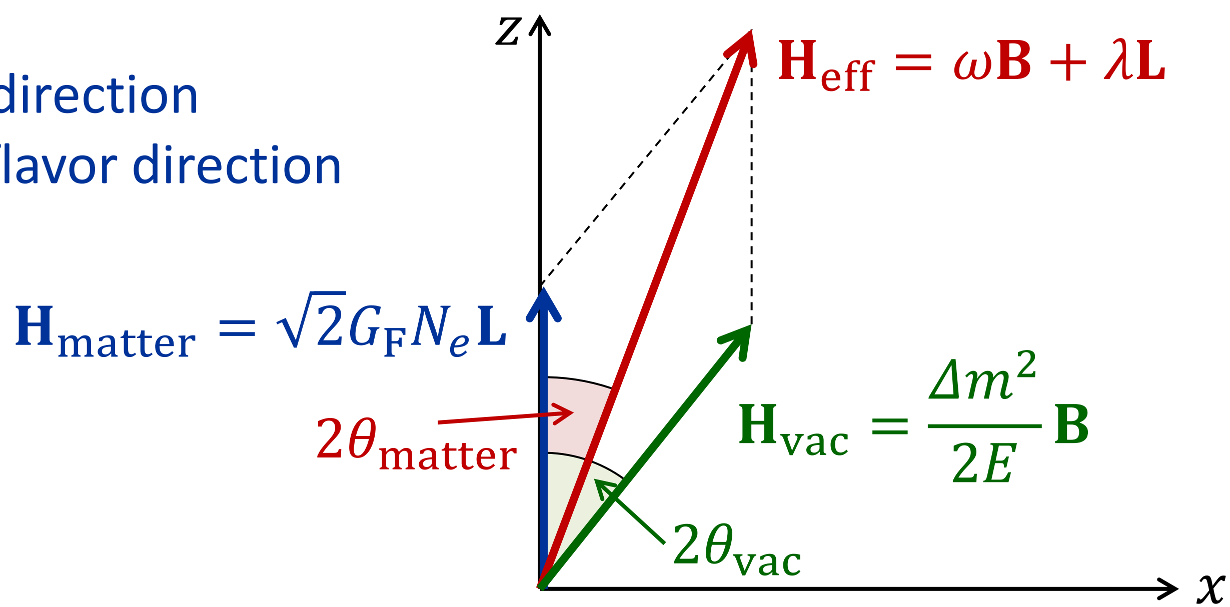
$$\mathrm{i}\partial_t \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \left[\frac{\Delta m^2}{2E}\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} + \sqrt{2}G_F \begin{pmatrix} N_e - \frac{N_n}{2} & 0 \\ 0 & -\frac{N_n}{2} \end{pmatrix}\right]\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}$$

Corresponding spin-precession equation

$$\dot{\mathbf{P}} = \underbrace{(\omega\mathbf{B} + \lambda\mathbf{L})}_{\mathbf{H}_{\mathrm{eff}}} \times \mathbf{P} \quad \text{with} \quad \omega = \Delta m^2/2E \quad \text{and} \quad \lambda = \sqrt{2}G_{\mathrm{F}}N_e$$

$\mathbf{B}$ unit vector in mass direction

$\mathbf{L} = \mathbf{e}_z$ unit vector in flavor direction

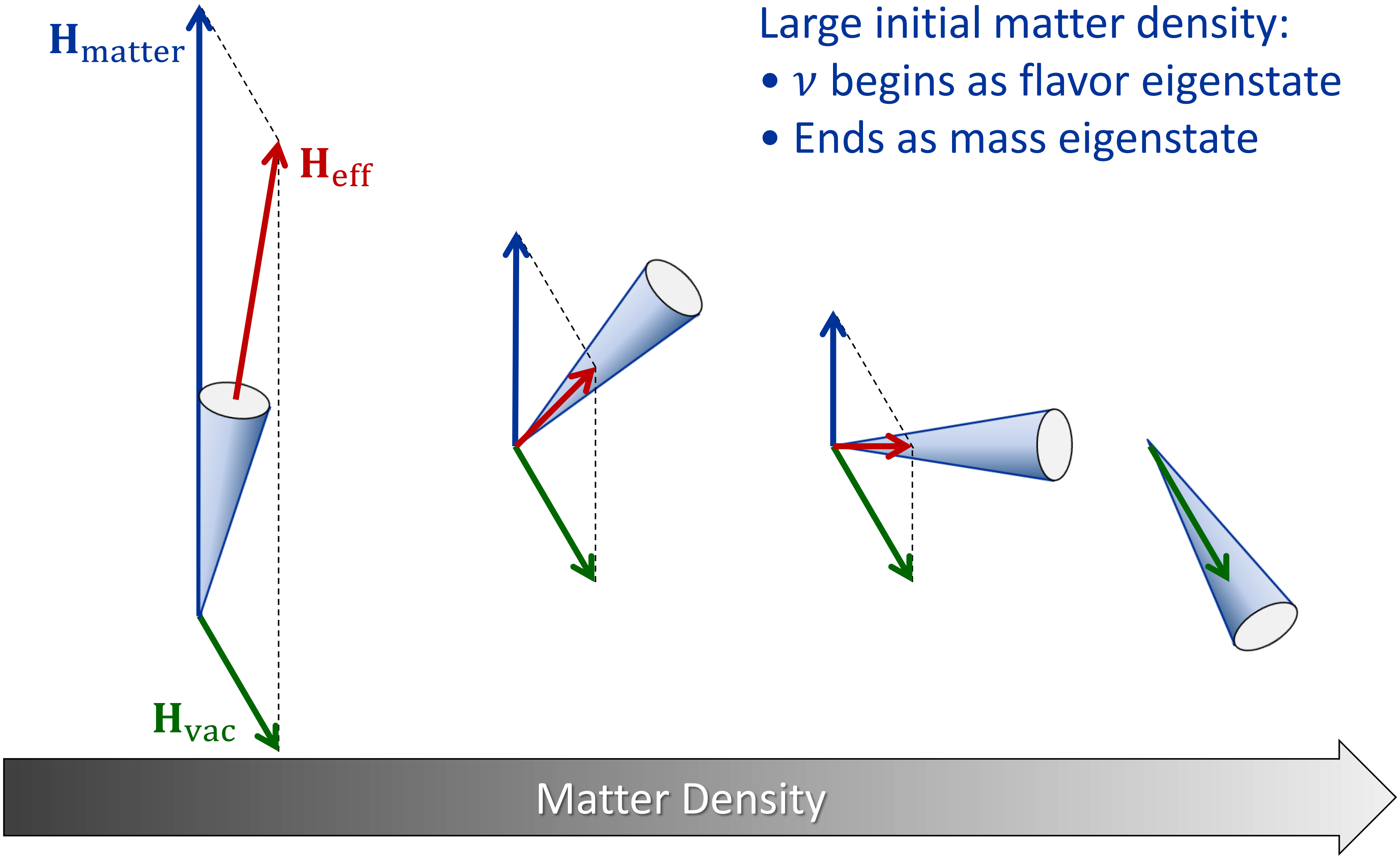


MSW Effect

Adiabatically decreasing density: Precession cone follows $\mathbf{H}_{\rm eff}$

Large initial matter density:

- ν begins as flavor eigenstate
- Ends as mass eigenstate



Adding Neutrino-Neutrino Interactions

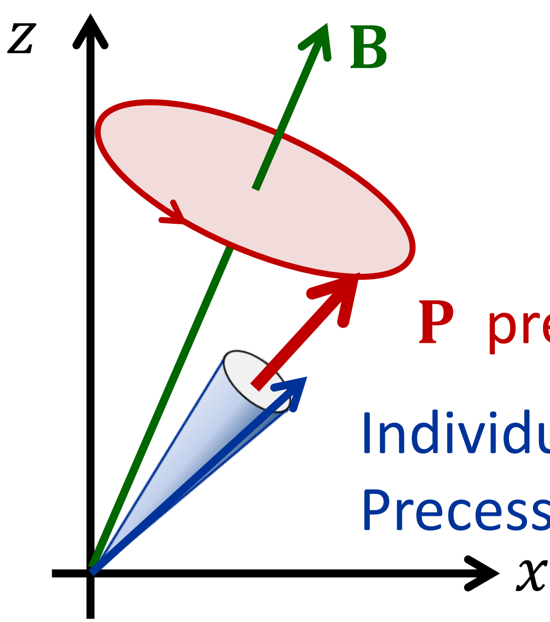
Precession equation for each ν mode with energy E, i.e. $\omega = \Delta m^2/2E$

$$\dot{\mathbf{P}}_\omega = \underbrace{(\omega\mathbf{B} + \lambda\mathbf{L} + \mu\mathbf{P})}_{\mathbf{H}_{\rm eff}} \times \mathbf{P}_\omega \quad \text{with} \quad \lambda = \sqrt{2}G_{\rm F}N_e \quad \text{and} \quad \mu = \sqrt{2}G_{\rm F}N_\nu$$

Total flavor spin of entire ensemble

$$\mathbf{P} = \sum_\omega \mathbf{P}_\omega \qquad \text{normalize} \qquad |\mathbf{P}_{\mathbf{t=0}}| = 1$$

Individual spins do not remain aligned – feel "internal" field $\mathbf{H}_{\nu\nu} = \mu\mathbf{P}$



Synchronized oscillations for large neutrino density $\mu \gg \delta\omega$

$\mathbf{P}$ precesses with $\omega_{\rm sync}$ for large ν density

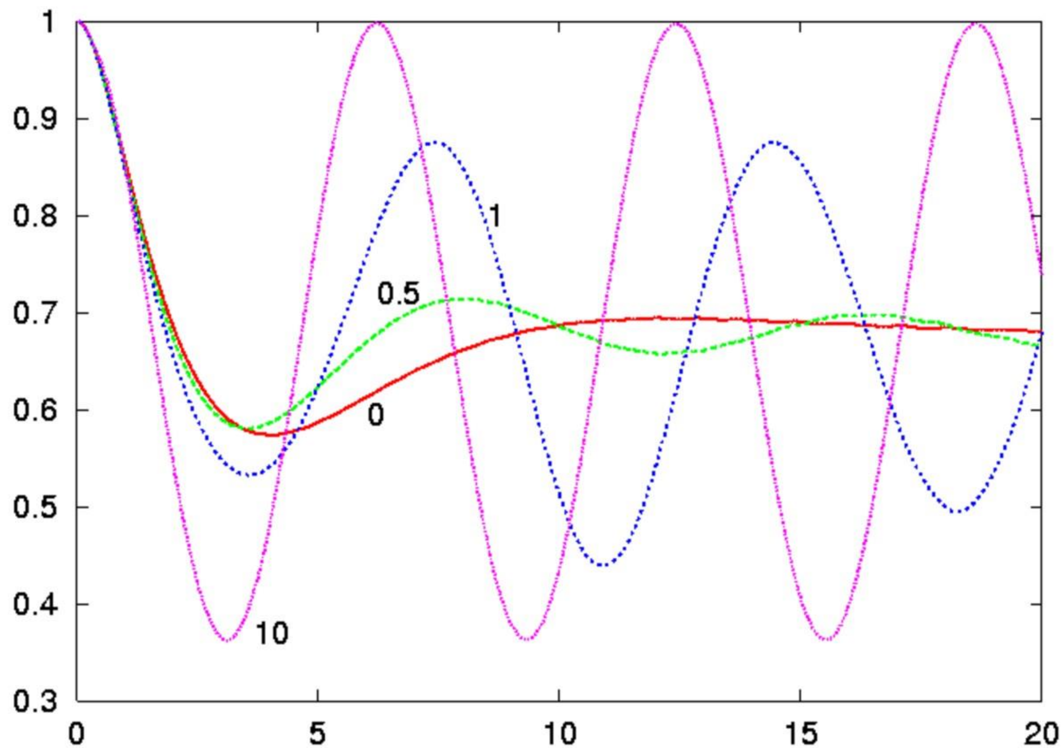
Individual $\mathbf{P}_\omega$ "trapped" on precession cones

Precess around $\mathbf{P}$ with frequency $\sim \mu$

Synchronising Oscillations by Neutrino Interactions

- Vacuum oscillation frequency depends on energy $\omega = \Delta m^2/2E$
- Ensemble with broad spectrum quickly decoheres kinematically
- ν-ν interactions "synchronize" the oscillations: $\omega_{\rm sync} = \langle \Delta m^2/2E \rangle$

Average e-flavor component of polarization vector



Time

Pastor, Raffelt & Semikoz, hep-ph/0109035

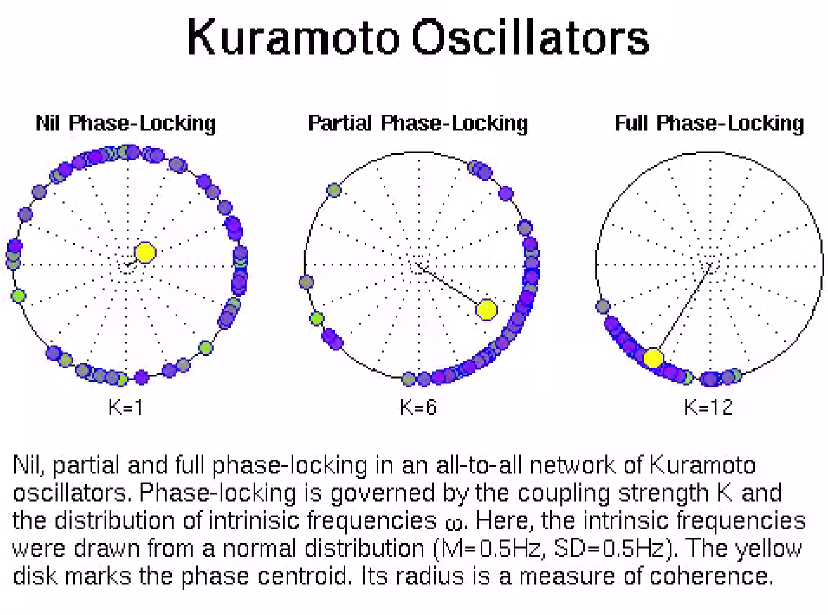
Connection to Kuramoto Model

Spontaneous emergence of flavor polarisation (“synchronisation”)?

No, initial polarisation can (partly) survive, not spontaneously appear.

J.Pantaleone, Stability of incoherence in an isotropic gas of oscillating neutrinos, PRD 58:073002 (1998)

Kuramoto model (1975) to mimic “synchronisation” of oscillators in nature (of order 10^4 citations in Google Scholar)



Animation from Wikipedia

N phase-coupled oscillators

$$\frac{d\phi_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\phi_j - \phi_i)$$

Neutrinos:
Essentially cosine coupling

Two Spins Interacting with a Dipole Force

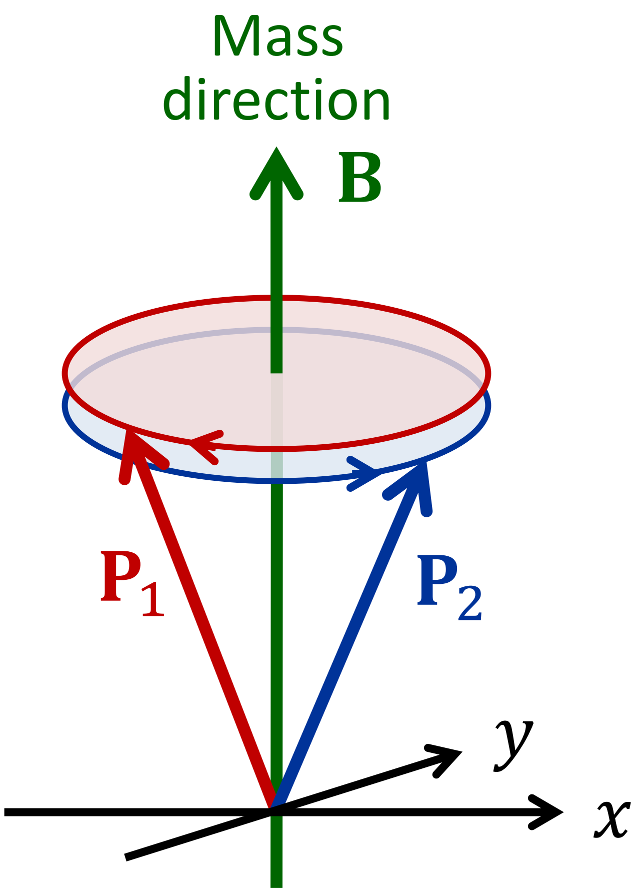
Simplest system showing ν-ν effects:
Isotropic neutrino gas with 2 energies E_1 and E_2, no ordinary matter

$$\dot{\mathbf{P}}_1 = (\omega_1 \mathbf{B} + \mu \mathbf{P}) \times \mathbf{P}_1 \quad \text{with} \quad \mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 \quad \text{and} \quad \omega_{1,2} = \Delta m^2 / 2E_{1,2}$$
$$\dot{\mathbf{P}}_2 = (\omega_2 \mathbf{B} + \mu \mathbf{P}) \times \mathbf{P}_2$$

Go to "co-rotating frame" around $\mathbf{B}$ direction

$$\dot{\mathbf{P}}_1 = (\omega_c \mathbf{B} - \omega \mathbf{B} + \mu \mathbf{P}) \times \mathbf{P}_1$$
$$\dot{\mathbf{P}}_2 = (\omega_c \mathbf{B} + \omega \mathbf{B} + \mu \mathbf{P}) \times \mathbf{P}_2$$

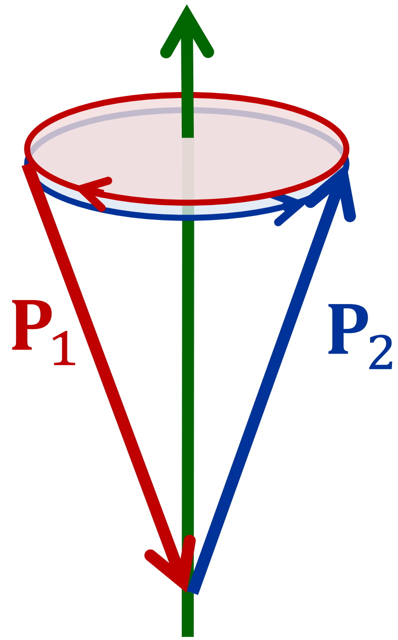
with $\omega_c = \frac{1}{2}(\omega_2 + \omega_1)$ and $\omega = \frac{1}{2}(\omega_2 - \omega_1)$

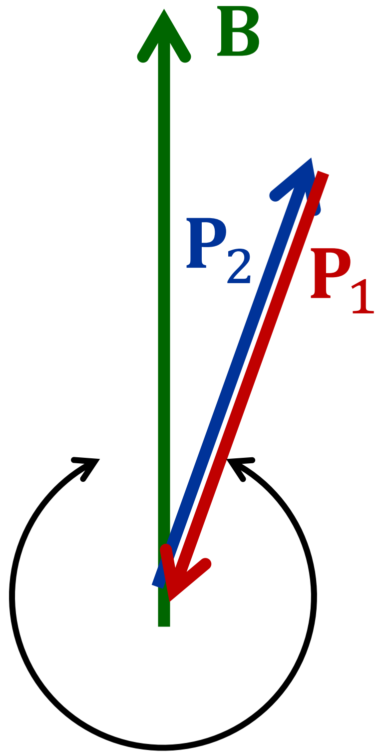


No interaction ($\mu = 0$)
$\mathrm{P}_{1,2}$ precess in opposite directions
Strong interactions ($\mu \to \infty$)
$\mathrm{P}_{1,2}$ stuck to each other
(no motion in co-rotating frame,
perfectly synchronized in lab frame)

Two Spins with Opposite Initial Orientation

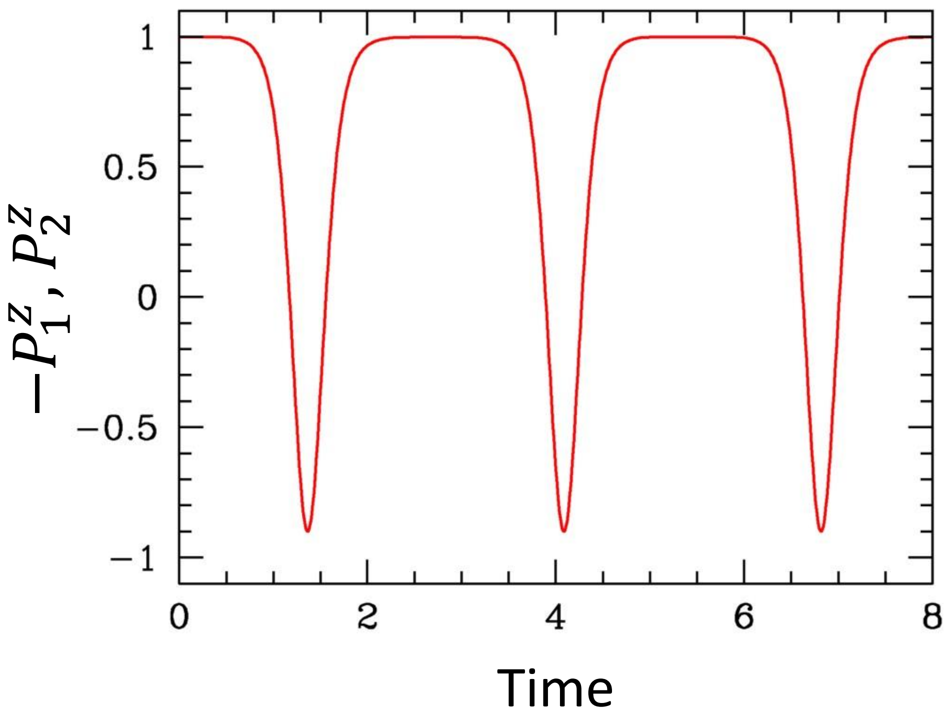
No interaction ($\mu = 0$)
Free precession in opposite directions





Strong interaction
($\mu \to \infty$)
Pendular motion

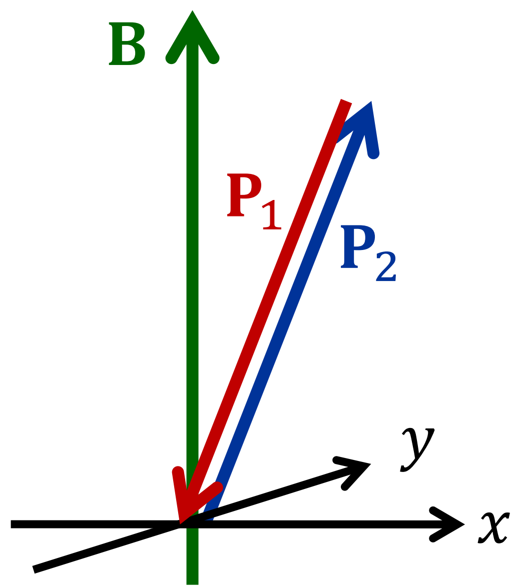
Even for very small mixing angle, large-amplitude flavor oscillations



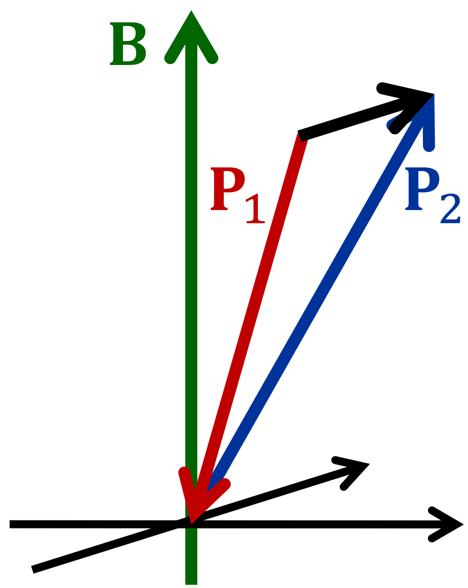
Instability in Flavor Space

Two-mode example in co-rotating frame, initially $\mathbf{P}_1 = \downarrow$, $\mathbf{P}_2 = \uparrow$ (flavor basis)

$$\dot{\mathbf{P}}_1 = [-\omega\mathbf{B} + \mu\,(\mathbf{P_1} + \mathbf{P_2})] \times \mathbf{P_1}$$
$$\dot{\mathbf{P}}_2 = [+\omega\mathbf{B} + \mu\,\underbrace{(\mathbf{P_1} + \mathbf{P_2})}_{\text{0 initially}}] \times \mathbf{P_2}$$



- Initially aligned in flavor direction and $\mathbf{P} = 0$
- Free precession $\pm\omega$



$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$

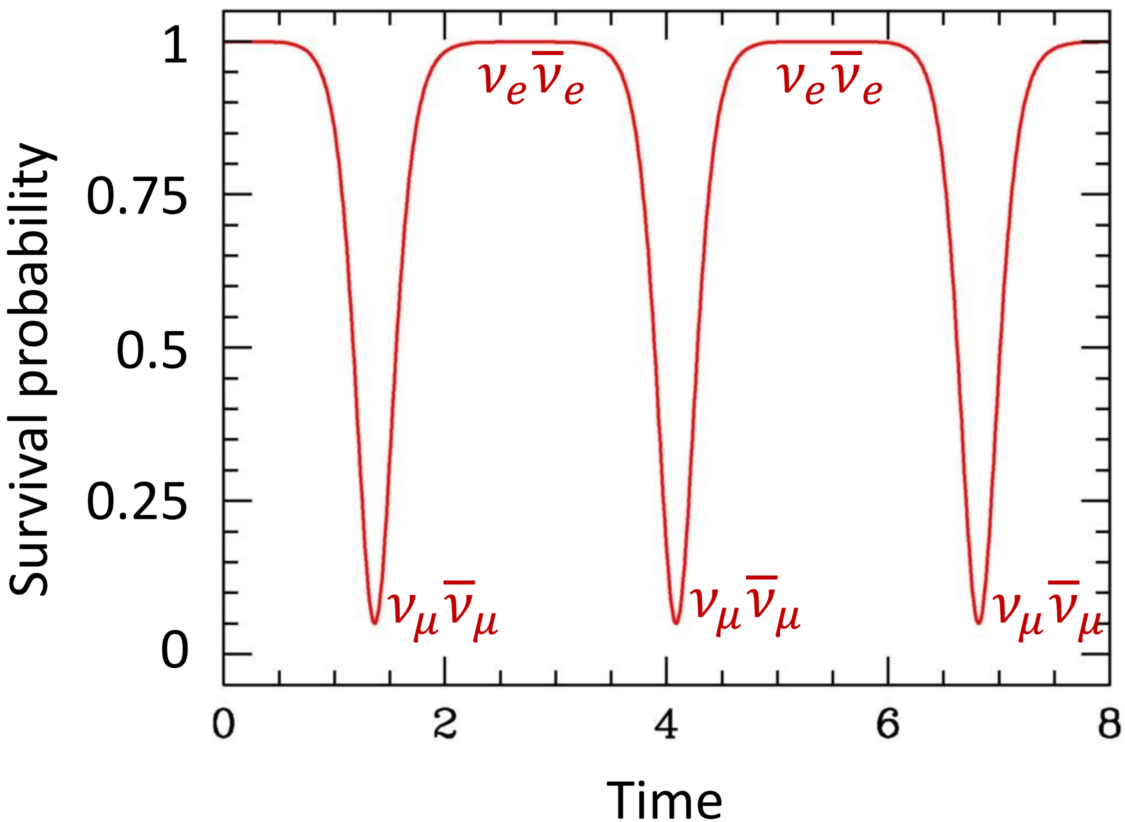
Matter effect transverse to mass and flavor directions

Both $\mathbf{P}_1$ and $\mathbf{P}_2$ tilt around $\mathbf{P}$ if μ is large

After a short time, transverse $\mathbf{P}$ develops by free precession

Collective Pair Annihilation

Gas of equal abundances of ν_e and $\overline{\nu}_e$, inverted mass hierarchy
Small effective mixing angle (e.g. made small by ordinary matter)



Dense neutrino gas unstable in flavor space: $\nu_e\overline{\nu}_e \leftrightarrow \nu_\mu\overline{\nu}_\mu$
Complete pair conversion even for a small mixing angle

Flavor Pendulum

Classical Hamiltonian for two spins interacting with a dipole force μ

$$H = \omega \mathbf{B} \cdot (\mathbf{P}_2 - \mathbf{P}_1) + \frac{\mu}{2} \mathbf{P}^2$$

Angular-momentum Poisson brackets

$$\{P_i, P_j\} = \epsilon_{ijk} P_k$$

Total angular momentum

$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$$

Precession equations of motion

$$\dot{\mathbf{P}}_{1,2} = (\mp \omega \mathbf{B} + \mu \mathbf{P}) \times \mathbf{P}_{1,2}$$

Lagrangian top (spherical pendulum with spin), moment of inertia I

$$H = \omega \mathbf{B} \cdot \mathbf{Q} + \frac{\mathbf{P}^2}{2I}$$

Total angular momentum $\mathbf{P}$, radius vector $\mathbf{Q}$, fulfilling

$$\{P_i, P_j\} = \epsilon_{ijk} P_k, \quad \{Q_i, Q_j\} = 0$$

$$\{P_i, Q_j\} = \epsilon_{ijk} Q_k$$

Pendulum EoMs

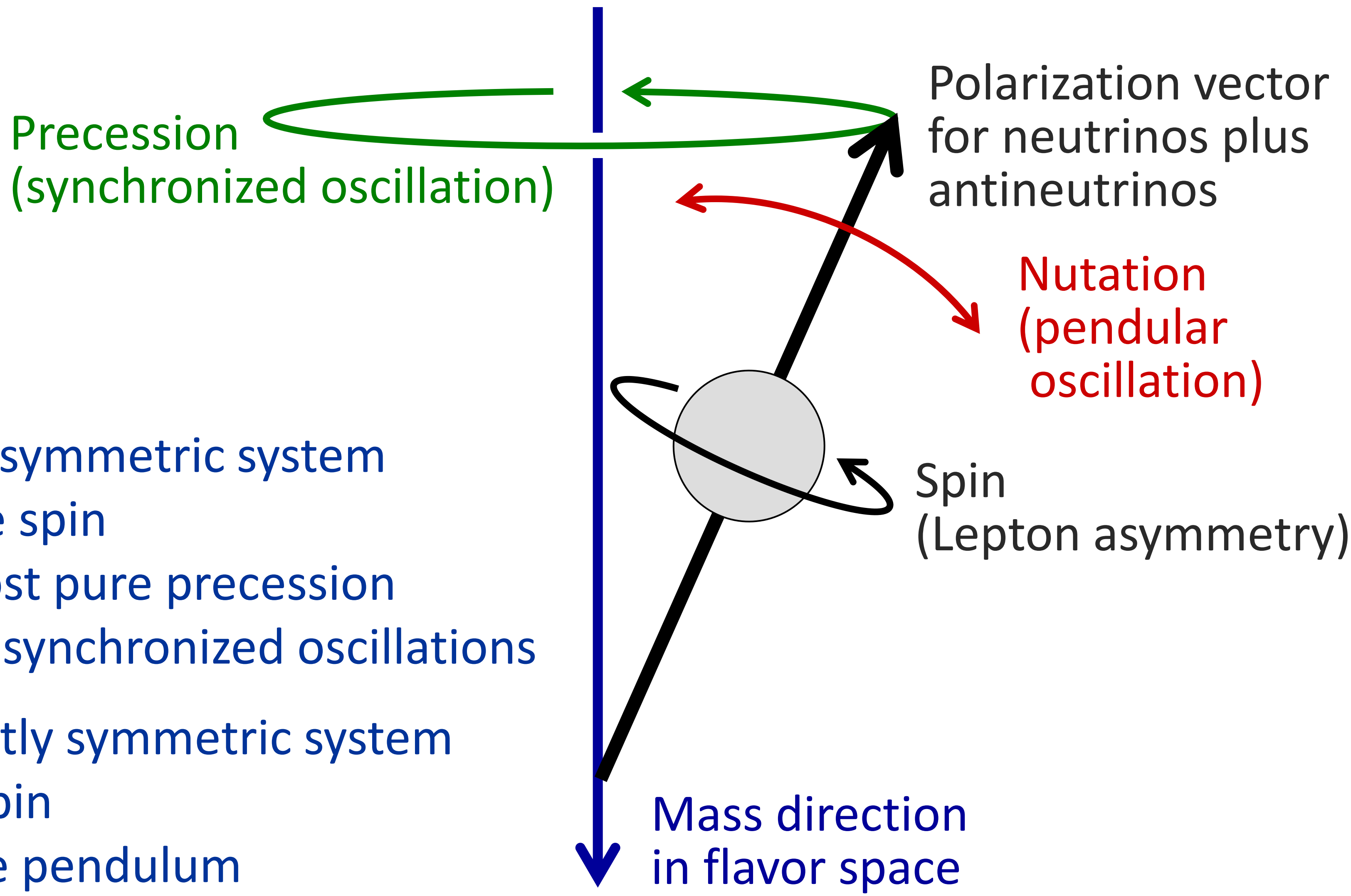
$$\dot{\mathbf{Q}} = I^{-1} \mathbf{P} \times \mathbf{Q} \quad \text{and} \quad \dot{\mathbf{P}} = \omega \mathbf{B} \times \mathbf{Q}$$

EoMs and Hamiltonians identical (up to a constant) with the identification

$$\mathbf{Q} = \mathbf{P}_2 - \mathbf{P}_1 - \frac{\omega}{\mu} \mathbf{B} \quad \text{and} \quad \mu = I^{-1}$$

Constants of motion: $\mathbf{P}_1^2$, $\mathbf{P}_2^2$, $\mathbf{B} \cdot \mathbf{P}$, $\mathbf{P} \cdot \mathbf{Q}$, $\mathbf{Q}^2$ and H

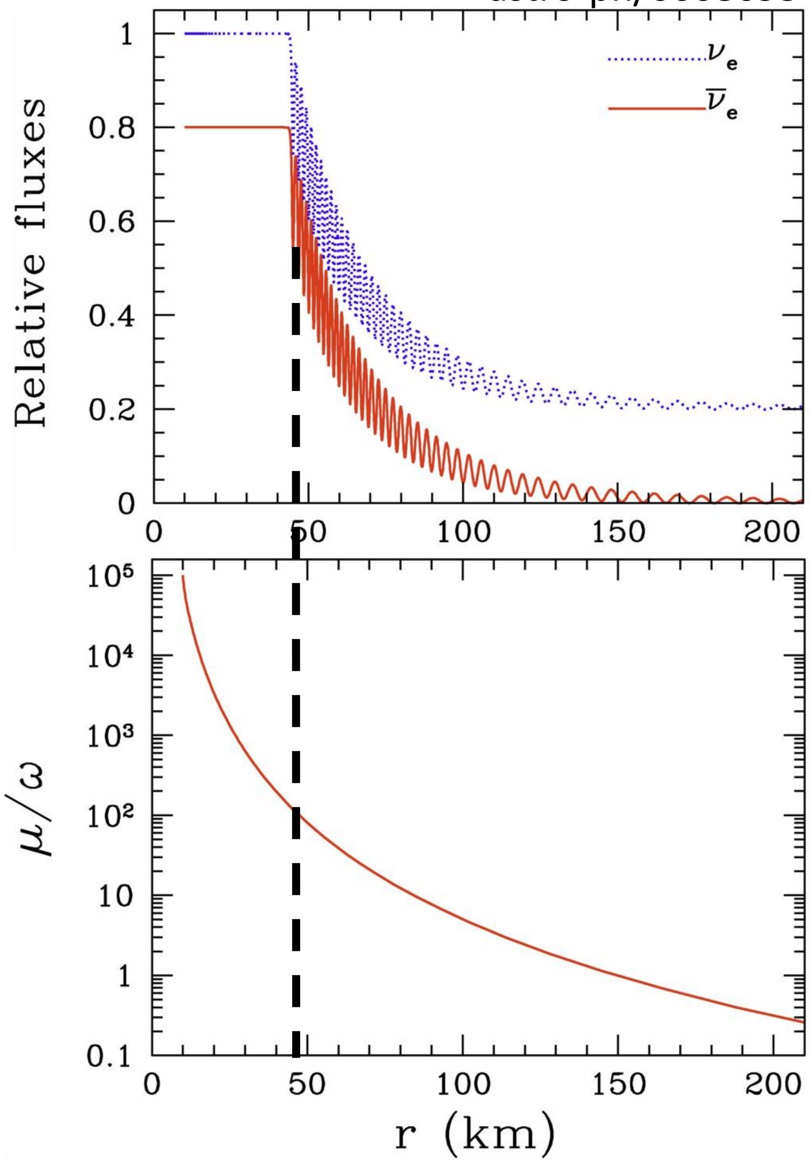
Pendulum in Flavor Space



- Very asymmetric system
  - Large spin
  - Almost pure precession
  - Fully synchronized oscillations
- Perfectly symmetric system
  - No spin
  - Plane pendulum

[Hannestad, Raffelt, Sigl, Wong: astro-ph/0608695]

Flavor Conversion in a Toy Supernova

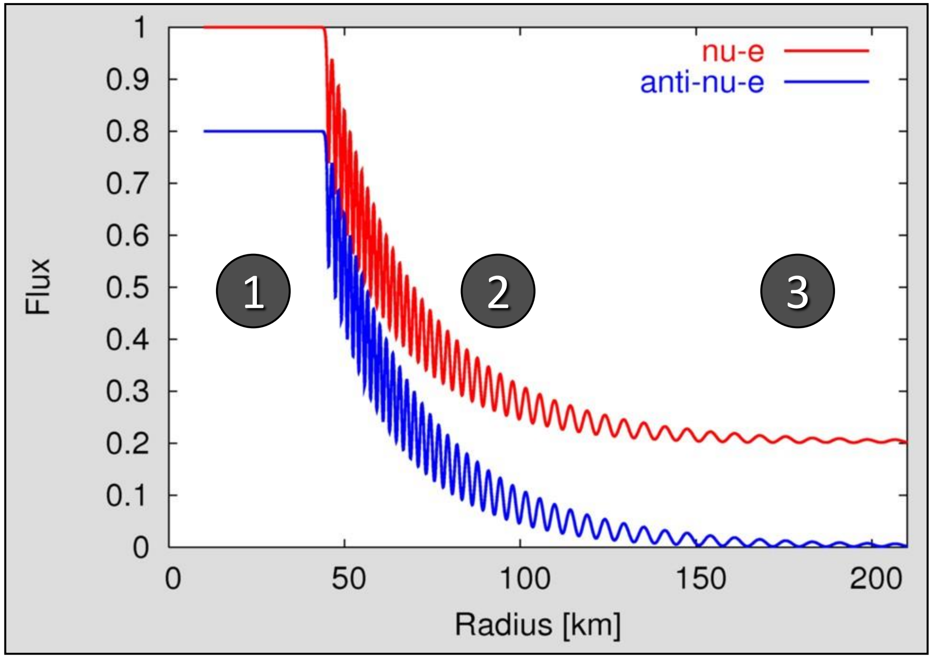


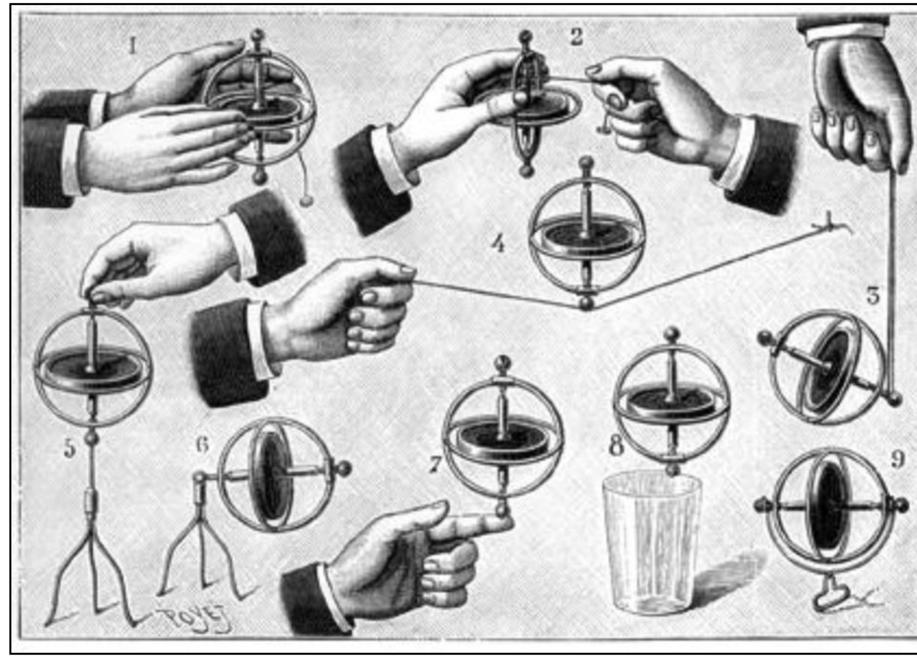
- Two modes with $\omega = \pm 0.3\ \mathrm{km}^{-1}$
- Assume 80% anti-neutrinos

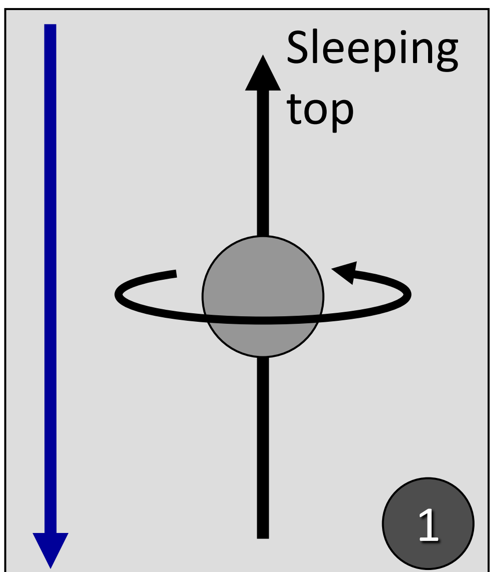
- Sharp onset radius
- Oscillation amplitude declining

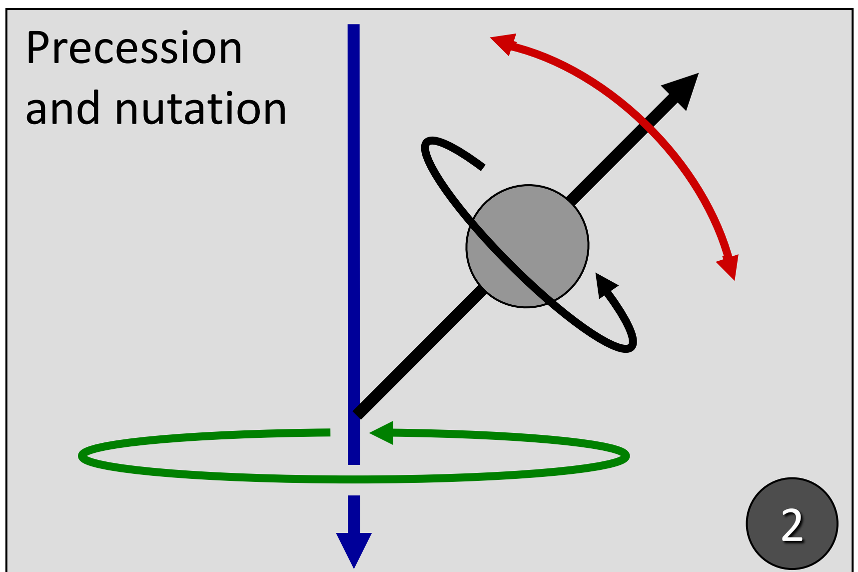
- Neutrino-neutrino interaction energy at nu sphere ($r = 10\ \mathrm{km}$)
  $$\mu = 0.3 \times 10^5\ \mathrm{km}^{-1}$$
- Falls off approximately as r^{-4} (geometric flux dilution and nus become more co-linear)

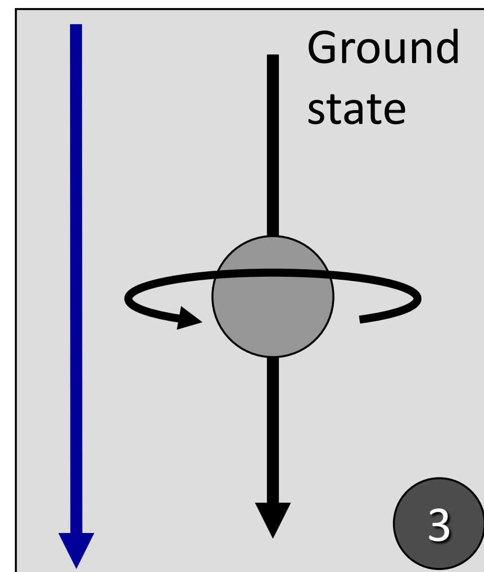
Neutrino Conversion and Flavor Pendulum











Fermi-Dirac Spectrum

Fermi-Dirac energy spectrum

$$\frac{dN}{dE} \propto \frac{E^2}{e^{E/T-\eta}+1}$$

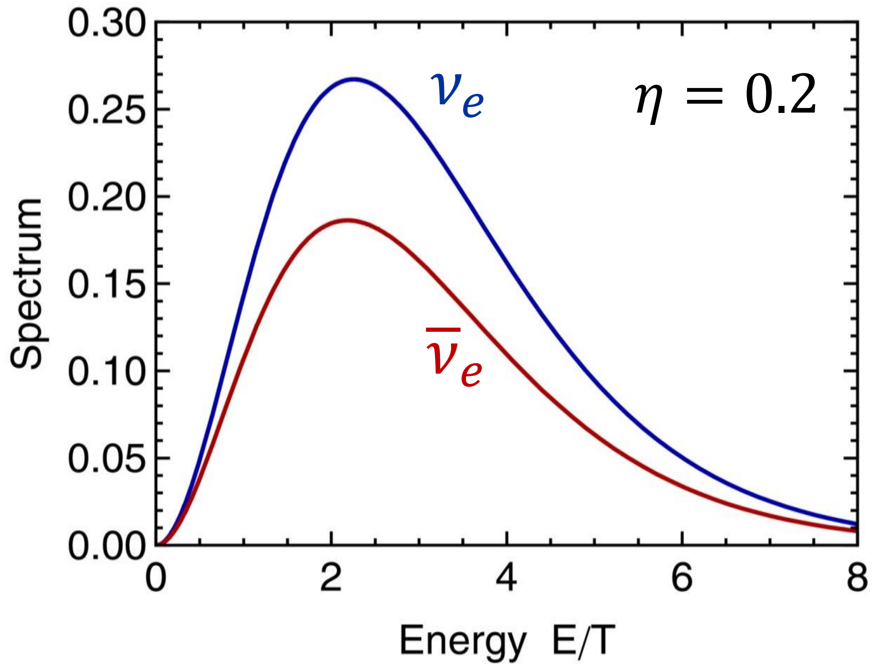
η degeneracy parameter, $-\eta$ for $\overline{\nu}$

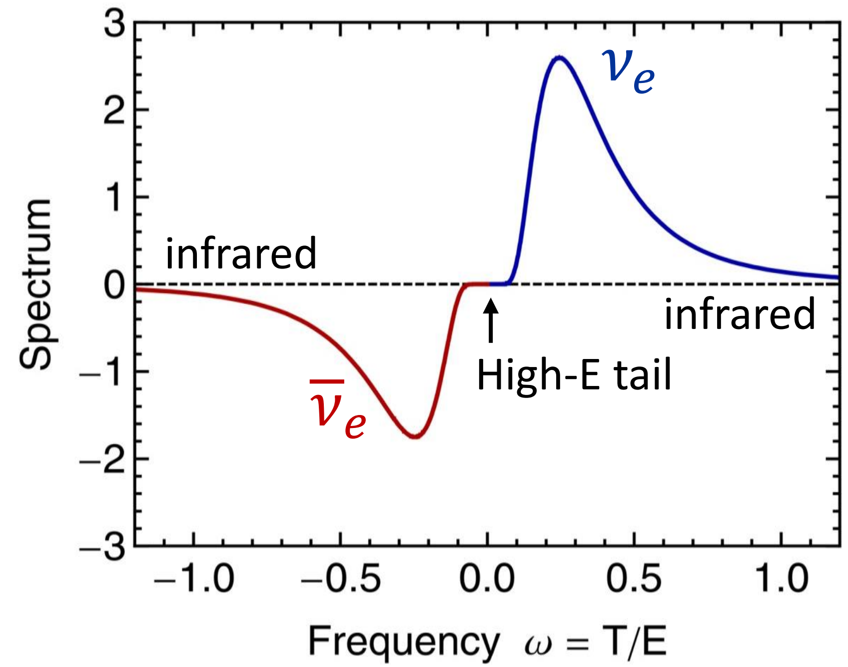
Same spectrum in terms of $\omega = T/E$

- Antineutrinos $E \to -E$
- and dN/dE negative (flavor isospin convention)

  $\omega > 0$: $\nu_e = \uparrow$ and $\nu_\mu = \downarrow$

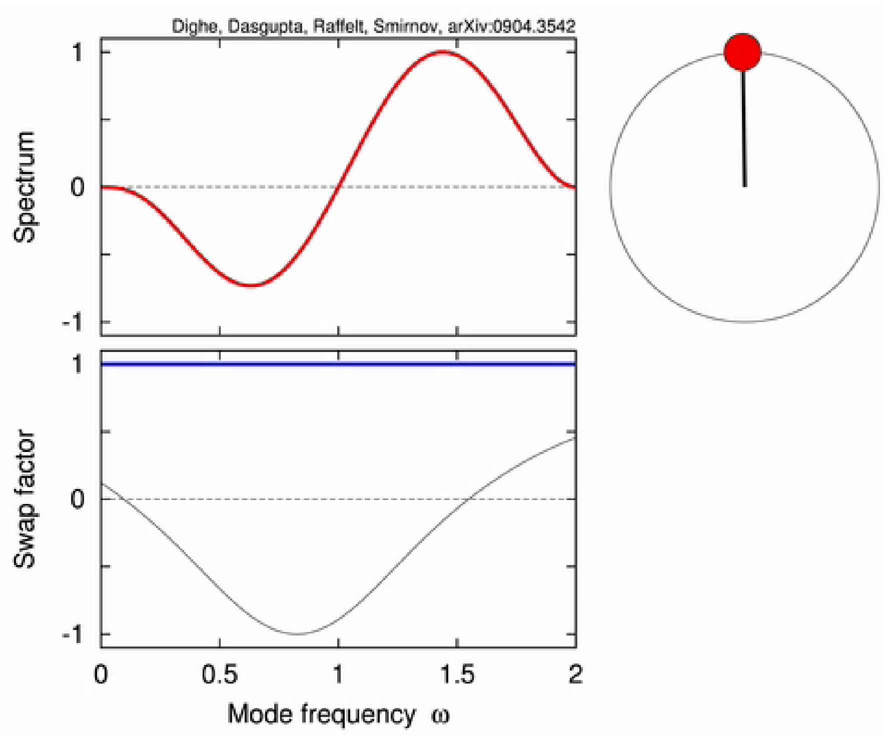
  $\omega < 0$: $\overline{\nu}_e = \downarrow$ and $\overline{\nu}_\mu = \uparrow$



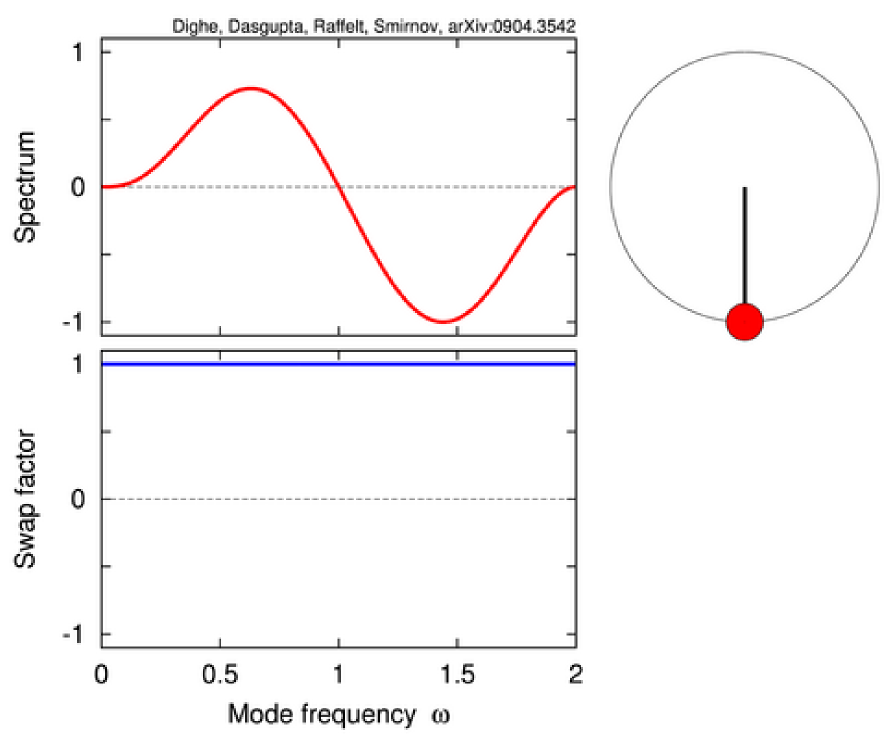


Flavor Pendulum

Single "positive" crossing
(potential energy at a maximum)



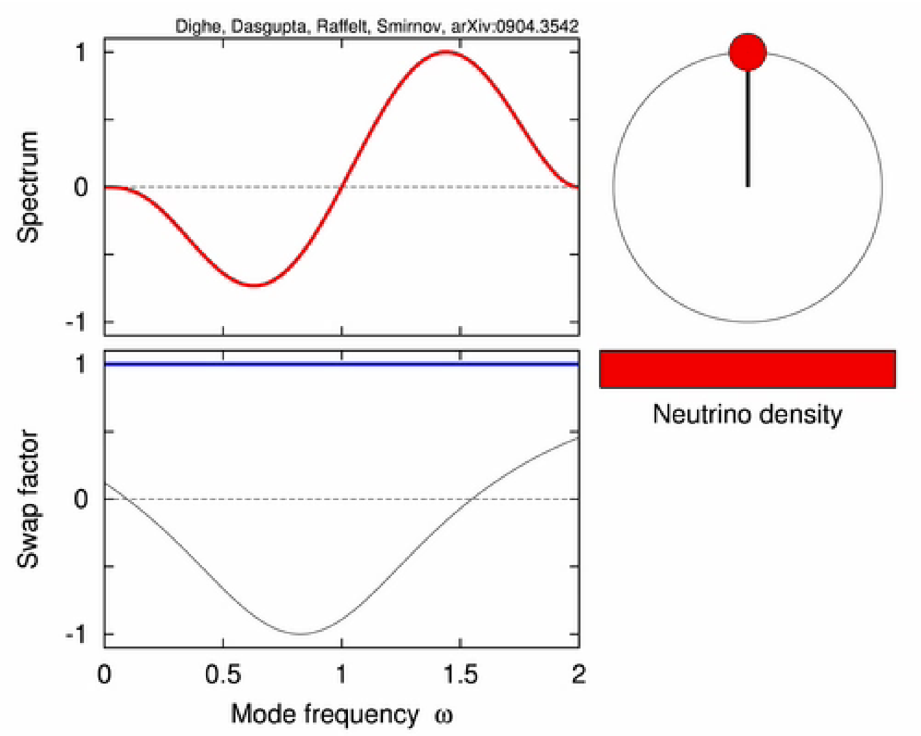
Single "negative" crossing
(potential energy at a minimum)



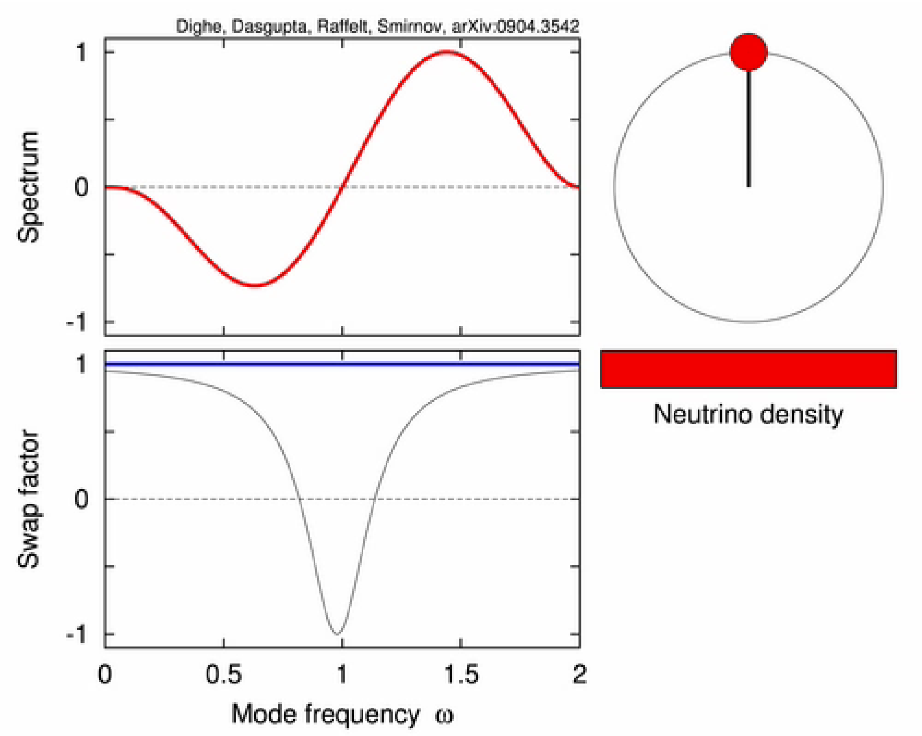
Dasgupta, Dighe, Raffelt & Smirnov, arXiv:0904.3542

Decreasing Neutrino Density

Certain initial neutrino density



Four times smaller initial neutrino density

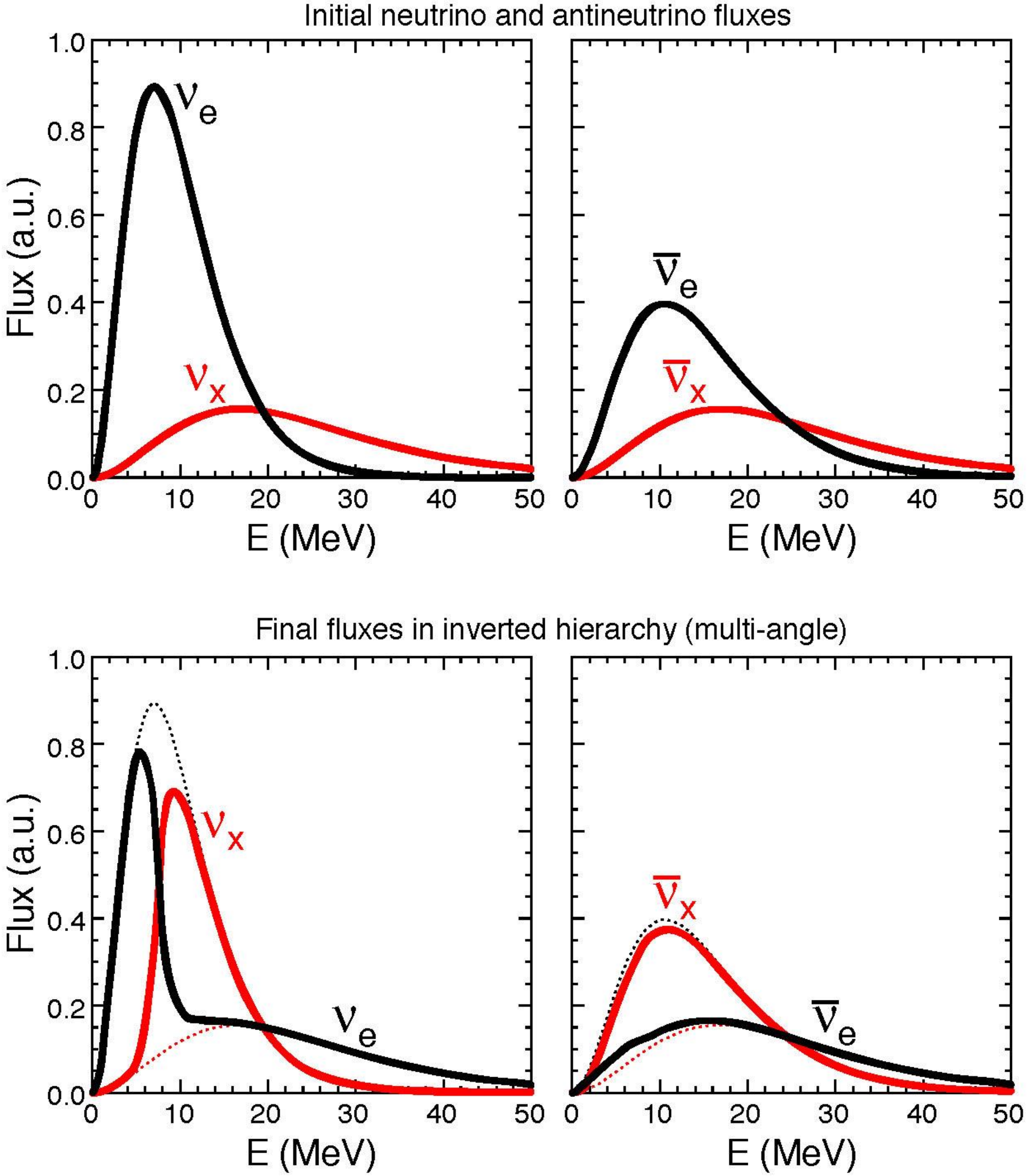


Dasgupta, Dighe, Raffelt & Smirnov, arXiv:0904.3542

Spectral Split

Initial fluxes at neutrino sphere

After collective trans-formation



Figures from Fogli, Lisi, Marrone & Mirizzi, arXiv:0707.1998

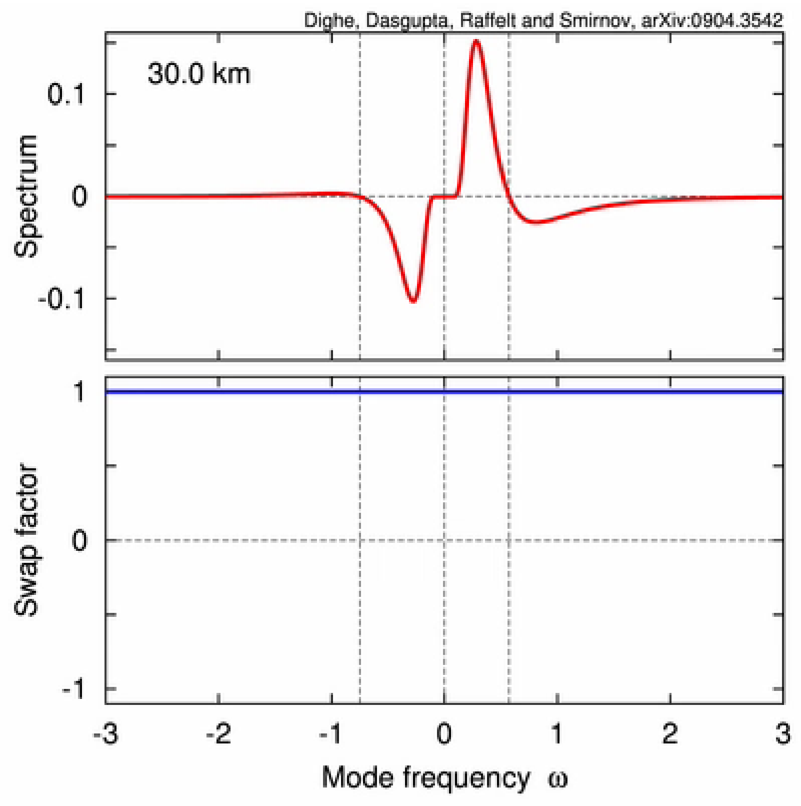
Explanations in

Raffelt & Smirnov arXiv:0705.1830 and 0709.4641

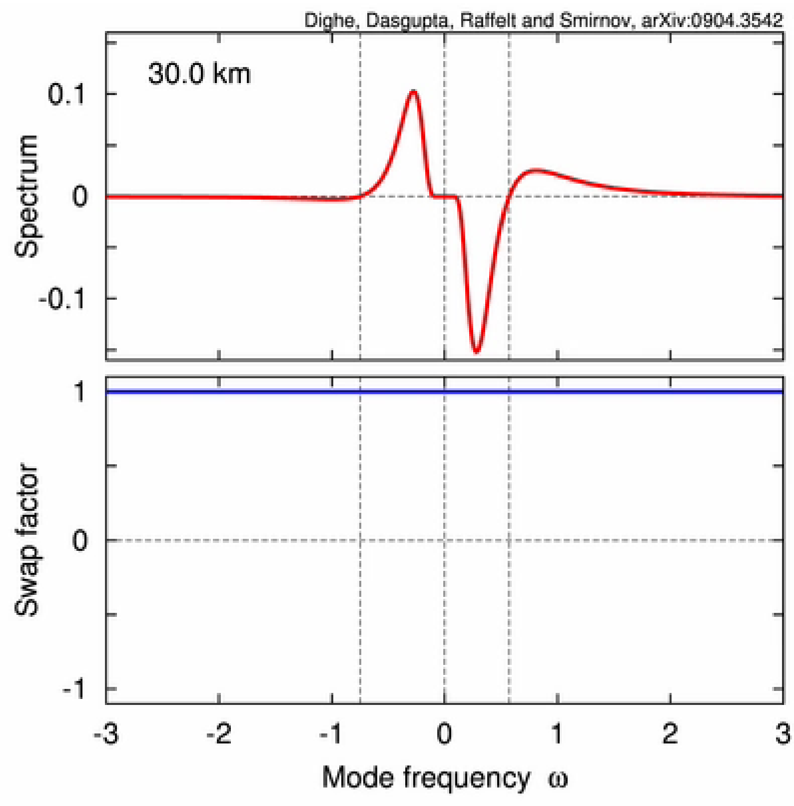
Duan, Fuller, Carlson & Qian arXiv:0706.4293 and 0707.0290

Supernova Cooling-Phase Example

Normal Ordering



Inverted Ordering



Dasgupta, Dighe, Raffelt & Smirnov, arXiv:0904.3542

Collective Nu Oscillations as a Many-Body Problem

Hamiltonian for interacting "flavor spins" (*classical* in mean-field approach)

$$H = \sum_{i=1}^{N} \omega_i \mathbf{B}\cdot\mathbf{P}_i + \sqrt{2}G_{\rm F}N_e\mathbf{L}\cdot\sum_{i=1}^{N}\mathbf{P}_i + \mu\sum_{i,j=1}^{N}\left(1-\cos\theta_{ij}\right)\mathbf{P}_i\cdot\mathbf{P}_j$$

Unit vector in mass direction

Unit vector in flavor direction

Multi-angle effects from current-current structure

"Spin-pairing H" for isotropic system (or single angle), ignoring matter effect

$$H = \sum_{i=1}^{N} \omega_i \mathbf{B}\cdot\mathbf{P}_i + \mu\mathbf{P}_{\rm tot}^2$$

BCS theory (using Anderson's pseudo-spin), nuclear physics, …

Integrable system (as many "Gaudin invariants" as spins)

→ Pehlivan, Balantekin, Kajino & Yoshida [arxiv:1105.1182] for introduction

N-mode coherent solutions ("Normal and anomalous solitons")

- Emil Yuzbashian, Phys. Rev. **B** 78, 184507 (2008) — Super-conductivity (BCS)
- Georg Raffelt, Phys. Rev. **D** 83, 105022 (2011) — Collective Nus

Classical Angular Momenta vs. Spin-1/2 States

Flavor pendulum arises from two coupled classical angular momentum vectors

$$\dot{\mathbf{P}}_1 = [-\omega\mathbf{B} + \mu\,(\mathbf{P_1} + \mathbf{P_2})] \times \mathbf{P_1}$$
$$\dot{\mathbf{P}}_2 = [+\omega\mathbf{B} + \mu\,(\mathbf{P_1} + \mathbf{P_2})] \times \mathbf{P_2}$$

Expectation values of flavor spins or average of many neutrinos (refractive limit!)

Two coupled spin ½ particles in external B-field (e.g. Zeeman effect in atoms)

$$H = \mathbf{B} \cdot (\omega_1 \mathbf{S}_1 + \omega_2 \mathbf{S}_2) + \mu\, \mathbf{S}_1 \cdot \mathbf{S}_2$$

Four quantum states – four eigenstates of Hamiltonian

↑↑, ↓↓ Strong coupling:
precession with a common frequency ("synchronisation")

↓↑, ↑↓ "pendular motion" not possible
at most four frequencies in the problem

Transition from two → many spin ½ particles:
Can we ignore "entanglement"?
Expectation value of product is product of expectation values?

Quantum Coherent Atomic Tunneling between Two Trapped Bose-Einstein Condensates

A. Smerzi,[1] S. Fantoni,[1,2] S. Giovanazzi,[1] and S. R. Shenoy[2]
[1]*International School of Advanced Studies and Istituto Nazionale di Fisica della Materia, via Beirut 2/4, I-34014, Trieste, Italy*
[2]*International Centre for Theoretical Physics, I-34100, Trieste, Italy*
(Received 15 May 1997)

We study the coherent atomic tunneling between two zero-temperature Bose-Einstein condensates (BEC) confined in a double-well magnetic trap. Two Gross-Pitaevskii equations for the self-interacting BEC amplitudes, coupled by a transfer matrix element, describe the dynamics in terms of the interwell phase difference and population imbalance. In addition to the anharmonic generalization of the familiar ac Josephson effect and plasma oscillations occurring in superconductor junctions, the nonlinear BEC tunneling dynamics sustains a self-maintained population imbalance: a novel "macroscopic quantum self-trapping" effect. [S0031-9007(97)04613-9]

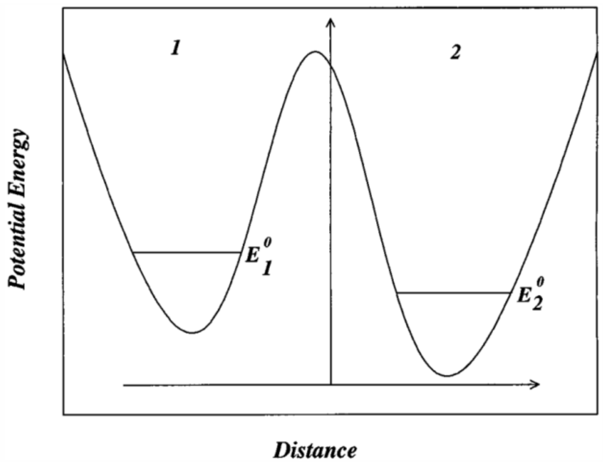


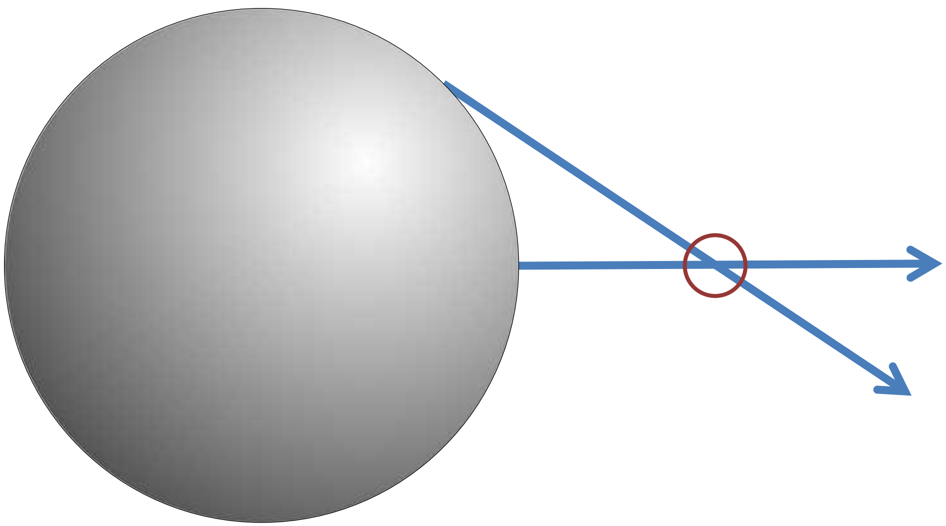
FIG. 1. The double-well trap for two Bose-Einstein condensates with $N_{1,2}$ and $E^0_{1,2}$ the number of particles and the zero-point energies in the trap 1, 2, respectively.

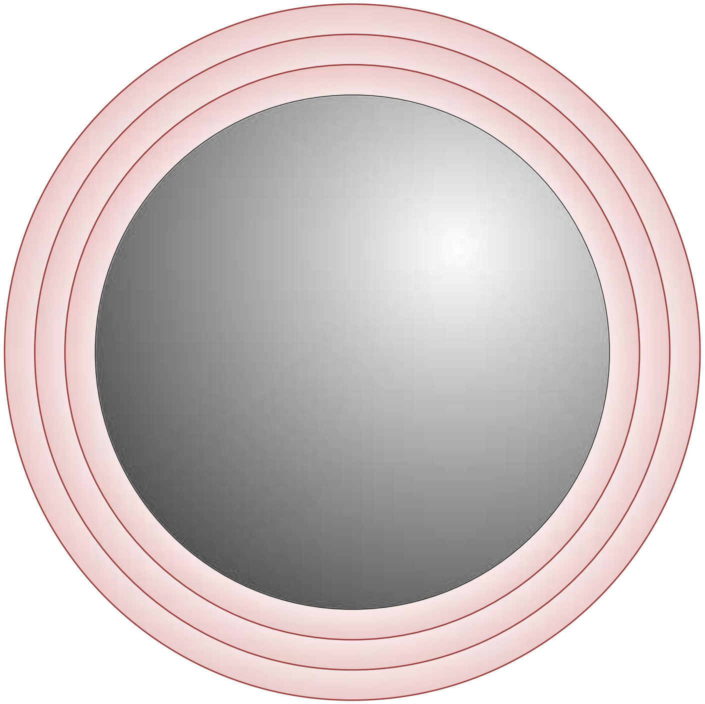
Population imbalance of interacting atomic BEC shows pendulum-like oscillations (Rabi oscillations for no interactions or few atoms)

For experimental test see:
Zibold, Nicklas, Gross & Oberthaler,
Classical Bifurcation at the Transition from Rabi to Josephson Dynamics, PRL 105:204101 (2010)

Correlated Trajectories vs. Field of Flavor Coherence

Assume globally spherically symmetric neutrino emission from SN core





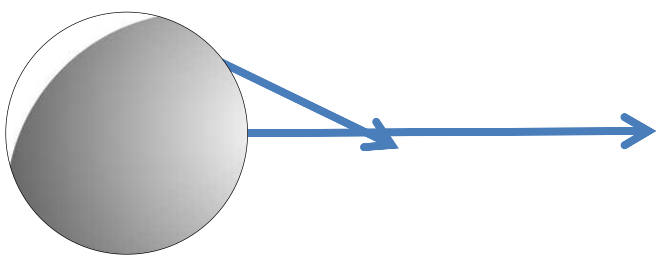
- Every ν meets every other ν at most once
- Nonlinear feedback on flavor evolution?

- Oscillating (or unstable) field $\varrho(r)$ of flavor coherence, acting back upon itself
- Do not worry about individual neutrinos

Evolution of the Questions

Bulb model of neutrino emission:

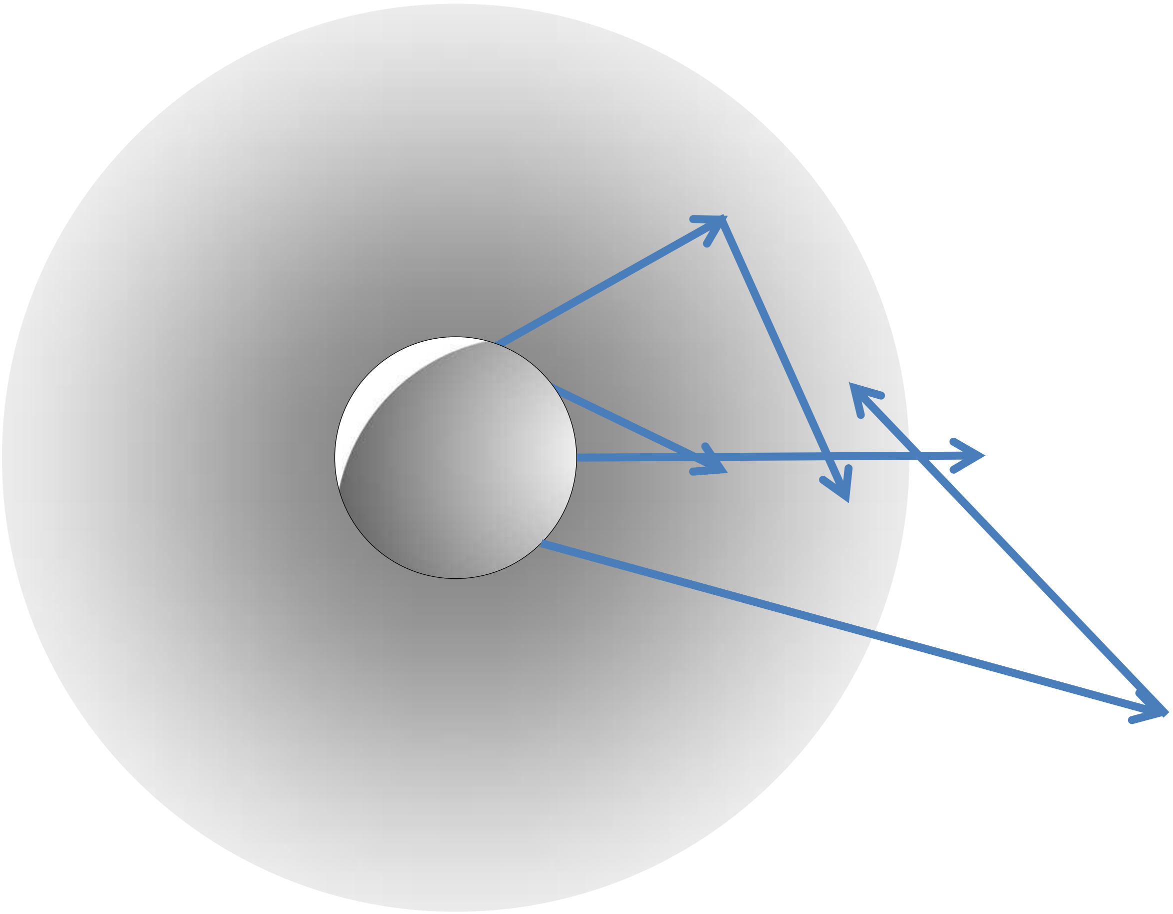
- Nu-nu interaction determined by aspect ratio of emission surface
- Instability as a function of radius – adiabatic conversion possible
- Flavor pendulum, spectral splits, multi-angle matter effect, three-flavor effects, …
- Spurious instabilities (need many angle bins in numerical studies)
- Instability in the transverse direction: Spontaneous symmetry breaking



Evolution of the Questions

Halo effect:

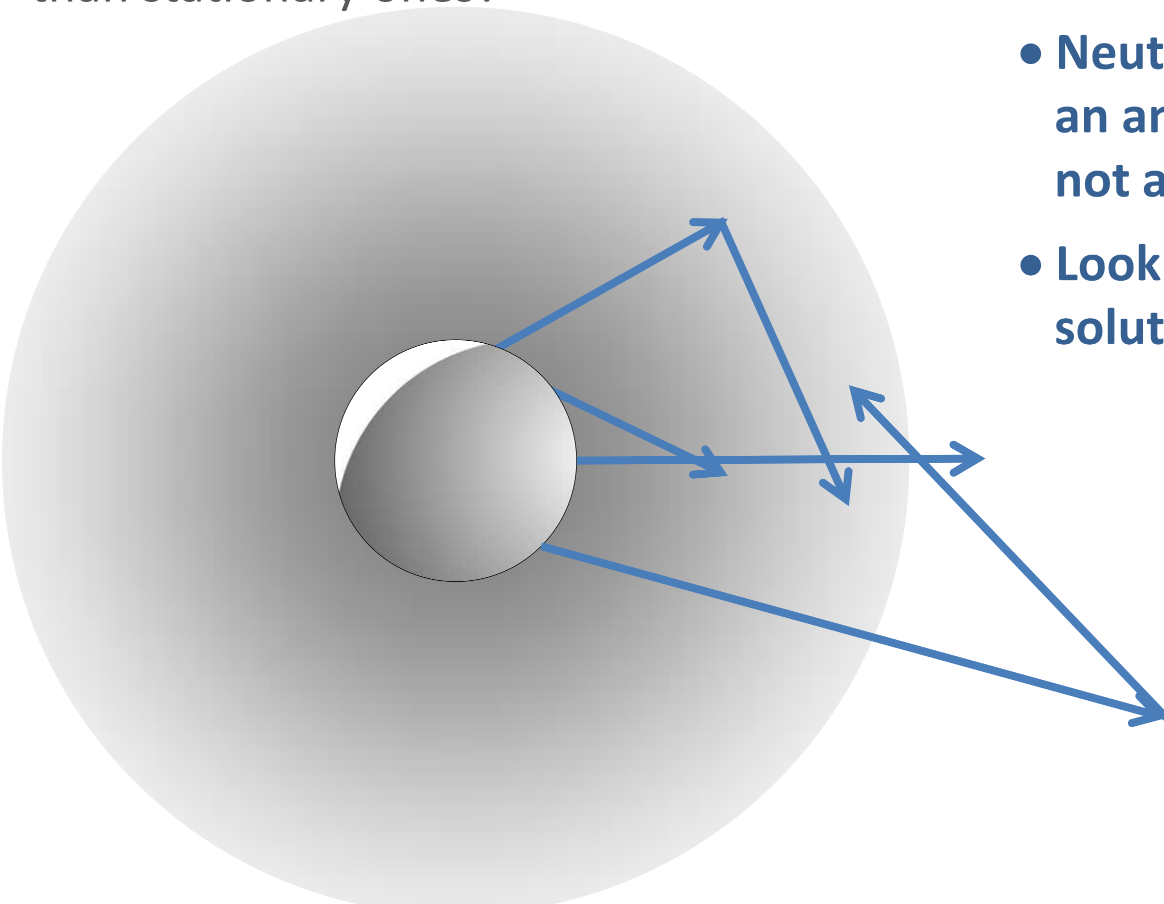
- Small re-scattered flux, much larger angular leverage
- Impact on collective effects?
- How to deal with backward flux?



Evolution of the Questions

Non-stationarity:

- Time-variation (of SN emission) in source region?
- Self-induced time variation, “pulsating modes” more unstable than stationary ones?



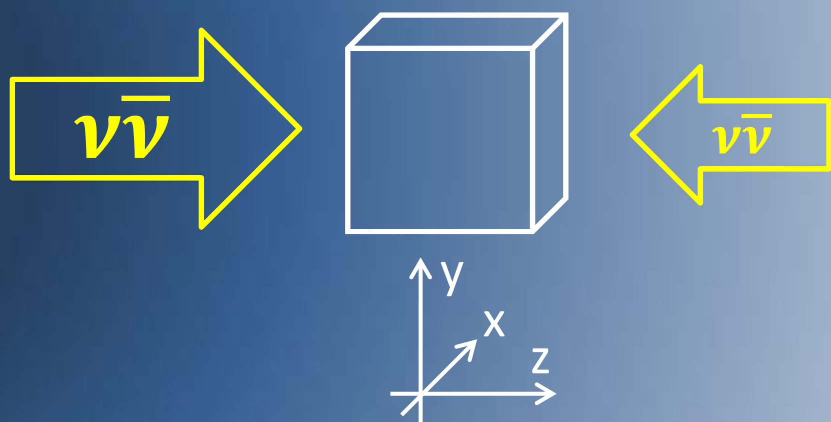
- **Neutrino field begins to look like an anisotropic “gas” or “medium”, not a “beam”**
- **Look for full space-time dependent solution $\varrho(t,\vec{x})$**

Neutrino gas in the near-free streaming regime

Small test volume for fast modes

- Homogeneous conditions
- Need $f_\nu(E,\theta,\varphi)$ for all species
- Large mean free path
- What is flavor evolution (t,x,y,z) ?

PNS



Shock

Linearised problem:
Study dispersion relation of
“flavor waves”

Solutions of Transport Equation

- Which collective and non-collective modes are supported by the neutrino medium?
- Which of them are effectively excited by the initial and/or boundary conditions?
- What is the outcome in the nonlinear regime?

In principle, follows from (e.g. numerical) solution of kinetic equation

First question can be addressed by normal mode analysis of linearised equations of motion

Fast Flavor Conversion

Flavor evolution governed by "Hamiltonian matrix" (here for 2 flavors)

$$\mathcal{H}_{\vec{p}}=\frac{\Delta m^2}{4E}\begin{pmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{pmatrix}+\mathcal{H}_{\rm mat}+\sqrt{2}G_{\rm F}\int\frac{d^3\vec{p}'}{(2\pi)^3}(1-\vec{v}\cdot\vec{v}')\begin{pmatrix}f_{\nu_e} & f_{\langle\nu_e|\nu_x\rangle} \\ f_{\langle\nu_x|\nu_e\rangle} & f_{\nu_x}\end{pmatrix}$$

Flavor conversion caused by off diagonals

Energy scales of the problem:

$\mu=\sqrt{2}\,G_{\rm F}\,n_{\nu\bar\nu}$ *Required for any collective effects*

$\omega_E=\Delta m^2/2E$ *Vacuum oscillation frequency*

$\lambda=\sqrt{2}\,G_{\rm F}\,n_{\rm e}$ *Matter effect*

Slow modes:

Require $\omega_E\neq 0$

Possible growth rate: $\kappa\sim\sqrt{\mu\omega_E}$, requires "crossing" of ω_E distribution

Fast modes:

Dynamical even for $\omega_E=0$

Growth rate: $\kappa\sim\sqrt{\mu\omega_E}$ slow growth

$\kappa\sim\mu$ fast growth, requires "crossing" of angle distribution

Linearisation for Fast Flavor Modes

Evolution equation: $iv^\alpha\partial_\alpha\varrho = [\mathcal{H},\varrho]$ with $v^\alpha = (1,\vec{v})$

Linearisation to find (propagating or unstable) collective modes:

$$\varrho = \begin{pmatrix} f_{\nu_e} & f_{\langle\nu_e|\nu_x\rangle} \\ f_{\langle\nu_x|\nu_e\rangle} & f_{\nu_x}\end{pmatrix} = \frac{f_{\nu_e}+f_{\nu_x}}{2} + \frac{f_{\nu_e}-f_{\nu_x}}{2}\begin{pmatrix} s & S \\ S^* & -s\end{pmatrix}$$

Field of flavor coherence

Linearised EOM for field of flavor coherence

$$iv^\alpha\partial_\alpha S_{\vec{p}} = \Bigg(\underbrace{\frac{\Delta m^2}{2E}}_{\text{Ignore for fast modes}} + v^\alpha\Lambda_\alpha\Bigg)S_{\vec{p}} - \mu v^\alpha \underbrace{\int\frac{d^3\vec{p}'}{(2\pi)^3}v'_\alpha\left(g_{\vec{p}'}S_{\vec{p}'} - \bar{g}_{\vec{p}'}\bar{S}_{\vec{p}'}\right)}_{\text{Same for all } E \text{ and } \nu \text{ and } \bar{\nu}}$$

Angle distribution of electron lepton number (ELN) carried by neutrinos

$$G_{\vec{v}} = \int\frac{dE\,E^2}{2\pi^2}\,\frac{f_{\nu_e,\vec{p}} - f_{\bar{\nu}_e,\vec{p}} - f_{\nu_x,\vec{p}} + f_{\bar{\nu}_x,\vec{p}}}{2}$$

Linearised EOM for field of flavor coherence

$$iv^\alpha(\partial_\alpha + i\Lambda_\alpha)S_{\vec{v}} = -\mu v^\alpha\int\frac{d\vec{v}'}{4\pi}v'_\alpha G_{\vec{v}'}S_{\vec{v}'}$$

Matter effect, “rotate away” by including it in derivative
if medium is homogeneous and stationary

Dispersion Relation for Fast Flavor Modes

Linearised EOM for field of flavor coherence – a wave equation

$$iv^{\alpha}\partial_{\alpha}S_{\vec{v}} = -\mu v^{\alpha}\int\frac{d\vec{v}'}{4\pi}v'_{\alpha}G_{\vec{v}'}S_{\vec{v}'}$$

Plane-wave ansatz

$$S_{\vec{v}}(t,\vec{r}) = Q_{\vec{v}}\big(\Omega,\vec{K}\big)\,e^{-i(\Omega t-\vec{K}\cdot\vec{r})}$$

EOM in Fourier space

$$\big(\Omega-\vec{v}\cdot\vec{K}\big)Q_{\vec{v}} = -\mu\int\frac{d\vec{v}'}{4\pi}(1-\vec{v}\cdot\vec{v}')\,G_{\vec{v}'}Q_{\vec{v}'}$$

Non-collective solutions:

$(\Omega,\vec{\mathrm{K}})$ real and “below the light cone”

$\big(\Omega-\vec{v}\cdot\vec{K}\big)=0$ for some mode $\vec{v}$

Dispersion relation:

$\Omega=\vec{v}\cdot\vec{K}$

for every $\vec{K}$ continuos infinity of frequencies

Collective solutions:

$\big(\Omega-\vec{v}\cdot\vec{K}\big)\neq 0$ for all $\vec{v}$ modes

$(\Omega,\vec{\mathrm{K}})$ real and “outside the light cone” or imaginary part

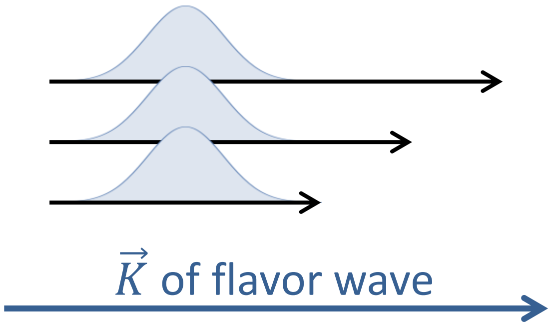
Eigenfunctions $Q_{\vec{v}}\propto\frac{\vec{a}\cdot\vec{v}+b}{\Omega-\vec{v}\cdot\vec{K}}$

Dispersion relation: $\det\Pi=0$

$$\Pi^{\mu\nu}=\eta^{\mu\nu}+\int\frac{d\vec{v}}{4\pi}\,G_{\vec{v}}\,\frac{v^{\mu}v^{\nu}}{\Omega-\vec{v}\cdot\vec{K}}$$

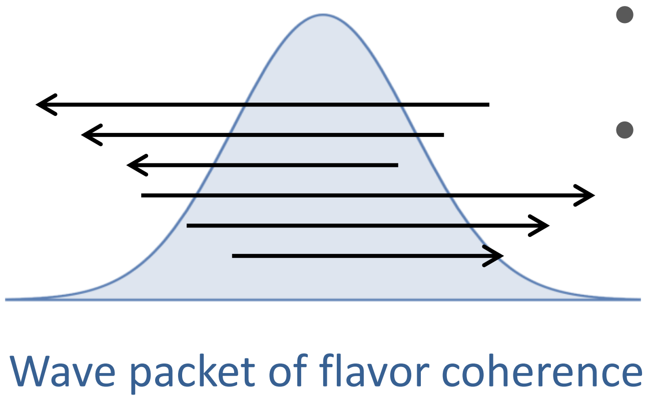
Fast Flavor Waves

Non-collective modes:



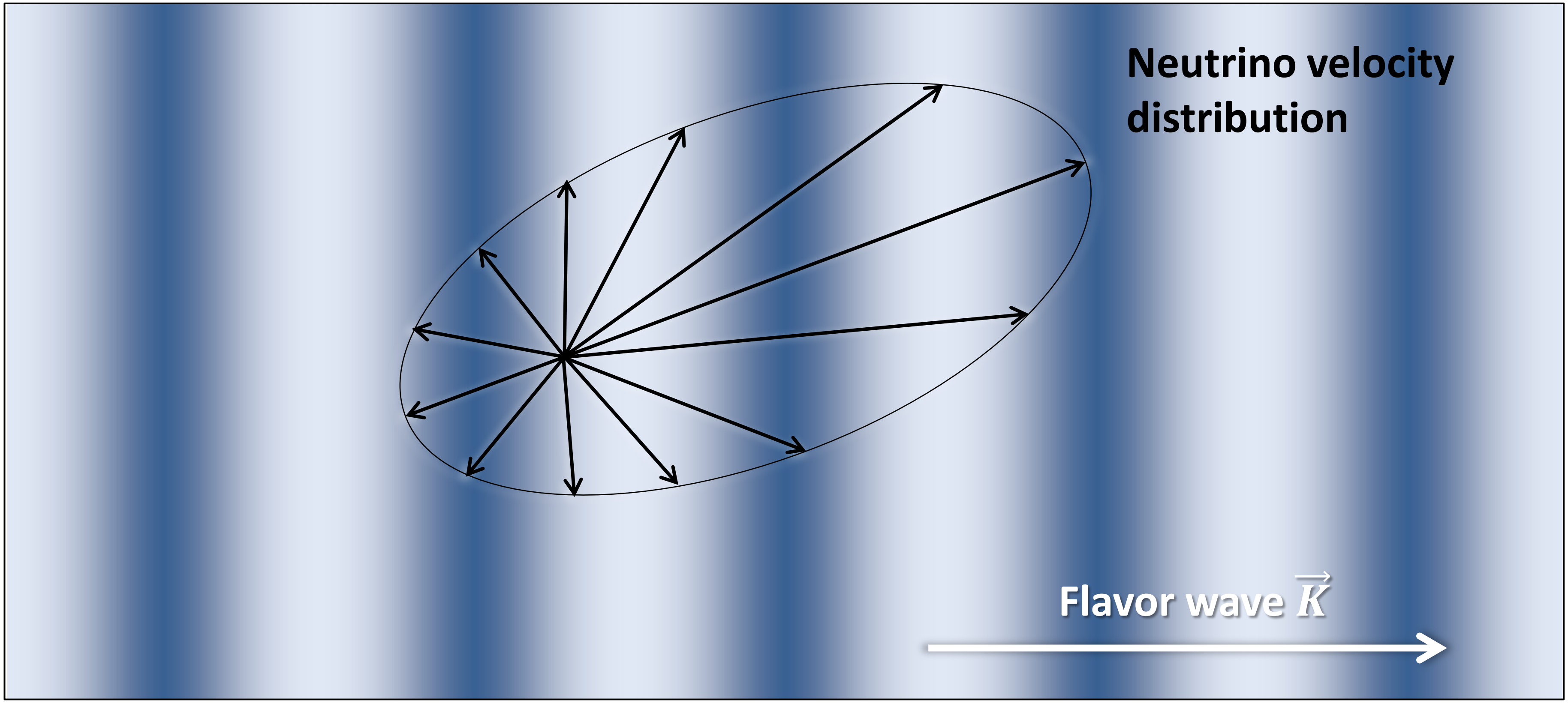
- Infinitely many neutrino velocity projections on $\vec{K}$
- Each carries along its initial flavor coherence
- Kinematical decoherence of initial wave packet
  (Does not happen in two-beam model)
- Fast dissipation of any initial wave packet

Collective modes



- Infinitely many neutrino velocity projections on $\vec{K}$
- Move through wave packet
  (here taken with vanishing central wave number)
- Wave packet moves in neutrino gas,
  independently of velocities of neutrino "beams"

Fast Flavor Waves



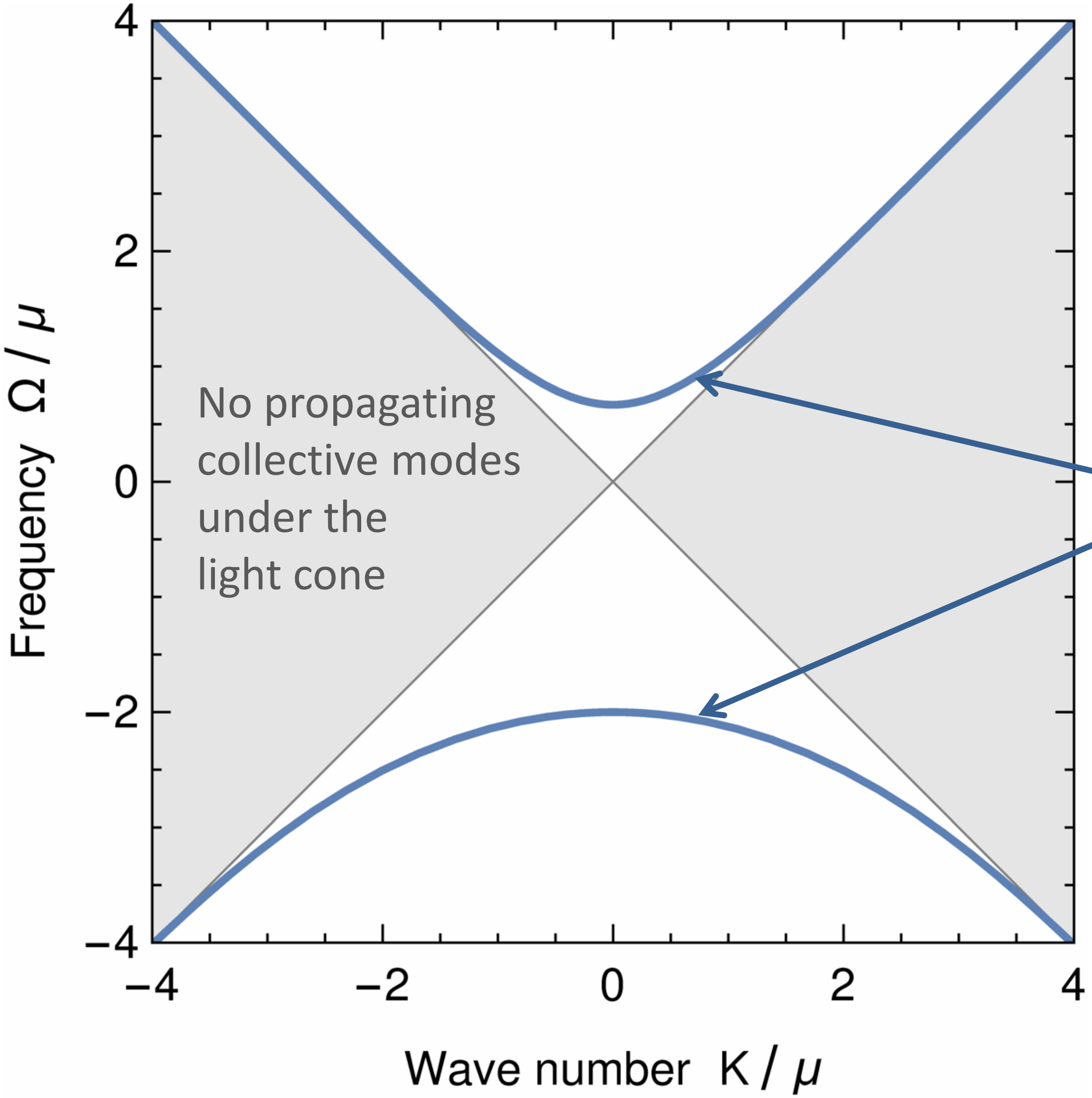
Non-collective modes: **Infinitely many** $\Omega = \vec{\mathrm{v}} \cdot \vec{K}$

- Flavor coherence carried by every neutrino mode separately
- Quick kinematical decoherence

Collective modes: $\Omega(\vec{K})$ according to collective dispersion relation

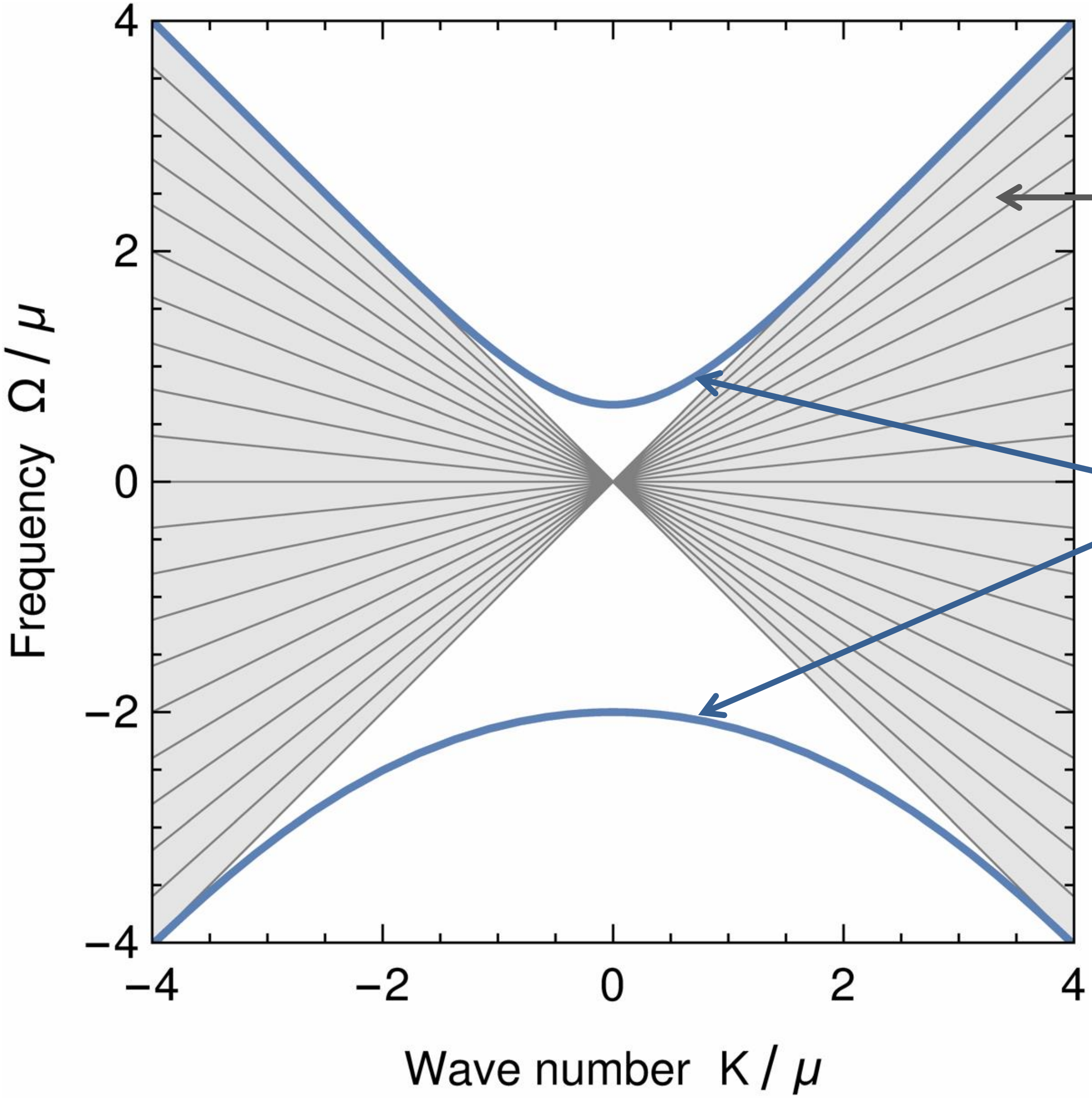
- Flavor wave (or wave packet) propagates and/or grows

Dispersion Relation for Isotropic Case



Collective modes

Dispersion Relation for Isotropic Case



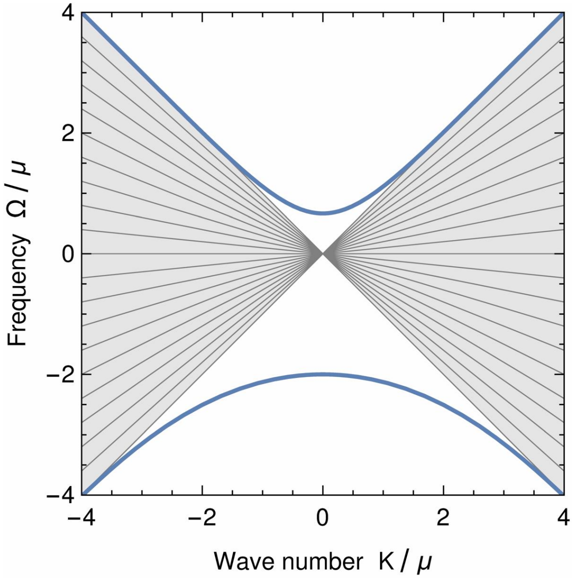
Non-collective modes with

$$\boldsymbol{\Omega = \vec{v} \cdot \vec{K}}$$

carried by neutrino "beams"

Collective modes

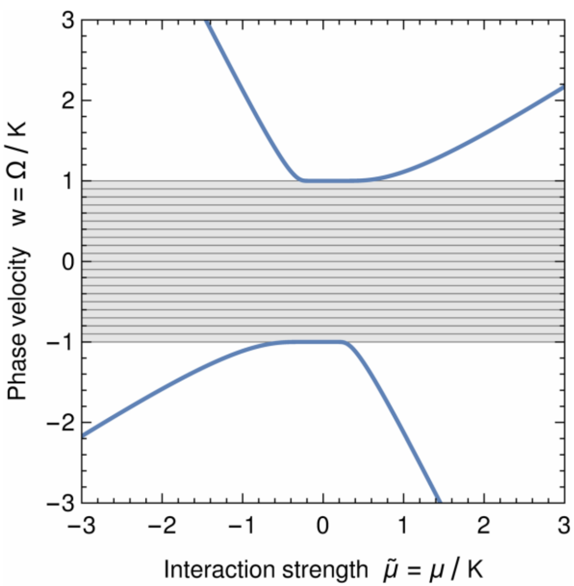
Dispersion Relation vs. Eigenvalues of Hamiltonian



Dispersion relation:

For fixed μ find $\Omega(K)$ from

$$\left(\Omega - \vec{v}\cdot\vec{K}\right)Q_{\vec{v}} = -\mu\int\frac{d\vec{v}'}{4\pi}\left(1-\vec{v}\cdot\vec{v}'\right)G_{\vec{v}'}Q_{\vec{v}'}$$



Eigenvalues of Hamiltonian:

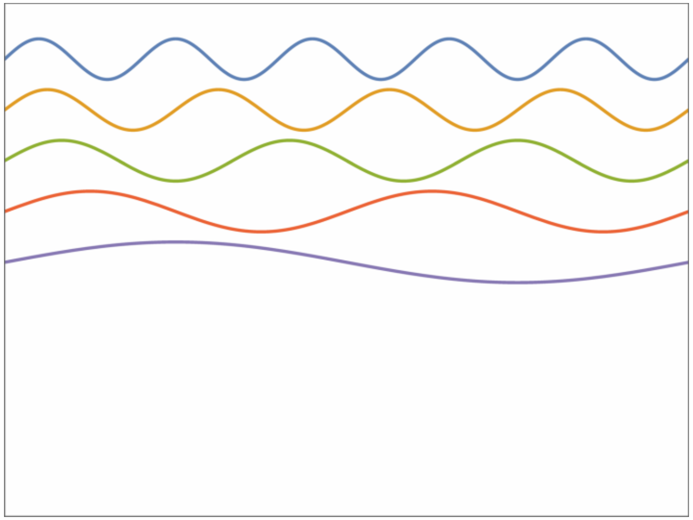
For fixed K find eigenvalues $w = \Omega(\mu)/K$ of $\mathcal{H}$

$$i\partial_t S_{\vec{v}}\left(t,\vec{K}\right) = \mathcal{H}\left(S_{\vec{v}}\right)$$

$$\mathcal{H}\left(S_{\vec{v}}\right) = \vec{v}\cdot\vec{K}\,S_{\vec{v}} - \mu\int\frac{d\vec{v}'}{4\pi}\left(1-\vec{v}\cdot\vec{v}'\right)G_{\vec{v}'}S_{\vec{v}'}$$

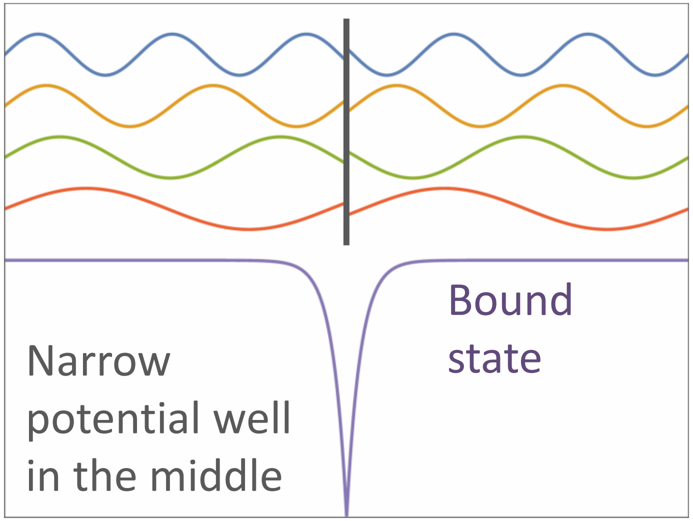
Bound vs Scattering States

← Electron waves in a box →



← Electron waves in a box →

Phase shift



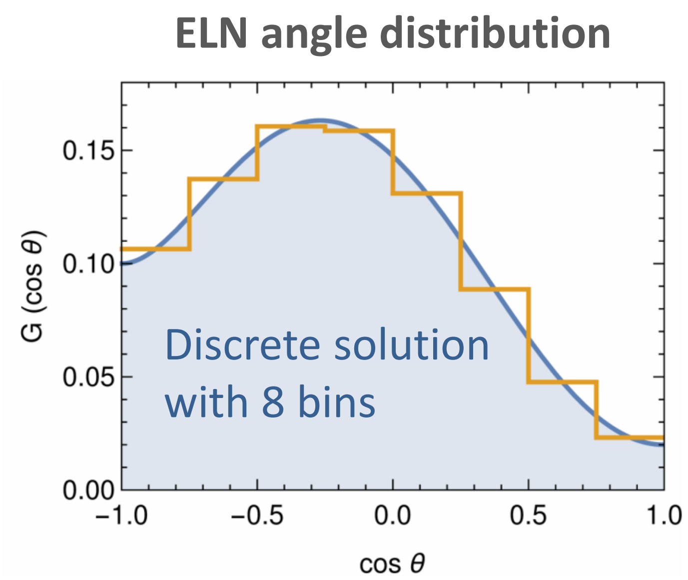
Continuum limit: Box size $\to \infty$

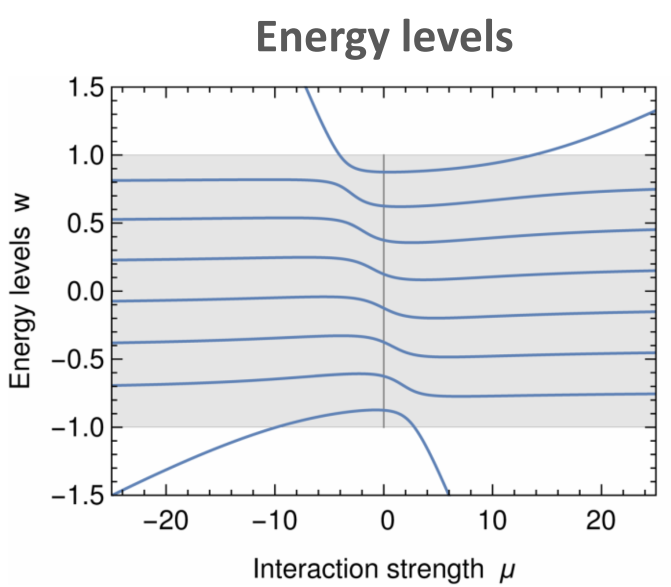
Continuum of scattering states

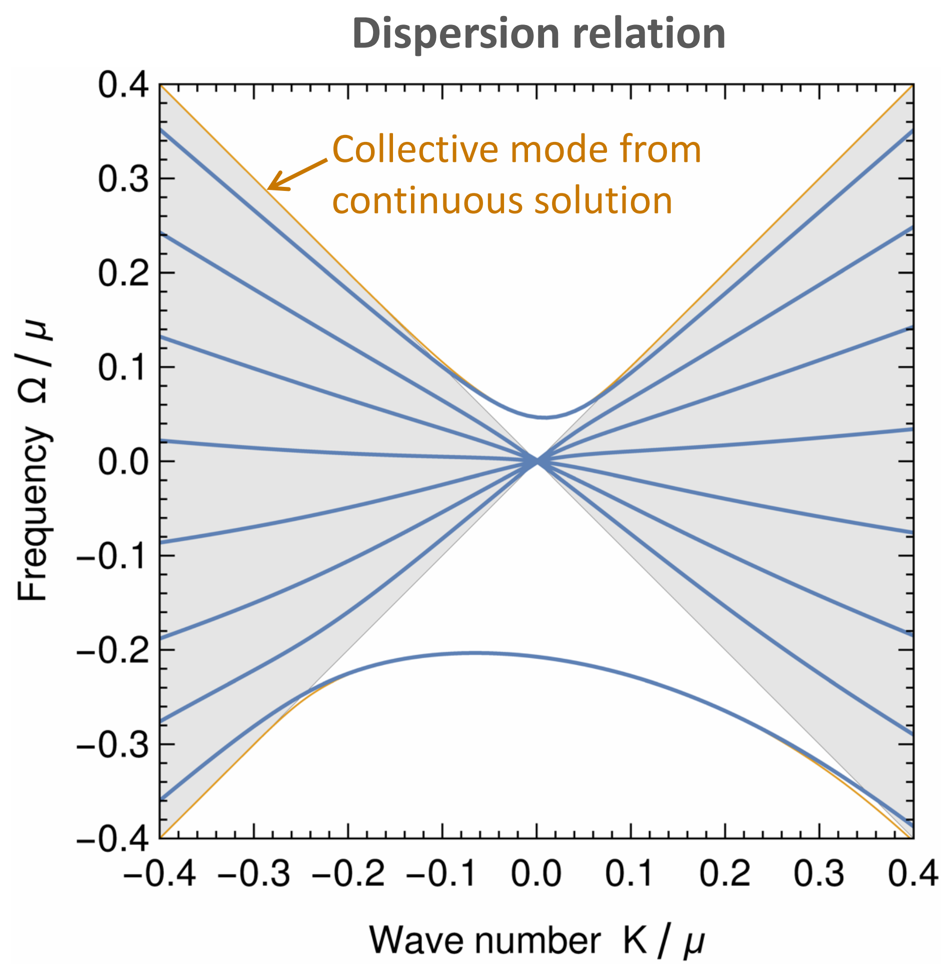
Continuum of scattering states
(with phase shifts) + Bound state

Non-collective modes ~ Scattering states
Collective modes ~ Bound states

No spectral crossing



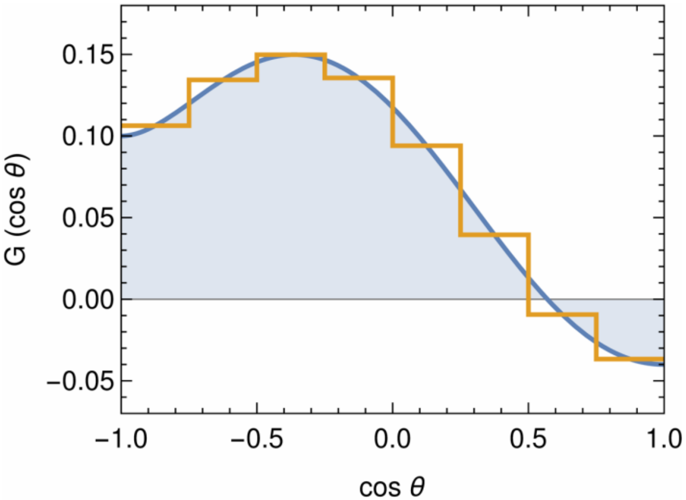




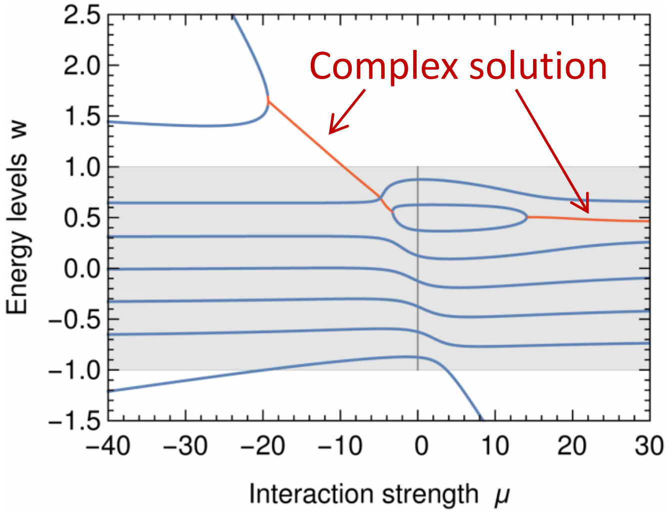
- **Two propagating collective modes**
- **“Peel off” from non-collective modes**

“Weak” spectral crossing

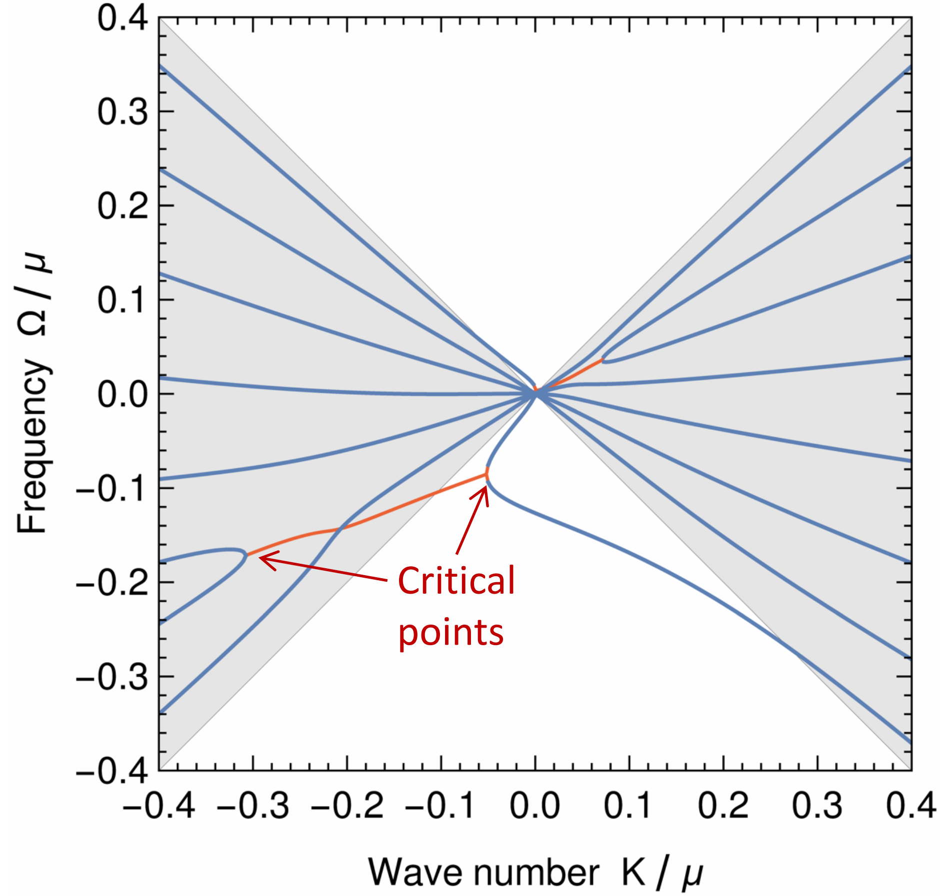
ELN angle distribution



Energy levels



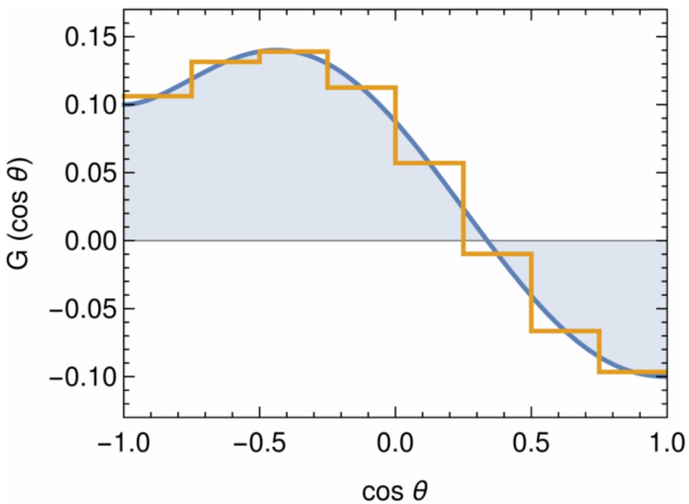
Dispersion relation



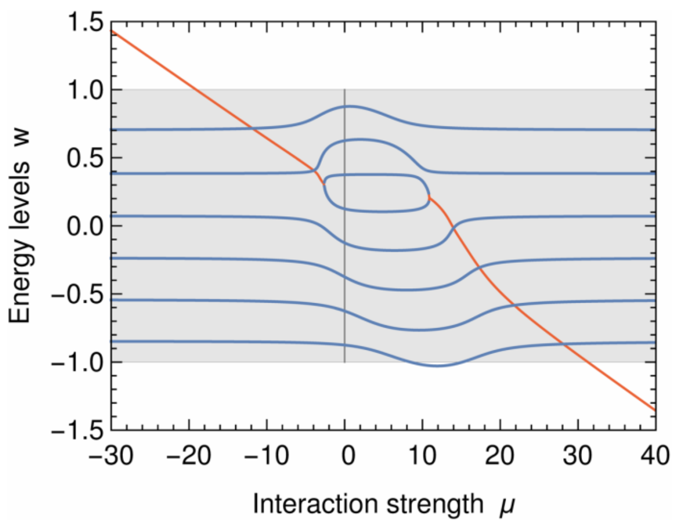
- **Propagating and unstable collective modes**
- **Unstable modes begin under the light cone**
- **No spurious instabilities in discrete case**

"Strong" spectral crossing

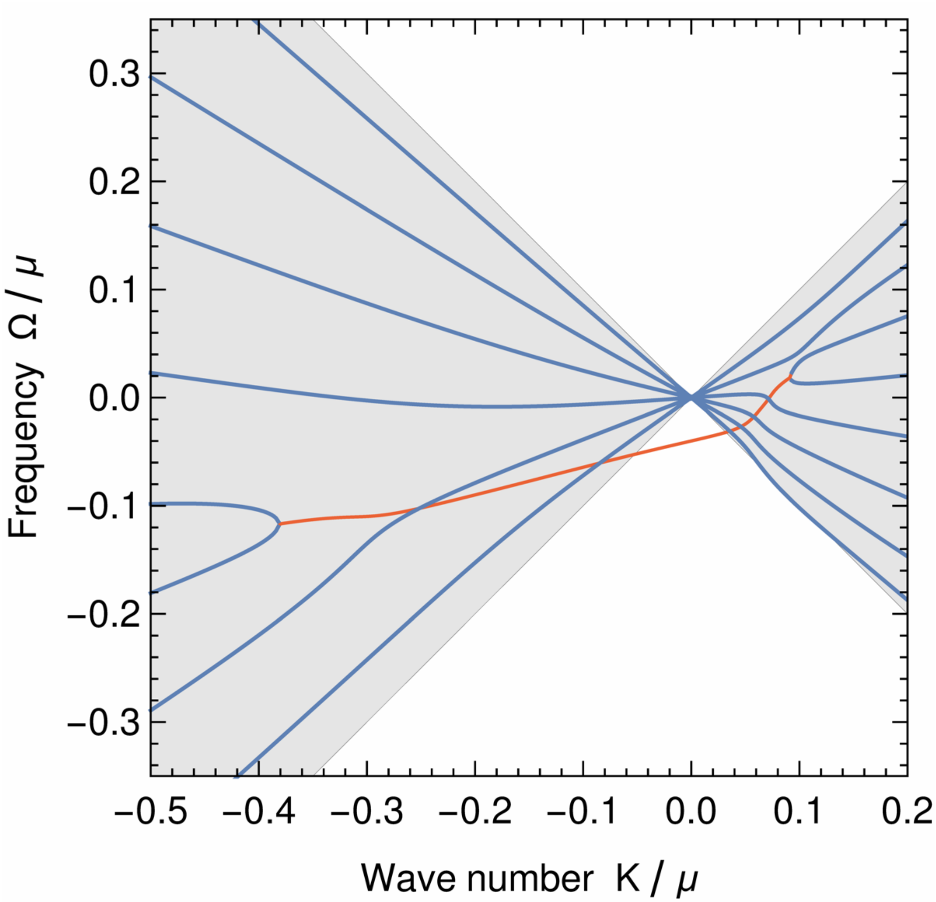
ELN angle distribution



Energy levels

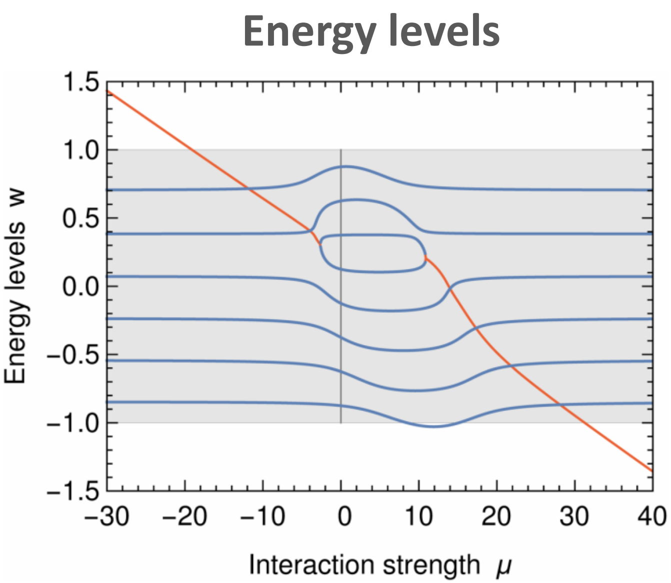


Dispersion relation

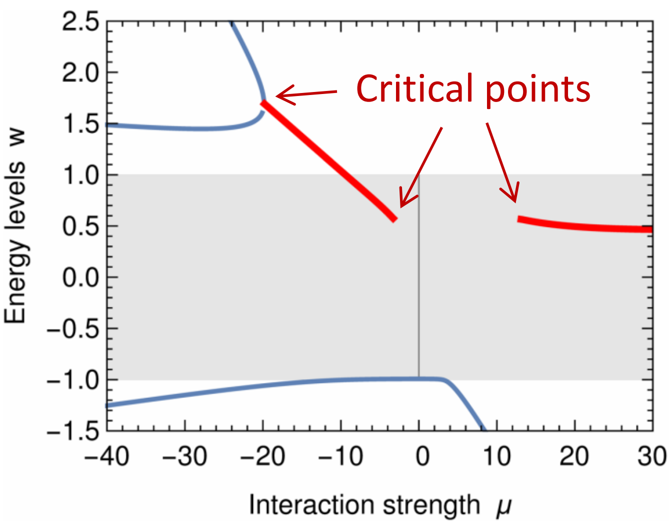


- **Only unstable collective modes**
- **Begin and end under the light cone**

Spectral crossing – Continuous Limit



- Solutions with complex eigenvalues appear as merging of two real eigenvalues
- With increasing μ must emerge below the light cone



Continuous limit of vanishing mode spacing:

- Critical points at $w = \cos(\theta_0)$, i.e. at crossing where $G(\cos\theta_0) = 0$
- Interaction strength $\mu_{1,2}$ of critical points follow easily from $G(\cos\theta)$
- Single crossing: Complex solution guaranteed
- Several crossings: Not guaranteed

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Fast Pairwise Conversion of Supernova Neutrinos: A Dispersion Relation Approach

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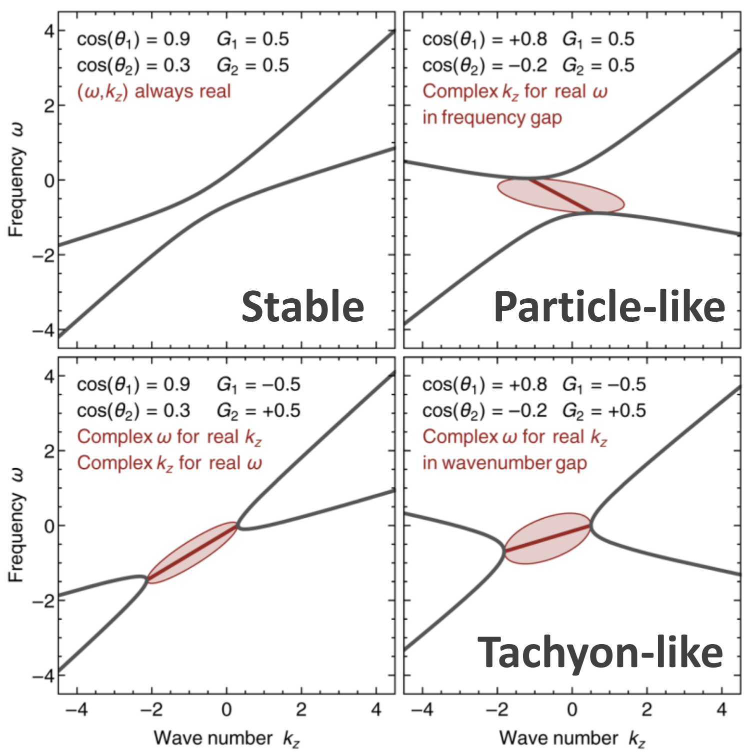
Classification of instabilities of "flavor waves" (Two-beam model)



Classification of instabilities of plasma waves (Two-beam model)

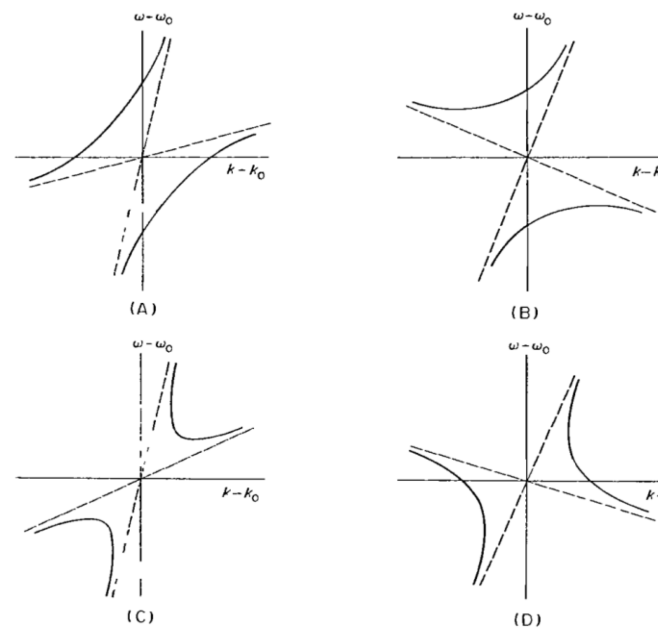
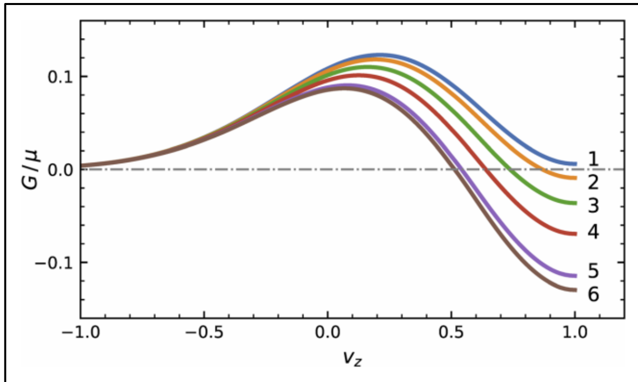


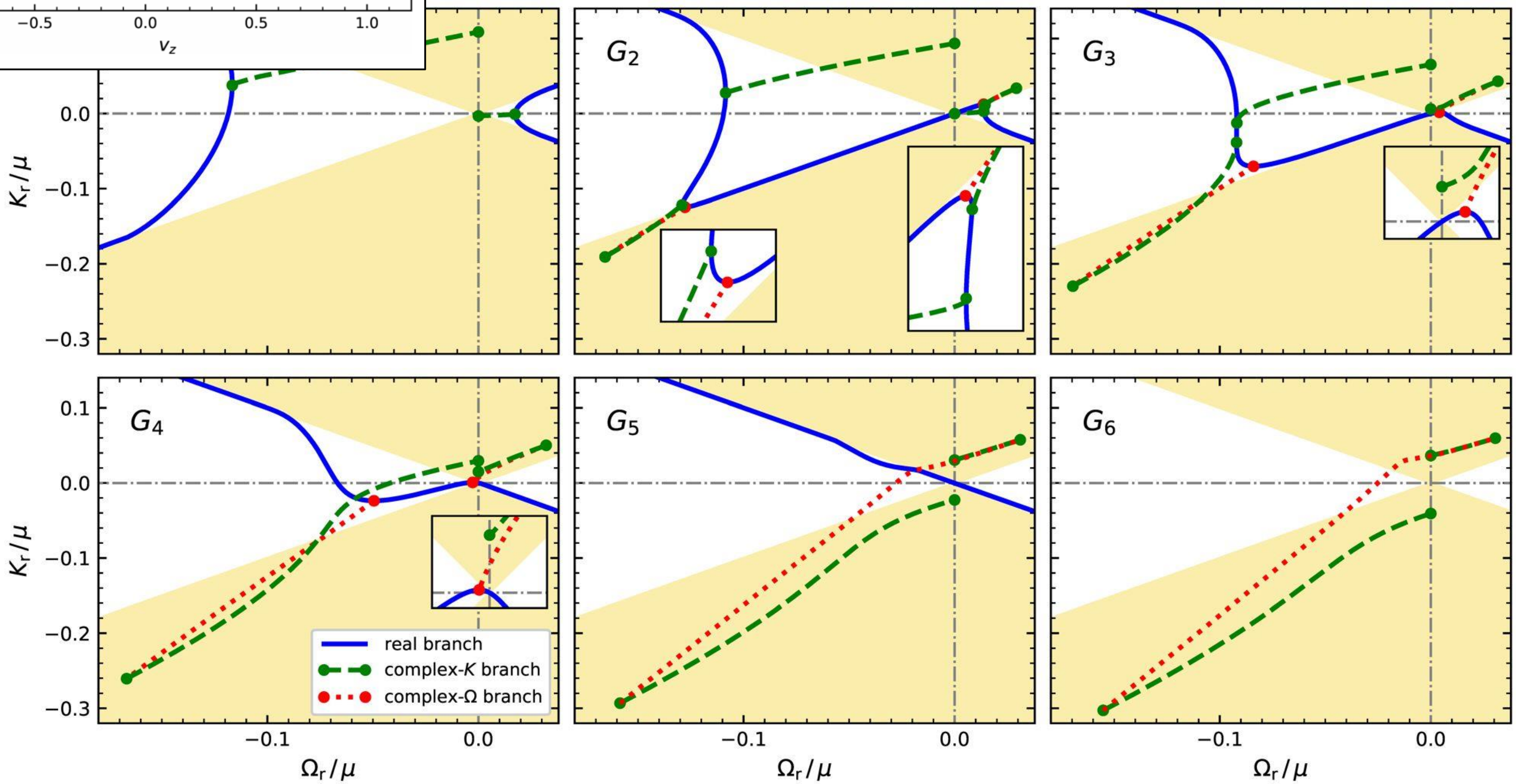
Fig. 23.

Landau & Lifshitz, Vol.10, Physical Kinetics
Chapter VI, Instability Theory

Dispersion Relation for Neutrino Flavor Waves



Dispersion relations for different angle distributions of neutrino gas



Yi, Ma, Martin & Duan, PRD 99 (2019) 063005 [arXiv:1901.01546]

Summary on Dispersion of Fast Flavor Waves

- Neutrino-neutrino interactions lead to emergence of collective modes of flavor coherence (propagating or unstable)
- Need not exist for every $\vec{K}/\mu$ (dispersion relations can end)
- Co-exist with non-collective modes
- "Wave packet of flavor coherence" dissipates by kinematical decoherence between non-collective modes
- Contains non-dissipating (propagating or growing) projection for sufficiently strong nu-nu interaction effect
- Explicit formulation of eigenfunctions for non-collective modes leads to simple identification of critical points
- Stable collective modes "peel off" from the light cone and exist only outside
- Unstable collective modes begin/end under the light cone from coalescence of non-collective modes

Capozzi, Raffelt & Stirner, arXiv:1906.08794, JCAP in press

Stability Criteria for Fast Flavor Waves

- Has only been investigated for axially symmetric distribution around K-vector
- One angle crossing guarantees instability
- Even number of crossings can be stable
- Odd number of crossings needed?
- General non-axisymmetric distribution?

Many Open Questions

Flavor evolution in dense neutrino flows still on the level of simplified toy models and parametric studies

- Kinetic equation in the mean-field limit justified?
- Realistic normal-mode analysis without symmetry assumptions?
- Realistic triggering of stable or unstable flavor waves?
- Do tachyonic modes lead to flavor equilibration?
- Realistic impact on SN explosion and nucleosynthesis?

For the past 25 years the same message:

"It is only the beginning.
A lot more work
ahead of us …"