# Fundamentals of collective neutrino oscillations

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Neutrino Quantum Kinetics in Dense Environments

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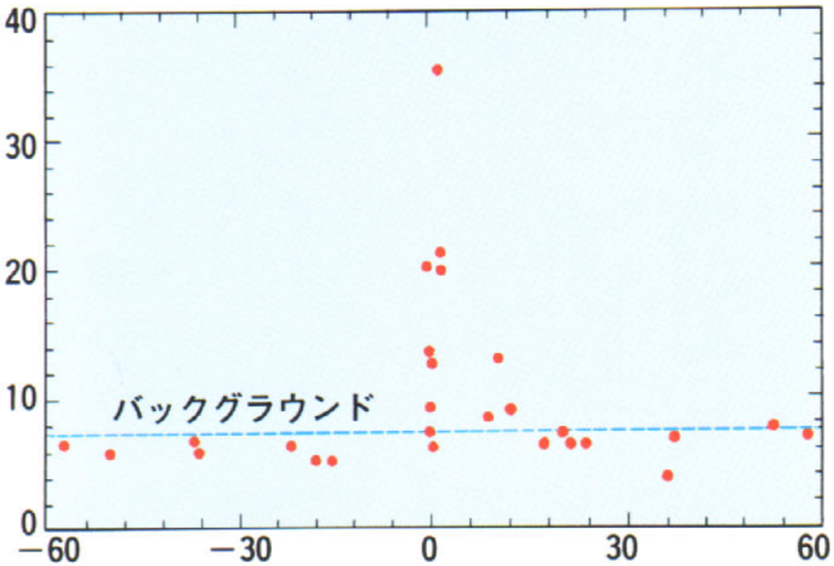
U.S. DEPARTMENT OF ENERGY | Office of Science

# N3AS Collaboration
# Network for Neutrinos, Nuclear Astrophysics, and Symmetries

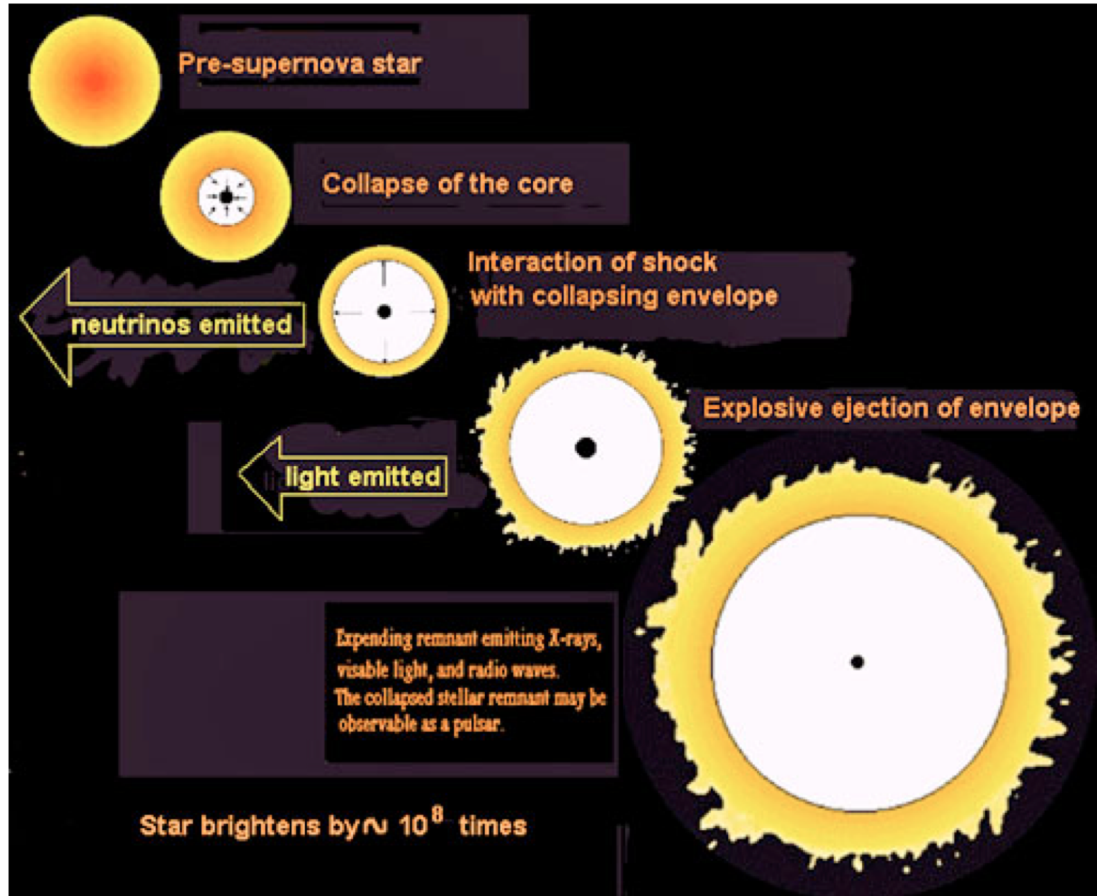
Multi-institutional network (3 centers and 8 sites) dedicated to recruiting and training postdocs and fostering collaborative efforts

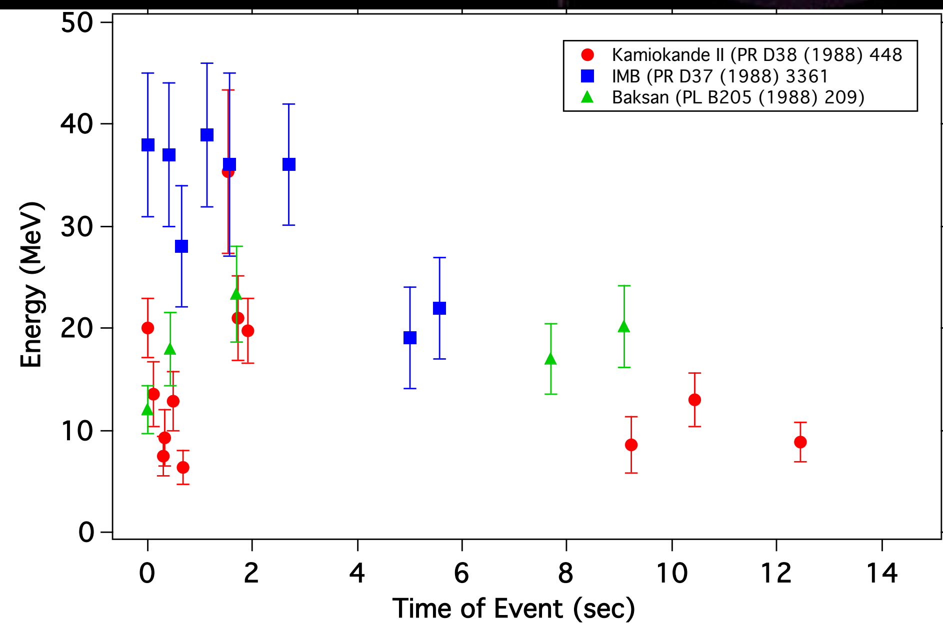
N3AS Postdocs: Jeff Berryman, Evan Grohs, Sophia Han, Amol Patwardhan, Sherwood Richers, Ermal Rrapaj, Manibrata Sen, and Xilu Wang

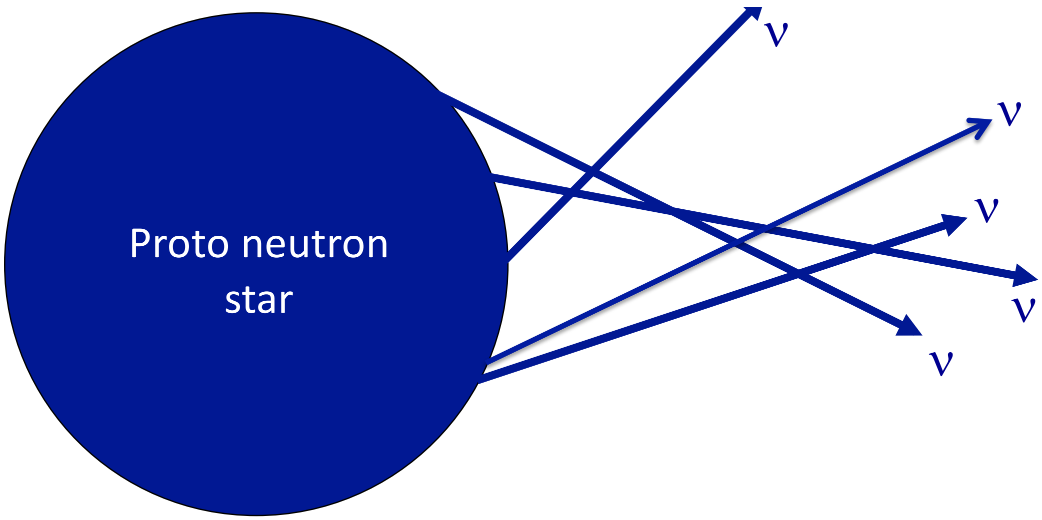
# Neutrinos from core-collapse supernovae




- $M_{prog} \geq 8\ M_{sun} \Rightarrow \Delta E \approx 10^{53}$ ergs $\approx 10^{59}$ MeV
- 99% of the energy is carried away by neutrinos and antineutrinos with $10 \leq E_\nu \leq 30$ MeV $\Rightarrow 10^{58}$ neutrinos






Energy released in a core-collapse SN: $\Delta E \approx 10^{53}$ ergs $\approx 10^{59}$ MeV
99% of this energy is carried away by neutrinos and antineutrinos!
~ $10^{58}$ Neutrinos!
This necessitates including the effects of νν interactions!

$$H = \sum a^{\dagger} a + \sum (1 - \cos\varphi) a^{\dagger} a^{\dagger} a a$$

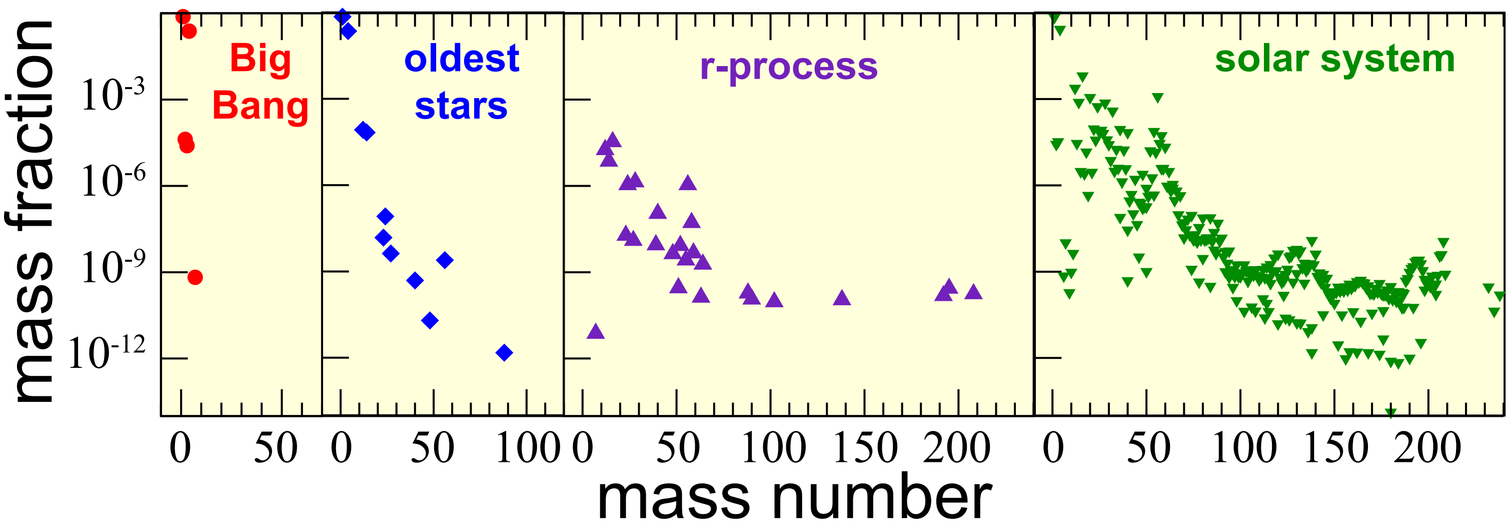
ν oscillations
MSW effect

neutrino-neutrino interactions

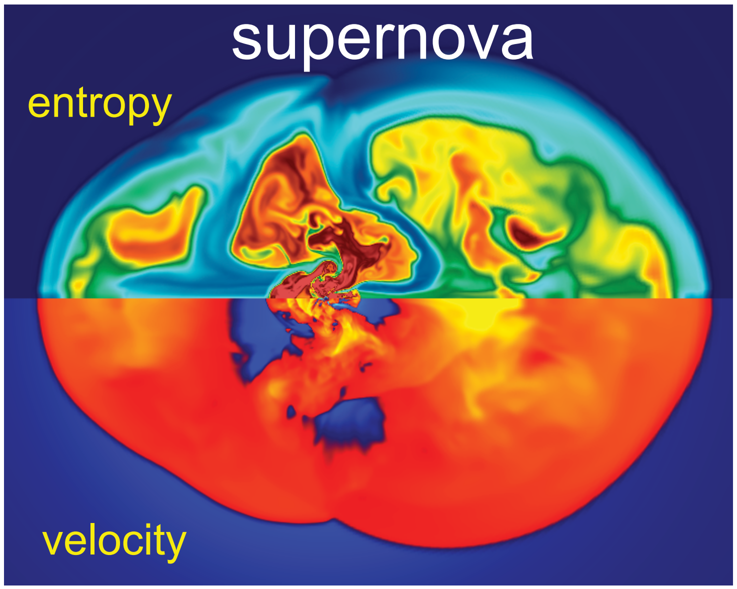
The second term makes the physics of a neutrino gas in a core-collapse supernova a very interesting many-body problem, driven by weak interactions.

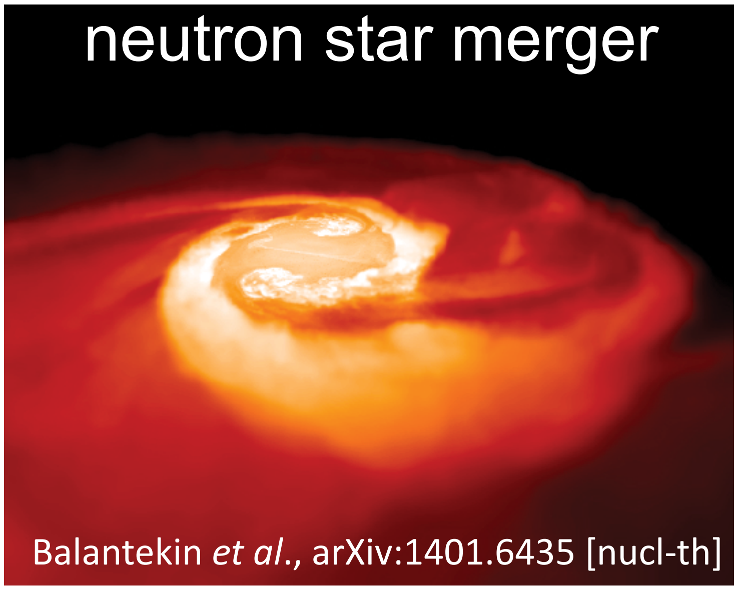
Neutrino-neutrino interactions lead to novel collective and emergent effects, such as conserved quantities and interesting features in the neutrino energy spectra (spectral "swaps" or "splits").

# The origin of elements




Neutrinos not only play a crucial role in the dynamics of these sites, but they also control the value of the electron fraction, the parameter determining the yields of the r-process.

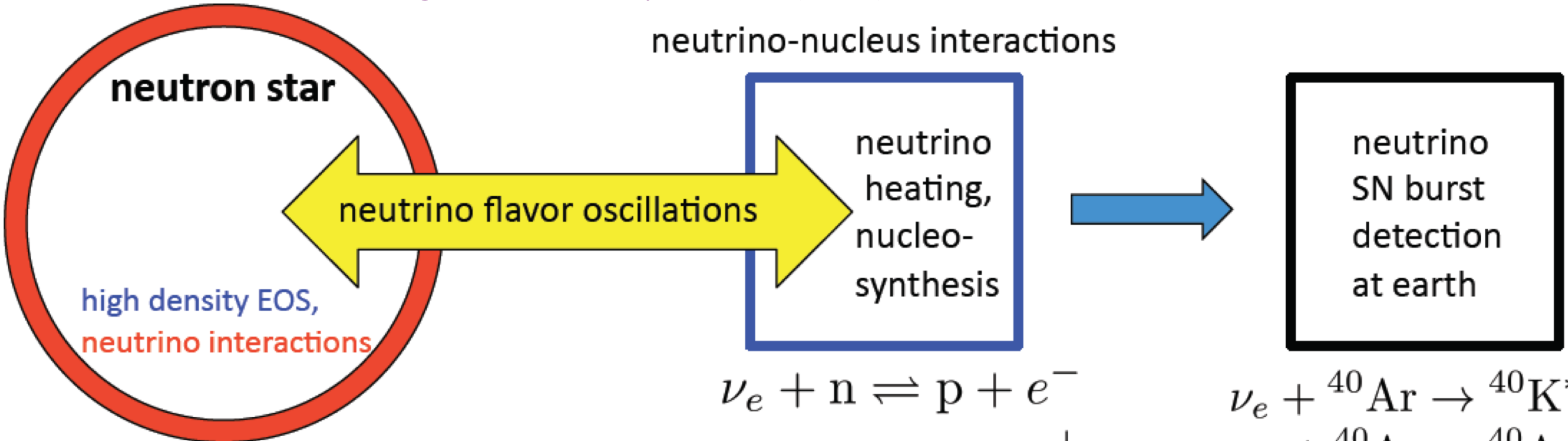



Possible sites for the r-process

Understanding a core-collapse supernova requires answers to a variety of questions some of which need to be answered, both theoretically and experimentally.

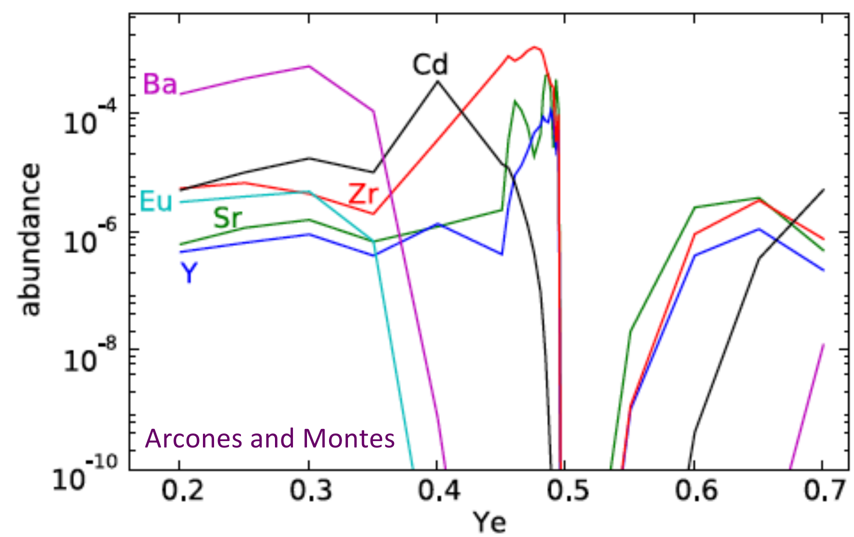
Balantekin and Fuller, Prog. Part. Nucl. Phys. **71** 162 (2013)




$$\nu_e + \mathrm{n} \rightleftharpoons \mathrm{p} + e^-$$
$$\bar{\nu}_e + \mathrm{p} \rightleftharpoons \mathrm{n} + e^+$$

$$\nu_e + {}^{40}\mathrm{Ar} \to {}^{40}\mathrm{K}^* + e^-$$
$$\nu_x + {}^{40}\mathrm{Ar} \to {}^{40}\mathrm{Ar}^* + \nu_x$$

$$Y_e = \frac{N_p}{N_p + N_n} = \frac{1}{1 + \lambda_p / \lambda_n}$$



# Collective Neutrino Oscillations

$$H = \sum_{\mathbf{p}} \omega_{\mathbf{p}} \vec{B} \cdot \vec{J}_{\mathbf{p}} + \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos \vartheta_{\mathbf{pq}}) \, \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$

$$\vec{B} = (0, 0, -1)_{\text{mass}} = (\sin 2\theta, 0, -\cos 2\theta)_{\text{flavor}}, \quad \omega_p = \frac{\delta m^2}{2p}$$

$$J_{\mathbf{p}}^{+} = a_1^{\dagger}(\mathbf{p}) a_2(\mathbf{p})$$
$$J_{\mathbf{p}}^{-} = a_2^{\dagger}(\mathbf{p}) a_1(\mathbf{p})$$
$$J_{\mathbf{p}}^{z} = \frac{1}{2} \left( a_1^{\dagger}(\mathbf{p}) a_1(\mathbf{p}) - a_2^{\dagger}(\mathbf{p}) a_2(\mathbf{p}) \right)$$

$$\begin{aligned} a_e(\mathbf{p}) &= \cos\theta \, a_1(\mathbf{p}) + \sin\theta \, a_2(\mathbf{p}) \\ a_x(\mathbf{p}) &= -\sin\theta \, a_1(\mathbf{p}) + \cos\theta \, a_2(\mathbf{p}) \end{aligned}$$

Two of the adiabatic eigenstates of this equation are easy to find in the single-angle approximation:

$$H = \sum_p \omega_p \vec{B} \cdot \vec{J}_p + \mu(r) \vec{J} \cdot \vec{J}$$

$$|j, +j\rangle = |N/2, N/2\rangle = |\nu_1, \ldots, \nu_1\rangle$$

$$|j, -j\rangle = |N/2, -N/2\rangle = |\nu_2, \ldots, \nu_2\rangle$$

$$E_{\pm N/2} = \mp \sum_p \omega_p \frac{N_p}{2} + \mu \frac{N}{2} \left( \frac{N}{2} + 1 \right)$$

To find the others will take a lot more work

# What is the mean-field approximation?

$$\left[\hat{O}_1, \hat{O}_2\right] \cong 0$$

$$\hat{O}_1\hat{O}_2 \approx \hat{O}_1\left\langle\hat{O}_2\right\rangle + \left\langle\hat{O}_1\right\rangle\hat{O}_2 - \left\langle\hat{O}_1\hat{O}_2\right\rangle$$

Expectation values should be calculated with a state $\left|\Psi\right\rangle$ chosen to satisfy:

$$\left\langle\hat{O}_1\hat{O}_2\right\rangle = \left\langle\hat{O}_1\right\rangle\left\langle\hat{O}_2\right\rangle$$

This reduces the two-body problem to a one-body problem:

$$a^\dagger a^\dagger a a \Rightarrow \left\langle a^\dagger a\right\rangle a^\dagger a + \left\langle a^\dagger a^\dagger\right\rangle a a + \text{h.c.}$$

$$\frac{\sqrt{2}G_F}{V}\sum_{\mathbf{p},\mathbf{q}}\left(1 - \cos\vartheta_{\mathbf{pq}}\right)\vec{J}_{\mathbf{p}}\cdot\vec{J}_{\mathbf{q}} \rightarrow \frac{\sqrt{2}G_F}{V}\sum_{\mathbf{p},\mathbf{q}}\left(1 - \cos\vartheta_{\mathbf{pq}}\right)\langle\vec{J}_{\mathbf{p}}\rangle\cdot\vec{J}_{\mathbf{q}}$$

# Mean field

## Neutrino-neutrino interaction

$$\bar{\psi}_{\nu L}\gamma^{\mu}\psi_{\nu L}\bar{\psi}_{\nu L}\gamma_{\mu}\psi_{\nu L} \Rightarrow \bar{\psi}_{\nu L}\gamma^{\mu}\psi_{\nu L}\left\langle \bar{\psi}_{\nu L}\gamma_{\mu}\psi_{\nu L}\right\rangle + \cdots$$

## Antineutrino-antineutrino interaction

$$\bar{\psi}_{\bar{\nu} R}\gamma^{\mu}\psi_{\bar{\nu} R}\bar{\psi}_{\bar{\nu} R}\gamma_{\mu}\psi_{\bar{\nu} R} \Rightarrow \bar{\psi}_{\bar{\nu} R}\gamma^{\mu}\psi_{\bar{\nu} R}\left\langle \bar{\psi}_{\bar{\nu} R}\gamma_{\mu}\psi_{\bar{\nu} R}\right\rangle + \cdots$$

## Neutrino-antineutrino interaction

$$\bar{\psi}_{\nu L}\gamma^{\mu}\psi_{\nu L}\bar{\psi}_{\bar{\nu} R}\gamma_{\mu}\psi_{\bar{\nu} R} \Rightarrow \bar{\psi}_{\nu L}\gamma^{\mu}\psi_{\nu L}\left\langle \bar{\psi}_{\bar{\nu} R}\gamma_{\mu}\psi_{\bar{\nu} R}\right\rangle + \cdots$$

Balantekin and Pehlivan, JPG **34**,1783 (2007)

## Neutrino-antineutrino can also have an additional mean field

$$\bar{\psi}_{\nu L}\gamma^{\mu}\psi_{\nu L}\bar{\psi}_{\bar{\nu} R}\gamma_{\mu}\psi_{\bar{\nu} R} \Rightarrow \bar{\psi}_{\nu L}\gamma^{\mu}\left\langle \psi_{\nu L}\bar{\psi}_{\bar{\nu} R}\gamma_{\mu}\right\rangle \psi_{\bar{\nu} R} + \cdots$$

However note that

$\left\langle \psi_{\nu L}\bar{\psi}_{\bar{\nu} R}\gamma_{\mu}\right\rangle \propto m_{\nu}$ (negligible if the medium is isotropic)

Fuller *et al.*
Volpe

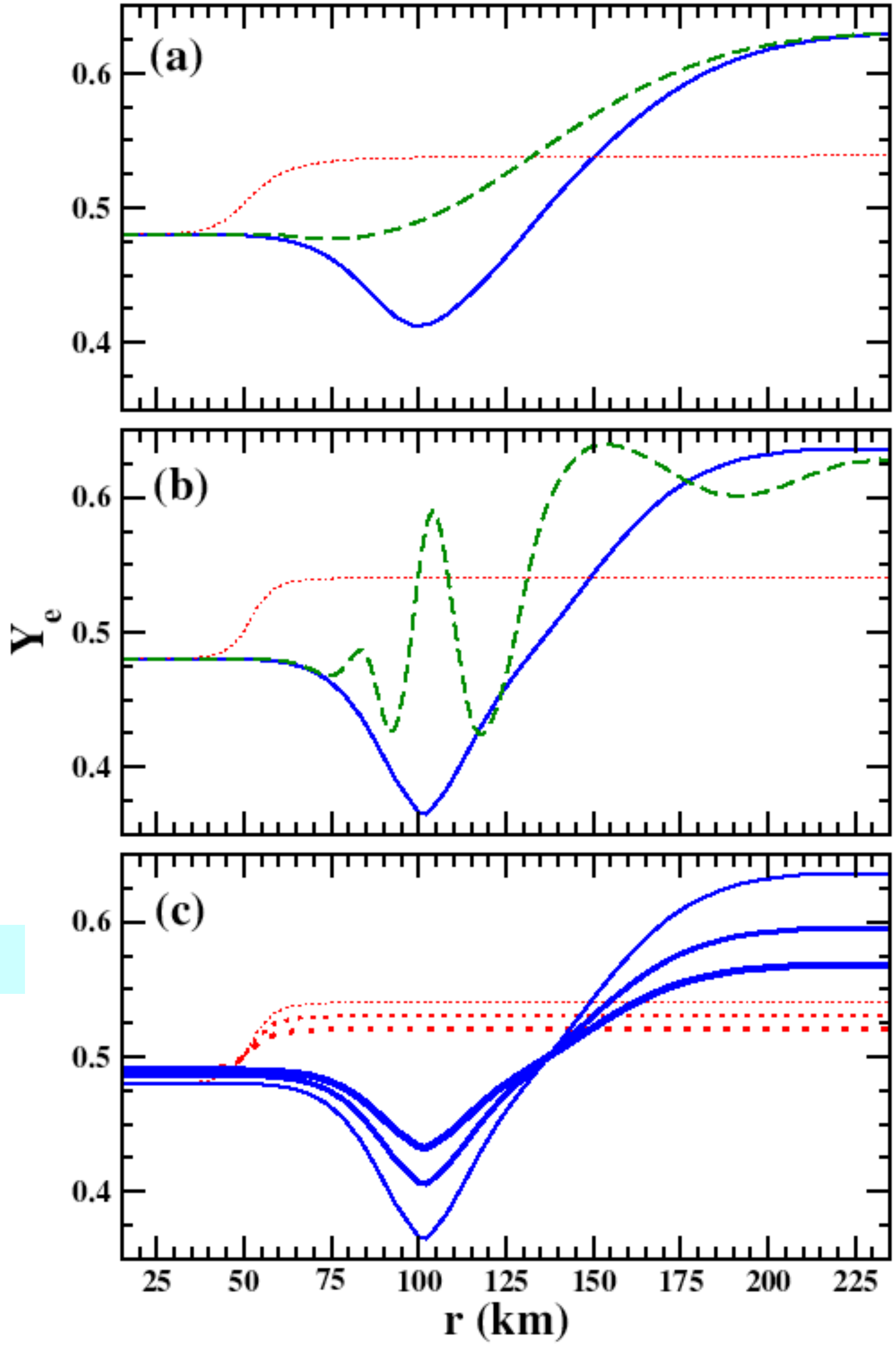
An example of an early mean field calculation

Equilibrium electron fraction with the inclusion of νν interactions

$L^{51}$ = 0.001, 0.1, 50

Balantekin and Yuksel, 2005

$X_\alpha$= 0, 0.3, 0.5 (thin, medium, thick lines)



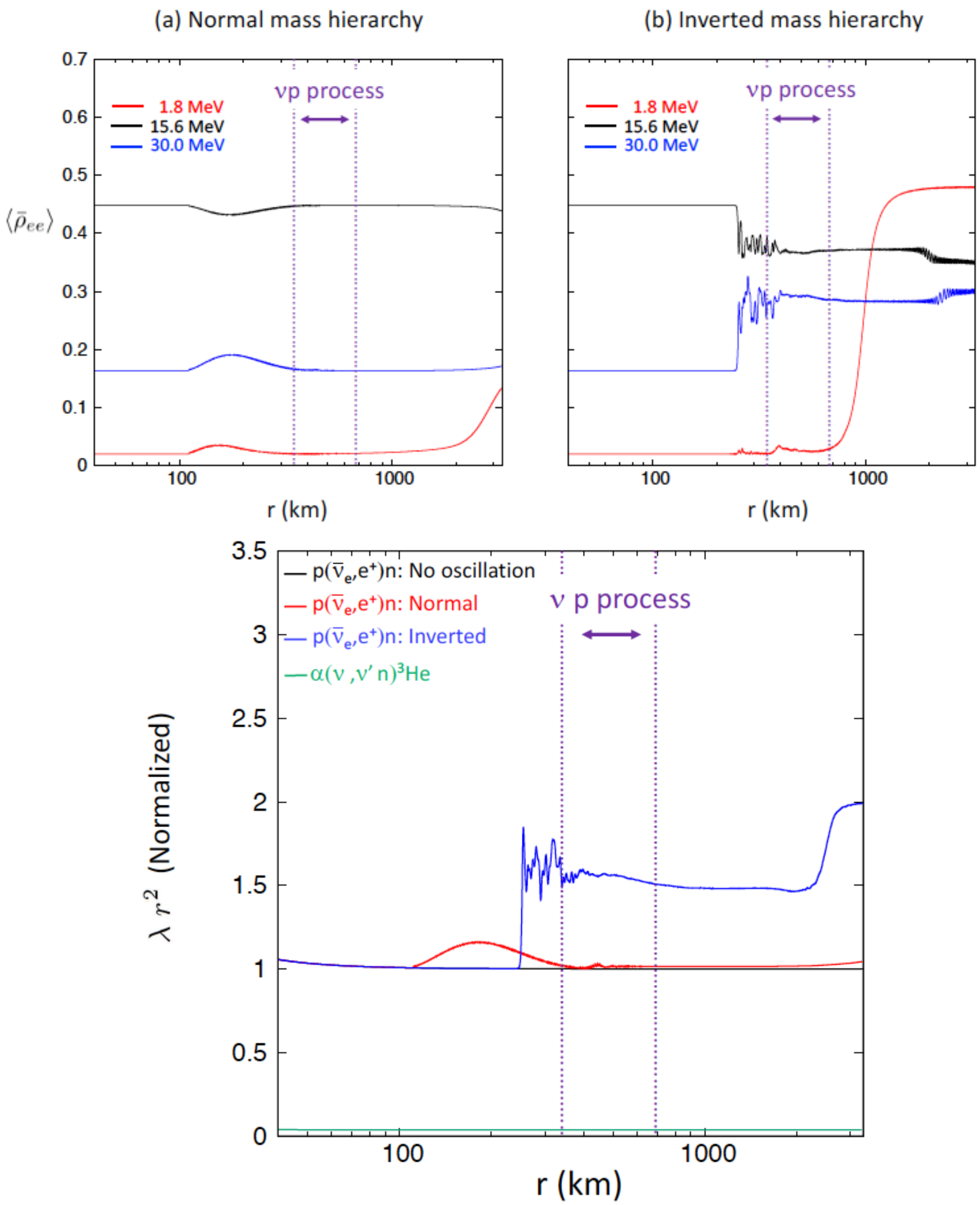

$\theta_{13}$~π/10

$\theta_{13}$~π/20

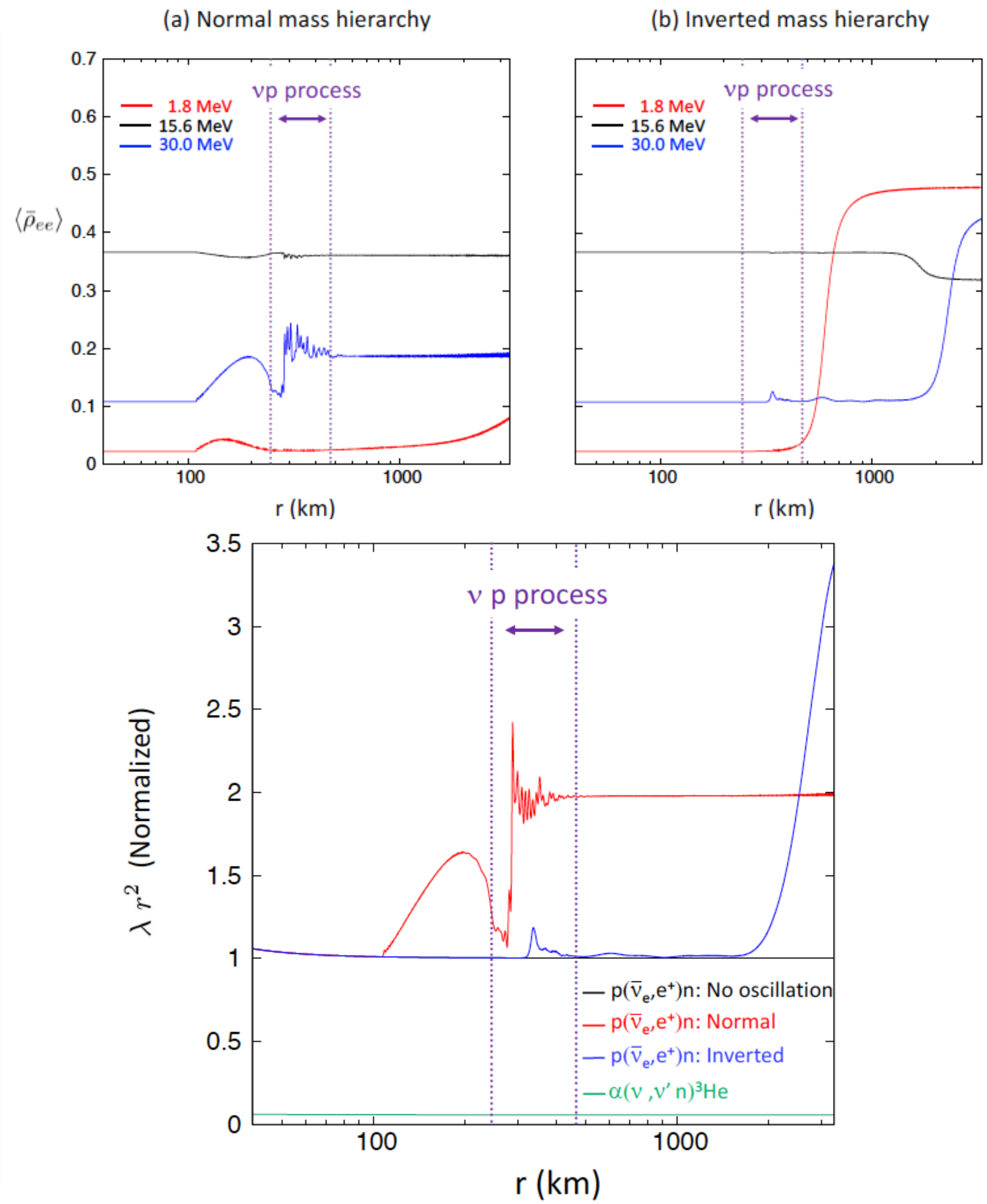
$\theta_{13}$~π/20 with α effect

# Collective Oscillations within mean field for the νp process

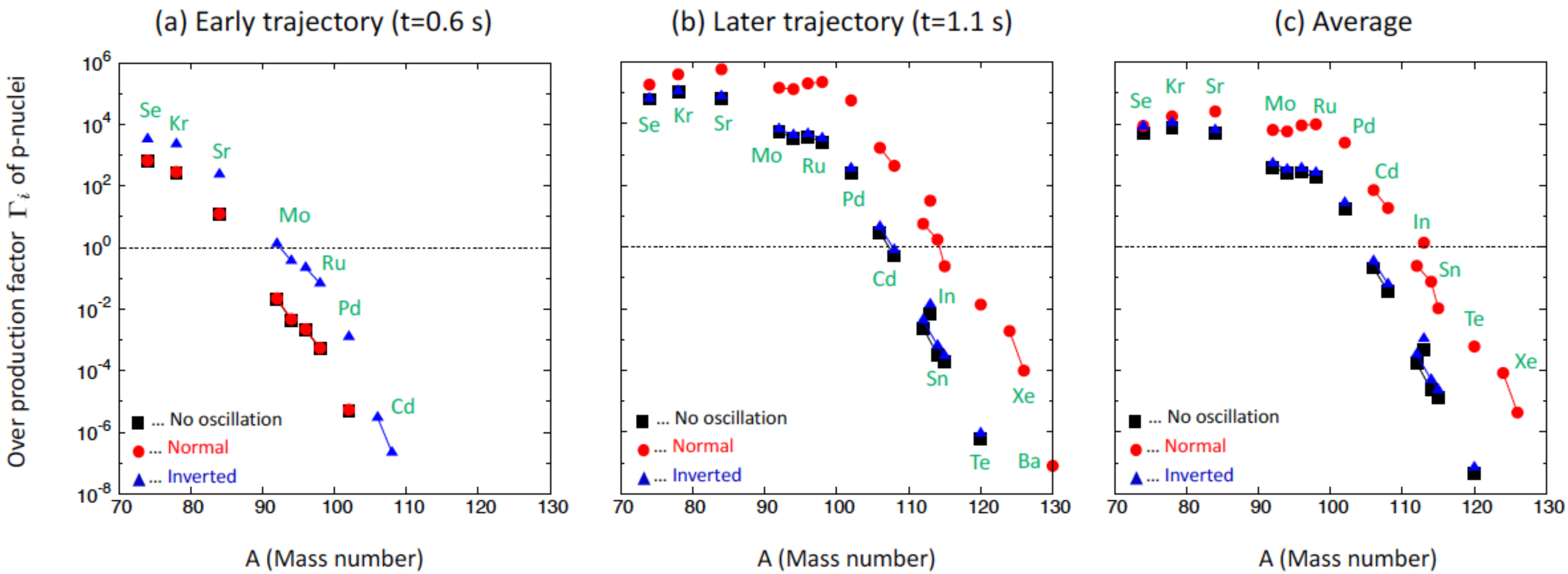
Sasaki *et al.,* Phys.Rev. D96 (2017) 043013




Early outflow (t=0.6 s.)




Later outflow (t=1.1 s.)




# Impact of the production of p-nuclei

Sasaki *et al.,* Phys.Rev. D96 (2017) 043013

See also Wu et al., Phys. Rev. D91 (2015) 065016

# This problem is “exactly solvable” in the single-angle approximation

$$H = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos\vartheta_{\mathbf{pq}}) \, \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$

$$H = \sum_p \omega_p \vec{B} \cdot \vec{J}_p + \mu(r) \vec{J} \cdot \vec{J}$$

# This problem is "exactly solvable" in the single-angle approximation

Pehlivan, Balantekin, Kajino, Yoshida, PRD**84** (2011) 065008

$$H = \sum_p \omega_p \vec{B} \cdot \vec{J}_p + \mu(r) \vec{J} \cdot \vec{J}$$

$$-\frac{1}{2\mu} - \sum_{p=1}^{M} \frac{j_p}{\omega_p - \zeta_\alpha} = \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^{\kappa} \frac{1}{\zeta_\alpha - \zeta_\beta}$$

$$|\zeta_1, \ldots, \zeta_\kappa\rangle = \mathcal{N}(\zeta_1, \ldots, \zeta_\kappa) \left( \prod_{\alpha=1}^{\kappa} S_\alpha^- \right) |j, +j\rangle$$

$$\vec{S}(\zeta_\alpha) \equiv \sum_p \frac{\vec{J}_p}{\omega_p - \zeta_\alpha}$$

$$E = E_{+N/2} + \sum_{\alpha=1}^{\kappa} \zeta_\alpha - \mu\kappa(N - \kappa + 1)$$

Recall that two of the adiabatic eigenstates of this equation are easy to find:

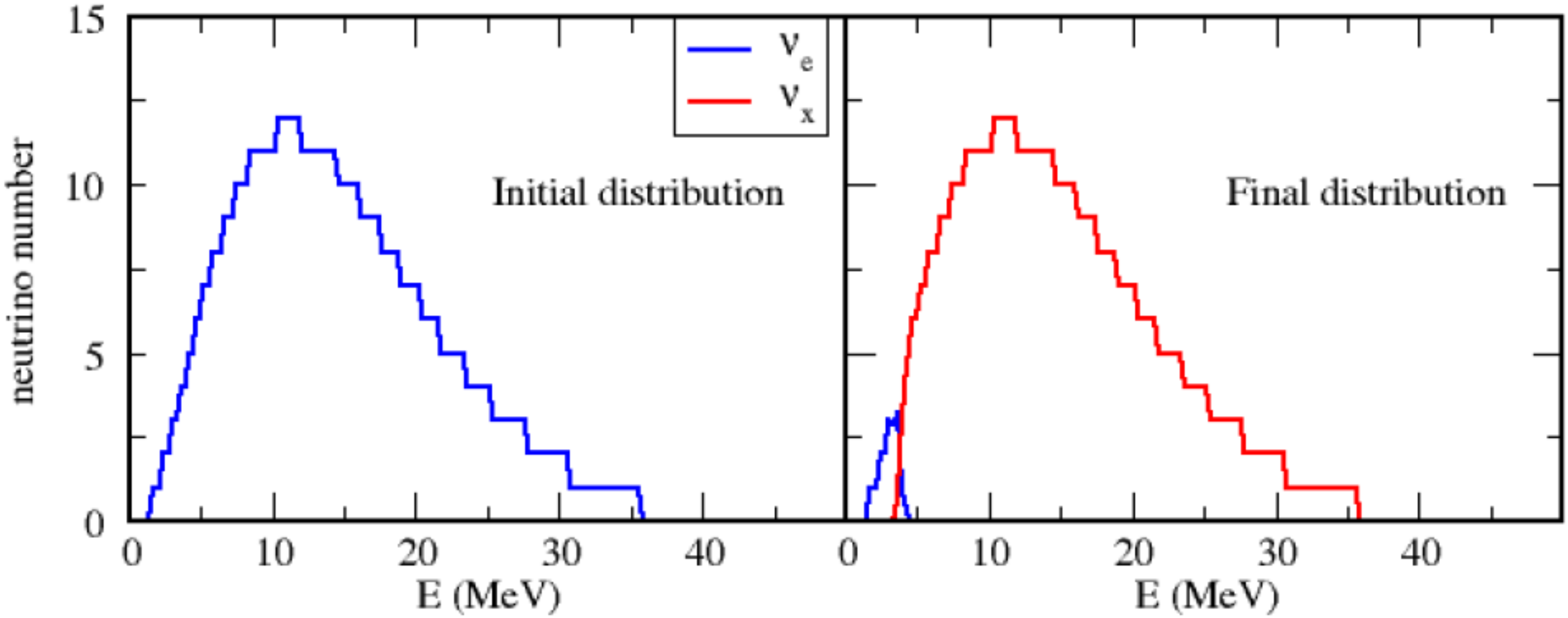
$$H = \sum_p \omega_p \vec{B} \cdot \vec{J}_p + \mu(r) \vec{J} \cdot \vec{J}$$

$$|j, +j\rangle = |N/2, N/2\rangle = |\nu_1, \ldots, \nu_1\rangle$$

$$|j, -j\rangle = |N/2, -N/2\rangle = |\nu_2, \ldots, \nu_2\rangle$$

$$E_{\pm N/2} = \mp \sum_p \omega_p \frac{N_p}{2} + \mu \frac{N}{2} \left( \frac{N}{2} + 1 \right)$$

# Away from the mean-field: First adiabatic solution of the *exact* many-body Hamiltonian

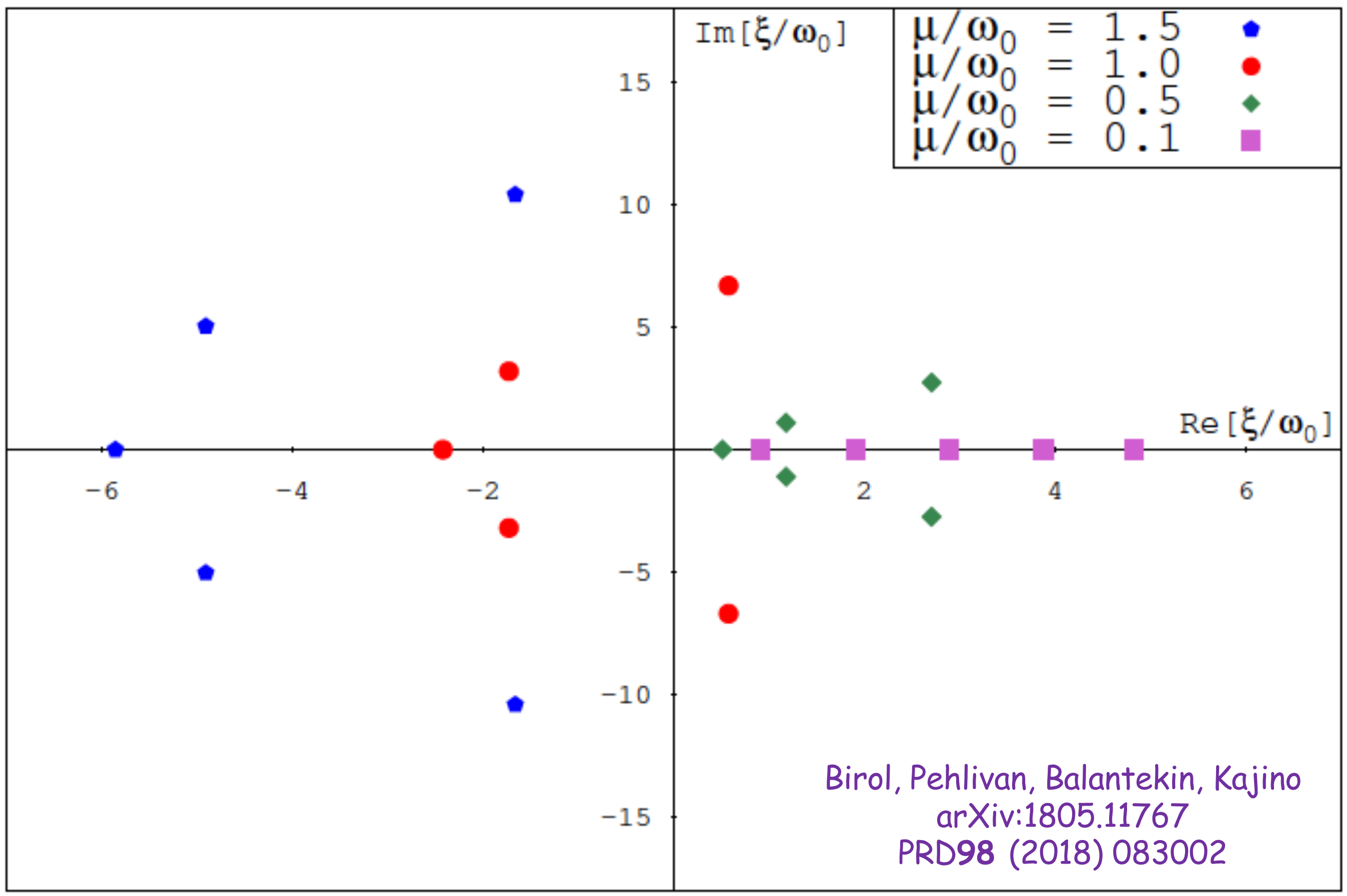



- Solutions of the Bethe ansatz equations for 250 neutrinos. Same behavior as the mean-field.
- Two flavors only
- Inverted hierarchy, no matter effect

2015

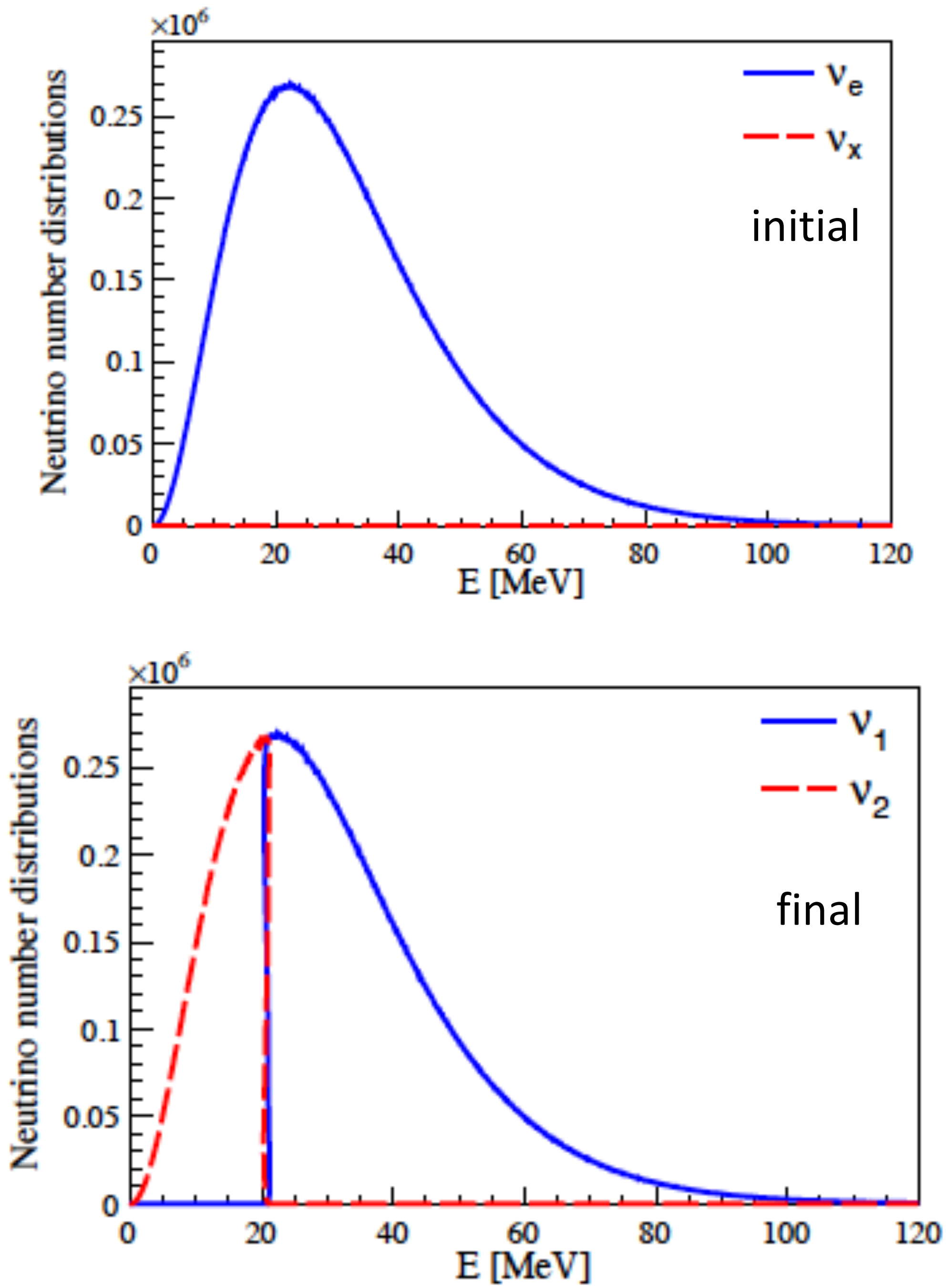
$$\mathbf{H} = \sum_{\omega} \omega \mathbf{B} \cdot \mathbf{J}_{\omega} + \mu(r) \mathbf{J} \cdot \mathbf{J}$$

$$|\psi\rangle_{\mu\to\infty} = \sum_{m=-n/2}^{m=+n/2} c_m \left|\frac{n}{2}, m\right\rangle \longrightarrow |\psi\rangle_{\mu\to 0} = \sum_m c_m \phi_m \left| \underbrace{\nu_1, \nu_1, \cdots, \nu_1}_{n/2+m}; \underbrace{\nu_2, \nu_2, \cdots, \nu_2}_{n/2-m} \right\rangle$$

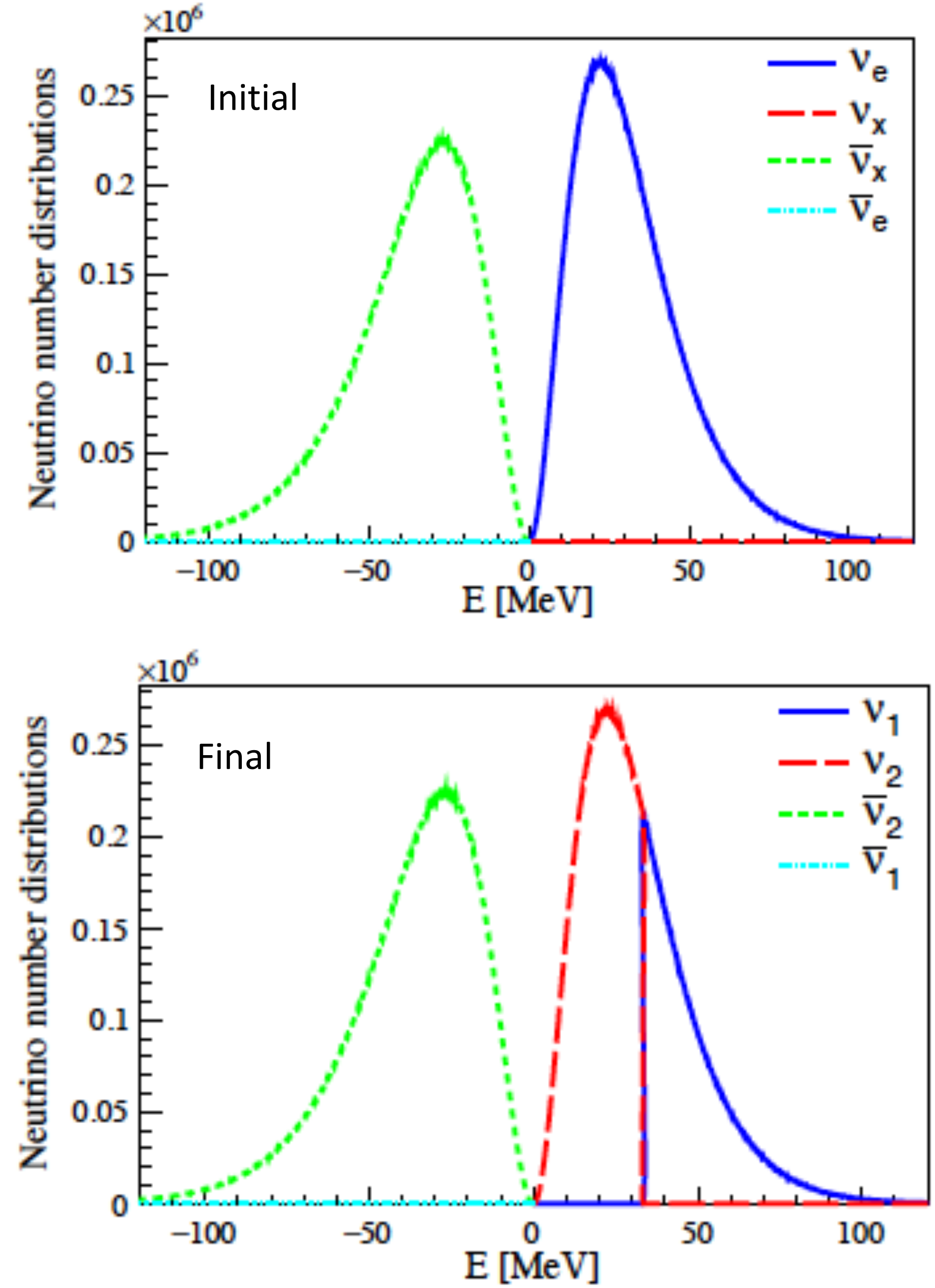


Adiabatic evolution of an initial thermal distribution (T = 10 MeV) of electron neutrinos. $10^8$ neutrinos distributed over 1200 energy bins with solar neutrino parameters and normal hierarchy.

Birol, Pehlivan, Balantekin, Kajino
arXiv:1805.11767
PRD**98** (2018) 083002



Adiabatic evolution of an initial thermal distribution of electron neutrinos (T=10 MeV) and antineutrinos of another flavor (T=12MeV). $10^8$ neutrinos distributed over 1200 energy bins both for neutrinos and antineutrinos with solar neutrino parameters and normal hierarchy.




Birol, Pehlivan, Balantekin, Kajino

# A more practical approach

$$\Lambda(\lambda) = \sum_{\alpha=1}^{\kappa} \frac{1}{\lambda - \zeta_\alpha}$$

$$\Lambda(\lambda)^2 + \Lambda'(\lambda) + \frac{1}{\mu}\Lambda(\lambda) = \sum_{q=1}^{M} 2j_q \frac{\Lambda(\lambda) - \Lambda(\omega_q)}{\lambda - \omega_q}$$

$$\Lambda_p^2 + (1 - 2j_p)\Lambda'_p + \frac{1}{\mu}\Lambda_p = \sum_{\substack{q=1 \\ q\neq p}}^{M} 2j_q \frac{\Lambda_p - \Lambda_q}{\omega_p - \omega_q}, \quad \Lambda_k = \Lambda(\omega_k)$$

$$E = E_{N/2} - 2\mu \sum_{p=1}^{M} j_p \omega_p \Lambda_p.$$

Patwardhan, Cervia, Balantekin, Phys.Rev. D99, 123013 (2019)

## These $\Lambda(\omega_p) = \Lambda_p$ are eigenvalues of the invariants, $h_p$:

$$h_p = -J_p^z + 2\mu \sum_{\substack{q=1 \\ q \neq p}}^{N} \frac{\mathbf{J}_p \cdot \mathbf{J}_q}{\omega_p - \omega_q}, \quad [h_p, h_q] = 0, p \neq q$$

$$\sum_p \omega_p h_p = -\sum_{p=1}^{N} \omega_p J_p^z + \mu \sum_{\substack{p,q=1 \\ p \neq q}}^{N} \mathbf{J}_p \cdot \mathbf{J}_q$$

$$h_p |\xi_1, \ldots, \xi_\kappa\rangle = \left( 2\mu \sum_{\substack{q=1 \\ q \neq p}}^{N} \frac{j_p j_q}{\omega_p - \omega_q} + j_p - 2\mu j_p \Lambda(\omega_p) \right) |\xi_1, \ldots, \xi_\kappa\rangle$$

Cervia, Patwardhan, Balantekin, arXiv:1905.00082

# Adiabatic Eigenstates

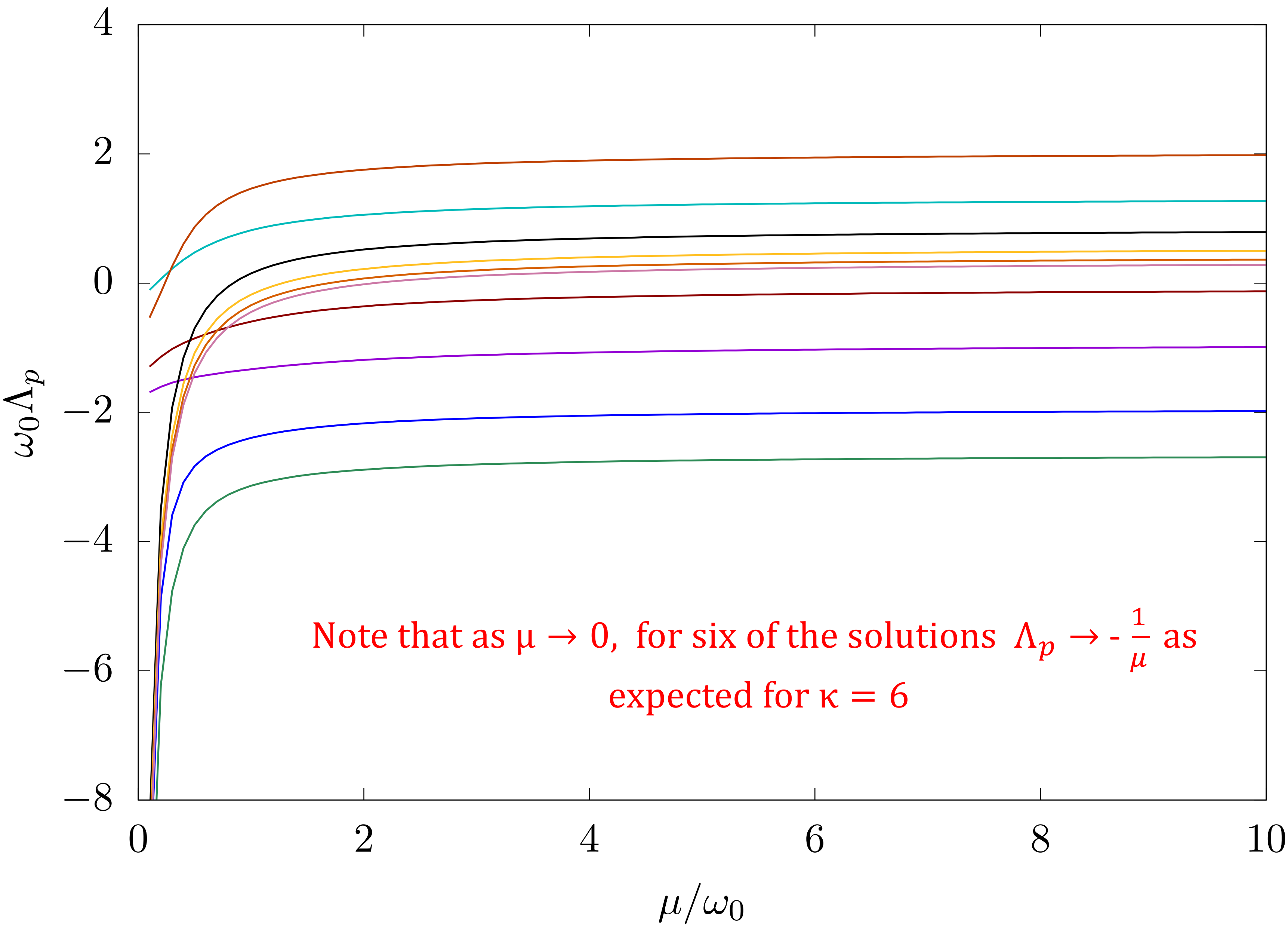
$$\Lambda(\lambda)^2 + \frac{1}{\mu}\Lambda(\lambda) = \sum_{q=1}^{M} 2j_q \frac{\Lambda(\lambda) - \Lambda(\omega_q)}{\lambda - \omega_q}$$

$$S_1^- S_2^- |j, +j\rangle = \frac{1}{2} \sum_{\substack{p,q=1 \\ p \neq q}}^{M} \left( \Lambda_p \Lambda_q + \frac{\Lambda_p - \Lambda_q}{\omega_p - \omega_q} \right) J_p^- J_q^- |j, +j\rangle$$

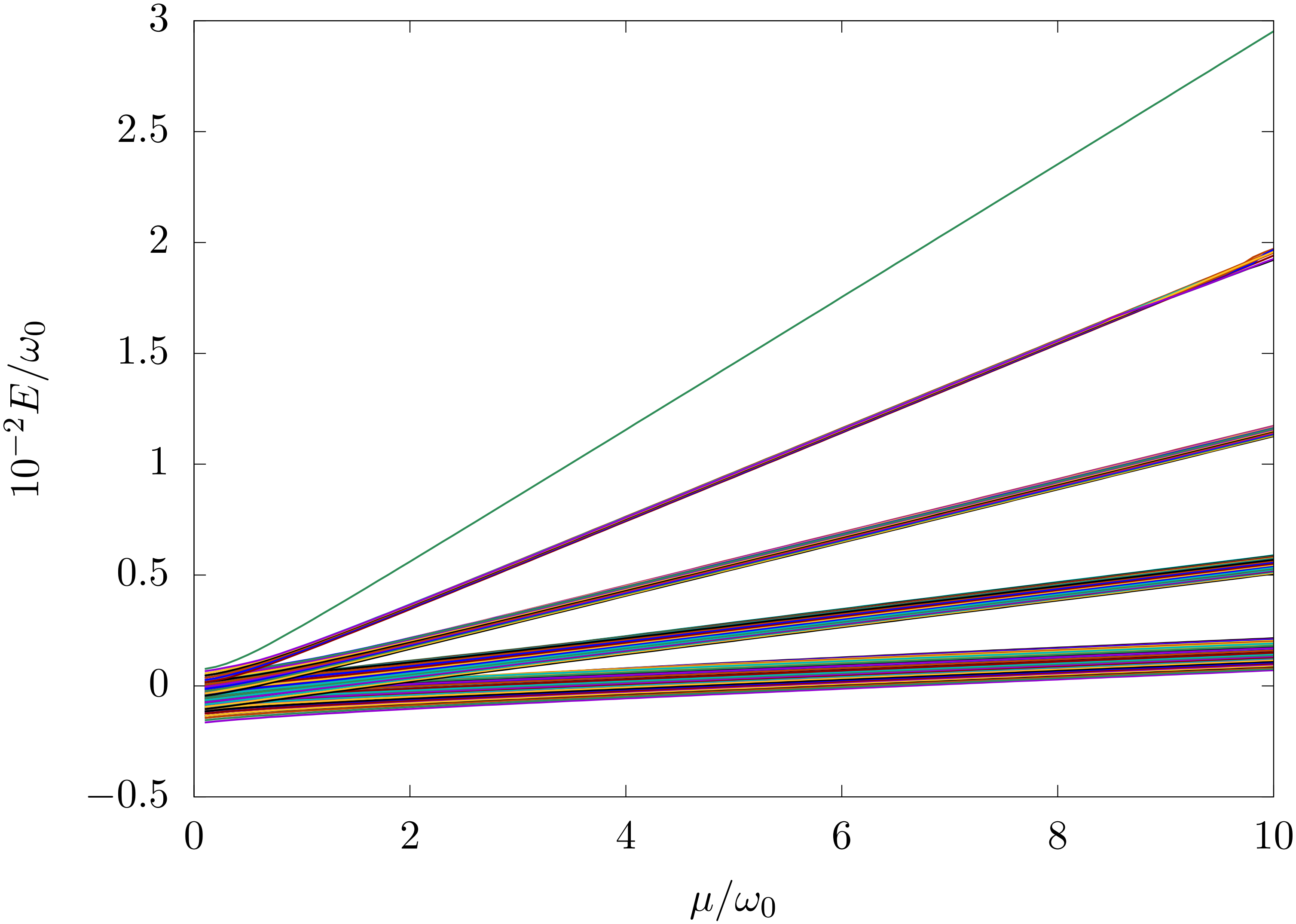
$$S_1^- S_2^- S_3^- |j, +j\rangle = \frac{1}{3!} \times$$

$$\sum_{\substack{p,q,r=1 \\ p,q,r \\ \text{distinct}}}^{M} \left[ \Lambda_p \Lambda_q \Lambda_r + 3\Lambda_r \frac{\Lambda_p - \Lambda_q}{\omega_p - \omega_q} + \frac{2}{\omega_p - \omega_q} \left( \frac{\Lambda_p - \Lambda_r}{\omega_p - \omega_r} - \frac{\Lambda_q - \Lambda_r}{\omega_q - \omega_r} \right) \right] J_p^- J_q^- J_r^- |j, +j\rangle$$

One of the 210 solutions for κ = 6 for 10 neutrinos in 10 bins with $\omega_p = p\omega_0$



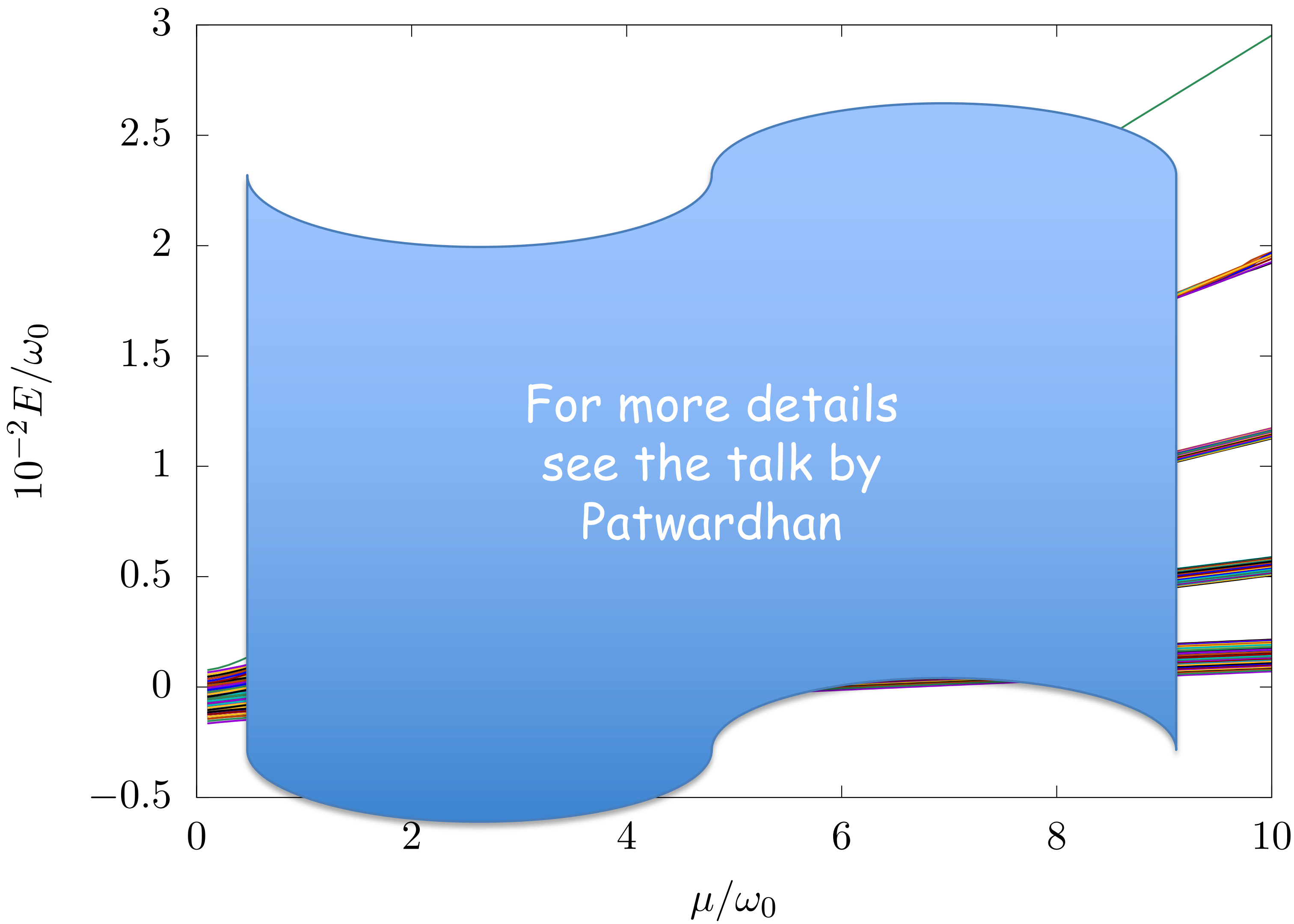
# Energy eigenvalues for a system with 10 neutrinos in 10 bins with $\omega_p = p\omega_0$ for $\kappa = 4$




$$|\zeta_1, \ldots, \zeta_\kappa\rangle = \mathcal{N}(\zeta_1, \ldots, \zeta_\kappa) \left[ \prod_{\alpha=1}^{\kappa} \left( \sum_p \frac{J_p^-}{\omega_p - \zeta_\alpha} \right) \right] |j, +j\rangle$$

Patwardhan, Cervia, Balantekin, Phys.Rev. D99, 123013 (2019)

## Energy eigenvalues for a system with 10 neutrinos in 10 bins with $\omega_p = p\omega_0$ for $\kappa = 4$




$$|\zeta_1, \ldots, \zeta_\kappa\rangle = \mathcal{N}(\zeta_1, \ldots, \zeta_\kappa) \left[ \prod_{\alpha=1}^{\kappa} \left( \sum_p \frac{J_p^-}{\omega_p - \zeta_\alpha} \right) \right] |j, +j\rangle$$

Patwardhan, Cervia, Balantekin, Phys.Rev. D99, 123013 (2019)

A system of N particles each of which can occupy k states (k = number of flavors)

**Exact Solution** → **Mean-field approximation**

Entangled and unentangled states → Only unentangled states

Dimension of Hilbert Space: $k^N$

Dimension of Hilbert Space: $kN$

# von Neumann entropy

$$S = -\,\mathrm{Tr}\,(\rho \log \rho)$$

| | Pure State | Mixed State |
|---|---|---|
| Density matrix | $\rho^2 = \rho$ | $\rho^2 \neq \rho$ |
| Entropy | $S = 0$ | $S \neq 0$ |

Polarization vector for a two-level system

$$\rho = \frac{1}{2}\left(\mathbb{I} + \vec{\sigma} \cdot \vec{P}\right)$$

Pick one of the neutrinos and introduce the reduced density matrix for this neutrino (with label "b")

$$\tilde{\rho} = \rho_b = \sum_{a,c,d,\dots} \langle \nu_a, \nu_c, \nu_d, \cdots | \rho | \nu_a, \nu_c, \nu_d, \cdots \rangle$$

# Partial Trace

$$\mathrm{Tr}_2 \begin{pmatrix} \rho_{\uparrow\uparrow,\uparrow\uparrow} & \rho_{\uparrow\uparrow,\uparrow\downarrow} & \rho_{\uparrow\uparrow,\downarrow\uparrow} & \rho_{\uparrow\uparrow,\downarrow\downarrow} \\ \rho_{\uparrow\downarrow,\uparrow\uparrow} & \rho_{\uparrow\downarrow,\uparrow\downarrow} & \rho_{\uparrow\downarrow,\downarrow\uparrow} & \rho_{\uparrow\downarrow,\downarrow\downarrow} \\ \rho_{\downarrow\uparrow,\uparrow\uparrow} & \rho_{\downarrow\uparrow,\uparrow\downarrow} & \rho_{\downarrow\uparrow,\downarrow\uparrow} & \rho_{\downarrow\uparrow,\downarrow\downarrow} \\ \rho_{\downarrow\downarrow,\uparrow\uparrow} & \rho_{\downarrow\downarrow,\uparrow\downarrow} & \rho_{\downarrow\downarrow,\downarrow\uparrow} & \rho_{\downarrow\downarrow,\downarrow\downarrow} \end{pmatrix}$$

# Partial Trace

$$\mathrm{Tr}_2 \begin{pmatrix} \rho_{\uparrow\uparrow,\uparrow\uparrow} & \rho_{\uparrow\uparrow,\uparrow\downarrow} & \rho_{\uparrow\uparrow,\downarrow\uparrow} & \rho_{\uparrow\uparrow,\downarrow\downarrow} \\ \rho_{\uparrow\downarrow,\uparrow\uparrow} & \rho_{\uparrow\downarrow,\uparrow\downarrow} & \rho_{\uparrow\downarrow,\downarrow\uparrow} & \rho_{\uparrow\downarrow,\downarrow\downarrow} \\ \rho_{\downarrow\uparrow,\uparrow\uparrow} & \rho_{\downarrow\uparrow,\uparrow\downarrow} & \rho_{\downarrow\uparrow,\downarrow\uparrow} & \rho_{\downarrow\uparrow,\downarrow\downarrow} \\ \rho_{\downarrow\downarrow,\uparrow\uparrow} & \rho_{\downarrow\downarrow,\uparrow\downarrow} & \rho_{\downarrow\downarrow,\downarrow\uparrow} & \rho_{\downarrow\downarrow,\downarrow\downarrow} \end{pmatrix}$$

$$= \begin{pmatrix} \rho_{\uparrow\uparrow,\uparrow\uparrow} + \rho_{\uparrow\downarrow,\uparrow\downarrow} & \rho_{\uparrow\uparrow,\downarrow\uparrow} + \rho_{\uparrow\downarrow,\downarrow\downarrow} \\ \rho_{\downarrow\uparrow,\uparrow\uparrow} + \rho_{\downarrow\downarrow,\uparrow\downarrow} & \rho_{\downarrow\uparrow,\downarrow\uparrow} + \rho_{\downarrow\downarrow,\downarrow\downarrow} \end{pmatrix}$$

# Partial Trace

$$\mathrm{Tr}_2 \begin{pmatrix} \rho_{\uparrow\uparrow,\uparrow\uparrow} & \rho_{\uparrow\uparrow,\uparrow\downarrow} & \rho_{\uparrow\uparrow,\downarrow\uparrow} & \rho_{\uparrow\uparrow,\downarrow\downarrow} \\ \rho_{\uparrow\downarrow,\uparrow\uparrow} & \rho_{\uparrow\downarrow,\uparrow\downarrow} & \rho_{\uparrow\downarrow,\downarrow\uparrow} & \rho_{\uparrow\downarrow,\downarrow\downarrow} \\ \rho_{\downarrow\uparrow,\uparrow\uparrow} & \rho_{\downarrow\uparrow,\uparrow\downarrow} & \rho_{\downarrow\uparrow,\downarrow\uparrow} & \rho_{\downarrow\uparrow,\downarrow\downarrow} \\ \rho_{\downarrow\downarrow,\uparrow\uparrow} & \rho_{\downarrow\downarrow,\uparrow\downarrow} & \rho_{\downarrow\downarrow,\downarrow\uparrow} & \rho_{\downarrow\downarrow,\downarrow\downarrow} \end{pmatrix}$$

$$= \begin{pmatrix} \rho_{\uparrow\uparrow,\uparrow\uparrow} + \rho_{\uparrow\downarrow,\uparrow\downarrow} & \rho_{\uparrow\uparrow,\downarrow\uparrow} + \rho_{\uparrow\downarrow,\downarrow\downarrow} \\ \rho_{\downarrow\uparrow,\uparrow\uparrow} + \rho_{\downarrow\downarrow,\uparrow\downarrow} & \rho_{\downarrow\uparrow,\downarrow\uparrow} + \rho_{\downarrow\downarrow,\downarrow\downarrow} \end{pmatrix}$$

# Partial Trace

$$\mathrm{Tr}_2 \begin{pmatrix} \rho_{\uparrow\uparrow,\uparrow\uparrow} & \rho_{\uparrow\uparrow,\uparrow\downarrow} & \rho_{\uparrow\uparrow,\downarrow\uparrow} & \rho_{\uparrow\uparrow,\downarrow\downarrow} \\ \rho_{\uparrow\downarrow,\uparrow\uparrow} & \rho_{\uparrow\downarrow,\uparrow\downarrow} & \rho_{\uparrow\downarrow,\downarrow\uparrow} & \rho_{\uparrow\downarrow,\downarrow\downarrow} \\ \rho_{\downarrow\uparrow,\uparrow\uparrow} & \rho_{\downarrow\uparrow,\uparrow\downarrow} & \rho_{\downarrow\uparrow,\downarrow\uparrow} & \rho_{\downarrow\uparrow,\downarrow\downarrow} \\ \rho_{\downarrow\downarrow,\uparrow\uparrow} & \rho_{\downarrow\downarrow,\uparrow\downarrow} & \rho_{\downarrow\downarrow,\downarrow\uparrow} & \rho_{\downarrow\downarrow,\downarrow\downarrow} \end{pmatrix}$$

$$= \begin{pmatrix} \rho_{\uparrow\uparrow,\uparrow\uparrow} + \rho_{\uparrow\downarrow,\uparrow\downarrow} & \rho_{\uparrow\uparrow,\downarrow\uparrow} + \rho_{\uparrow\downarrow,\downarrow\downarrow} \\ \rho_{\downarrow\uparrow,\uparrow\uparrow} + \rho_{\downarrow\downarrow,\uparrow\downarrow} & \rho_{\downarrow\uparrow,\downarrow\uparrow} + \rho_{\downarrow\downarrow,\downarrow\downarrow} \end{pmatrix}$$

Pick one of the neutrinos and introduce the reduced density matrix for this neutrino (with label "b")

$$\tilde{\rho} = \rho_b = \sum_{a,c,d,\ldots} \langle \nu_a, \nu_c, \nu_d, \cdots | \rho | \nu_a, \nu_c, \nu_d, \cdots \rangle$$

$$S = -\mathrm{Tr}\left(\tilde{\rho}\ \log \tilde{\rho}\right)$$

$$\tilde{\rho} = \frac{1}{2}(\mathbb{I} + \vec{\sigma} \cdot \vec{P})$$

$$S = -\frac{1-|\vec{P}|}{2} \log\left(\frac{1-|\vec{P}|}{2}\right) - \frac{1+|\vec{P}|}{2} \log\left(\frac{1+|\vec{P}|}{2}\right)$$

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Entanglement entropy

$$S = -\mathrm{Tr}\left(\tilde{\rho}\, \log \tilde{\rho}\right)$$

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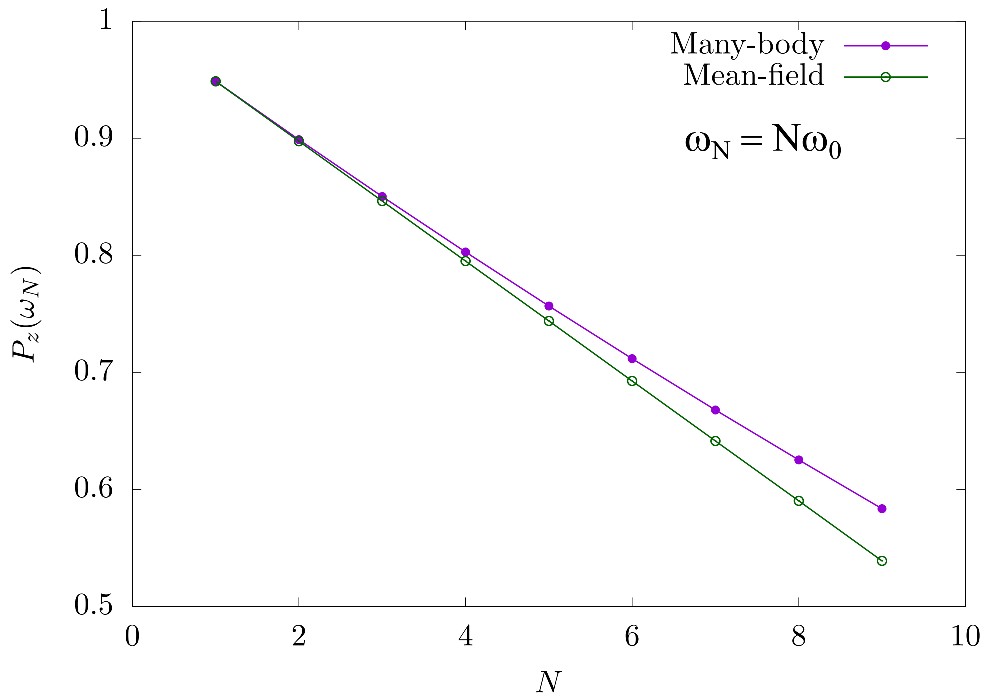
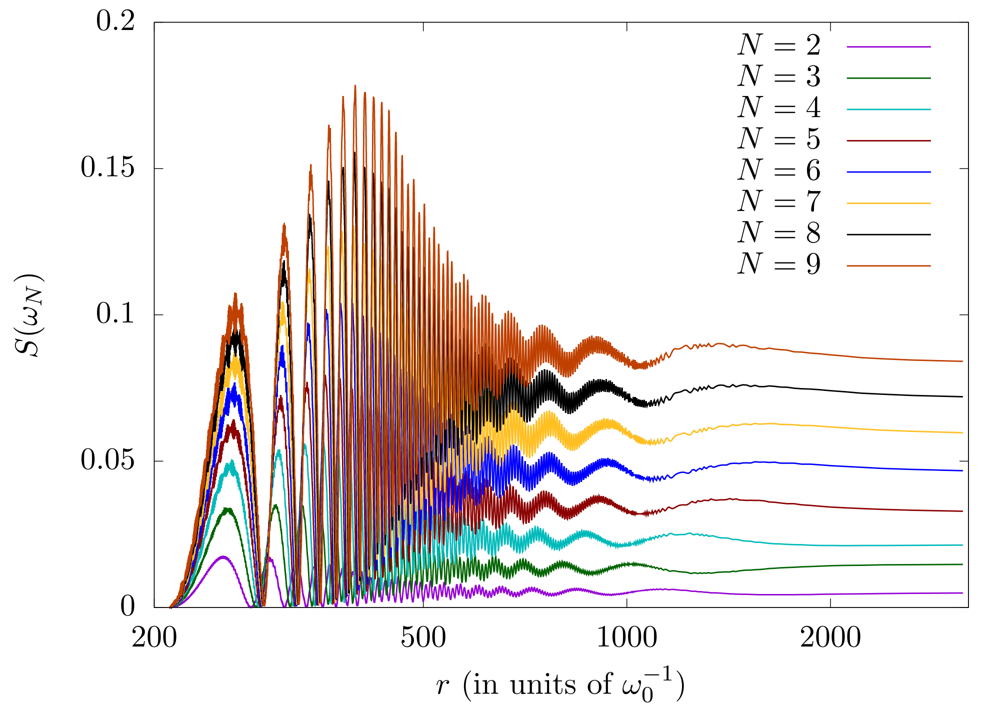
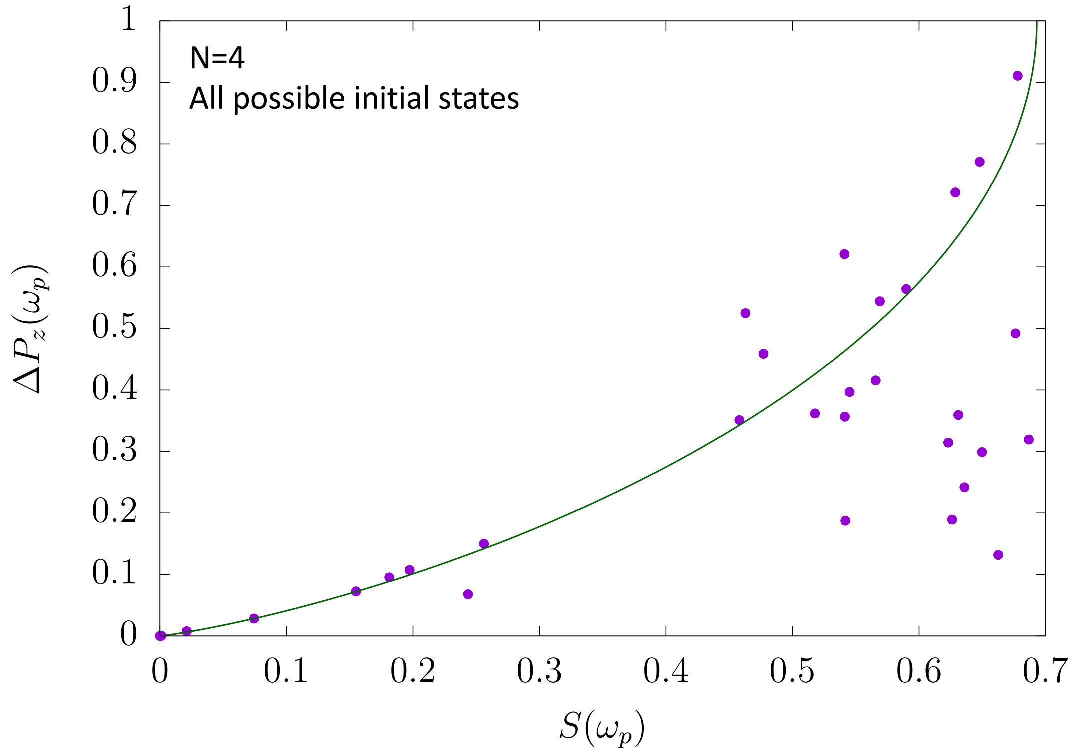


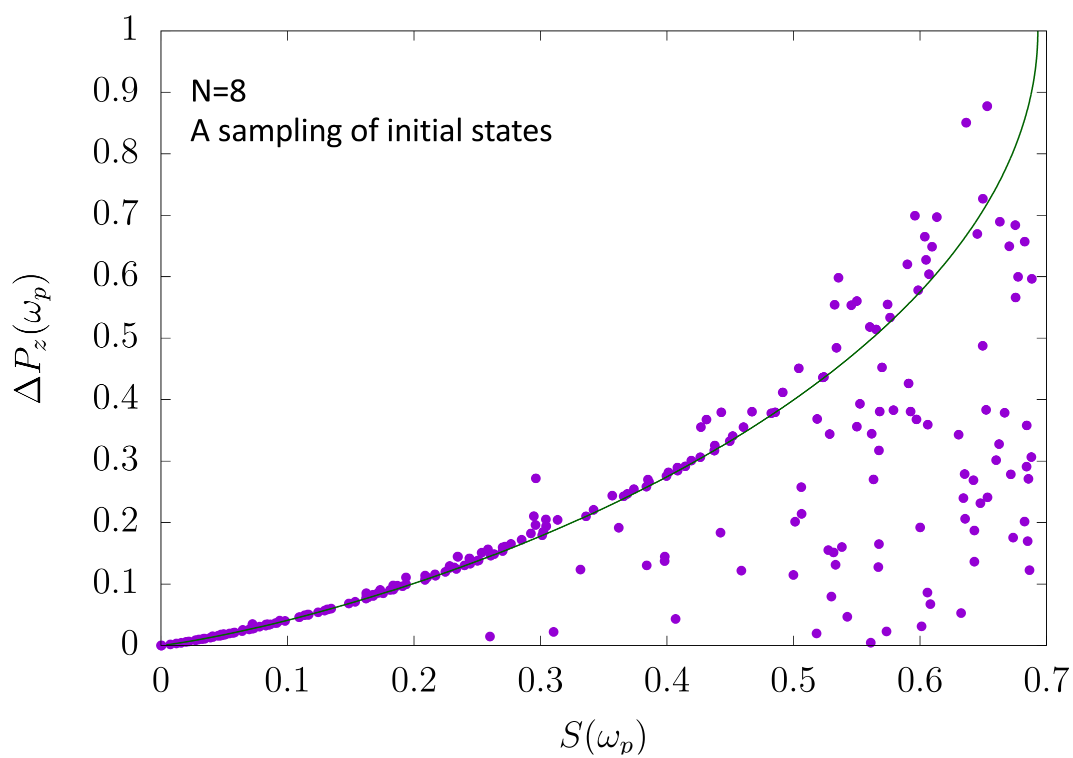




Initial state:
all electron neutrinos

Note: S = 0 for mean-field approximation

Cervia, Patwardhan, Balantekin, Coppersmith, Johnson, arXiv:1908.03511




# Correlation between entanglement entropy and deviation from the mean-field

$$\Delta P_z(\omega_p) = \left| P_z^{MF}(\omega_p) - P_z^{MB}(\omega_p) \right|$$

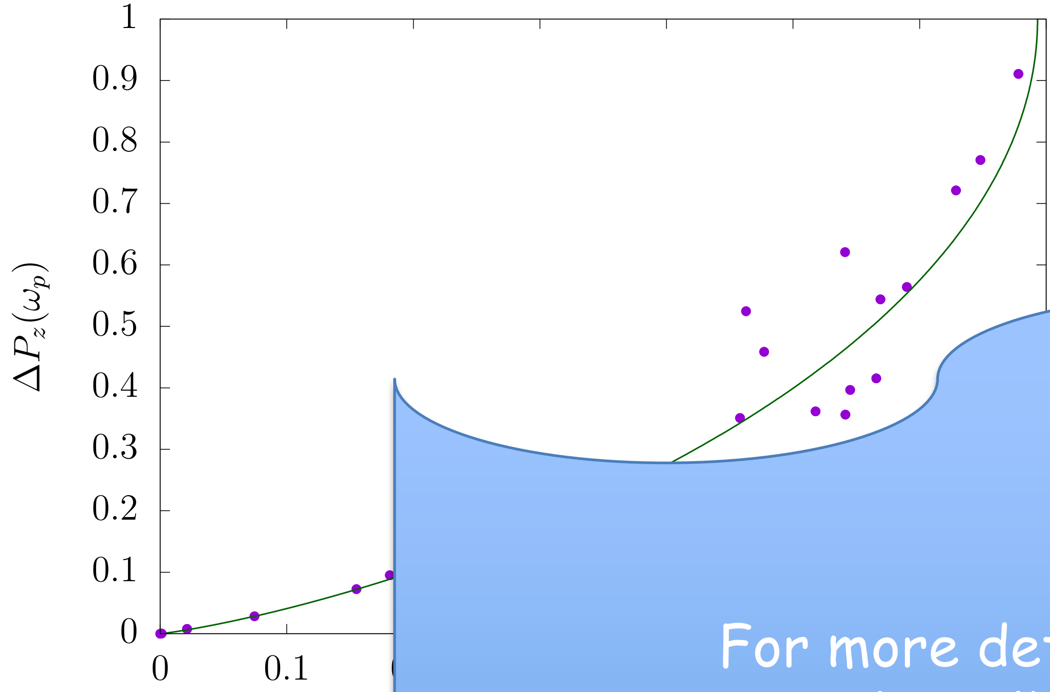
$$\tilde{\rho} = \frac{1}{2}(\mathbb{I} + \vec{\sigma} \cdot \vec{P})$$

$$S = -\frac{1-P}{2}\log\left(\frac{1-P}{2}\right) - \frac{1+P}{2}\log\left(\frac{1+P}{2}\right)$$

$$P = |\vec{P}| = \sqrt{P_x^2 + P_y^2 + P_z^2}$$

Cervia, Patwardhan, Balantekin, Coppersmith, Johnson, arXiv:1908.03511

Correlation between entanglement entropy and deviation from the mean-field




$^F(\omega_p) - P_z^{MB}(\omega_p)|$

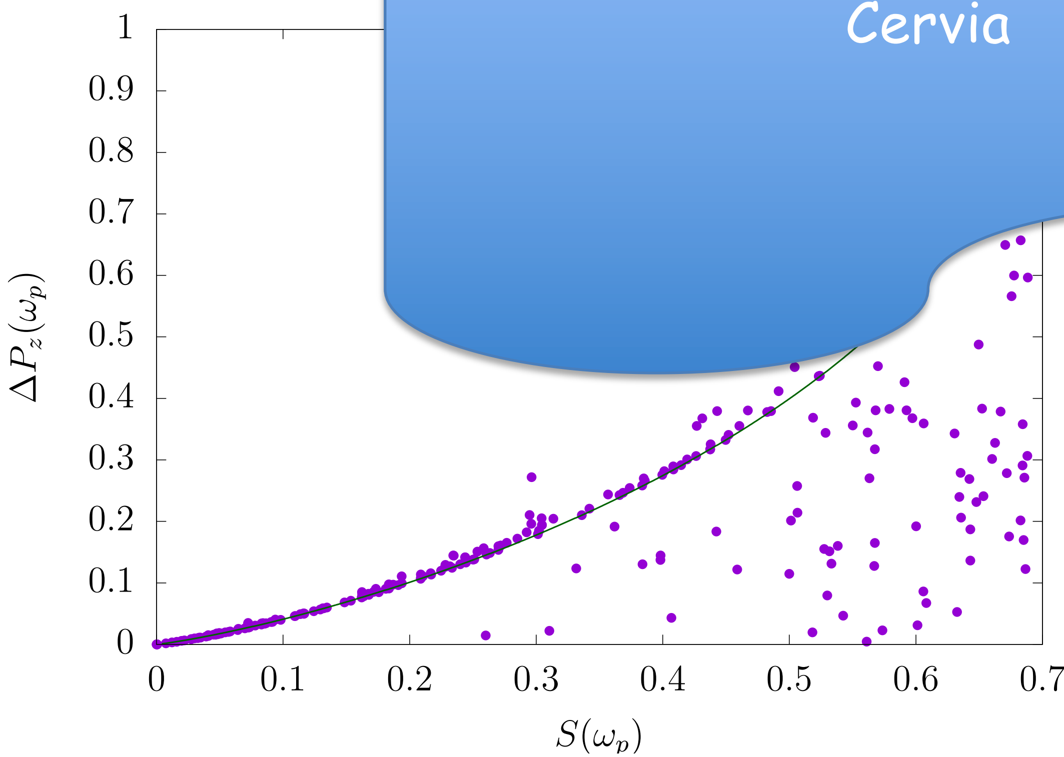
$\mathbb{I} + \vec{\sigma}\cdot\vec{P})$

$\frac{P}{} \log\left(\frac{1-P}{2}\right)$

$\frac{P}{} \log\left(\frac{1+P}{2}\right)$

$$P = |\vec{P}| = \sqrt{P_x^2 + P_y^2 + P_z^2}$$

For more details see the talk by Cervia




Cervia, Patwardhan, Balantekin, Coppersmith, Johnson, arXiv:1908.03511

## Example: analytically solvable N=2 case

$$H|\psi\rangle = \begin{pmatrix} -\Omega + 2\mu & 0 & 0 & 0 \\ 0 & -\eta + \mu & \mu & 0 \\ 0 & \mu & \eta + \mu & 0 \\ 0 & 0 & 0 & \Omega + 2\mu \end{pmatrix} \begin{pmatrix} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{pmatrix}$$

$$\Omega \equiv (\omega_1 + \omega_2)/2$$

$$\eta \equiv (\omega_1 - \omega_2)/2$$

# Example: analytically solvable N=2 case

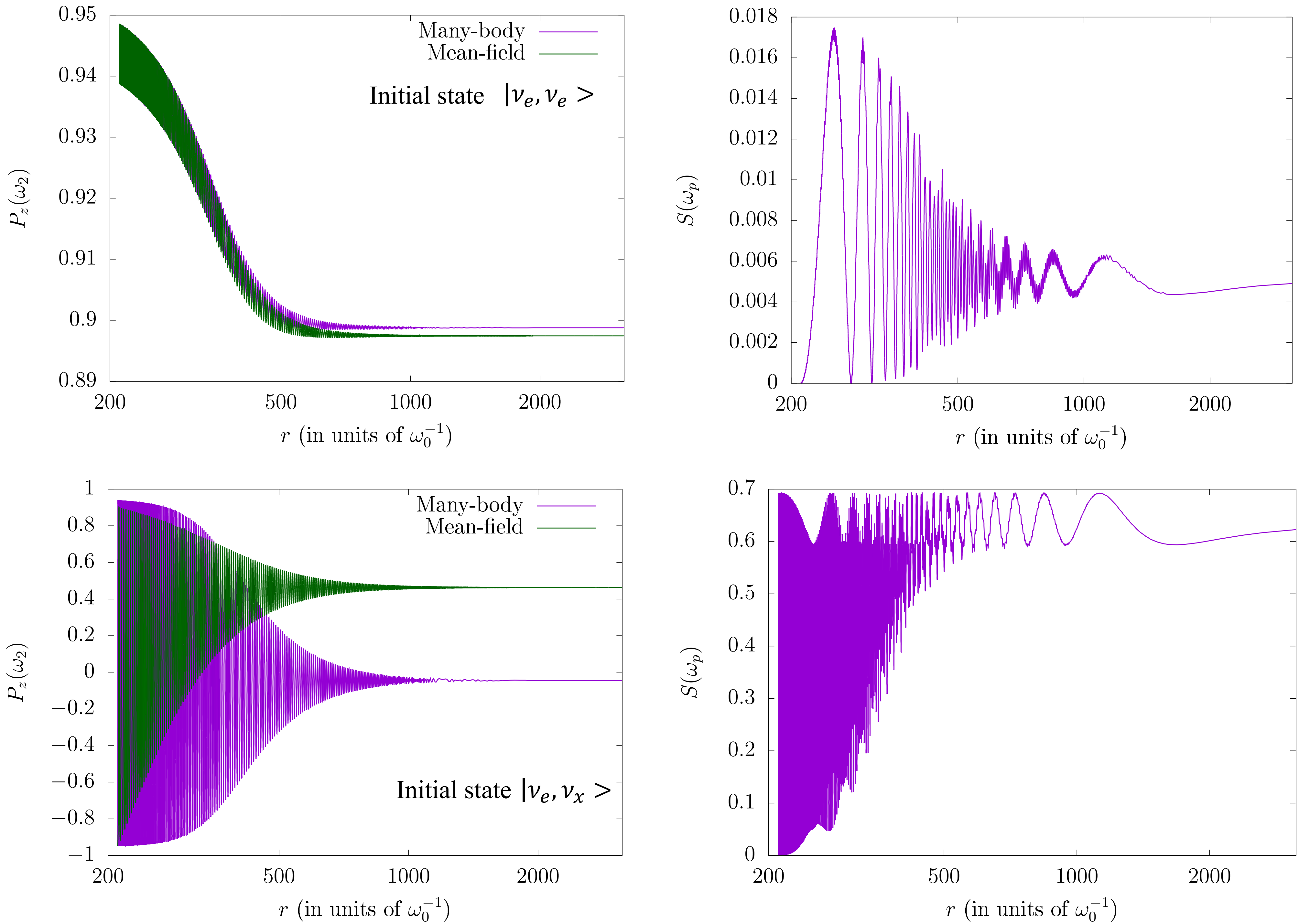
## Initial state: two electron neutrinos

$$P_z = \pm \cos 2\theta + \frac{1}{2} \sin^2 2\theta \left[ \frac{\eta\mu_0 - \eta\mu \cos 2Q}{\sqrt{(\mu^2 + \eta^2)(\mu_0^2 + \eta^2)}} \right],$$

$$P^2 = 1 - \frac{1}{4} \sin^4 2\theta \left[ \left( \cos 2R - \frac{\eta^2 + \mu_0 \mu \cos 2Q}{\sqrt{(\mu_0^2 + \eta^2)(\mu^2 + \eta^2)}} \right)^2 + \left( \sin 2R - \frac{\mu \sin 2Q}{\sqrt{\mu^2 + \eta^2}} \right)^2 \right]$$

$$R = \int_{t_0}^{t} \mu \, dt'$$

$$Q = \int_{t_0}^{t} \sqrt{\mu^2 + \eta^2} \, dt'$$



Many-body
Mean-field
Initial state $|\nu_e, \nu_e>$
$P_z(\omega_2)$
$S(\omega_p)$
$r$ (in units of $\omega_0^{-1}$)
Initial state $|\nu_e, \nu_x>$

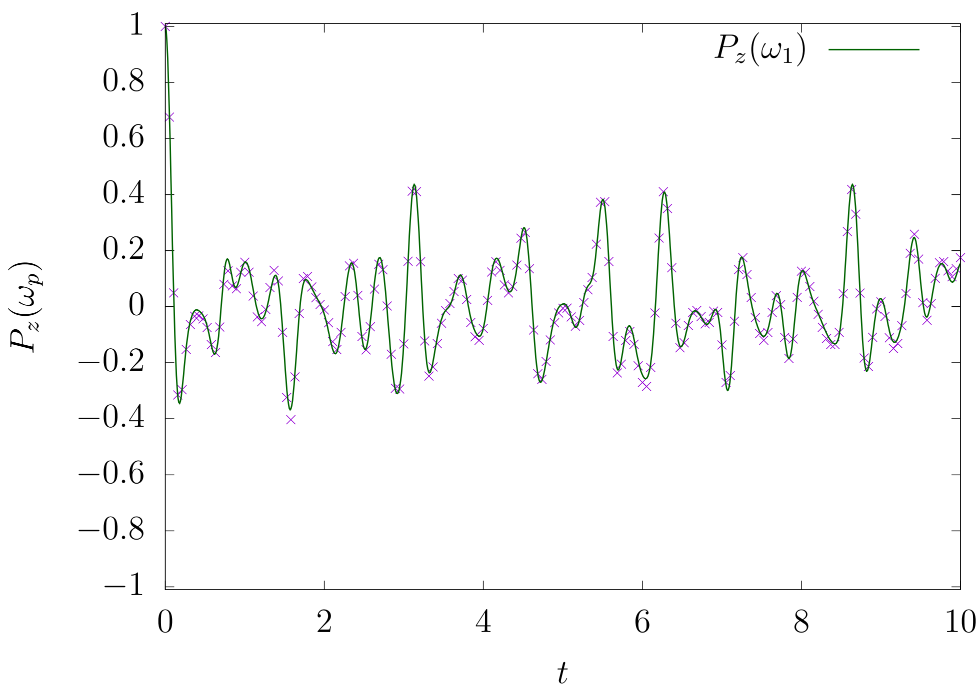
## Direct time evolution of the many-body system in discretized momentum space

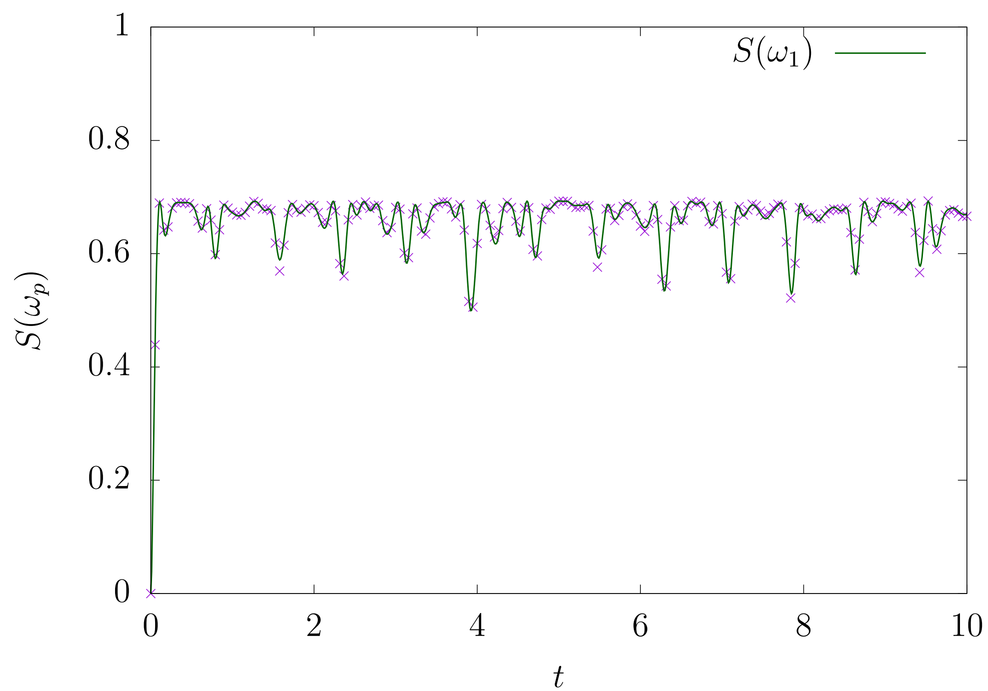
Rrapaj, arXiv:1905.13335

For details see Rrapaj's talk

# Comparison between direct time evolution in the single-angle limit and the Richardson-Gaudin method

Initial state: $2\nu_e, 2\nu_\mu, 2\bar{\nu}_e, 2\bar{\nu}_\mu$




Solid line: Richardson-Gaudin method (Cervia, et al.)
Crosses: Direct time evolution (Rrapaj)

# CONCLUSIONS

- Full many-body Hamiltonian describing collective neutrino oscillations is exactly solvable in the single-angle approximation in the sense that eigenvalues and eigenvectors can be expressed in terms of solutions of a Bethe ansatz equations.
- However, obtaining solutions of these Bethe ansatz equations is notoriously difficult for neutrinos in more than three energy bins.

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- However, obtaining solutions of these Bethe ansatz equations is notoriously difficult for neutrinos in more than three energy bins.
- Many-body literature contains several prescriptions with varying degrees of numerical difficulty to obtain eigenvalues.
- We provided an approach to calculate eigenstates (crucial for astrophysical calculations) and demonstrated its numerical feasibility.

# CONCLUSIONS

- We started comparing with mean-field calculations. We see some differences between mean-field and adiabatic many-body solutions for systems with a small number of neutrinos. There is a strong dependence of these differences on the initial conditions. These differences are correlated with the degree of entanglement between individual neutrinos.
- Calculations performed using the mean-field approximation have revealed a lot of interesting physics about collective behavior of neutrinos in astrophysical environments. Here we have explored possible scenarios where further interesting features can arise by going beyond this approximation.


Thank you very much!