Eigenvalues & eigenstates of the many-body collective neutrino oscillation problem

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References

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Bedtime reading

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Today's talk

- Collective neutrino oscillations as a many-body problem
- Eigenvalues and eigenstates of neutrino many-body Hamiltonian, for a two-flavor, single-angle system
- Talks by Michael Cervia and Ermal Rrapaj:
  flavor evolution of a neutrino many-body system, and comparison with the mean-field description

Outline

1. Many-body treatment of neutrino oscillations
2. Solving the Bethe Ansatz equations
3. Eigenvalues and Eigenstates

Neutrino oscillations: flavor/mass isospin operators

- Denote Fermionic operators for neutrino flavor/mass states as $a_\alpha(\mathbf{p})$, $a_j(\mathbf{p})$, where $\alpha = e, x$, and $j = 1, 2$

$$\begin{aligned} a_e(\mathbf{p}) &= \cos\theta\, a_1(\mathbf{p}) + \sin\theta\, a_2(\mathbf{p}) \\ a_x(\mathbf{p}) &= -\sin\theta\, a_1(\mathbf{p}) + \cos\theta\, a_2(\mathbf{p}) \end{aligned}$$

- Introduce the mass-basis isospin operators

$$J^+_\mathbf{p} = a^\dagger_1(\mathbf{p}) a_2(\mathbf{p}) \ , \qquad J^-_\mathbf{p} = a^\dagger_2(\mathbf{p}) a_1(\mathbf{p}) \ ,$$
$$J^z_\mathbf{p} = \frac{1}{2}\left(a^\dagger_1(\mathbf{p}) a_1(\mathbf{p}) - a^\dagger_2(\mathbf{p}) a_2(\mathbf{p})\right) \ ,$$

which obey the usual $SU(2)$ commutation relations

$$[J^+_\mathbf{p}, J^-_\mathbf{q}] = 2\delta_{\mathbf{pq}} J^z_\mathbf{p} \ , \qquad [J^z_\mathbf{p}, J^\pm_\mathbf{q}] = \pm\delta_{\mathbf{pq}} J^\pm_\mathbf{p} .$$

Neutrino oscillations: many-body Hamiltonian

- Vacuum oscillations:

$$H_{\text{vac}} = \sum_{\mathbf{p}} \left(\frac{m_1^2}{2p} a_1^\dagger(\mathbf{p}) a_1(\mathbf{p}) + \frac{m_2^2}{2p} a_2^\dagger(\mathbf{p}) a_2(\mathbf{p}) \right)$$
$$= \sum_{\omega} \omega \vec{B} \cdot \vec{J}_\omega \ ,$$

where $\omega = \frac{\delta m^2}{2p}$, $\quad \vec{J}_\omega = \sum_{p = \frac{\delta m^2}{2\omega}} \vec{J}_{\mathbf{p}}$, $\quad$ and

$\vec{B} = (0, 0, -1)_{\text{mass}} = (\sin 2\theta, 0, -\cos 2\theta)_{\text{flavor}}$.

- Neutrino-neutrino interactions

$$H_{\nu\nu} = \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos\vartheta_{\mathbf{pq}}) \, \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}} \ .$$

Neutrino Hamiltonian: single-angle approximation

- Suitable averaging over the angle $\vartheta_{\mathbf{pq}}$ to simplify the problem

$$H_{\nu\nu} \approx \frac{\sqrt{2}G_F}{V}\langle(1-\cos\vartheta_{\mathbf{pq}})\rangle \vec{J}\cdot\vec{J} \qquad \text{(includes } J_p^2 \text{ terms)}$$

$$\equiv \mu(r)\vec{J}\cdot\vec{J}, \qquad \text{where } \vec{J} = \sum_\omega \vec{J}_\omega$$

- Combined neutrino Hamiltonian (vacuum + self-interactions)

$$H_\nu = \sum_\omega \omega\vec{B}\cdot\vec{J}_\omega + \mu(r)\vec{J}\cdot\vec{J}.$$

Mean-field (random phase) approximation

- This method yields the effective one-particle neutrino Hamiltonian

$$H \sim H^{\text{RPA}} = \sum_{\omega} \omega \vec{B} \cdot \vec{J}_{\omega} + \mu \vec{P} \cdot \vec{J} \,,$$

where $\vec{P}_{\omega} = 2\langle \vec{J}_{\omega} \rangle$ is the "Polarization vector", and $\vec{P} = \sum_{\omega} \vec{P}_{\omega}$

- The self-consistency requirement of the mean-field approach then implies that $\vec{P}_{\omega}$ must satisfy

$$\frac{d}{dt}\vec{P}_{\omega} = (\omega \vec{B} + \mu \vec{P}) \times \vec{P}_{\omega}$$

Many-body eigenstates: limiting μ cases

- Many-body neutrino Hamiltonian (two-flavor, single-angle):

$$H_\nu = \sum_{p=1}^{M} \omega_p \vec{B} \cdot \vec{J}_p + \mu(r) \vec{J} \cdot \vec{J},$$

where p is an index for the ωs in the system, M in number

- As $\mu \to 0$, eigenstates are simply tensor products of single-particle vacuum mass eigenstates, e.g.,

$$|\nu_1\, \nu_1\, \nu_1\, \ldots\rangle, \quad |\nu_2\, \nu_1\, \nu_1\, \ldots\rangle, \quad |\nu_1\, \nu_2\, \nu_1\, \ldots\rangle, \quad \ldots$$

- As $\mu \to \infty$, the total isospin states $|j, m\rangle_{\text{mass/flavor}}$ are eigenstates, with eigenvalues $\mu\, j(j+1)$

Many-body eigenstates: Richardson-Gaudin procedure

- For $0 < \mu < \infty$, the extremal states $|\nu_1, \ldots, \nu_1\rangle$ and $|\nu_2, \ldots, \nu_2\rangle$ are eigenstates, with energies

$$E_{(\pm N/2)} = \mp \sum_{p=1}^{M} \omega_p \frac{N_p}{2} + \mu \frac{N}{2} \left(\frac{N}{2} + 1 \right),$$

N_p the number of neutrinos at frequency ω_p, and $N = \sum_p N_p$.

- Other eigenstates can be constructed from the extremal states by applying Gaudin operators

$$S^{\pm}(\zeta_\alpha) = \sum_{p=1}^{M} \frac{J_p^{\pm}}{\omega_p - \zeta_\alpha},$$

where ζ_α are parameters to be determined

Many-body eigenstates: Bethe Ansatz equations

- Bethe Ansatz: $|\zeta_\alpha\rangle \equiv S^-(\zeta_\alpha)|\frac{N}{2}, \frac{N}{2}\rangle$ is an eigenstate of H_ν if ζ_α satisfies the equation

$$-\frac{1}{2\mu} - \sum_{p=1}^{M} \frac{j_p}{\omega_p - \zeta_\alpha} = 0,$$

where $j_p = N_p/2$

- More generally, $|\zeta_1, \ldots, \zeta_\kappa\rangle \equiv S^-(\zeta_1)\ldots S^-(\zeta_\kappa)|\frac{N}{2}, \frac{N}{2}\rangle$ is an eigenstate of the Hamiltonian with the energy

$$E(\zeta_1, \ldots, \zeta_\kappa) = E_{N/2} + \sum_{\alpha=1}^{\kappa} \zeta_\alpha - \kappa\mu(N - \kappa + 1),$$

if $\zeta_1, \ldots, \zeta_\kappa$ obey the system of equations

$$-\frac{1}{2\mu} - \sum_{p=1}^{M} \frac{j_p}{\omega_p - \zeta_\alpha} = \sum_{\substack{\beta=1 \\ (\beta\neq\alpha)}}^{\kappa} \frac{1}{\zeta_\alpha - \zeta_\beta} \, .$$

Outline

Finding roots of Bethe Ansatz equations

- For a given κ, the Bethe Ansatz (BA) equations are a system of simultaneous equations in κ complex variables $\{\zeta_1, \ldots, \zeta_\kappa\}$. Singularities present as ζ_αs approach one another
- To avoid singularities, can be converted to polynomial equations where the order of each polynomial is $M + \kappa - 2$ (M is the number of frequencies in the system)
- To obtain a complete set of eigenstates, one must solve BA equations $\forall\, \kappa \in \{1, \ldots, N\}$: a different set of equations must be solved for each κ

An alternative approach: the ‘Lambda’ method

- Defining $\Lambda(\lambda) = \sum_{\alpha=1}^{\kappa} \frac{1}{\lambda - \zeta_\alpha}$, one can convert BA equations into an ODE, which thereafter can be reduced to another algebraic system by taking the limit $\lambda \to \omega_q$

- One obtains the following set of equations for each ω_q:

$$\Lambda^2(\omega_q) + (1 - 2j_q)\Lambda'(\omega_q) + \frac{1}{\mu}\Lambda(\omega_q) = 2\sum_{\substack{p=1 \\ p\neq q}}^{M} j_p \frac{\Lambda(\omega_q) - \Lambda(\omega_p)}{\omega_q - \omega_p}.$$

- In particular, if $j_q = 1/2$, the $\Lambda'(\omega_q)$ term disappears and the equation becomes purely algebraic in all the $\Lambda(\omega_p)$. If $j_q > 1/2$, taking derivatives of the ODE w.r.t λ before taking the $\lambda \to \omega_q$ limit gives additional equations which can be used to eliminate all $\Lambda'(\omega_p), \Lambda''(\omega_p), \ldots$, etc.

The $j_q = 1/2$ case: algebraic Λ equations

- Denote $\Lambda_q = \Lambda(\omega_q)$ and define $\tilde{\Lambda}_q = \mu\Lambda_q$. The case where $j_q = 1/2 \ \forall q$ (i.e., only one neutrino at each ω) reduces to

$$\tilde{\Lambda}_q^2 + \tilde{\Lambda}_q = \mu \sum_{\substack{p=1 \\ p\neq q}}^{N} \frac{\tilde{\Lambda}_q - \tilde{\Lambda}_p}{\omega_q - \omega_p}.$$

- System of size N represented by N coupled polynomial equations, of quadratic order regardless of N or κ. In fact, the same set of equations admits solutions for all $\kappa = 0, \ldots, N$. And there are no singularities.

- Solutions corresponding to a particular κ satisfy

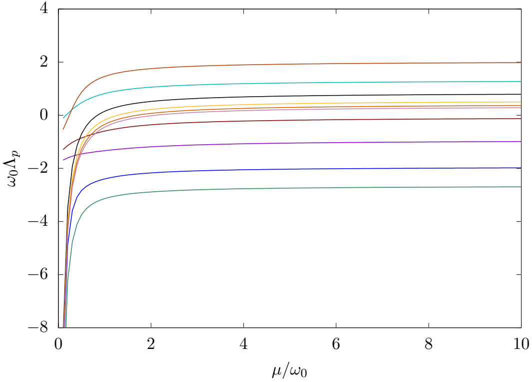
$$\sum_p \tilde{\Lambda}_p = -\kappa,$$

Solving Λ equations by homotopy continuation

$$\tilde{\Lambda}_q^2 + \tilde{\Lambda}_q = \mu \sum_{\substack{p=1 \\ p\neq q}}^{N} \frac{\tilde{\Lambda}_q - \tilde{\Lambda}_p}{\omega_q - \omega_p}.$$

- In the $\mu = 0$ limit, each $\tilde{\Lambda}_q$ can take the value 0 or -1, giving a total of 2^N solutions. In each solution, the number of $\tilde{\Lambda}_q$s that are -1 is the κ of that particular solution.
- Each of the 2^N solutions for $\mu > 0$ can be constructed incrementally, starting from the correponding $\mu = 0$ solution:
  - Use each $\tilde{\Lambda}_q$ solution obtained at $\mu = \mu_n$ to construct a starting guess for the corresponding solution at $\mu = \mu_n + \delta\mu$
  - Improve the starting guess using iterative numerical methods such as Newton-Raphson

Ten neutrino system: $\{\Lambda(\omega_q)\}$ solutions



[Patwardhan, Cervia, Balantekin, Phys. Rev. D 99 123013 (2019)]

Figure: A solution $\{\Lambda(\omega_1), \ldots, \Lambda(\omega_{10})\}$ as a function of μ, for a system with ten neutrinos, distributed uniformly across frequencies $\omega_q = q\omega_0$. Shown here is one sample solution out of the 1024 for this system.

Outline

Energy eigenvalues from Λs

- Recall that energy eigenvalues given by:

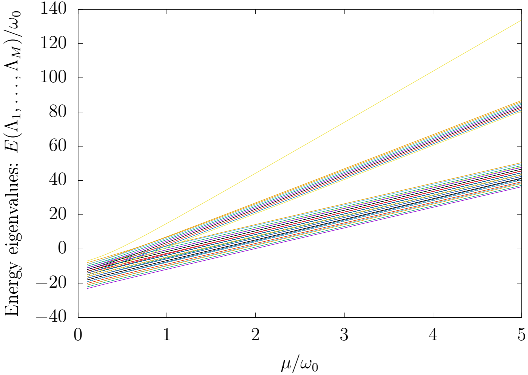
$$E(\zeta_1, \ldots, \zeta_\kappa) = E_{N/2} + \sum_{\alpha=1}^{\kappa} \zeta_\alpha - \kappa\mu(N - \kappa + 1)$$

- In terms of $\tilde{\Lambda}_q$, these may be rewritten as

$$E(\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_N) = -\sum_p \frac{\omega_p}{2} + \mu \frac{N}{2}\left(\frac{N}{2} + 1\right) - \sum_p \omega_p \tilde{\Lambda}_p$$

- Instructive to group energy eigenvalues according to κ: solutions are eigenstates of J_z, with eigenvalue $m = N/2 - \kappa$. Within each κ, energy eigenvalues split into branches at large μ. Each branch associates to a particular $|j, m\rangle$ as $\mu \to \infty$, with $j = |m|, \ldots, N/2$.

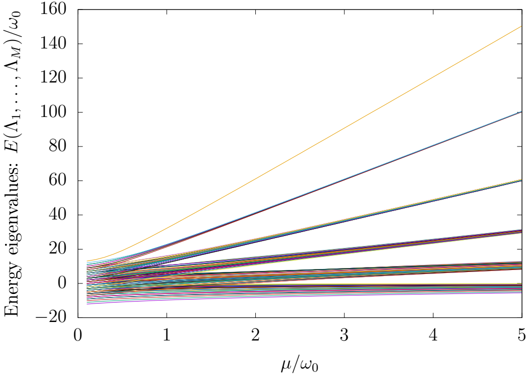
Ten neutrino system: energy eigenvalues



[Patwardhan, Cervia, Balantekin, Phys. Rev. D 99 123013 (2019)]

Figure: Energy eigenvalues corresponding to all $\kappa = 2$ $(m = 3)$ solutions of the BA equations, for an $N = 10$ neutrino system, as functions of μ.

Ten neutrino system: energy eigenvalues



[Patwardhan, Cervia, Balantekin, Phys. Rev. D 99 123013 (2019)]

Figure: Energy eigenvalues corresponding to all $\kappa = 5$ $(m = 0)$ solutions of the BA equations, for an $N = 10$ neutrino system, as functions of μ.

Energy eigenstates from Λs

- The process of calculating the eigenstates from the Λs involves first calculating the power sums $P_f = \sum_{\alpha=1}^{\kappa} (S_\alpha^-)^f$, where $S_\alpha^- \equiv S^-(\zeta_\alpha)$ are the Gaudin operators defined earlier. In terms of Λs, the power sums are given by

$$P_f = \sum_{\alpha=1}^{\kappa} (S_\alpha^-)^f = \sum_{i=1}^{\kappa} \sum_{p_1=1}^{M} \cdots \sum_{p_f=1}^{M} \frac{J_{p_1}^- \cdots J_{p_f}^-}{(\omega_{p_1} - \zeta_\alpha) \ldots (\omega_{p_f} - \zeta_\alpha)}$$

$$= \sum_{p_1=1}^{M} \cdots \sum_{p_f=1}^{M} J_{p_1}^- \cdots J_{p_f}^- \sum_{m=1}^{f} \Lambda_{p_m} \prod_{\substack{l=1 \\ l \neq m}}^{f} \frac{1}{\omega_{p_l} - \omega_{p_m}}.$$

- For an eigenstate with a particular κ, the power sums P_f for $f = 1, \ldots, \kappa$ are required

Energy eigenstates from Λs

- Recognize that

$$|\zeta_1, \ldots, \zeta_\kappa\rangle = e_\kappa(S_1^-, \ldots, S_\kappa^-) \left| \frac{N}{2}, \frac{N}{2} \right\rangle$$

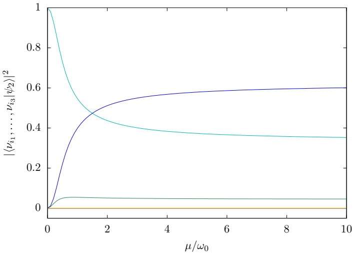
where e_κ is the κ-th elementary symmetric polynomial

- The elementary symmetric polynomials may be calculated recursively from the power sums using Newton's identities:

$$e_k = \frac{1}{k} \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} \, P_i,$$

for $k = 1, \ldots, \kappa$. Calculating e_κ for a particular set of Λs immediately yields the corresponding eigenstate

Energy eigenstates: examples



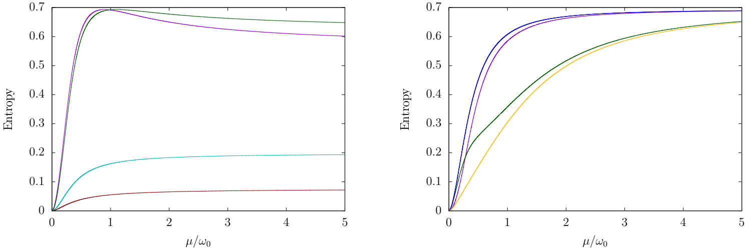
[Patwardhan, Cervia, Balantekin, Phys. Rev. D 99 123013 (2019)]

Figure: Overlaps, $|\langle\nu_{i_1},\ldots,\nu_{i_N}|\psi_n\rangle|^2$, of particular energy eigenstates $|\psi_n\rangle$ with the mass basis states (with $i_1,\ldots,i_N = 1,2$), as functions of μ. Left: a particular eigenstate with $\kappa = 1$, of an $N = 3$ system. Right: a particular eigenstate with $\kappa = 2$, of an $N = 4$ system. Observe that, for $\mu > 0$, a state with a certain κ has ${}^N C_\kappa$ nontrivial components.

Conclusions

- Calculations of collective neutrino flavor evolution typically rely on a 'mean-field', i.e., effective one-particle description
- Important to test the efficacy and/or limitations of the mean-field by performing many-body calculations
- Evolution in the many-body case can be studied by calculating the eigenvalues and eigenvectors of the Hamiltonian by solving the Bethe Ansatz equations
- For certain simple systems, qualititive differences in flavor evolution observed between many-body and mean-field treatments, resulting from entangled states which are absent in the mean-field limit—talks by Michael and Ermal to follow

Entanglement: a preview



[Cervia et al., arxiv:1908.03511]

Figure: Entropy of entanglement between the neutrino at frequency ω_4 and the rest of the ensemble, for all eigenstates of an $N = 4$ neutrino system, corresponding to $\kappa = 1$ (left) and 2 (right).

The N3AS collaboration

- Network in Neutrinos, Nuclear Astrophysics, and Symmetries
- Multi-institutional network (3 centers + 8 sites) dedicated to recruiting and training postdocs, fostering collaborative efforts, and advancing research in the following areas:
  - Neutrino physics and astrophysics
  - Dense matter
  - Dark matter
- Funded by National Science Foundation (NSF) and Heising-Simons Foundation
- https://n3as.wordpress.com/

Simple case: two neutrinos, $j_1 = j_2 = 1/2$

Defining $\Lambda_p \equiv \Lambda(\omega_p)$ and $\eta = \omega_2 - \omega_1$, the equations are:

$$\Lambda_1^2 + \frac{1}{\mu}\Lambda_1 = \frac{\Lambda_1 - \Lambda_2}{-\eta}$$

$$\Lambda_2^2 + \frac{1}{\mu}\Lambda_2 = \frac{\Lambda_2 - \Lambda_1}{\eta}.$$

The following solutions can be obtained analytically

$$\Lambda_1 = \Lambda_2 = 0, \quad \text{or} \quad \Lambda_1 = \Lambda_2 = -\frac{1}{\mu}, \quad \text{or}$$

$$\Lambda_1 = -\left(\frac{1}{2\mu} + \frac{1}{\eta}\right) \pm \frac{1}{2}\sqrt{\frac{1}{\mu^2} + \frac{4}{\eta^2}}$$

$$\Lambda_2 = -\left(\frac{1}{2\mu} - \frac{1}{\eta}\right) \mp \frac{1}{2}\sqrt{\frac{1}{\mu^2} + \frac{4}{\eta^2}}.$$

The three sets of solutions correspond to $\kappa = 0, 2,$ and 1, respectively