Eigenvalues & eigenstates of the many-body collective neutrino oscillation problem

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References

Amol V. Patwardhan, Michael J. Cervia, and A. B. Balantekin

Michael J. Cervia, Amol V. Patwardhan, and A. B. Balantekin

Michael J. Cervia, Amol V. Patwardhan, A. B. Balantekin,
S. N. Coppersmith, and Calvin W. Johnson
arXiv:1908.03511

Ermal Rrapaj
arXiv:1905.13335
Bedtime reading

Savas Birol, Y. Pehlivan, A. B. Balantekin, and T. Kajino

Y. Pehlivan, A. B. Balantekin, Toshitaka Kajino, and Takashi Yoshida

Alexandre Faribault, Omar El Araby, Christoph Sträter, and Vladimir Gritsev

Pieter W. Claeys
arxiv:1809.04447 (PhD thesis)
Today’s talk

- Collective neutrino oscillations as a many-body problem

- Eigenvalues and eigenstates of neutrino many-body Hamiltonian, for a two-flavor, single-angle system

- Talks by Michael Cervia and Ermal Rrapaj: flavor evolution of a neutrino many-body system, and comparison with the mean-field description
Outline

1. Many-body treatment of neutrino oscillations
2. Solving the Bethe Ansatz equations
3. Eigenvalues and Eigenstates
Neutrino oscillations: flavor/mass isospin operators

- Denote Fermionic operators for neutrino flavor/mass states as $a_{\alpha}(p)$, $a_j(p)$, where $\alpha = e, x$, and $j = 1, 2$

$$a_e(p) = \cos \theta a_1(p) + \sin \theta a_2(p)$$
$$a_x(p) = -\sin \theta a_1(p) + \cos \theta a_2(p)$$

- Introduce the mass-basis isospin operators

$$J^+_p = a_1^\dagger(p) a_2(p), \quad J^-_p = a_2^\dagger(p) a_1(p),$$
$$J^z_p = \frac{1}{2} \left( a_1^\dagger(p) a_1(p) - a_2^\dagger(p) a_2(p) \right),$$

which obey the usual $SU(2)$ commutation relations

$$[J^+_p, J^-_q] = 2\delta_{pq} J^z_p, \quad [J^z_p, J^\pm_q] = \pm \delta_{pq} J^\pm_p.$$
Neutrino oscillations: many-body Hamiltonian

- **Vacuum oscillations:**
  \[
  H_{\text{vac}} = \sum_p \left( \frac{m_1^2}{2p} a_1^\dagger(p) a_1(p) + \frac{m_2^2}{2p} a_2^\dagger(p) a_2(p) \right) \\
  = \sum_\omega \omega \vec{B} \cdot \vec{J}_\omega ,
  \]
  where \( \omega = \frac{\delta m^2}{2p} \), \( \vec{J}_\omega = \sum_p \vec{J}_p \), and
  \[
  \vec{B} = (0, 0, -1)_{\text{mass}} = (\sin 2\theta, 0, -\cos 2\theta)_{\text{flavor}}.
  \]

- **Neutrino-neutrino interactions**
  \[
  H_{\nu\nu} = \frac{\sqrt{2}G_F}{\sqrt{V}} \sum_{p,q} \left( 1 - \cos \vartheta_{pq} \right) \vec{J}_p \cdot \vec{J}_q .
  \]
Neutrino Hamiltonian: single-angle approximation

- Suitable averaging over the angle $\theta_{pq}$ to simplify the problem

$$H_{\nu\nu} \approx \frac{\sqrt{2}G_F}{V} \langle (1 - \cos \theta_{pq}) \rangle \vec{J} \cdot \vec{J}$$

(includes $J^2_p$ terms)

$$\equiv \mu(r) \vec{J} \cdot \vec{J},$$

where $\vec{J} = \sum \omega \vec{J}_\omega$

- Combined neutrino Hamiltonian (vacuum + self-interactions)

$$H_\nu = \sum \omega B \cdot \vec{J}_\omega + \mu(r) \vec{J} \cdot \vec{J}.$$
Mean-field (random phase) approximation

- This method yields the effective one-particle neutrino Hamiltonian

\[ H \sim H_{\text{RPA}} = \sum_{\omega} \omega \vec{B} \cdot \vec{J}_\omega + \mu \vec{P} \cdot \vec{J}, \]

where \( \vec{P}_\omega = 2\langle \vec{J}_\omega \rangle \) is the “Polarization vector”, and \( \vec{P} = \sum_\omega \vec{P}_\omega \)

- The self-consistency requirement of the mean-field approach then implies that \( \vec{P}_\omega \) must satisfy

\[ \frac{d}{dt} \vec{P}_\omega = (\omega \vec{B} + \mu \vec{P}) \times \vec{P}_\omega \]
Many-body eigenstates: limiting $\mu$ cases

- Many-body neutrino Hamiltonian (two-flavor, single-angle):

$$H_\nu = \sum_{p=1}^{M} \omega_p \vec{B} \cdot \vec{J}_p + \mu(r) \vec{J} \cdot \vec{J},$$

where $p$ is an index for the $\omega$s in the system, $M$ in number

- As $\mu \to 0$, eigenstates are simply tensor products of single-particle vacuum mass eigenstates, e.g.,

$$|\nu_1 \nu_1 \nu_1 \ldots\rangle, \quad |\nu_2 \nu_1 \nu_1 \ldots\rangle, \quad |\nu_1 \nu_2 \nu_1 \ldots\rangle, \quad \ldots$$

- As $\mu \to \infty$, the total isospin states $|j, m\rangle_{\text{mass/flavor}}$ are eigenstates, with eigenvalues $\mu j(j+1)$
Many-body eigenstates: Richardson-Gaudin procedure

- For $0 < \mu < \infty$, the extremal states $|\nu_1, \ldots, \nu_1\rangle$ and $|\nu_2, \ldots, \nu_2\rangle$ are eigenstates, with energies

$$E(\pm N/2) = \mp \sum_{p=1}^{M} \omega_p \frac{N_p}{2} + \mu \frac{N}{2} \left( \frac{N}{2} + 1 \right),$$

where $N_p$ is the number of neutrinos at frequency $\omega_p$, and $N = \sum_p N_p$.

- Other eigenstates can be constructed from the extremal states by applying Gaudin operators

$$S^\pm(\zeta_\alpha) = \sum_{p=1}^{M} \frac{J^\pm_p}{\omega_p - \zeta_\alpha},$$

where $\zeta_\alpha$ are parameters to be determined.
Many-body eigenstates: Bethe Ansatz equations

- **Bethe Ansatz:** $|\zeta_\alpha\rangle \equiv S^{-}(\zeta_\alpha)|\frac{N}{2}, \frac{N}{2}\rangle$ is an eigenstate of $H_\nu$ if $\zeta_\alpha$ satisfies the equation

$$-\frac{1}{2\mu} - \sum_{p=1}^{M} \frac{j_p}{\omega_p - \zeta_\alpha} = 0,$$

where $j_p = N_p/2$

- More generally, $|\zeta_1, \ldots, \zeta_\kappa\rangle \equiv S^{-}(\zeta_1) \ldots S^{-}(\zeta_\kappa)|\frac{N}{2}, \frac{N}{2}\rangle$ is an eigenstate of the Hamiltonian with the energy

$$E(\zeta_1, \ldots, \zeta_\kappa) = E_{N/2} + \sum_{\alpha=1}^{\kappa} \zeta_\alpha - \kappa \mu (N - \kappa + 1),$$

if $\zeta_1, \ldots, \zeta_\kappa$ obey the system of equations

$$-\frac{1}{2\mu} - \sum_{p=1}^{M} \frac{j_p}{\omega_p - \zeta_\alpha} = \sum_{\beta=1}^{\kappa} \sum_{(\beta \neq \alpha)}^{\kappa} \frac{1}{\zeta_\alpha - \zeta_\beta}.$$
Outline

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Finding roots of Bethe Ansatz equations

For a given $\kappa$, the Bethe Ansatz (BA) equations are a system of simultaneous equations in $\kappa$ complex variables $\{\zeta_1, \ldots, \zeta_\kappa\}$. Singularities present as $\zeta_\alpha$’s approach one another.

To avoid singularities, can be converted to polynomial equations where the order of each polynomial is $M + \kappa - 2$ ($M$ is the number of frequencies in the system).

To obtain a complete set of eigenstates, one must solve BA equations $\forall \kappa \in \{1, \ldots, N\}$: a different set of equations must be solved for each $\kappa$. 
An alternative approach: the ‘Lambda’ method

- Defining $\Lambda(\lambda) = \sum_{\alpha=1}^{\kappa} \frac{1}{\lambda - \zeta_{\alpha}}$, one can convert BA equations into an ODE, which thereafter can be reduced to another algebraic system by taking the limit $\lambda \to \omega_q$

- One obtains the following set of equations for each $\omega_q$:

$$\Lambda^2(\omega_q) + (1 - 2j_q)\Lambda'(\omega_q) + \frac{1}{\mu}\Lambda(\omega_q) = 2\sum_{\begin{array}{c}p=1 \\ p \neq q \end{array}}^{M} j_p \frac{\Lambda(\omega_q) - \Lambda(\omega_p)}{\omega_q - \omega_p}.$$

- In particular, if $j_q = 1/2$, the $\Lambda'(\omega_q)$ term disappears and the equation becomes purely algebraic in all the $\Lambda(\omega_p)$. If $j_q > 1/2$, taking derivatives of the ODE w.r.t $\lambda$ before taking the $\lambda \to \omega_q$ limit gives additional equations which can be used to eliminate all $\Lambda'(\omega_p), \Lambda''(\omega_p), \ldots$, etc.
The $j_q = 1/2$ case: algebraic $\Lambda$ equations

- Denote $\Lambda_q = \Lambda(\omega_q)$ and define $\tilde{\Lambda}_q = \mu \Lambda_q$. The case where $j_q = 1/2 \quad \forall q$ (i.e., only one neutrino at each $\omega$) reduces to

$$\tilde{\Lambda}_q^2 + \tilde{\Lambda}_q = \mu \sum_{\substack{p=1 \\text{to} \ N \ \text{if} \ p \neq q}} \frac{\tilde{\Lambda}_q - \tilde{\Lambda}_p}{\omega_q - \omega_p}.$$ 

- System of size $N$ represented by $N$ coupled polynomial equations, of quadratic order regardless of $N$ or $\kappa$. In fact, the same set of equations admits solutions for all $\kappa = 0, \ldots, N$. And there are no singularities.

- Solutions corresponding to a particular $\kappa$ satisfy

$$\sum_{p} \tilde{\Lambda}_p = -\kappa,$$
Solving $\Lambda$ equations by homotopy continuation

$$\tilde{\Lambda}_q^2 + \tilde{\Lambda}_q = \mu \sum_{\substack{p=1 \\ p \neq q}}^N \frac{\tilde{\Lambda}_q - \tilde{\Lambda}_p}{\omega_q - \omega_p}.$$ 

- In the $\mu = 0$ limit, each $\tilde{\Lambda}_q$ can take the value 0 or $-1$, giving a total of $2^N$ solutions. In each solution, the number of $\tilde{\Lambda}_q$s that are $-1$ is the $\kappa$ of that particular solution.

- Each of the $2^N$ solutions for $\mu > 0$ can be constructed incrementally, starting from the corresponding $\mu = 0$ solution:
  - Use each $\tilde{\Lambda}_q$ solution obtained at $\mu = \mu_n$ to construct a starting guess for the corresponding solution at $\mu = \mu_n + \delta \mu$
  - Improve the starting guess using iterative numerical methods such as Newton-Raphson
Ten neutrino system: \( \{ \Lambda(\omega_q) \} \) solutions

Figure: A solution \( \{ \Lambda(\omega_1), \ldots, \Lambda(\omega_{10}) \} \) as a function of \( \mu \), for a system with ten neutrinos, distributed uniformly across frequencies \( \omega_q = q \omega_0 \). Shown here is one sample solution out of the 1024 for this system.
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Energy eigenvalues from $\Lambda$s

- Recall that energy eigenvalues given by:

$$E(\zeta_1, \ldots, \zeta_\kappa) = E_{N/2} + \sum_{\alpha=1}^{\kappa} \zeta_\alpha - \kappa \mu (N - \kappa + 1)$$

- In terms of $\tilde{\Lambda}_q$, these may be rewritten as

$$E(\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_N) = -\sum_p \frac{\omega_p}{2} + \mu \frac{N}{2} \left( \frac{N}{2} + 1 \right) - \sum_p \omega_p \tilde{\Lambda}_p$$

- Instructive to group energy eigenvalues according to $\kappa$: solutions are eigenstates of $J_z$, with eigenvalue $m = \frac{N}{2} - \kappa$. Within each $\kappa$, energy eigenvalues split into branches at large $\mu$. Each branch associates to a particular $|j, m\rangle$ as $\mu \to \infty$, with $j = |m|, \ldots, N/2$. 
Ten neutrino system: energy eigenvalues


**Figure:** Energy eigenvalues corresponding to all $\kappa = 2$ ($m = 3$) solutions of the BA equations, for an $N = 10$ neutrino system, as functions of $\mu$. 
Ten neutrino system: energy eigenvalues

![Graph showing energy eigenvalues as functions of $\mu/\omega_0$ for different $\kappa = 5$ solutions of the BA equations, for an $N = 10$ neutrino system.]


**Figure:** Energy eigenvalues corresponding to all $\kappa = 5$ ($m = 0$) solutions of the BA equations, for an $N = 10$ neutrino system, as functions of $\mu$. 
Energy eigenstates from $\Lambda$s

The process of calculating the eigenstates from the $\Lambda$s involves first calculating the power sums $P_f = \sum_{\alpha=1}^{\kappa} (S^-_\alpha)^f$, where $S^-_\alpha \equiv S^- (\zeta_\alpha)$ are the Gaudin operators defined earlier. In terms of $\Lambda$s, the power sums are given by

$$P_f = \sum_{\alpha=1}^{\kappa} (S^-_\alpha)^f = \sum_{i=1}^{\kappa} \sum_{p_1=1}^{M} \cdots \sum_{p_f=1}^{M} \frac{J^-_{p_1} \cdots J^-_{p_f}}{(\omega_{p_1} - \zeta_\alpha) \cdots (\omega_{p_f} - \zeta_\alpha)} \sum_{m=1}^{f} \Lambda_{p_m} \prod_{l=1}^{f} \frac{1}{\omega_{p_l} - \omega_{p_m}}.$$

For an eigenstate with a particular $\kappa$, the power sums $P_f$ for $f = 1, \ldots, \kappa$ are required.
Energy eigenstates from $\Lambda$s

- Recognize that

$$|\zeta_1, \ldots, \zeta_\kappa\rangle = e_\kappa(S_1^-, \ldots, S_\kappa^-) \left| \frac{N}{2}, \frac{N}{2} \right\rangle$$

where $e_\kappa$ is the $\kappa$-th elementary symmetric polynomial

- The elementary symmetric polynomials may be calculated recursively from the power sums using Newton’s identities:

$$e_k = \frac{1}{k} \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} P_i,$$

for $k = 1, \ldots, \kappa$. Calculating $e_\kappa$ for a particular set of $\Lambda$s immediately yields the corresponding eigenstate.
Energy eigenstates: examples

Figure: Overlaps, $|\langle \nu_{i_1}, \ldots, \nu_{i_N} | \psi_n \rangle |^2$, of particular energy eigenstates $|\psi_n\rangle$ with the mass basis states (with $i_1, \ldots, i_N = 1, 2$), as functions of $\mu$. Left: a particular eigenstate with $\kappa = 1$, of an $N = 3$ system. Right: a particular eigenstate with $\kappa = 2$, of an $N = 4$ system. Observe that, for $\mu > 0$, a state with a certain $\kappa$ has $^N C_\kappa$ nontrivial components.
Conclusions

- Calculations of collective neutrino flavor evolution typically rely on a ‘mean-field’, i.e., effective one-particle description.

- Important to test the efficacy and/or limitations of the mean-field by performing many-body calculations.

- Evolution in the many-body case can be studied by calculating the eigenvalues and eigenvectors of the Hamiltonian by solving the Bethe Ansatz equations.

- For certain simple systems, qualitative differences in flavor evolution observed between many-body and mean-field treatments, resulting from entangled states which are absent in the mean-field limit—talks by Michael and Ermal to follow.
Entanglement: a preview

[Cervia et al., arxiv:1908.03511]

Figure: Entropy of entanglement between the neutrino at frequency $\omega_4$ and the rest of the ensemble, for all eigenstates of an $N = 4$ neutrino system, corresponding to $\kappa = 1$ (left) and 2 (right).
The N3AS collaboration

- Network in Neutrinos, Nuclear Astrophysics, and Symmetries
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  - Neutrino physics and astrophysics
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- Funded by National Science Foundation (NSF) and Heising-Simons Foundation
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Simple case: two neutrinos, $j_1 = j_2 = 1/2$

Defining $\Lambda_p \equiv \Lambda(\omega_p)$ and $\eta = \omega_2 - \omega_1$, the equations are:

\begin{align*}
\Lambda_1^2 + \frac{1}{\mu} \Lambda_1 &= \frac{\Lambda_1 - \Lambda_2}{-\eta} \\
\Lambda_2^2 + \frac{1}{\mu} \Lambda_2 &= \frac{\Lambda_2 - \Lambda_1}{\eta}.
\end{align*}

The following solutions can be obtained analytically:

\begin{align*}
\Lambda_1 &= \Lambda_2 = 0, \quad \text{or} \quad \Lambda_1 &= \Lambda_2 = -\frac{1}{\mu}, \quad \text{or} \\
\Lambda_1 &= -\left(\frac{1}{2\mu} + \frac{1}{\eta}\right) \pm \frac{1}{2} \sqrt{\frac{1}{\mu^2} + \frac{4}{\eta^2}} \\
\Lambda_2 &= -\left(\frac{1}{2\mu} - \frac{1}{\eta}\right) \pm \frac{1}{2} \sqrt{\frac{1}{\mu^2} + \frac{4}{\eta^2}}.
\end{align*}

The three sets of solutions correspond to $\kappa = 0, 2, \text{ and } 1$, respectively.