Lecture notes - Minimal energy cost of entanglement extraction

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Abstract

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1 Introduction

1.1 What is in this lecture for you?

This Friday morning lecture covers a recent paper by Lucas Hackl and myself [1]: The minimal energy cost for extracting entanglement from the ground state of a system of modes with a quadratic Hamiltonian.[1] Our solution is very much based on the geometrical structures underlying Gaussian states. Therefore, I want to use this lecture to give you an introduction to the geometry of Gaussian states, and to present some practical and abstract aspects and methods along the way. My hope is that you will find them both practical and esthetically pleasing, and get excited to address related questions and use similar methods in your research. In addition to the references already mentioned in earlier lectures, some relevant textbooksin this direction are [2, 3, 4, 5], and further references as cited in our article.

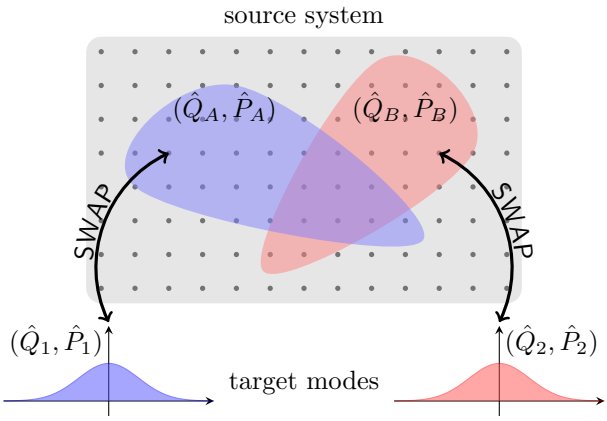

Figure 1: General framework of entanglement extraction from a system of bosonic modes. Two target modes get entangled by swapping their states with a pair of entangled modes in the source system. [1]

1.2 What is entanglement extraction?

Entanglement is a resource which plays a key role all over different quantum information processing tasks. Hence, in technological implementations, we need to be able to provide it, e.g., at the input ports of a quantum computer. The standard way to get input qubits entangled is to make them interact with each other. In situations where this is not possible, be it due to engineering or spacetime constraints, alternative approaches are needed.

One such alternative is the idea to extract entanglement from a source system. The ground state of many complex quantum systems, from many body systems to quantum fields, exhibit entanglement

[1] To be specific, we find the minimum increase in the energy expectation value of a system of bosonic or fermionic modes with a quadratic Hamiltonian, when a pair of modes in a pure entangled state is swapped out of the system and brought into a product state.

between spatially separated regions. Various extraction protocols have been studied in recent years, e.g., in the context of *entanglement harvesting* from quantum fields. However, only last year Beny et al. identified the energy cost of such entanglement extraction protocols as a so far unexplored question [6].

Aside from potential future applications, this question is very interesting from a theoretical perspective given how much investigating the entanglement structure of complex quantum systems has contributed to our understanding of them.

The model for which we here derive the minimum energy cost of entanglement extraction could be called *partner mode extraction*. In this framework we consider the source system to consist of a system of N modes with a quadratic Hamiltonian. (The modes can be bosonic or fermionic, both cases are covered in [1]. For the purpose of this lecture we restrict attention to bosonic modes.) We assume the system to be in its ground state initially.

In this source system we pick two source modes A and B which are entangled. This entanglement we want to extract from the system. To this end we assume that we have two target modes, which initially are in a product state. The extraction then consists of swapping the state of the target modes with the state of the source modes in the system.

After the swap operation, the source system is not in its ground state any longer, thus its energy expectation value has increased. This increase in the energy expectation value is what we refer to as the energy cost and calculate in this work. (For practical implementations, in particular with a low number of repetitions, other figures of merits such as the variance also are important.)

We restrict our choice of source modes to be a pair of partner modes. These are pair of modes which are in a product state with the rest of the system, while being in a pure two-mode entangled state with each other. We conjecture that this choice minimises the energy cost (for reasons to be discussed later).

Being very technical, and discarding the motivation of entanglement extraction, what we calculate is the minimum increase in energy expectation of a quadratic Hamiltonian when a pair of modes that are in a pure entangled state when the system is in its overall ground state, are replaced by a product state.

1.3 Warm-up example: Dilute boson gas

Consider the following Hamiltonian of a weakly interacting dilute Boson gas (see, e.g., [7]),

$$\hat{H} = \sum_{k\neq 0} \omega_k a_k^\dagger a_k + \gamma_k \left(a_k^\dagger a_{-k}^\dagger + a_k a_{-k} \right) \tag{1}$$

with $\omega_k = \omega_{-k}$ and $\gamma_k = \gamma_{-k}$, and $0 < \gamma_k < \omega_k/2$ such that the Hamiltonian is bounded from below. The sum runs over all modes with non-zero momentum, which have very low occupation numbers, but it excludes the zero momentum mode, which is macroscopically occupied. In this way, the Hamiltonian describes the interaction of excitations in the higher modes with the condensate of particles in the zero-mode.

The structure of the Hamiltonian is such that it couples mode pairs of opposite momentum. To make this most evident, we can use $a_k = \frac{1}{\sqrt{2}} \left(\hat{q}_k + \mathrm{i}\hat{p}_k \right)$ and rewrite the Hamiltonian as

$$\begin{aligned} \hat{H} &= \sum_{k>0} \hat{H}_k - \sum_{k\neq 0} \frac{\omega_k}{2} \\ &= \sum_{k>0} \frac{\omega_k}{2} \left(\hat{q}_k^2 + \hat{p}_k^2 + \hat{q}_{-k}^2 + \hat{p}_{-k}^2 \right) + 2\gamma_k \left(\hat{q}_k \hat{q}_{-k} - \hat{p}_k \hat{p}_{-k} \right) - \sum_{k\neq 0} \frac{\omega_k}{2}. \end{aligned} \tag{2}$$

Each operator $\hat{H}_k$ represents a two-mode Hamiltonian represented by the matrix

$$\mathbf{h}_k = \begin{pmatrix} \omega_k & 0 & 2\gamma_k & 0 \\ 0 & \omega_k & 0 & -2\gamma_k \\ 2\gamma_k & 0 & \omega_k & 0 \\ 0 & -2\gamma_k & 0 & \omega_k \end{pmatrix}, \tag{3}$$

which has excitation energy $\epsilon = \sqrt{\omega_k^2 - 4\gamma_k^2}$ in both its eigenmodes. The ground state of $\hat{H}$ is thus a product state of two-mode states where each pair of modes with opposite momentum is in the state with covariance matrix

$$\mathbf{G}_k = \frac{1}{\sqrt{\omega_k^2 - 4\gamma_k^2}} \begin{pmatrix} \omega_k & 0 & -2\gamma_k & 0 \\ 0 & \omega_k & 0 & 2\gamma_k \\ -2\gamma_k & 0 & \omega_k & 0 \\ 0 & 2\gamma_k & 0 & \omega_k \end{pmatrix}, \tag{4}$$

which means that they are in a two-mode squeezed state with entanglement entropy

$$S = s_b\left(1/\sqrt{1-4\gamma_k^2/\omega_k^2}\right),$$

where $s_b(x) = \frac{x+1}{2}\log\left(\frac{x+1}{2}\right) - \frac{x-1}{2}\log\left(\frac{x-1}{2}\right)$.

If we want to extract that entanglement from the gas, swap the state of such a mode pair onto another system and leave the modes of the gas behind in a product state instead - by how much do we increase the energy expectation value of the gas?

We are restricted to leave behind a product state of the two modes. Hence we should leave behind the modes in the ground states of the Hamiltonian restricted to the single modes. The single mode Hamiltonians are represented by the diagonal blocks of $\mathbf{h}_k$, i.e., the two-by-two matrices with ω_k on their diagonal. These each have ground state energy $\omega_k/2$. Subtracting the ground state energy of $\mathbf{h}_k$ we find that the energy of the gas increases by

$$\Delta E = \omega_k - \sqrt{\omega_k^2 - 4\gamma_k^2} = \omega_k\left(1-\sqrt{1-4\gamma_k^2/\omega_k^2}\right). \tag{5}$$

This simple examples captures all the essentials of the result presented in this lecture. This is because the pairing of entangled modes that we see between opposite momentum moes here is a general property of pure Gaussian states: In all pure Gaussian states, every mode has one partner mode with which it is in a pure two-mode squeezed state. Therefore, our next step is to discuss the entanglement structure of pure Gaussian states and how to construct the partner mode of a given mode. This we will then use to derive the general minimal energy cost for the extraction of partner modes.

2 A (hint of the) geometry of Gaussian states

For calculations, by hand or computer, we typically use matrices and vectors to represent Gaussian states. In fact, many text books only cover matrix and vector notation. However, sometimes the matrices can obscure the appealing mathematical structures underneath the formalism. Therefore, in this section, let us briefly review the symplectic geometry underlying *bosonic* Gaussian states, and introduce the abstract index notation. This provides

- a geometric way to understand and think about Gaussian states,
- a formalism which seamlessly carries over to fermionic Gaussian states,
- a powerful and practical formalism for analytic calculations, in which we can derive
- basis-independent expressions safely.

2.1 Phase space and co-phase space

The symplectic formalism for Gaussian states is based on the formalism of classical mechanics. There the phase space of a system corresponding to N modes is the $2N$-dimensional vector space V which is spanned by N generalised coordinates and their conjugate momenta $\{q_1, p_1, ..., q_n, p_n\}$. Linear observables of the system are linear, real-valued functions of the coordinates, i.e., they are elements of the co-vector space $V^* = \{v : V \to \mathbb{R} |\, v \text{ linear }\}$.

In **abstract index notation**, we denote elements of the vector space as objects with one index upstairs, and elements of the covector space with one index downstairs.

$$\alpha \in V \leftrightarrow \alpha^a, \qquad x \in V^* \leftrightarrow v_a \tag{6}$$

The rule of abstract index notation is that every index which appears one time downstairs and one time upstairs is contracted (summed over). (In fact, if an index appears more than twice or appears two times upstairs or downstairs, something is wrong!) Hence, in abstract index notation using a covector to map a vector to a number reads

$$x(\alpha) \Leftrightarrow x_a\alpha^a = \alpha^a x_a. \tag{7}$$

Now let us consider quantum Gaussian states. Also here, linear observables are represented by elements of V^*. The **quantization map** maps covectors to the corresponding quantum-mechanical operators. It is a linear map:

$$\begin{aligned} \hat{\xi} : V^* &\to \mathcal{L}(\mathcal{H}) \\ x_a &\mapsto x_a \hat{\xi}^a \end{aligned} \tag{8}$$

Since the quantization map acts on covectors, we denote it with an index upstairs. And we denote it with a hat, because it returns an operator acting on the Hilbert space of the system. (This means, objects with hats do not commute and we need to keep track of their order when manipulating expressions.)

With respect to a basis of operators $(\hat{q}_1, \hat{p}_1, ..., \hat{q}_n, \hat{p}_n)$ of the system, the quantization map is represented by an operator-valued vector $\underline{\hat{\xi}}$.

$$\hat{\xi}^a \equiv \underline{\hat{\xi}} = \begin{pmatrix} \hat{q}_1 \\ \hat{p}_1 \\ ... \\ \hat{q}_n \\ \hat{p}_n \end{pmatrix}. \tag{9}$$

2.2 Commutator, covariances and hamiltonians as two-forms

The symplectic form of the system, the covariance matrix of a state and also quadratic Hamiltonians all are represented by two-forms acting on the phase and the co-phase space respectively.

The **symplectic form** is a non-degenerate anti-symmetric two-form on the co-phase space

$$\begin{aligned} \Omega : V^* \times V^* &\to \mathbb{R} \\ (u, v) &\mapsto \Omega(u, v) = -\mathrm{i} \left[u_a \hat{\xi}^a, v_b \hat{\xi}^b \right] \end{aligned} \tag{10}$$

which yields the commutator between the linear observables associated with the covectors. Hence, if two linear observables commute, then the corresponding covectors lie in the symplectic complement of each other. In abstract index notation it has two indices upstairs Ω^{ab}, and we have

$$\left[\hat{\xi}^a, \hat{\xi}^b \right] = \mathrm{i}\Omega^{ab}. \tag{11}$$

Picking a **symplectic basis** in V^*, i.e., a basis with respect to which Ω is represented by a matrix in the standard form

$$\Omega^{ab} \equiv \oplus_{i=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{12}$$

is equivalent to choosing a set of canonically commuting operators $(\hat{q}_1, \hat{p}_1, ..., \hat{q}_n, \hat{p}_n)$ in the quantization map. The symplectic form being non-degenerate, it both induces a map from covectors to vectors $v_a \mapsto \Omega^{ab} v_b$, and it has an inverse two-form $\Omega^{-1} : V \times V \to \mathbb{R}$ which satisfies $\Omega^{ab} \Omega^{-1}_{bc} = \delta^a{}_c$. (Note, that the latter is the symplectic form on phase space of classical mechanics.)

The **symplectic group** consists of all linear maps $M : V \to V$ which preserve the symplectic form.[2]

$$\Omega(v, w) = \Omega(M^\intercal(v), M^\intercal(w)). \tag{13}$$

The abstract index notation, where the linear map on V reads $M^a{}_b$, this condition reads

$$M^a{}_b \Omega^{bc} M^{\intercal d}_c = \Omega^{ad}. \tag{14}$$

Acting with $(M^\intercal)_a{}^b$ on covectors preserves the commutator between the associated linear observables.

The **covariance matrix** of a state $|\psi\rangle$ is represented by a symmetric two-form which gives the correlation of the linear observables associated with the covectors.

$$\begin{aligned} G : V^* \times V^* &\to \mathbb{R} \\ (u, v) &\mapsto \langle\psi| \left(u_a \hat{\xi}^a v_b \hat{\xi}^b + v_b \hat{\xi}^b u_a \hat{\xi}^a \right) |\psi\rangle \end{aligned} \tag{15}$$

[2]The transpose $M^\intercal$ of a linear map is defined by $M^\intercal(v)(\alpha) = v(M(\alpha))$ for all $v \in V^*, \alpha \in V$.

In abstract index notation this reads

$$G^{ab} = \langle\psi| \left(\hat{\xi}^a\hat{\xi}^b + \hat{\xi}^b\hat{\xi}^a\right) |\psi\rangle \,. \tag{16}$$

This means we can interpret the covariance matrix of a state as a metric on the co-phase space - linear observables that are not correlated are then represented by orthognal covectors.

For a **pure Gaussian state** Williamson's theorem guarantees that there is a symplectic basis with respect to which G is represented by the identity matrix $G^{ab} \equiv \mathbb{I}_{2N}$, i.e., G is a product state where these N modes each are in their vacuum state. (Note that sometimes the covariance matrix of a pure state is defined to be half of G.)

A general **quadratic Hamiltonian** $\hat{H}$ is of the form

$$\hat{H} = \frac{1}{2}h_{ab}\hat{\xi}^a\hat{\xi}^b + f_a\hat{\xi}^a. \tag{17}$$

For the purpose of the scope of this lecture we will chose the linear part $\hat{\xi}^a f_a$ to vanish (which can always be achieved by a linear shift of the basis operators and does not change the entanglement properties of the state). The quadratic part of the Hamiltonian is given by a symmetric two-form on phase space.

$$h : V \times V \to \mathbb{R} \tag{18}$$

If the Hamiltonian is bounded from below, Williamson's theorem ensures that there exists a symplectic basis such that h_{ab} is represented by a block-diagonal matrix

$$h \equiv \oplus_{i=1}^{N} \begin{pmatrix} \omega_i & 0 \\ 0 & \omega_i \end{pmatrix}, \tag{19}$$

where $\omega_i \in \mathbb{R}^+$ are the symplectic eigenvalues or **excitation energies** of $\hat{H}$. The symplectic eigenvalues arise from the eigenvalues of the linear map $K^a{}_b = \Omega^{ac}h_{cb} : V \to V$ which are $\{\pm\mathrm{i}\omega_i\}$ for $i = 1, ..., N$. The ground state of $\hat{H}$ is a Gaussian state, in the energy eigenbasis it takes the form of the identity matrix.

The **expectation value** of $\hat{H}$ (with $f_a = 0$) in a Gaussian state with covariance matrix G^{ab} is

$$\left\langle \hat{H} \right\rangle = \frac{1}{4}\,\mathrm{Tr}\left(h_{ab}G^{ba}\right) = \frac{1}{4}h_{ab}G^{ab}. \tag{20}$$

2.3 Complex linear structure

In the abstract index notation it is straightforward to concatenate vectors, maps and two-forms to define new objects. If the index structure of the expression is valid we can be sure that the definition is well-behaved and basis-independent. Let us here make use of this to introduce the **linear complex structure** of a Gaussian state as

$$J^a{}_b := -G^{ac}\Omega^{-1}_{cb}. \tag{21}$$

The index structure shows that $J : V \to V$ is a linear map acting on the phase space. It has a row of interesting properties, and just in the next section we will use it to construct partner modes.

- It is called a linear complex structure because if G is a pure Gaussian state, then $J^2 = -\mathbb{I}$.
- It contains all information about the state's covariance matrix, since $G = -J\Omega$. Thus J is used to derive the entanglement structure of Gaussian states.
- It can also be expressed as

  $$J^a{}_b = \Omega^{ac}G^{-1}_{cb}. \tag{22}$$

  This identity is easily checked using the matrix representation with respect to the symplectic basis where $G^{ab} \equiv \mathbb{I}$ is given by the identity matrix.

3 Partner modes and entanglement structure of Gaussian states

Consider a system of N bosonic modes which is in a pure state $|\Psi\rangle$. In this system we can define a single mode A by picking two linear observables which obey the canonical commutation relations

$$\left[\hat{Q}_A, \hat{P}_A\right] = \mathrm{i}. \tag{23}$$

It is has been shown that it is possible to find a **partner mode** $\bar{A}$ of the mode A, such that $|\Psi\rangle$ factors into a product state between the two-mode subspace $A\bar{A}$ and the rest of the system [8].

$$|\Psi\rangle = |\psi\rangle_{A\bar{A}} \otimes |0\rangle_R \tag{24}$$

Hence, if $|\Psi\rangle$ is a Gaussian state, there is a symplectic basis such that G is represented by the matrix

$$G \equiv \left(\begin{array}{cc|cc|c} \cosh 2r & 0 & \sinh 2r & 0 & 0 \\ 0 & \cosh 2r & 0 & -\sinh 2r & 0 \\ \hline \sinh 2r & 0 & \cosh 2r & 0 & 0 \\ 0 & -\sinh 2r & 0 & \cosh 2r & 0 \\ \hline 0 & 0 & 0 & 0 & \mathbb{I}_{N-2} \end{array}\right). \tag{25}$$

The basis corresponds to a decomposition $A \oplus \bar{A} \oplus R$ of the system's phase space, i.e, the first two columns refer to the mode A and the second two to the partner mode $\bar{A}$.

The **von Neumann entropy** S_A between the partner modes A and $\bar{A}$ is given by [9, 10]

$$S_A = s_b(\cosh 2r_i), \quad s_b(x) = \left(\frac{x+1}{2}\right)\log\left(\frac{x+1}{2}\right) - \left(\frac{x-1}{2}\right)\log\left(\frac{x-1}{2}\right). \tag{26}$$

For a given mode A, we can use the simple and appealing **partner mode formula** to construct $\bar{A}$. We assume that the linear observables defining the mode

$$\hat{Q}_a = x_a\hat{\xi}^a, \qquad \hat{P}_a = k_a\hat{\xi}^a \tag{27}$$

bring the single-mode covariance matrix into the standard form we see on the first diagonal block[3], i.e.,

$$G^{ab}x_a x_b = G^{ab}k_a k_b = \cosh 2r, \quad G^{ab}x_a k_b = 0. \tag{28}$$

Then the partner mode is spanned by the two quadrature operators $\hat{Q}_{\bar{A}} = \bar{x}_a\hat{\xi}^a$ and $\hat{P}_{\bar{A}} = \bar{k}_a\hat{\xi}^a$ with

$$\bar{x}_a = \coth(2r)x_a + \frac{1}{\sinh 2r}\left(J^\intercal\right)_a{}^c k_c, \tag{29}$$

$$\bar{k}_a = -\coth(2r)k_a + \frac{1}{\sinh 2r}\left(J^\intercal\right)_a{}^c x_c. \tag{30}$$

(Note that the transpose of J , i.e., $\left(J^\intercal\right)_a{}^b = \Omega^{-1}_{ac}G^{cb} = -G^{-1}_{ac}\Omega^{cb}$, is a linear map $V^* \to V^*$.)

Using the identities $(J^\intercal)^2 = -\mathbb{I}$ and $G\Omega^{-1}G = -\Omega$, we can check that the covectors $(x, k, \bar{x}, \bar{k})$ bring the upper right 4×4-block of G in the form above, i.e., that

$$G^{ab}\bar{x}_a\bar{x}_b = G^{ab}\bar{k}_a\bar{k}_b = \cosh 2r, \quad G^{ab}x_a\bar{x}_b = -G^{ab}k_a\bar{k}_b = \sinh 2r, \tag{31}$$

$$G^{ab}k_a\bar{x}_b = G^{ab}x_a\bar{k}_b = G^{ab}\bar{x}_a\bar{k}_b = 0. \tag{32}$$

How do we know, that the modes A and $\bar{A}$ share no correlations with the rest of the system? To see this, let x'_a and k'_a define another mode in the system, i.e., $\Omega^{ab}x'_a k'_b = 1$. Then

$$\Omega^{ab}x'_a\bar{x}_b = \coth(2r)\Omega^{ab}x'_a x_b + \frac{G^{ad}x'_a k_d}{\sinh 2r}, \qquad \Omega^{ab}k'_a\bar{x}_b = \coth(2r)\Omega^{ab}k'_a x_b + \frac{G^{ad}k'_a k_d}{\sinh 2r}, \tag{33}$$

$$\Omega^{ab}x'_a\bar{k}_b = -\coth(2r)\Omega^{ab}x'_a k_b + \frac{G^{ad}x'_a x_d}{\sinh 2r}, \qquad \Omega^{ab}x'_a\bar{x}_b = -\coth(2r)\Omega^{ab}k'_a k_b + \frac{G^{ad}k'_a x_d}{\sinh 2r}. \tag{34}$$

This means that if the mode commutes with the original mode, i.e.,

$$\Omega(x, x') = \Omega(x, k') = \Omega(k, x') = \Omega(k, k') = 0, \tag{35}$$

[3] This form can always be obtained by a single-mode symplectic transformation.

then the commutator with the partner mode is proportional to the correlation with the original mode. In particular, this warrants that G^{ab} can be cast into the standard form (25).

From the partner mode construction one can derive the **entanglement structure of pure Gaussian states**: For arbitrary bipartitions the state factors into a product of entangled mode pairs [11, 1].

To see this, consider a system of N modes which we bipartite into two subsystems of N_A and N_B modes, with $N_A \leq N_B$. Then by combining the normal modes $(q_1^A, p_1^A, \cdots, q_{N_A}^A, p_{N_A}^A)$ of A, i.e., the modes which bring the partial covariance matrix of A into normal form, and the normal modes $(q_1^B, p_1^B, \cdots, q_{N_B}^B, p_{N_B}^B)$ of B, we obtain a symplectic basis of the whole system in which G^{ab} is represented by

$$G \equiv \left(\begin{array}{ccc|ccc|ccc} \mathbf{ch}_1 & \cdots & 0 & \mathbf{sh}_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{ch}_{N_A} & 0 & \cdots & \mathbf{sh}_{N_A} & 0 & \cdots & 0 \\ \hline \mathbf{sh}_1 & \cdots & 0 & \mathbf{ch}_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{sh}_{N_A} & 0 & \cdots & \mathbf{ch}_{N_A} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \mathbb{I}_2 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \mathbb{I}_2 \end{array}\right), \tag{36}$$

built from the 2×2-matrices, with $r_i \geq 0$[4],

$$\mathbf{ch}_i = \begin{pmatrix} \cosh 2r_i & 0 \\ 0 & \cosh 2r_i \end{pmatrix}, \quad \mathbf{sh}_i = \begin{pmatrix} \sinh 2r_i & 0 \\ 0 & -\sinh 2r_i \end{pmatrix}. \tag{37}$$

4 Energy cost of partner mode extraction

We can now derive the least amount of energy that one needs to inject into a system with a quadratic Hamiltonian, when extracting a pair of partner modes from its ground state. In fact, it turns out that at its core this problem effectively simplifies to solving the case of a source system of only 2 modes.

4.1 Partner mode extraction is a two-mode problem

Recall that our overall framework was the following: We are given a source system S consisting of N modes, which is in the ground state of its quadratic Hamiltonian $\hat{H} = \frac{1}{2} h_{ab} \hat{\xi}^a \hat{\xi}^b$. We then pick a pair of partner modes A and $\bar{A}$ in the system, and swap the state of these modes with the states of two external target modes.

Initially the system is in its ground state $|0\rangle$ which we know is a product state between the mode pair and the rest of the system. After the swap, the two partner modes are left behind in a product state.

$$|0\rangle\langle 0|_S = |\psi\rangle\langle\psi|_{A\bar{A}} \otimes |0'\rangle\langle 0'|_R \overset{SWAP}{\longrightarrow} \rho_A \otimes \sigma_{\bar{A}} \otimes |0'\rangle\langle 0'|_R \tag{38}$$

By how much increases the energy expectation value of the system under this process?

$$\Delta E = \Delta \left\langle \hat{H} \right\rangle = \mathrm{Tr}\left(\rho_A \otimes \sigma_{\bar{A}} \otimes |0'\rangle\langle 0'|_R \, \hat{H}\right) - \mathrm{Tr}\left(|\psi\rangle\langle\psi|_{A\bar{A}} \otimes |0'\rangle\langle 0'|_R \, \hat{H}\right) = ...? \tag{39}$$

We can split the Hamiltonian into three terms,

$$\hat{H} = \hat{H}_R + \hat{H}_{A\bar{A},R} + \hat{H}_{A\bar{A}} \tag{40}$$

separating the part of the Hamiltonian acting only on the rest of the system, the part coupling the rest of the system and the partner modes, and the part acting only on the partner modes. Then we see that:

- $\hat{H}_R$ contributes to both terms equally because the state of R is unchanged
- $\left\langle \hat{H}_{A\bar{A},R} \right\rangle = 0$ in both states because they both are product states without correlations between $A\bar{A}$ and R

[4]Note, that $r_i = 0$ means the two modes are actually not entangled.

- hence $\hat{H}_{A\bar{A}}$ determines the energy cost alone

$$\Delta E = \Delta \left\langle \hat{H} \right\rangle = \Delta \left\langle \hat{H}_{A\bar{A}} \right\rangle = \mathrm{Tr}\left(\rho_A \otimes \sigma_{\bar{A}} \hat{H}_{A\bar{A}}\right) - \mathrm{Tr}\left(|\psi\rangle\langle\psi|_{A\bar{A}}\, \hat{H}_{A\bar{A}}\right) \tag{41}$$

Furthermore, it is clear which state $\rho_A \otimes \sigma_{\bar{A}}$ minimizes the energy cost: We need to choose the product of the ground states of the single mode restrictions of

$$\hat{H}_{A\bar{A}} = \hat{H}_A + \hat{H}_{\bar{A}} + \hat{H}_{A,\bar{A}}. \tag{42}$$

That is we choose

$$\rho_A = |g_A\rangle\langle g_A|\,, \quad \sigma_{\bar{A}} = |g_{\bar{A}}\rangle\langle g_{\bar{A}}| \tag{43}$$

where $|g_A\rangle$ is the ground state of $\hat{H}_A$ and $|g_{\bar{A}}\rangle$ of $\hat{H}_{\bar{A}}$, both of which are Gaussian states.

This shows that to calculate the energy cost for extracting the partner modes A and $\bar{A}$ we only need to know the two-mode quadratic Hamiltonian $\hat{H}_{A\bar{A}}$. Hence, once we know the minimum energy cost for a source system of only two modes, in larger source systems we only need to optimize over the choice of partner modes such that the restricted two-mode Hamiltonian yields the lowest energy cost possible.

Just as one would assume, the lowest possible energy cost can be achieved if the space spanned by A and $\bar{A}$ coincides with the space of the two energy eigenmodes of $\hat{H}$ with the lowest excitation energies.

4.2 Connecting energy eigenmodes to partner modes

In order to calculate and to minimize the energy cost we need to relate the partner modes that we are extracting to the eigenmodes of the Hamiltonian $\hat{H}_{A\bar{A}}$.

Because in the basis given by the partner modes $(x_a, k_a, \bar{x}_a, \bar{k}_a)$ we know that the covariance matrix of the initial state is given by

$$G^{ab} \equiv_{A\bar{A}} \left(\begin{array}{cc|cc} \cosh 2r & 0 & \sinh 2r & 0 \\ 0 & \cosh 2r & 0 & -\sinh 2r \\ \hline \sinh 2r & 0 & \cosh 2r & 0 \\ 0 & -\sinh 2r & 0 & \cosh 2r \end{array}\right), \quad h_{ab} \equiv_{A\bar{A}} ...? \tag{44}$$

but we do not know which matrix represents h_{ab}. We do know however that there is a symplectic basis of eigenmodes of $\hat{H}_{A\bar{A}}$. In this basis we have

$$G^{ab} \equiv_H \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \quad h_{ab} \equiv_H \left(\begin{array}{cc|cc} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_1 & 0 & 0 \\ \hline 0 & 0 & \epsilon_2 & 0 \\ 0 & 0 & 0 & \epsilon_2 \end{array}\right). \tag{45}$$

Since both bases are symplectic, there is a symplectic transformation connecting them. We will derive the most general form of such a transformation between the energy eigenmodes to a pair of partner modes with squeezing parameter r. Then we can pick the optimal one to minimise the energy cost.

To get from energy modes to partner modes, we work our way backwards and first apply the inverse squeezing operation to the partner modes

$$S^a{}_b \equiv_{A\bar{A}} \begin{pmatrix} \cosh r & 0 & -\sinh r & 0 \\ 0 & \cosh r & 0 & \sinh r \\ -\sinh r & 0 & \cosh r & 0 \\ 0 & \sinh r & 0 & \cosh r \end{pmatrix} \tag{46}$$

This symplectic transformation maps the partner modes to a basis of modes in which $G^{ab} \equiv \mathbb{I}$ is represented by the identity matrix, just as it is in the energy eigenbasis. However, we still do not know the form of h_{ab}.

What we do know is that the unsqueezed modes and the energy eigenbasis are connected by a symplectic transformation $M^a{}_b$ which leaves the shape of $G \equiv \mathbb{I}$ invariant. That means M also lies in the orthogonal group. As such, it can be written as the exponential $M^a{}_b = \exp(K)^a{}_b$ of a generator $K^a{}_b : V \to V$. In order for M to be both orthogonal and symplectic, the generator has to obey two conditions:

$$M^a{}_b G^{bc} (M^\intercal)_c{}^d = G^{ad} \;\Rightarrow\quad K^a{}_b G^{bc} + G^{ab} (K^\intercal)_b{}^c = 0 \tag{47}$$

$$M^a{}_b\Omega^{bc}(M^\intercal)_c{}^d = G^{ad} \;\Rightarrow\quad K^a{}_b\Omega^{bc} - \Omega^{ab}(K^\intercal)_b{}^c = 0 \tag{48}$$

There is a two-parameter family of such matrices, which in turn means that we obtain a two-parameter family of possible $M(\theta,\phi)$ of possible transformations between the energy eigenmodes and unsqueezed modes.

4.3 Optimal partner modes

Now we know that the any possible pair of partner modes is connected to the energy eigenmodes by a transformation of the form

$$N^a{}_b = S^a{}_c M(\theta,\phi)^c{}_b. \tag{49}$$

We can use the explicit expressions for $M(\theta,\phi)$ and S to calculate the energy increase ΔE as a function of the parameters. As worked out in our paper [1], ΔE is minimal if

$$M(\theta,\phi) \equiv_H \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{50}$$

i.e., the optimal partner modes are obtained by taking the odd and even combinations of the energy eigenmodes and then perform two-mode squeezing on these modes where the amount of entanglement between the modes is controlled by the squeezing parameter r.

4.4 Minimal energy cost

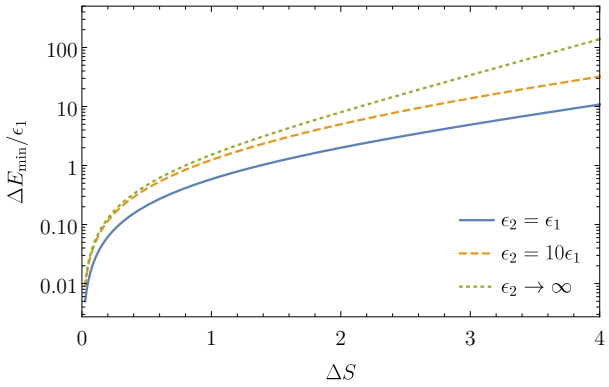

Figure 2: Minimal energy cost $\Delta E_{\min}$ as a function of the extracted entanglement entropy ΔS from a bosonic, quadratic two-mode Hamiltonian with excitation energies ϵ_1 and ϵ_2. [1]

The minimal energy cost for the extraction of a partner mode pair with an amount of entanglement set by the squeezing parameter r is found to be

$$\Delta E_{\min} = \frac{1}{2}\left(\sqrt{\epsilon_1^2 + \epsilon_2^2 + 2\epsilon_1\epsilon_2\cosh 4r} - \epsilon_1 - \epsilon_2\right), \tag{51}$$

where ϵ_1, ϵ_2 are the excitation energies of $\hat{H}_{A\bar{A}}$. In the limit of a degenerate Hamiltonian the energy cost goes down to

$$E_{\min} \overset{\epsilon_2\to\epsilon_1}{\longrightarrow} 2\epsilon_1 \sinh^2(r). \tag{52}$$

Even as the value of the second energy level diverges, the energy cost is upper bounded by

$$E_{\min} \overset{\epsilon_2\to\infty}{\longrightarrow} \epsilon_1 \left(\sinh 2r\right)^2. \tag{53}$$

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