ANALYTIC BOUNDARIES OF THE EFTHEDRON AND ITS DE-PROJECTION

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[A] "The EFT hedron" N. Arkani-Hamed (IAS), Tzu-Chen Huang (Caltech), Y-T H 2012.15849

[B] "Into the EFThedron and UV constraints from IR consistency" Li-Yuan Chiang, Wei Li, He-Chen Wen, Laurentiu Rodina, Y-T H 2105.02862 & to appear

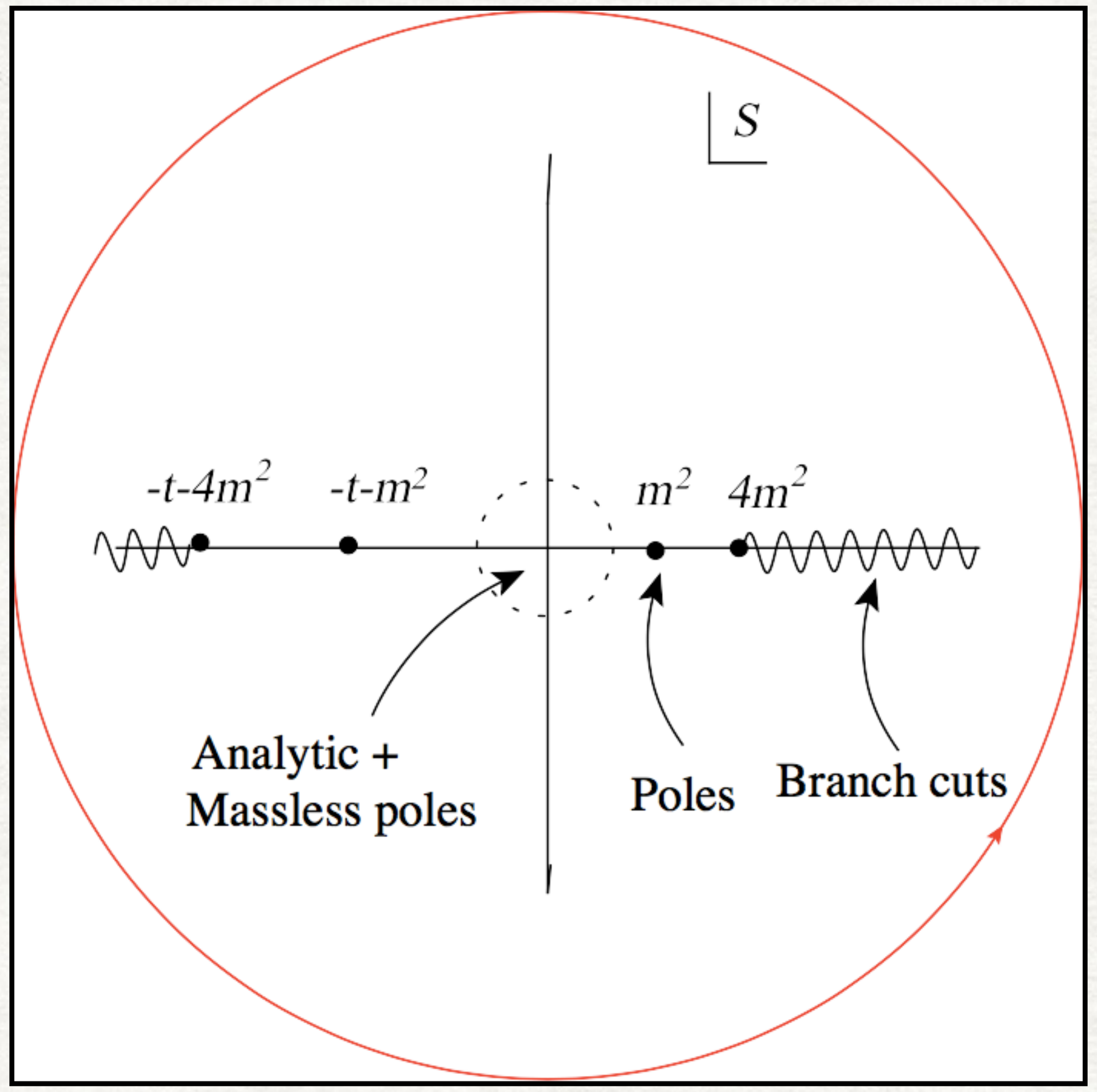
The s-MATRIX (bootstrap) reload

Carving out the space of consistent theory through the space of consistent M(s,t) assuming

- For t<0, |M(s,t)/s^2| -> 0 at high energies

$$0 = \frac{1}{2\pi i}\int_{\infty} \frac{ds'}{s'} \frac{M(s',t)}{(s'-s_1)(s'-s_2)}$$

- M(s,t) is analytic away from the real axes for t<< M^2



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Impose Unitarity + Locality + Crossing

manifest Local+Crossing ansatz -> impose non-perturbative unitarity $|S_\ell(s)| \leq 1$

Paulos, Penedones, Toledo, van Rees, Vieira 1708.06765
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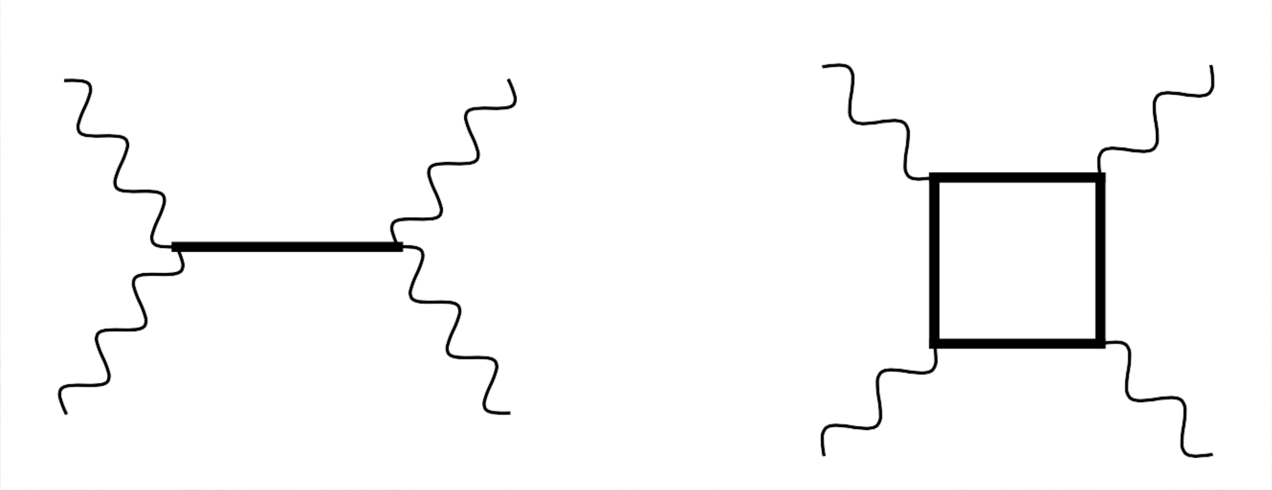
The space is characterized by a positive geometry:

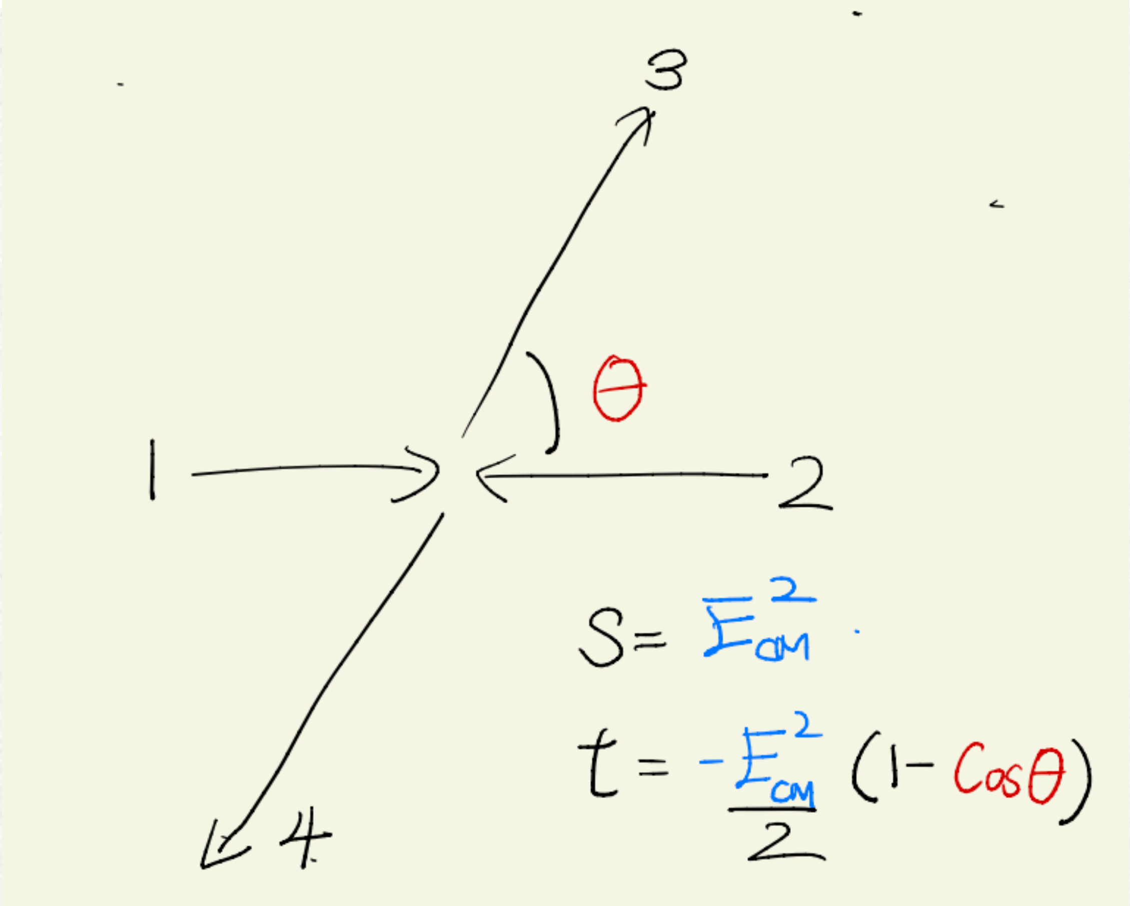
The convex hull of product moment curves

The space we are interested corresponds to the space of effective field theories (EFT) that descends from a UV completion



$$\mathcal{L} = \partial\phi\partial\phi + g_0\phi^4 + g_2(\partial\phi)^2\phi^2 + g_4(\partial\phi)^2(\partial\phi)^2 + g_6(\partial\phi)^2(\partial\phi)^2 \cdots$$

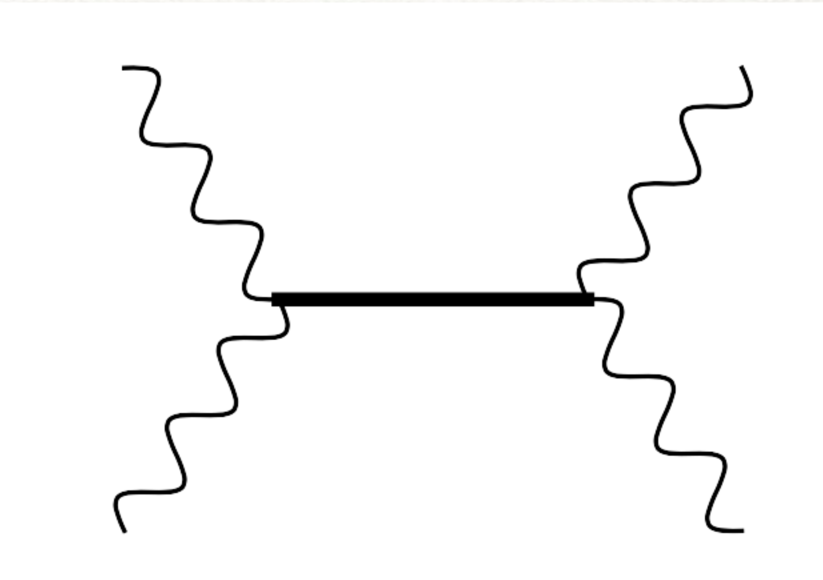
$$M^{IR}(s,t) = \{massless\ poles\} + \sum_{k,q} g_{k,q} s^{k-q} t^q,$$



The S-matrix at low energy is a polynomial i.e. expanded in small E. The Taylor coefficients identified with the Wilson coefficients of the EFT

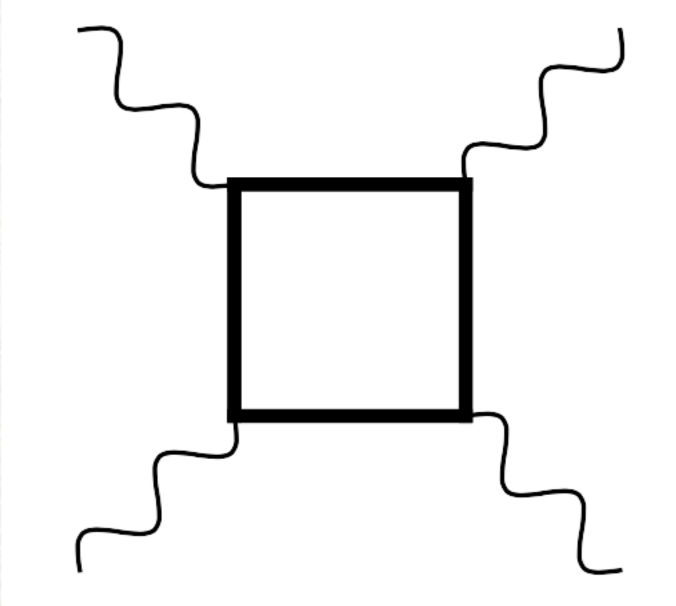
U(1) Linear sigma model

$$\mathcal{L} = \frac{1}{2}(\partial_\mu h)^2 - \frac{m_h^2}{2}h^2 + \left(1+\frac{h}{v}\right)^2 \frac{1}{2}(\partial\pi\cdot\partial\pi)$$



$$M(s,t) = -\frac{\lambda}{8m_h^2}\left(\frac{s^2}{s-m_h^2}+\frac{t^2}{t-m_h^2}+\frac{u^2}{u-m_h^2}\right) \longrightarrow M^{\rm IR}(s,t) = \frac{\lambda}{8m_h^2}\left(\frac{s^2+t^2+u^2}{m_h^2}+\frac{s^3+t^3+u^3}{m_h^4}+\cdots\right)$$

Massive loop



$$M(s,t) = \lambda^4 \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-m_X^2][(\ell-p_1)^2-m_X^2][(\ell-p_1-p_2)^2-m_X^2][(\ell+p_4)^2-m_X^2]} + perm(2,3,4)\,.$$

$$I_4[s,t] = \frac{1}{(4\pi)^2}\frac{uv}{8\beta_{uv}}\Bigg\{2\log^2\left(\frac{\beta_{uv}+\beta_u}{\beta_{uv}+\beta_v}\right) + \log\left(\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u}\right)\log\left(\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v}\right) - \frac{\pi^2}{2} + \sum_{i=u,v}\left[2\mathrm{Li}_2\left(\frac{\beta_i-1}{\beta_{uv}+\beta_i}\right) - 2\mathrm{Li}_2\left(-\frac{\beta_{uv}-\beta_i}{\beta_i+1}\right) - \log^2\left(\frac{\beta_i+1}{\beta_{uv}+\beta_i}\right)\right]\Bigg\}$$

where $u = -\frac{4m_X^2}{s}$ and $v = -\frac{4m_X^2}{t}$, and

$$\beta_u = \sqrt{1+u}, \quad \beta_v = \sqrt{1+v}, \quad \beta_{uv} = \sqrt{1+u+v}\,.$$

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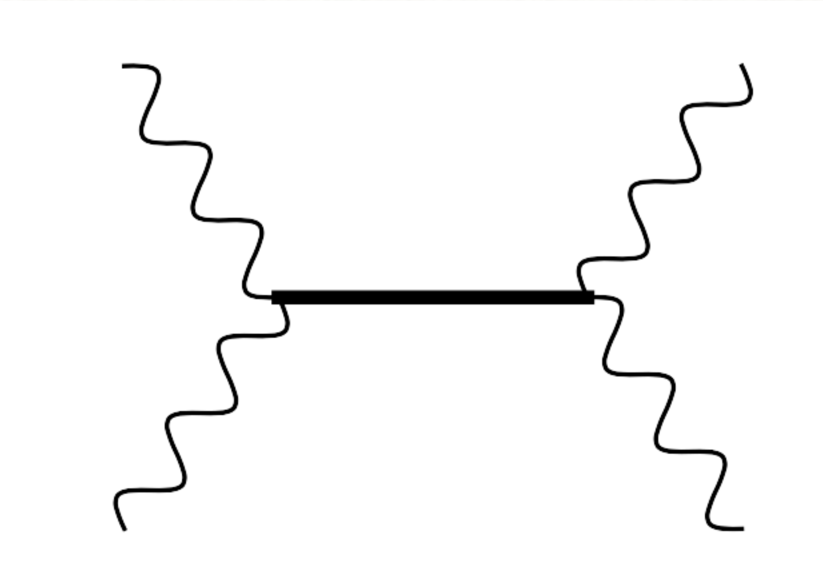

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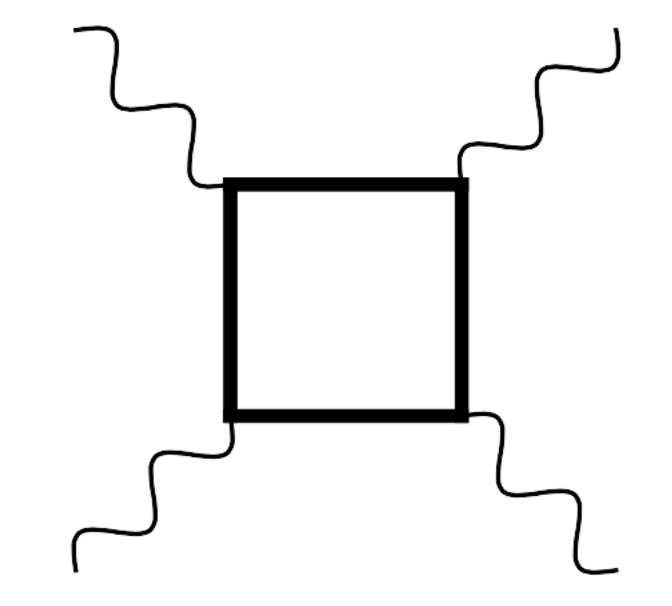

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$$\downarrow$$

$$M^{\rm IR}(s,t) = \frac{g^4}{2m_X^4}\left(1+\frac{1}{5!}\frac{\sigma_2}{m_X^4}+\frac{20}{7!3}\frac{\sigma_3}{m_X^6}+\frac{2}{7!3}\frac{\sigma_2^2}{m_X^8}+\frac{1}{6!33}\frac{\sigma_3\sigma_2}{m_X^{10}}+\cdots\right)$$

where $\sigma_n = s^n+t^n+u^n$

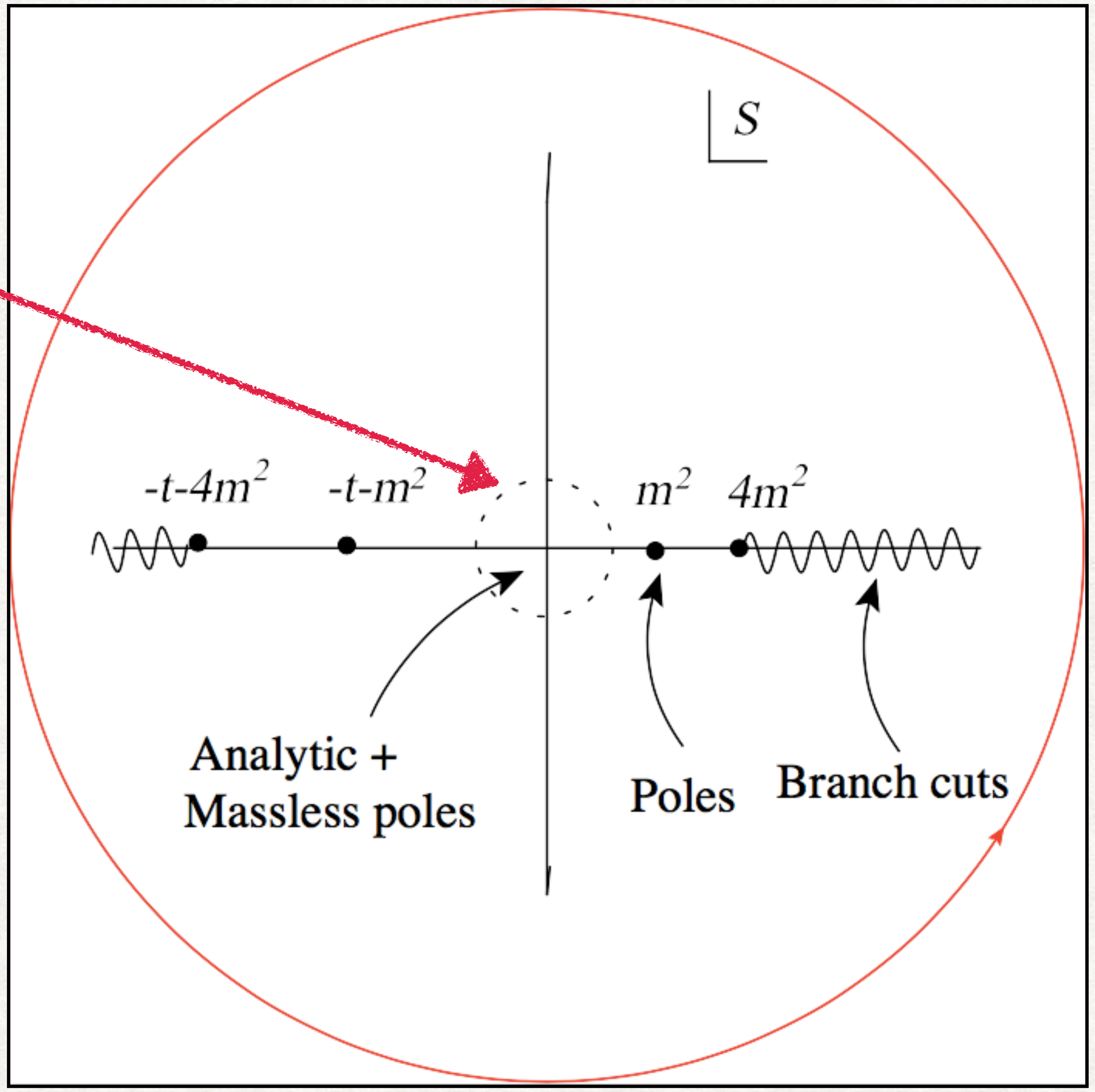
Dispersion relations relates the IR to the UV

Unitarity and Causality tells us that M(s,t)< s^2 at large s for t<0

$$I = \frac{i}{2\pi}\int_{\infty} \frac{ds}{s^{n+1}} M(s,0)$$

=0 for n>1

$$M^{IR}(s,t) = \{massless\ poles\} + \sum_{k,q} g_{k,q} s^{k-q} t^q,$$



The vanishing of the contour tell us that the Taylor coefficients (g) of the EFT is completely controlled by the discontinuity

$$\sum_{q=0}^{\infty} g_{n+q,q} t^q = \int \frac{ds}{s^{n+1}} Dis[M(s,t)]$$

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Unitarity and Causality tells us that M(s,t)< s^2 at large s for t>0

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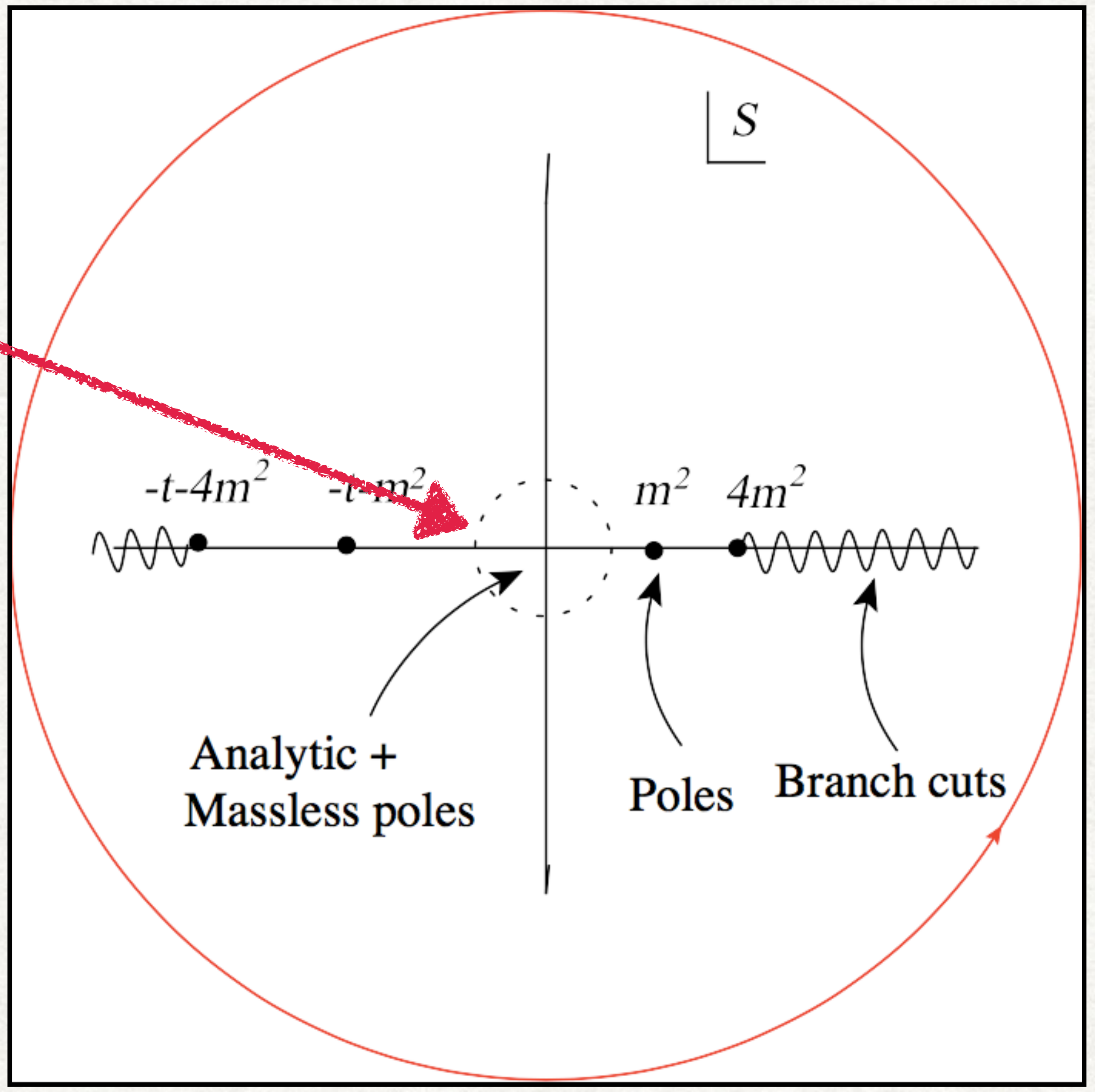

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$$g_{k,q} = -\frac{1}{q!}\frac{\partial^q}{\partial t^q}\left(\sum_a \frac{Res_{s=m_a^2}M(s,t)}{(m_a^2)^{k-q+1}} + \int_{4m_a^2}\frac{ds'}{s'^{k-q+1}}DisM(s,t)\right)\Bigg|_{t=0} + \{u\}$$

$$g_{k,q} = \frac{1}{q!}\frac{d^q}{dt^q}\left(\sum_a \frac{\mathsf{p}_a G_{\ell_a}(1+2\frac{t}{m_a^2})}{(m_a^2)^{k-q+1}} + \sum_b \int ds' \mathsf{p}_{b,\ell}(s')\frac{G_\ell(1+2\frac{t}{s'})}{(s')^{k-q+1}} + \{u\}\right)\Bigg|_{t=0}$$

positive

The Wilson coefficients are given by a convex hull

$$g_{k,q} = \frac{1}{q!}\frac{d^q}{dt^q}\left(\sum_a \frac{\mathsf{p}_a G_{\ell_a}(1+2\frac{t}{m_a^2})}{(m_a^2)^{k-q+1}} + \sum_b \int ds' \mathsf{p}_{b,\ell}(s') \frac{G_\ell(1+2\frac{t}{s'})}{(s')^{k-q+1}} + \{u\}\right)\Bigg|_{t=0}$$

The expansion in t is governed by the Taylor expansion of the Gegenbauer polynomial.

A. J. Tolley, Z. Y. Wang and S. Y. Zhou, "New positivity bounds from full crossing symmetry," [arXiv:2011.02400 [hep-th]].

S. Caron-Huot and V. Van Duong, "Extremal Effective Field Theories," [arXiv:2011.02957 [hep-th]].

B. Bellazzini, "Softness and amplitudes' positivity for spinning particles," JHEP **02**, 034 (2017) doi:10.1007/JHEP02(2017)034 [arXiv:1605.06111 [hep-th]];
C. de Rham, S. Melville, A. J. Tolley and S. Y. Zhou, "Positivity bounds for scalar field theories," Phys. Rev. D **96**, no.8, 081702 (2017) doi:10.1103/PhysRevD.96.081702 [arXiv:1702.06134 [hep-th]];
C. de Rham, S. Melville, A. J. Tolley and S. Y. Zhou, "UV complete me: Positivity Bounds for Particles with Spin," JHEP **03**, 011 (2018) doi:10.1007/JHEP03(2018)011 [arXiv:1706.02712 [hep-th]];
A. Sinha and A. Zahed, "Crossing Symmetric Dispersion Relations in QFTs," [arXiv:2012.04877 [hep-th]].

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Let's look closer at the s-channel contribution in detail. Focusing on D=4

$$g^s_{k,q} = \sum_a p_a \frac{v_{\ell_a,q}}{m_a^{2(k+1)}}$$

$$G_\ell(1+2\delta) = v_{\ell,0}+v_{\ell,1}\delta+\cdots v_{\ell,\ell}\delta^\ell = \sum_{q=0}^{\ell} v_{\ell,q}\delta^q$$

We have a product geometry

$$\begin{pmatrix} g_{0,0} & & & \\ g_{1,0} & g_{1,1} & & \\ g_{2,0} & g_{2,1} & g_{2,2} & \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} = \sum_a p_a \begin{pmatrix} \frac{1}{m_a^2} \\ \frac{1}{m_a^4} \\ \frac{1}{m_a^6} \\ \frac{1}{m_a^8} \\ \vdots \end{pmatrix} \otimes (v_{\ell_a,0}, v_{\ell_a,1}, v_{\ell_a,2}, \cdots)$$

Note that v is a simple polynomial in J

$$v_{\ell,q} = \frac{2^q}{q!(2-q)!}\frac{(\alpha)_{\ell+q}}{\prod_{a=1}^q(\alpha+2a-1)} = \frac{\prod_{a=0}^q(J-a(a-1))}{(q!)^2} \qquad J = \ell(\ell+1)$$

After a linear transformation $\vec{a}^T = \vec{g}^T\mathbf{G}$ we have

$$\begin{pmatrix} a_{0,0} & & & \\ a_{1,0} & a_{1,1} & & \\ a_{2,0} & a_{2,1} & a_{2,2} & \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} = \sum_a p_a \begin{pmatrix} \frac{1}{m_a^2} \\ \frac{1}{m_a^4} \\ \frac{1}{m_a^6} \\ \frac{1}{m_a^8} \\ \vdots \end{pmatrix} \otimes (1, J, J^2, J^3, \cdots)$$

at the core, the couplings → are governed by the hull of product moments

The linear transformed EFT couplings $\vec{a}^T = \vec{g}^T \mathbf{G}$ live inside our favorite product geometry !

$$\begin{pmatrix} a_{0,0} & & & \\ a_{1,0} & a_{1,1} & & \\ a_{2,0} & a_{2,1} & a_{2,2} & \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} = \sum_a p_a \begin{pmatrix} \frac{1}{m_a^2} \\ \frac{1}{m_a^4} \\ \frac{1}{m_a^6} \\ \frac{1}{m_a^8} \\ \vdots \end{pmatrix} \otimes (1, J, J^2, J^3, \cdots)$$

An inverse transform rotates us back to the physical coupling

"a" geometry

$$a_{k,q} = \sum p \frac{J^q}{m^k}$$

Rotation →

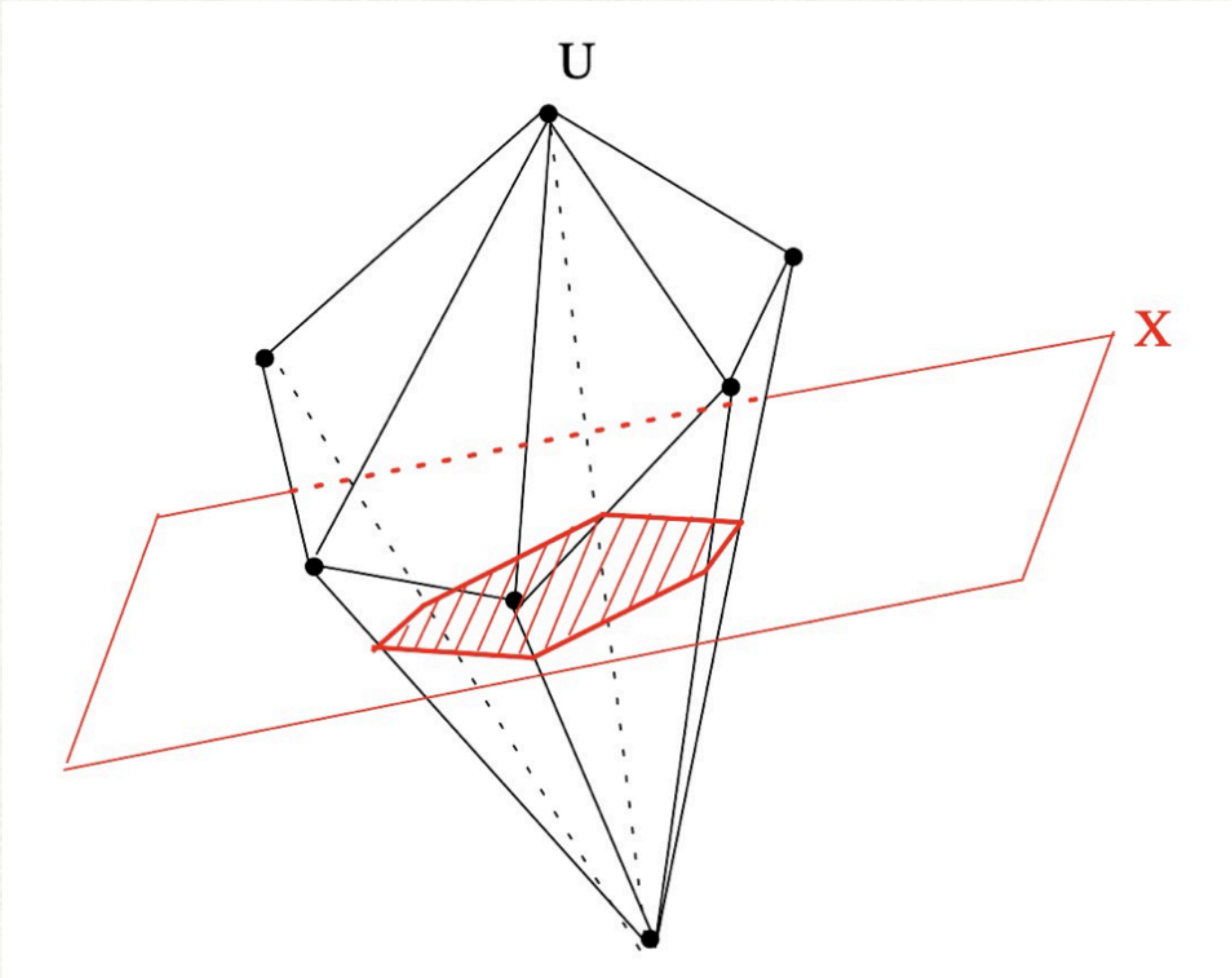
s-EFThedron

$$g^s_{k,q} = L(a_{k,q})$$

↓ **Intersection**

cyclic plane

$$n(g^s_{k,q}) = 0$$



We can further impose cyclic symmetry

$$M(s,t) - M(t,s) = 0$$

$$g_{k,q} = g_{k,k-q}$$

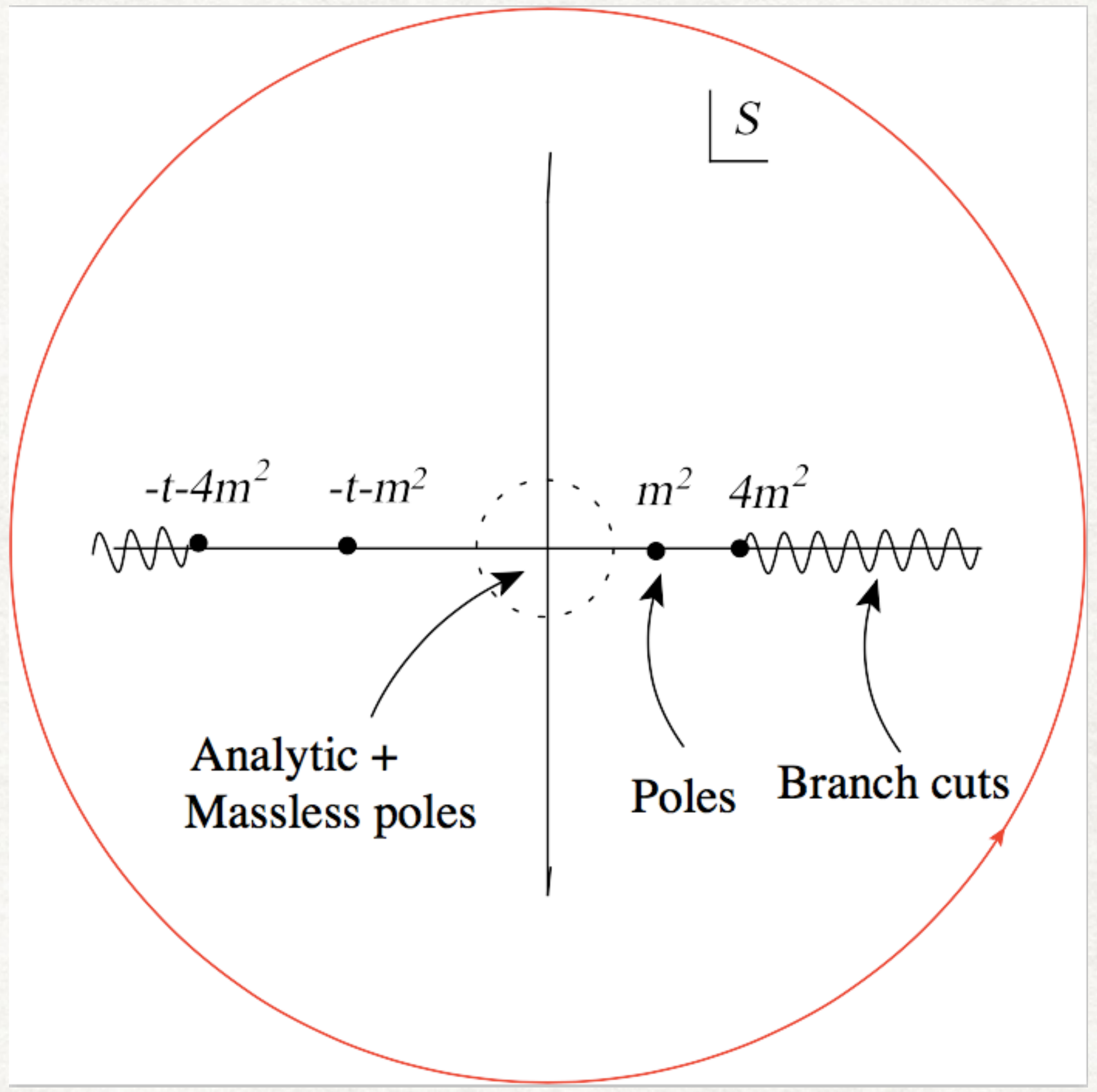
The Full-EFThedron

So far we've only considered s-channel singularities, in general at fixed t, we will have both s,u thresholds

Consider a,b -> a,b

$$M^{\mathrm{IR}}(a_1, b_2, b_2, a_3) = \sum_{k,q} g_{k,q} z^{k-q} t^q$$

$$s = -t/2 + z, \; u = -t/2 - z$$



once again the dispersion relations leads to

$$\sum_{k,q} g_{k,q} z^{k-q} t^q = -\sum_i p_i \, P_{\ell_i}\left(1+\frac{2t}{m_i^2}\right)\left(\frac{1}{-\frac{t}{2} - z - m_i^2} + \frac{1}{-\frac{t}{2} + z - m_i^2}\right)$$

expanding in z, t

$$g_{k,q} = \sum_i p_i \frac{u_{\ell_i,k,q}}{m_i^{2(k+1)}}$$

$$\vec{u}_{\ell,k} = \begin{pmatrix} u_{\ell,k,0} \\ 0 \\ u_{\ell,k,2} \\ \vdots \\ 0 \\ u_{\ell,k,k} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{(k-1)_2}{2}\frac{1}{2^2} & (k-1)_1 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{(1)_k}{k!}2^{-k} & -\frac{(1)_{k-1}}{k-1!}2^{2-k} & \frac{(1)_{k-2}}{k-2!}2^{4-k} & \cdots & -2^{k-2} & 1 \end{pmatrix} \begin{pmatrix} v_{\ell,0} \\ v_{\ell,1} \\ v_{\ell,2} \\ \vdots \\ v_{\ell,k-1} \\ v_{\ell,k} \end{pmatrix}$$

The u vectors are simply a projection from v

The Full-EFThedron

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Consider a,b -> a,b

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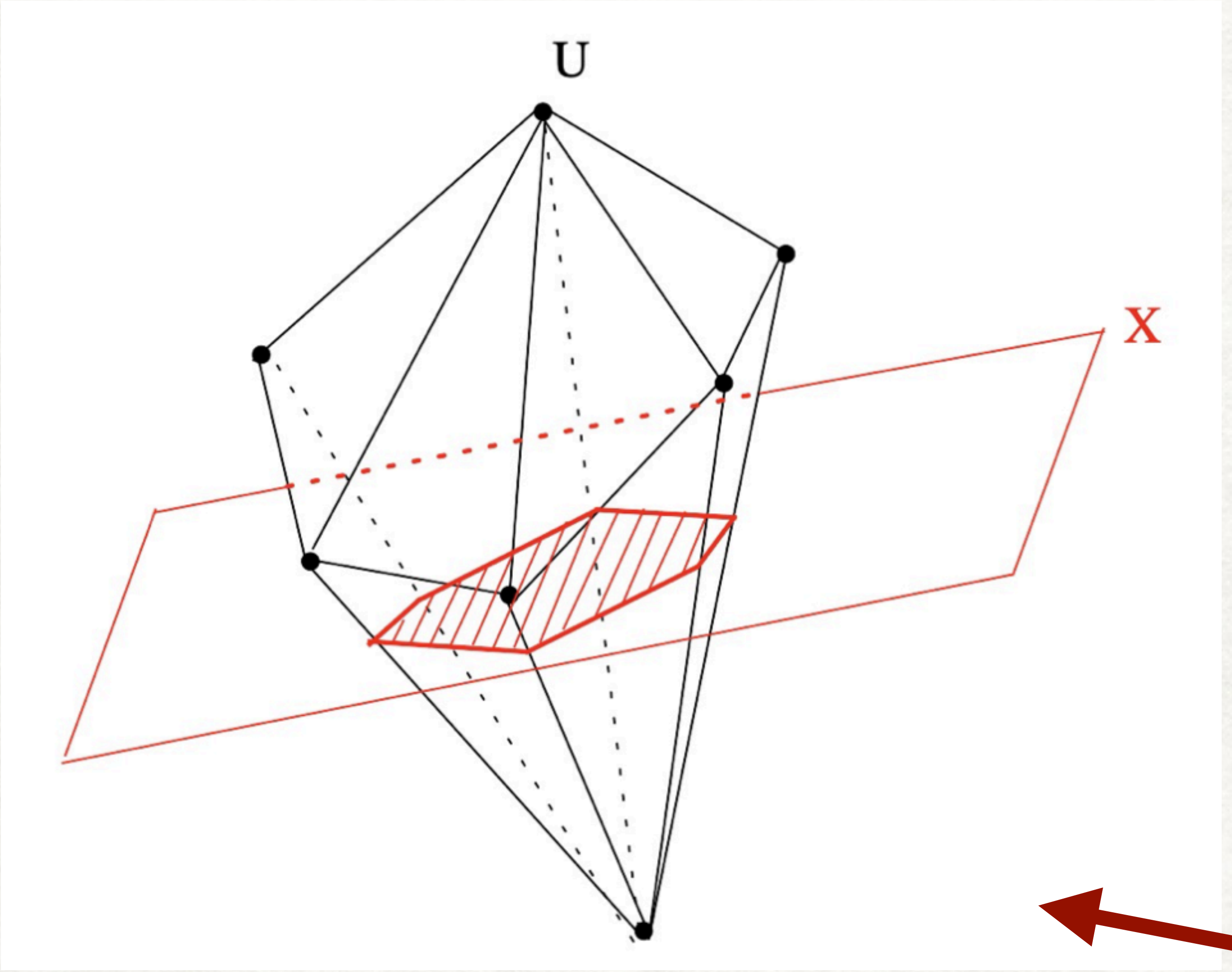
a-geometry $a_{k,q} = \sum p \frac{J^{2q}}{m^{2k+2}}$ —Rotation→ *s*-EFThedron $g^{s}_{k,q} = \sum p \frac{v_q}{m^{2k}}$ —Projection→ *su*-EFThedron $g^{su}_{k,q} = \sum p \frac{u_{k,q}}{m^{2k}}$

s-EFThedron —Intersection→ cyclic plane $n(g^s) = 0$

su-EFThedron —Intersection→ permutation plane $n(g^{su}) = 0$



For identical scalars we further impose permutation invariance

$$\mathbf{X}_{\mathrm{perm}} \to M(s,t) = M(s, -t-s) = M(-t-s, t)$$

The stem of this talk: the convex hull of product moment curve

$$[y]_{d+1\times d+1} = \begin{pmatrix} y^{(0,0)} & y^{(0,1)} & \dots & y^{(0,d)} \\ y^{(1,0)} & y^{(1,1)} & \dots & y^{(1,d)} \\ \vdots & \vdots & \vdots & \vdots \\ y^{(d,0)} & y^{(d,1)} & \dots & y^{(d,d)} \end{pmatrix} \in \sum_i c_i \begin{pmatrix} 1 \\ x_i \\ x_i^2 \\ \vdots \\ x_i^d \end{pmatrix} \begin{pmatrix} 1 \\ \tilde{x}_i \\ \tilde{x}_i^2 \\ \vdots \\ \tilde{x}_i^d \end{pmatrix}^T , \quad \forall c_i > 0$$

The two moments may not be identical

(a) $x \in \mathbb{R}_+$ (b) $x \in [0,1]$ (c) $x \in \mathbb{N}$

We would like to find the boundary of this hull

$$f[y^{(m,n)}] \geq 0$$

carves out the image of the hull in the space of $[y]_{d+1\times d+1}$

In literature this is called the multivariate moment problem: what are the sufficient conditions on $[y]_{d+1\times d+1}$ such that a solution in the form of the RHS exists.

Begin with a single moment

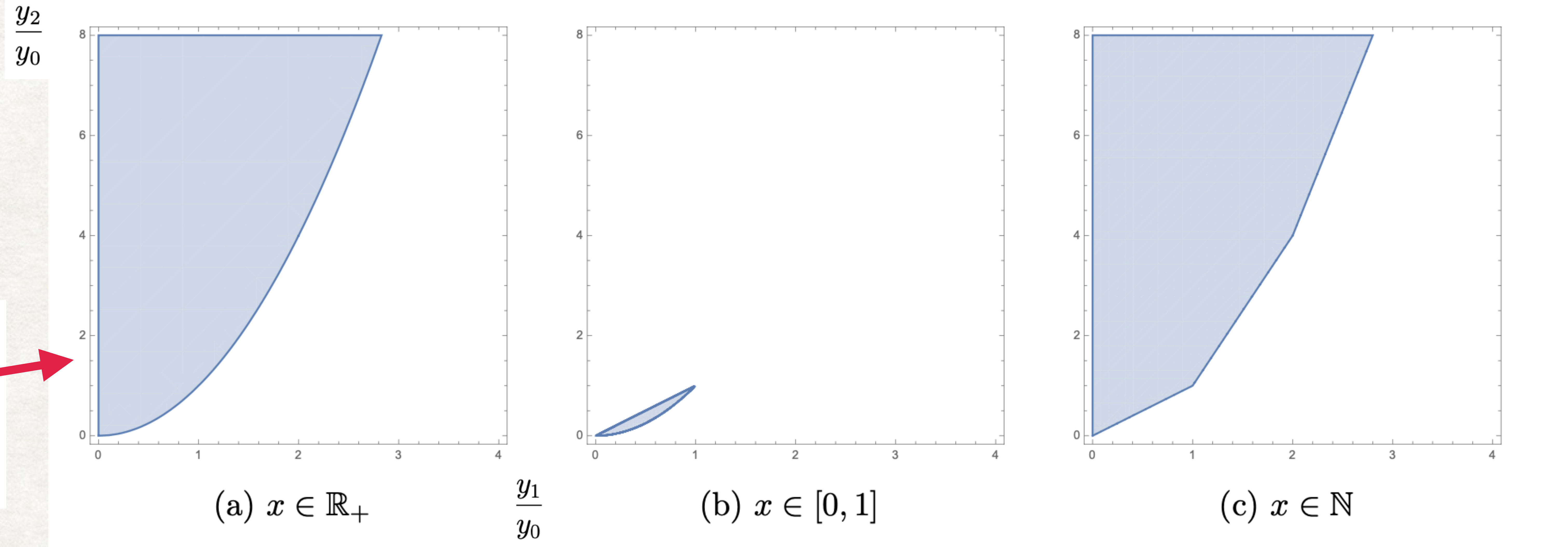
$$\vec{y} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_d \end{pmatrix} \in \sum_i c_i \begin{pmatrix} 1 \\ x_i \\ x_i^2 \\ \vdots \\ x_i^d \end{pmatrix}, \quad \forall c_i > 0$$

If we are only assuming c>0 the relevant space is projective.

Start with **P**^{2}

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} \in \sum_i c_i \begin{pmatrix} 1 \\ x_i \\ (x_i)^2 \end{pmatrix} \qquad c_i,\ x_i \geq 0.$$

$$y_0,\ y_1,\quad \det\begin{pmatrix} y_0 & y_1 \\ y_1 & y_2 \end{pmatrix} > 0$$



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Start with $\mathbf{P}^2$ the space is carved out by the positivity of the leading principle minors of the Hankel and (shifted) Hankel matrix

$$\begin{pmatrix} y_0 & y_1 \\ y_1 & y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix}$$

Generalize to arbitrary dimensions

$$K_n[\vec{y}] \equiv \begin{pmatrix} y_0 & y_1 & \cdots & y_n \\ y_1 & y_2 & \cdots & y_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ y_n & y_{n+1} & \cdots & y_{2n} \end{pmatrix} \rightarrow \quad \text{Det}\,(K_n[\vec{y}]) \geq 0, \quad \text{Det}\left(K_n^{\text{shift}}[\vec{y}]\right) \equiv \text{Det}\,(K_n[\vec{y}])\,|_{y_i \to y_{i+1}} \geq 0$$

$$K_n[\vec{y}] \equiv \begin{pmatrix} y_0 & y_1 & \cdots & y_n \\ y_1 & y_2 & \cdots & y_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ y_n & y_{n+1} & \cdots & y_{2n} \end{pmatrix}$$

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Substituting the hull we can see that these boundaries form a hierarchal complex

$$\vec{y} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_d \end{pmatrix} \in \sum_i c_i \begin{pmatrix} 1 \\ x_i \\ x_i^2 \\ \vdots \\ x_i^d \end{pmatrix}$$

$$\text{Det}\,(K_n[\vec{y}]) = \sum_{\{i_1,i_2,\cdots,i_{n+1}\}} \left[\prod_{a=1}^{n+1} c_{i_a}\right] \prod_{1\leq a<b<n+1} (x_{i_a} - x_{i_b})^2$$

$$\text{Det}\left(K_n^{\text{shift}}[\vec{y}]\right) = \sum_{\{i_1,i_2,\cdots,i_{n+1}\}} \left[\prod_{a=1}^{n+1} c_{i_a} x_{i_a}\right] \prod_{1\leq a<b<n+1} (x_{i_a} - x_{i_b})^2$$

vanishes if there are less than n+1 elements

furthermore $\text{Det}\,K_n^{\text{shift}} = 0, \quad \text{Det}\,K_n \neq 0$ if there is at most n elements + origin

Thus

$$\text{Det}\,K_0 \subset \text{Det}\,K_0^{\text{shift}} \subset \cdots \subset \text{Det}\,K_{n-1} \subset \text{Det}\,K_{n-1}^{\text{shift}} \subset \text{Det}\,K_n \subset \text{Det}\,K_n^{\text{shift}}$$

successive vanishing of the Hankel determinant represents the reduction of rank

The positivity conditions on the Hankel matrices can be easily understood as follows: identifying

$$K(y) = \sum_i c_i \mathbf{x}_i \mathbf{x}_i^T = \sum_i c_i \begin{pmatrix} 1 \\ x_i \\ x_i^2 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 & x_i & x_i^2 & \cdots \end{pmatrix},$$

$$\longrightarrow \quad \mathbf{v}^T K \mathbf{v} = \sum_i c_i (\mathbf{v}^T \mathbf{x}_i)^2 \geq 0\,.$$

Positive for any vector v implies the principle minors are positive

To see why this maybe sufficient condition, consider the infinite dimensional limit. With $f(x) = \mathbf{v}^T \mathbf{x}$ we see that the positivity of the Hankel matrix is equivalent to

$$\mathbb{E}[f(x)^2] \equiv \sum_i c_i f(x_i)^2 \geq 0.$$

By considering the a function $f(x) = \mathbf{v}^T \mathbf{x}$ sharply peaked around some xi, we can solve for pi and a solution exists.

The positivity conditions on the Hankel matrices can be easily understood as follows: writing

$$K(y) = \sum_i c_i \mathbf{x}_i \mathbf{x}_i^T = \sum_i c_i \begin{pmatrix} 1 \\ x_i \\ x_i^2 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 & x_i & x_i^2 & \cdots \end{pmatrix},$$

$\longrightarrow$

$$\mathbf{v}^T K \mathbf{v} = \sum_i c_i (\mathbf{v}^T \mathbf{x}_i)^2 \geq 0\,.$$

Positive for any vector v implies the principle minors are positive

For bounded moments we further have, say $a < x_i < b$ then

$$\mathbb{E}\left[f(x)^2 (x-a)(b-x)\right] = \mathbf{v}^T \left(\sum_i c_i (x_i - a)(b - x_i) \mathbf{x}_i \mathbf{x}_i^T \right) \mathbf{v} \geq 0$$

Thus we require instead the following to be PSD

$$\sum_i c_i (x_i - a)(b - x_i) \begin{pmatrix} 1 \\ x_i \\ x_i^2 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 & x_i & x_i^2 & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} (-y_2 + (a+b)y_1 - ab y_0) & (-y_3 + (a+b)y_2 - ab y_1) & \cdots \\ (-y_3 + (a+b)y_2 - ab y_1) & (-y_4 + (a+b)y_3 - ab y_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0.$$

The positivity conditions on the Hankel matrices can be easily understood as follows:

For bounded moments we further have, say $a < x_i < b$ then

$$\mathbf{v}^T \left(\sum_i c_i (x_i - a)(b - x_i) \mathbf{x}_i \mathbf{x}_i^T \right) \mathbf{v} \geq 0$$

For half moment problem $a = 0, b \to \infty.$

$$\mathbb{E}\left[f(x)^2 x\right] \geq 0 \rightarrow \mathbf{v}^T \left(\sum_i c_i (x_i) \mathbf{x}_i \mathbf{x}_i^T \right) \mathbf{v} \Rightarrow \begin{pmatrix} y_1 & y_2 & y_3 & \cdots \\ y_2 & y_3 & y_4 & \cdots \\ y_3 & y_4 & y_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \geq 0,$$

Positive for any vector v implies the principle minors of the shifted Hankel is positive

For Hausdorff moment problem $a = 0, b = 1$

$$\mathbb{E}[(1-x) f(x)^2] \geq 0 \Rightarrow \begin{pmatrix} (y_0 - y_1) & (y_1 - y_2) & (y_2 - y_3) & \cdots \\ (y_1 - y_2) & (y_2 - y_3) & (y_3 - y_4) & \cdots \\ (y_2 - y_3) & (y_3 - y_4) & (y_4 - y_5) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \geq 0$$

The twisted Hankel is positive

The positivity conditions on the Hankel matrices can be easily understood as follows:

For bounded moments we further have, say $a < x_i < b$ then

$$\mathbf{v}^T \left(\sum_i c_i (x_i - a)(b - x_i) \mathbf{x}_i \mathbf{x}_i^T \right) \mathbf{v} \geq 0$$

For discrete moment problem $x \in \{s_i\}$

$$(a) \quad \mathbb{E}\left[f(x)^2 (x - s_1)\right] \geq 0 \Rightarrow \begin{pmatrix} (y_1 - s_1 y_0) & (y_2 - s_1 y_1) & \cdots \\ (y_2 - s_1 y_1) & (y_3 - s_1 y_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0.$$

$$(b) \quad \mathbb{E}\left[f(x)^2 (x - s_i)(x - s_{i+1})\right] \cdots \geq 0$$

$$\Rightarrow \begin{pmatrix} (y_2 - (s_i + s_{i+1}) y_1 + s_i s_{i+1} y_0) & (y_3 - (y_i + s_{i+1}) y_2 + s_i s_{i+1} y_1) & \cdots \\ (y_3 - (s_i + s_{i+1}) y_2 + s_i s_{i+1} y_1) & (y_4 - (y_i + s_{i+1}) y_3 + s_i s_{i+1} y_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \cdots \geq 0.$$

$$(c) \quad \mathbb{E}\left[f(x)^2 (s_n - x)\right] \geq 0 \Rightarrow \begin{pmatrix} (-y_1 + s_n y_0) & (-y_2 + s_n y_1) & \cdots \\ (-y_2 + s_n y_1) & (-y_3 + s_n y_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0.$$

$$\frac{1}{(s_{i+1} - s_i)} \det \begin{pmatrix} y_0 & 1 & 1 \\ y_1 & s_i & s_{i+1} \\ y_2 & s_i^2 & s_{i+1}^2 \end{pmatrix}$$

The boundaries of cyclic polytope

$$[y]_{d+1\times d+1} = \begin{pmatrix} y^{(0,0)} & y^{(0,1)} & \dots & y^{(0,d)} \\ y^{(1,0)} & y^{(1,1)} & \dots & y^{(1,d)} \\ \vdots & \vdots & \vdots & \vdots \\ y^{(d,0)} & y^{(d,1)} & \dots & y^{(d,d)} \end{pmatrix} \in \sum_i c_i \begin{pmatrix} 1 \\ x_i \\ x_i^2 \\ \vdots \\ x_i^d \end{pmatrix} \begin{pmatrix} 1 \\ \tilde{x}_i \\ \tilde{x}_i^2 \\ \vdots \\ \tilde{x}_i^d \end{pmatrix}^T , \quad \forall\, c_i > 0$$

This suggests generalized Hankel matrix

$$K(y) = \sum_i p_i \begin{pmatrix} 1 \\ x_i \\ \tilde{x}_i \\ x_i^2 \\ x_i\tilde{x}_i \\ \tilde{x}_i^2 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 & x_i & \tilde{x}_i & x_i^2 & x_i\tilde{x}_i & \tilde{x}_i^2 & \cdots \end{pmatrix} = \begin{pmatrix} y^{(0,0)} & y^{(1,0)} & y^{(0,1)} & \dots \\ y^{(1,0)} & y^{(2,0)} & y^{(1,1)} & \dots \\ y^{(0,1)} & y^{(1,1)} & y^{(0,2)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

the convex hull for the full moment implies

$$\boxed{K(y) = \begin{pmatrix} y_{0,0} & y_{1,0} & y_{0,1} & \cdots \\ y_{1,0} & y_{2,0} & y_{1,1} & \cdots \\ y_{0,1} & y_{1,1} & y_{0,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \geq 0}$$

This suggests generalized Hankel matrix

$$K(y) = \sum_i p_i \begin{pmatrix} 1 \\ x_i \\ \tilde{x}_i \\ x_i^2 \\ x_i\tilde{x}_i \\ \tilde{x}_i^2 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 & x_i & \tilde{x}_i & x_i^2 & x_i\tilde{x}_i & \tilde{x}_i^2 & \cdots \end{pmatrix} = \begin{pmatrix} y^{(0,0)} & y^{(1,0)} & y^{(0,1)} & \cdots \\ y^{(1,0)} & y^{(2,0)} & y^{(1,1)} & \cdots \\ y^{(0,1)} & y^{(1,1)} & y^{(0,2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Half-moment $x, \tilde{x} \in \mathbb{R}^+$

$$K(y), \quad K^{\text{shift},x}(y) \equiv K(y)|_{y^{(m,n)} \to y^{(m+1,n)}}, \quad K^{\text{shift},\tilde{x}}(y), \quad K^{\text{shift},x,\tilde{x}}(y) \geq 0$$

Bounded moments $x \in [0,1]$

$$K(y) \geq 0, \quad K^{\text{shift},x}(y) \geq 0, \quad K^{\text{twist,x}}(y) \equiv M(y) - M^{\text{shift,x}}(y) \geq 0$$

Discrete moments $\tilde{x} \in \{s_i\}$

$$K(y) \geq 0$$

$$\begin{pmatrix} (y_{0,1} - s_1 y_{0,0}) & (y_{1,1} - s_1 y_{1,0}) & \cdots \\ (y_{1,1} - s_1 y_{1,0}) & (y_{2,1} - s_1 y_{2,0}) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} (-y_{0,1} + s_n y_{0,0}) & (-y_{1,1} + s_n y_{1,0}) & \cdots \\ (-y_{1,1} + s_n y_{1,0}) & (-y_{2,1} + s_n y_{2,0}) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0$$

$$\begin{pmatrix} (y_{0,2} - (s_i + s_{i+1}) y_{0,1} + s_i s_{i+1} y_{0,0}) & (y_{1,2} - (s_i + s_{i+1}) y_{1,1} + s_i s_{i+1} y_{1,0}) & \cdots \\ (y_{1,2} - (s_i + s_{i+1}) y_{1,1} + s_i s_{i+1} y_{1,0}) & (y_{2,2} - (s_i + s_{i+1}) y_{2,1} + s_i s_{i+1} y_{2,0}) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \geq 0$$

The Full-EFThedron

$$M(s,t) = \cdots + g_{3,0}s^3 + g_{3,1}s^2t + g_{4,0}s^4 + g_{4,1}s^3t + g_{4,2}s^2t^2 \cdots \qquad (D^6\phi^4,\ D^8\phi^4)$$

The a-geometry boundaries

$$a_{3,1} \geq 0$$

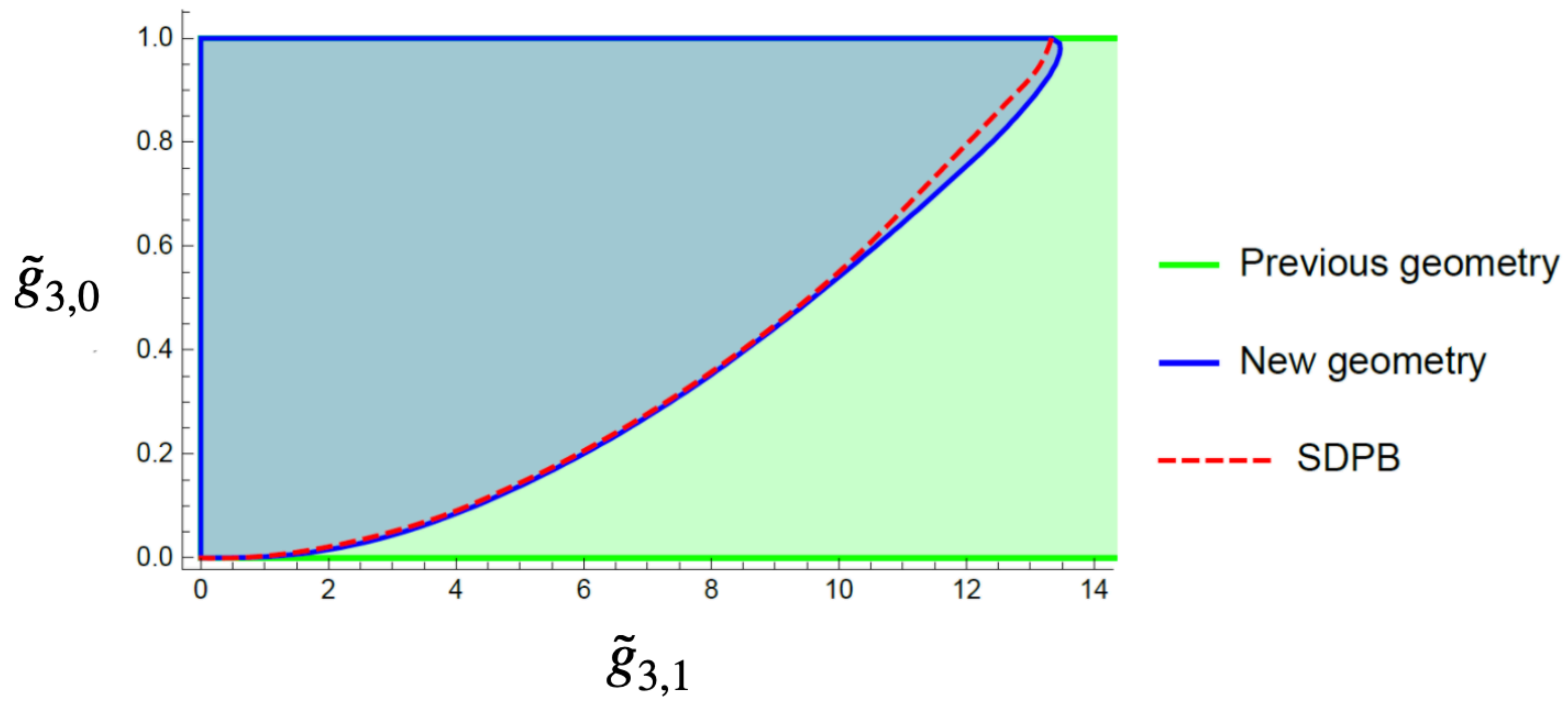
$$a_{2,0} - a_{4,0} \geq 0$$

$$\begin{pmatrix} a_{2,0} & a_{3,1} \\ a_{3,1} & a_{4,2} \end{pmatrix} \geq 0$$

$$\det \begin{pmatrix} a_{4,0} & 1 & 1 \\ a_{4,1} & 6 & 20 \\ a_{4,2} & 36 & 400 \end{pmatrix} \geq 0$$

$$\det \begin{pmatrix} a_{2,0} & a_{3,0} & a_{3,1} \\ a_{3,0} & a_{4,0} & a_{4,1} \\ a_{3,1} & a_{4,1} & a_{4,2} \end{pmatrix} \geq 0$$

$$\vec{a}^T = \vec{g}^T \mathbf{G}$$



The Full-EFThedron

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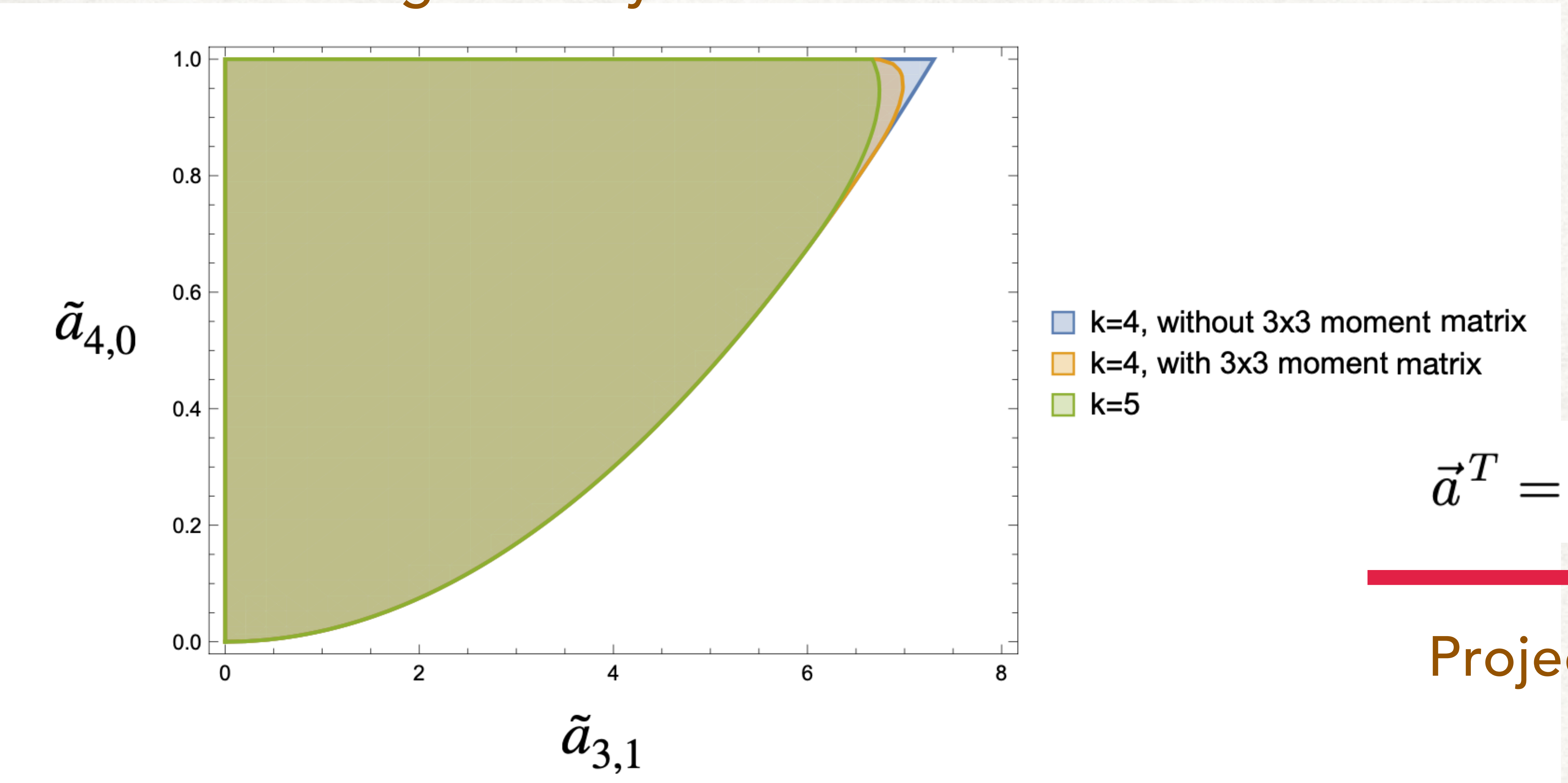
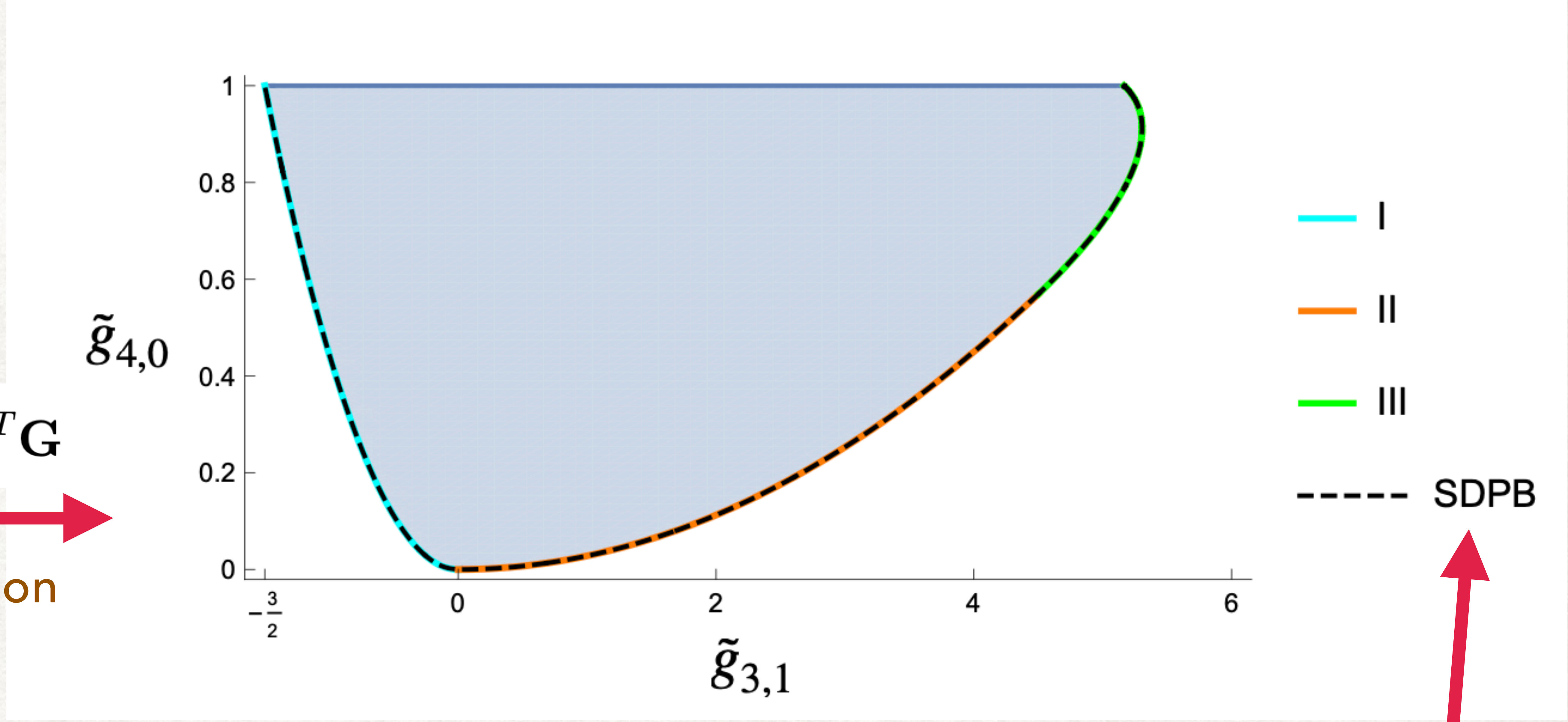
The a-geometry boundaries



$$\vec{a}^T = \vec{g}^T \mathbf{G}$$

Projection



S. Caron-Huot and V. Van Duong, "Extremal Effective Field Theories," [arXiv:2011.02957 [hep-th]].

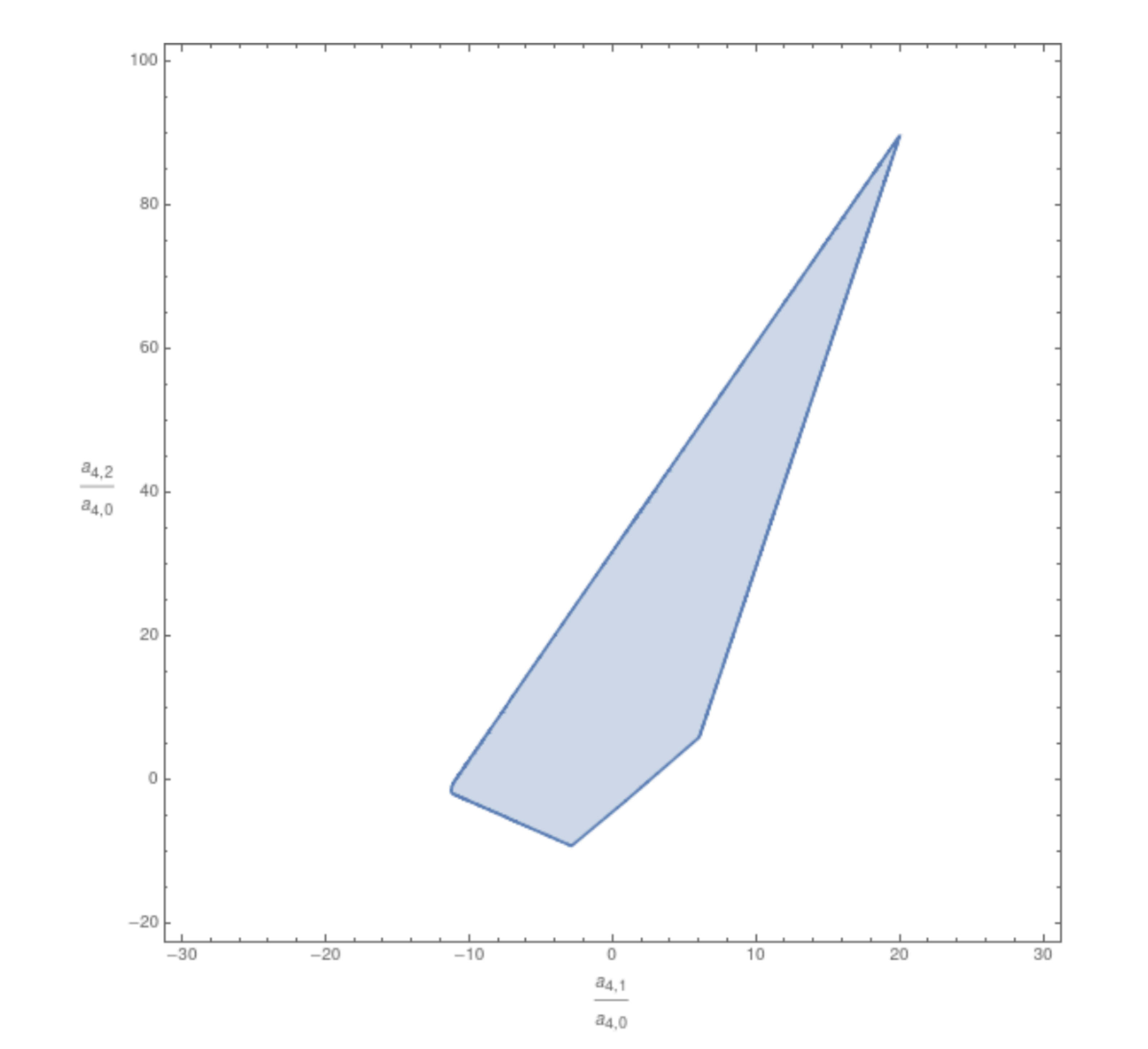
$$\begin{aligned}
&\text{Region I:} && \tilde{g}_{31} = -\frac{3}{2}\sqrt{\tilde{g}_{40}}, && 0 \le \tilde{g}_{40} \le 1\\
&\text{Region II:} && \tilde{g}_{31} = \frac{1}{2}\sqrt{\frac{427}{3}\tilde{g}_{40}}, && 0 \le \tilde{g}_{40} \le \frac{243}{427}\\
&\text{Region III:} && \tilde{g}_{31} = \frac{30}{7}\tilde{g}_{40} + \frac{37}{42}\sqrt{\tilde{g}_{40}\,(21 - 20\tilde{g}_{40})}, && \frac{243}{427} \le \tilde{g}_{40} \le 1
\end{aligned}$$

We can generalize to spinning external states $M(+h,+h,-h,-h)$

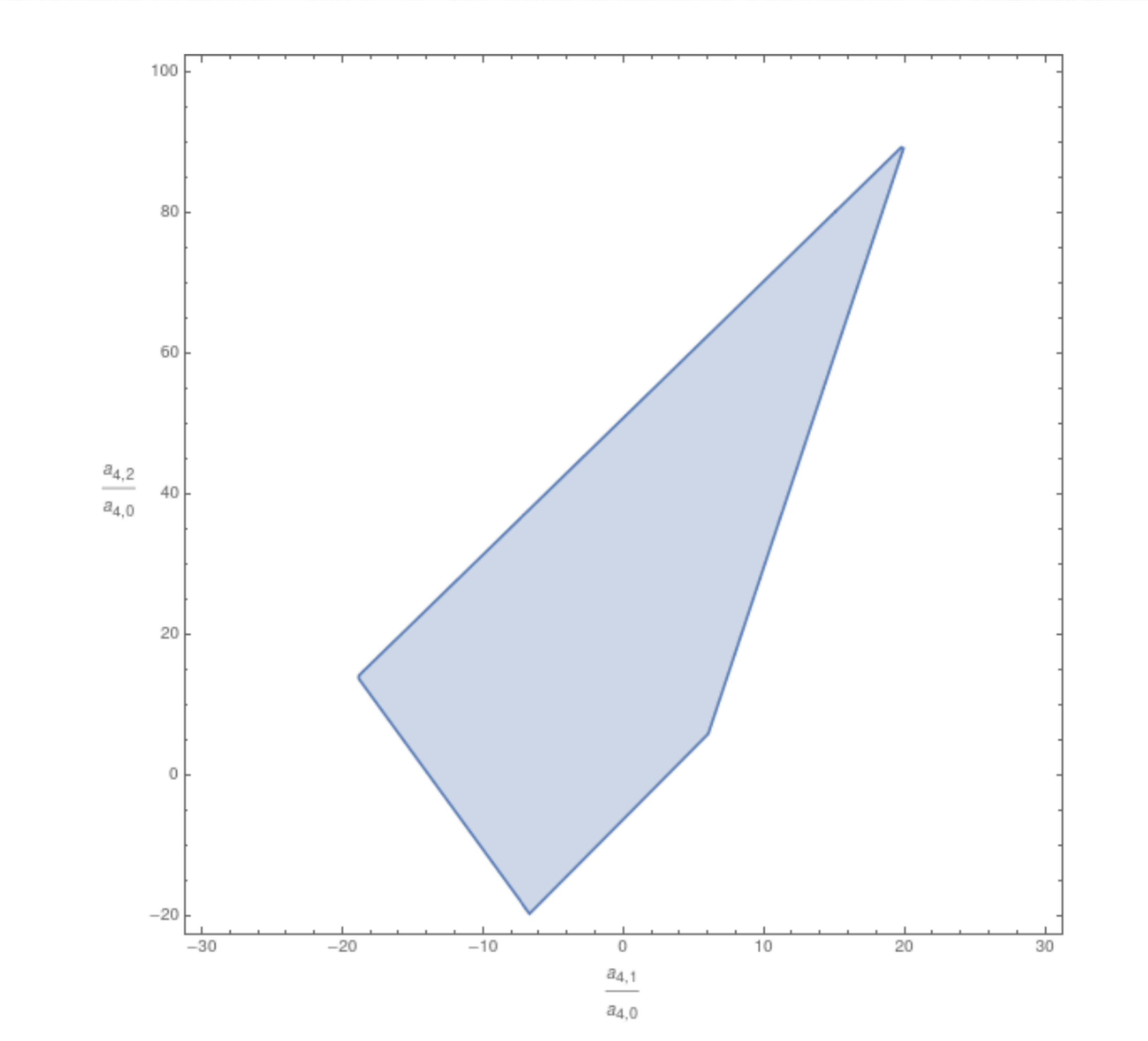
leads to EFTs of photons and gravitons

$$[12]^{2h}\langle 34\rangle^{2h}(\sum_{k,q} a_{k,q}s^{k-q}t^q) = -[12]^{2h}\langle 34\rangle^{2h}\left(\sum_{\ell_a\geq 0} p_i \frac{d^{\ell_i=even}_{0,0}(\theta)}{s-m_i^2} + \sum_{\ell_j\geq 2h} p_j \frac{\tilde{d}^{\ell_j}_{2h,2h}(\theta)}{-t-s-m_j^2}\right)$$

D^8F^4



D^8R^4



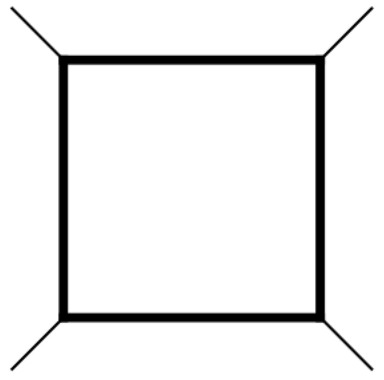
Explicit EFTs

$$g_{5,0}s^5+g_{5,1}s^4t+g_{5,2}s^3t^2+g_{5,3}s^2t^3 = s^5+xs^4t+ys^3t^2+ys^2t^3$$

- (a) The tree-level exchange of a massive Higgs in the linear Sigma model

$$-\frac{s}{s-m^2}-\frac{t}{t-m^2}\bigg|_{m\to\infty} = \cdots + \frac{1}{m^{10}}(s^5+t^5)+\cdots \qquad (10.17)$$

- (b) The one-loop contribution of a massive scalar X coupled to a massless scalar ϕ via $X^2\phi$. The one-loop integrand is simply the massive box, whose low energy expansion is:



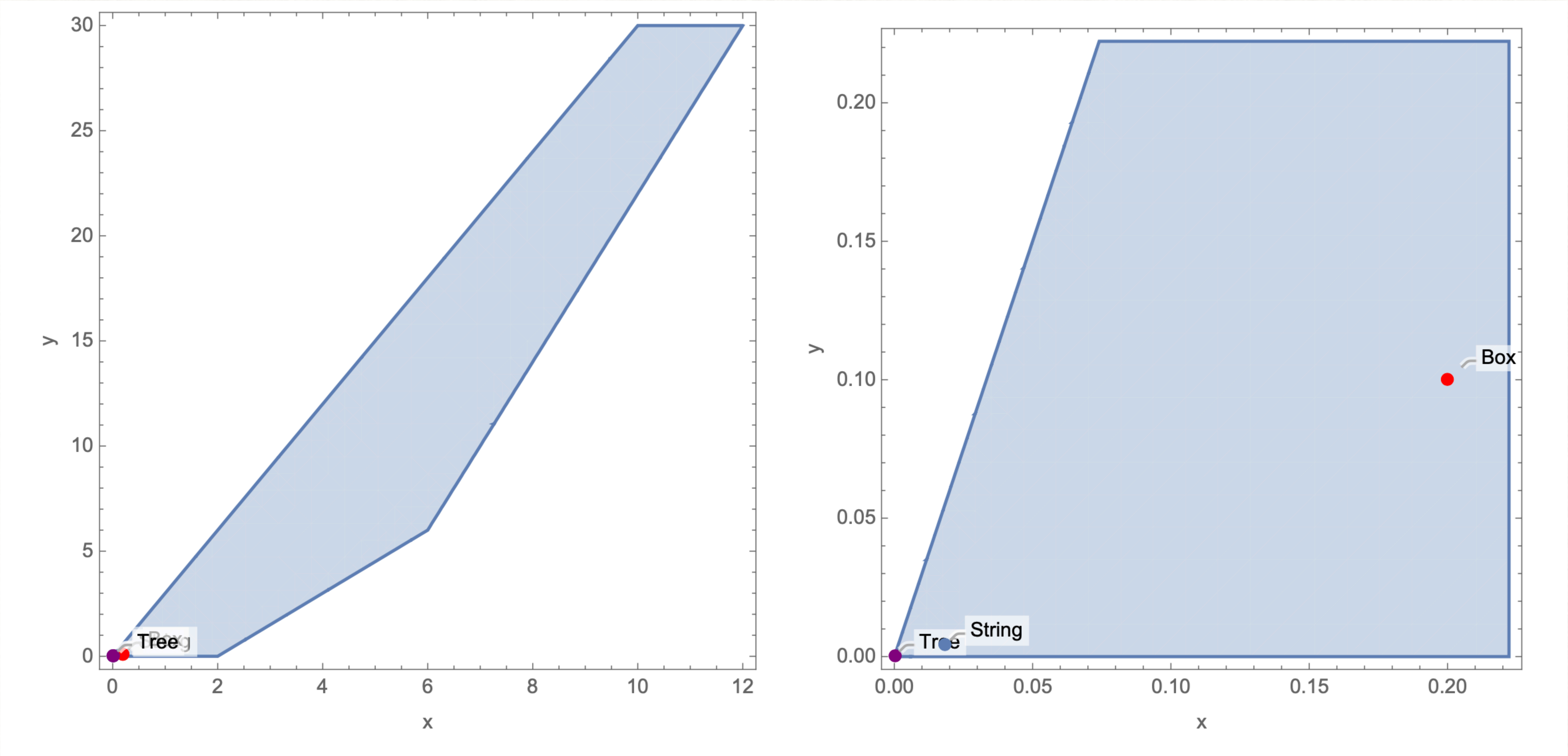
$$\bigg|_{m\to\infty} = \cdots + \frac{(s^5+\frac{1}{5}s^4t+\frac{1}{10}s^3t^2+\frac{1}{10}s^2t^3+\frac{1}{5}st^4+t^5)}{1153152m^{14}\pi^2}+\cdots \qquad (10.18)$$

- (c) The type-I stringy completion of bi-adjoint scalar theory:

$$-\frac{\Gamma[-\alpha' s]\Gamma[-\alpha' t]}{\Gamma[1-\alpha' s-\alpha' t]}\bigg|_{\alpha'\to 0} = \cdots + \alpha'^5\left[\zeta_7 s^5 + \left(-\frac{\pi^4\zeta_3}{90}-\frac{\pi^2\zeta_5}{6}+3\zeta_7\right)s^4t + \left(-\frac{\pi^4\zeta_3}{72}-\frac{\pi^2\zeta_5}{3}+5\zeta_7\right)s^3t^2 + (s\leftrightarrow t)\right]+\cdots \qquad (10.19)$$

For example

$$g_{5,0}s^5+g_{5,1}s^4t+g_{5,2}s^3t^2+g_{5,3}s^2t^3 = s^5+xs^4t+ys^3t^2+ys^2t^3$$



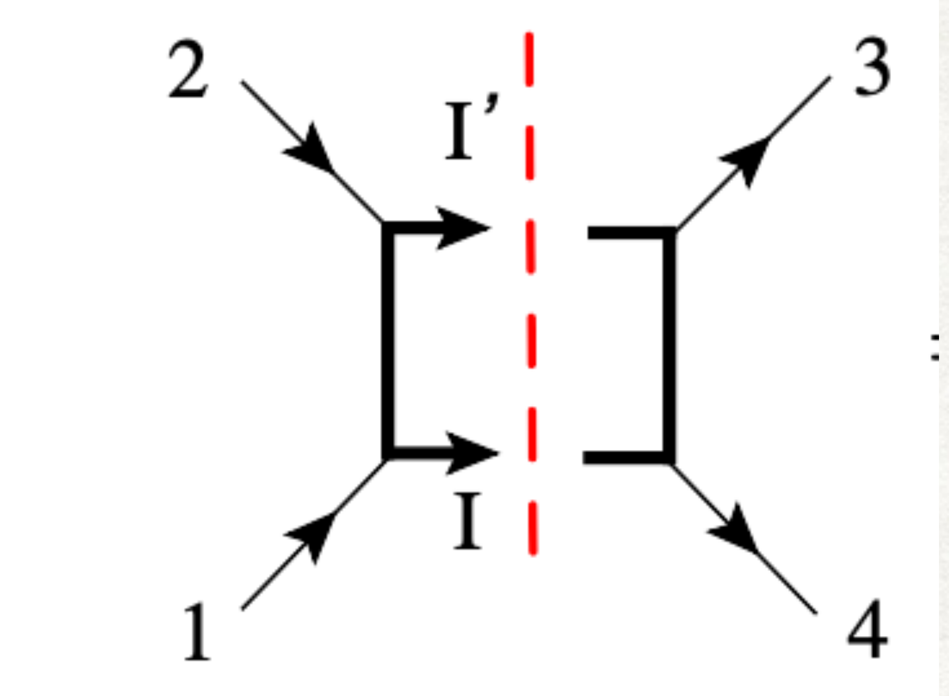
Low Spin-Dominance

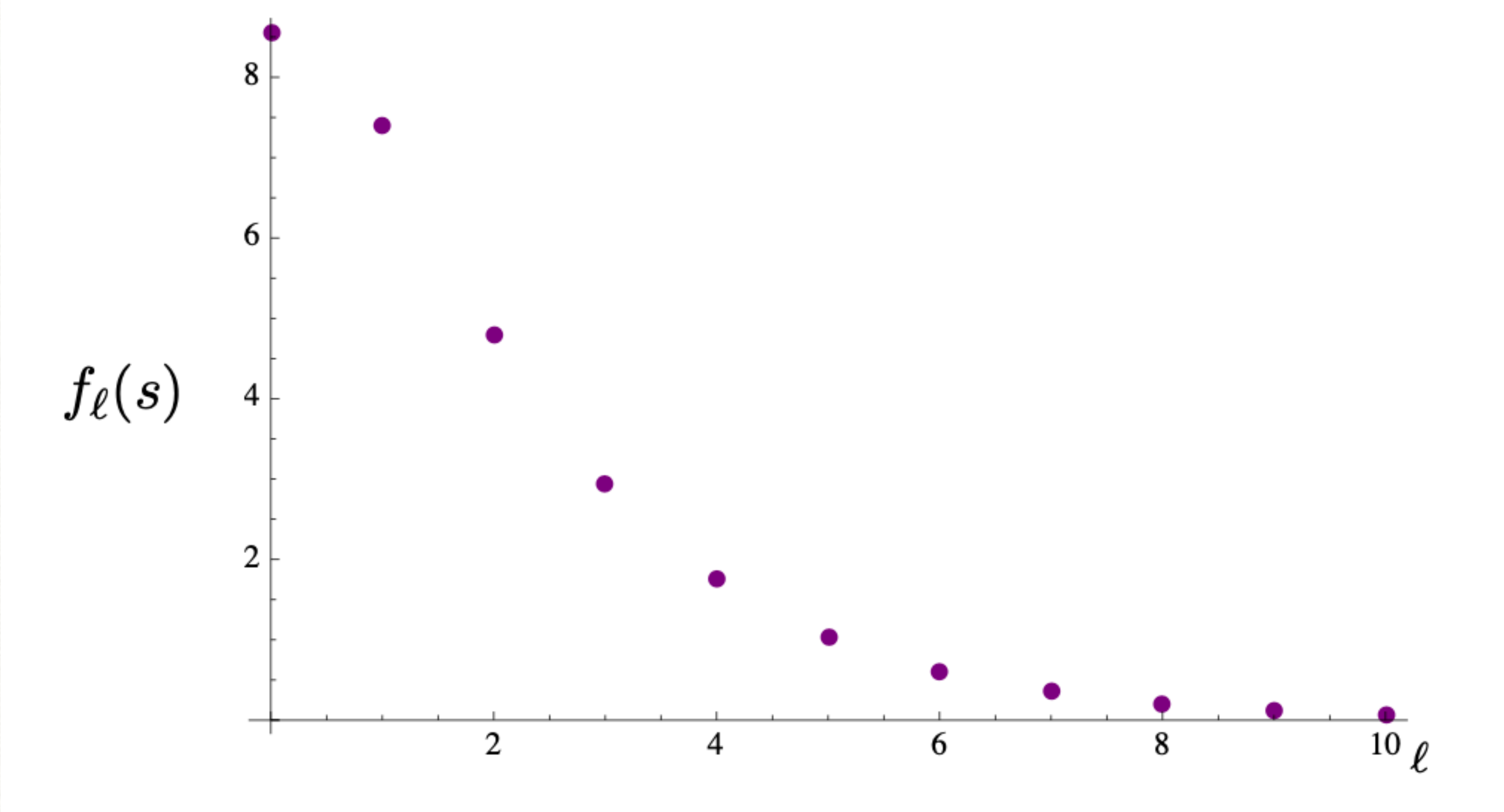
Recall that in the defining the EFThedron, we used

$$g_{k,q} = \sum_a \mathsf{p}_a \frac{2^q u^{\alpha}_{\ell_a,k,q}}{(m_a^2)^{k+1}} \qquad \mathsf{p}_a > 0$$

We only require this to be positive

But for any generic consistent UV completion





$$\langle \hat{p}_{in}|T^\dagger T|\hat{p}_{out}\rangle = \int_{4m^2}^{\infty} ds \frac{4J_s}{s^2} \sum_\ell \mathsf{p}_\ell(s) \frac{2}{2\ell+1} G_\ell^{\frac{1}{2}}(\cos\theta),$$

$$\mathsf{p}_\ell(s) \equiv |f_\ell(s)|^2$$

Suppressed at large spins !

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Recall that in the defining the EFThedron, we used

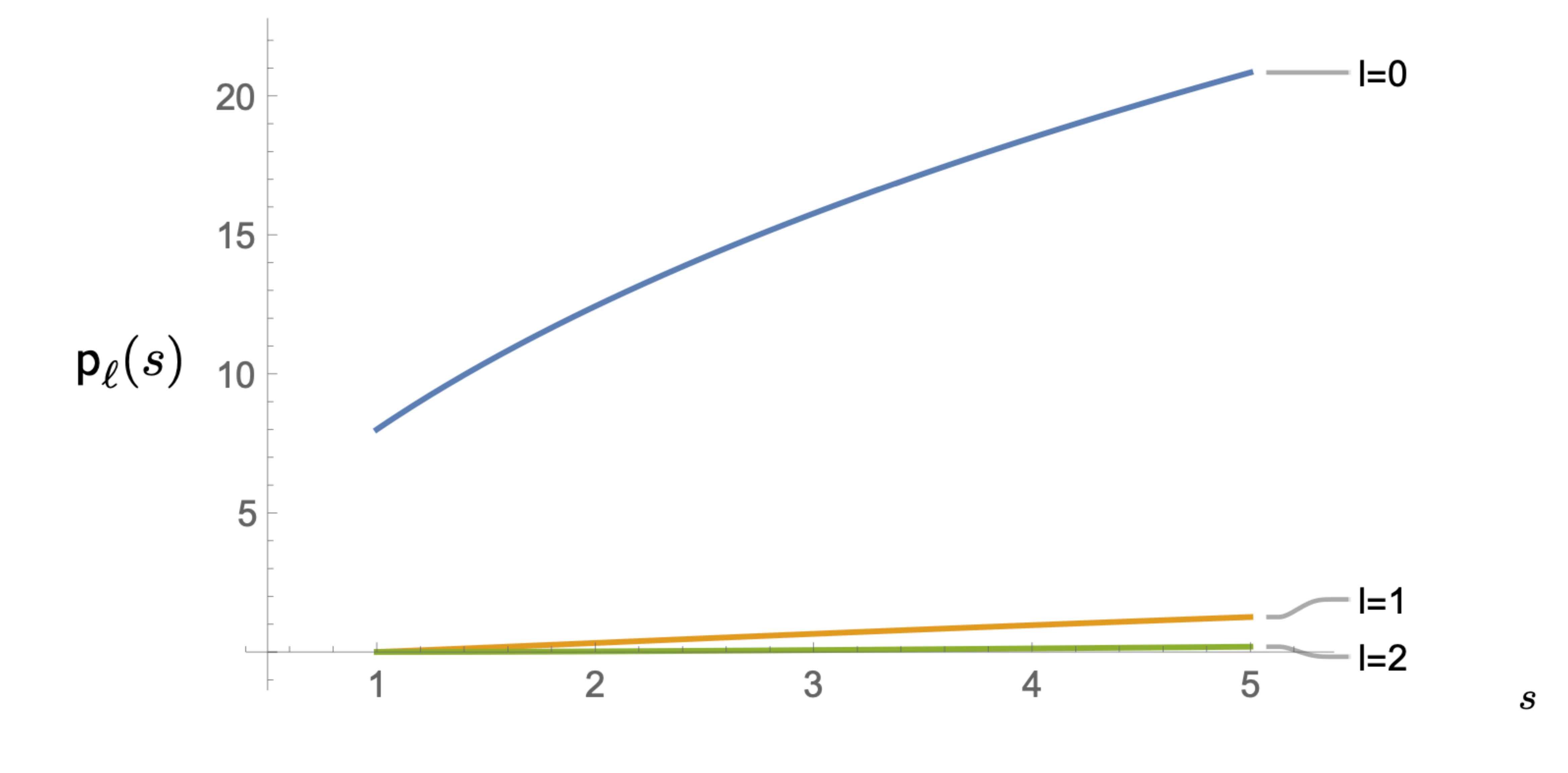
$$g_{k,q} = \sum_a \mathsf{p}_a \frac{2^q u^{\alpha}_{\ell_a,k,q}}{(m_a^2)^{k+1}} \qquad \mathsf{p}_a > 0$$

We only require this to be positive

But for any generic consistent UV completion

Open string

$\ell \backslash n$	1	2	3	4	5
0	1				$\frac{1}{11880}$
1		$\frac{1}{14}$		$\frac{1}{924}$	
2			$\frac{1}{84}$		$\frac{25}{39312}$
3				$\frac{2}{693}$	
4					$\frac{125}{144144}$



Suppressed at large spins !

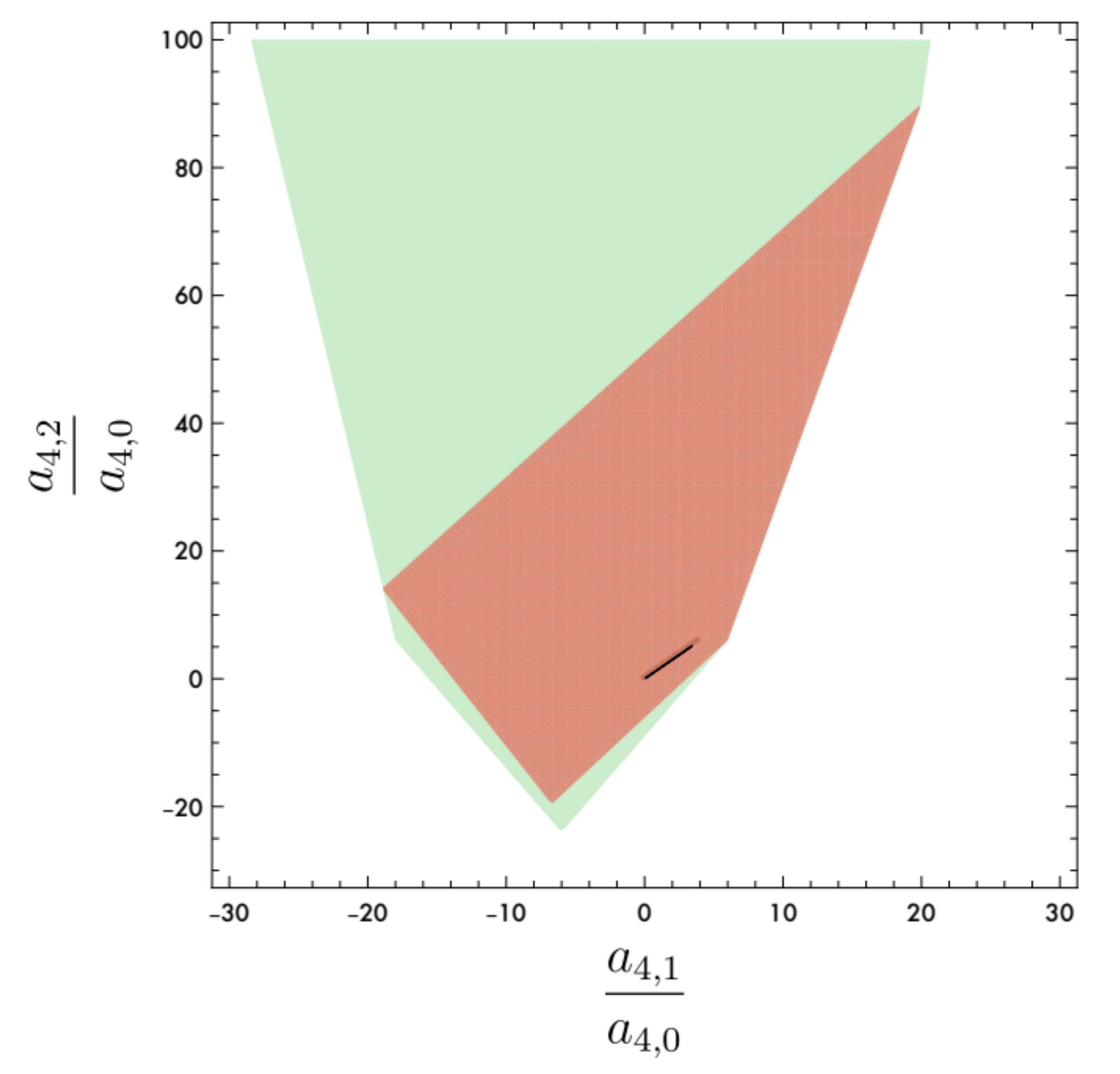
See also Z. Bern D. Kosmopoulos, A Zhiboedov 2103.12729

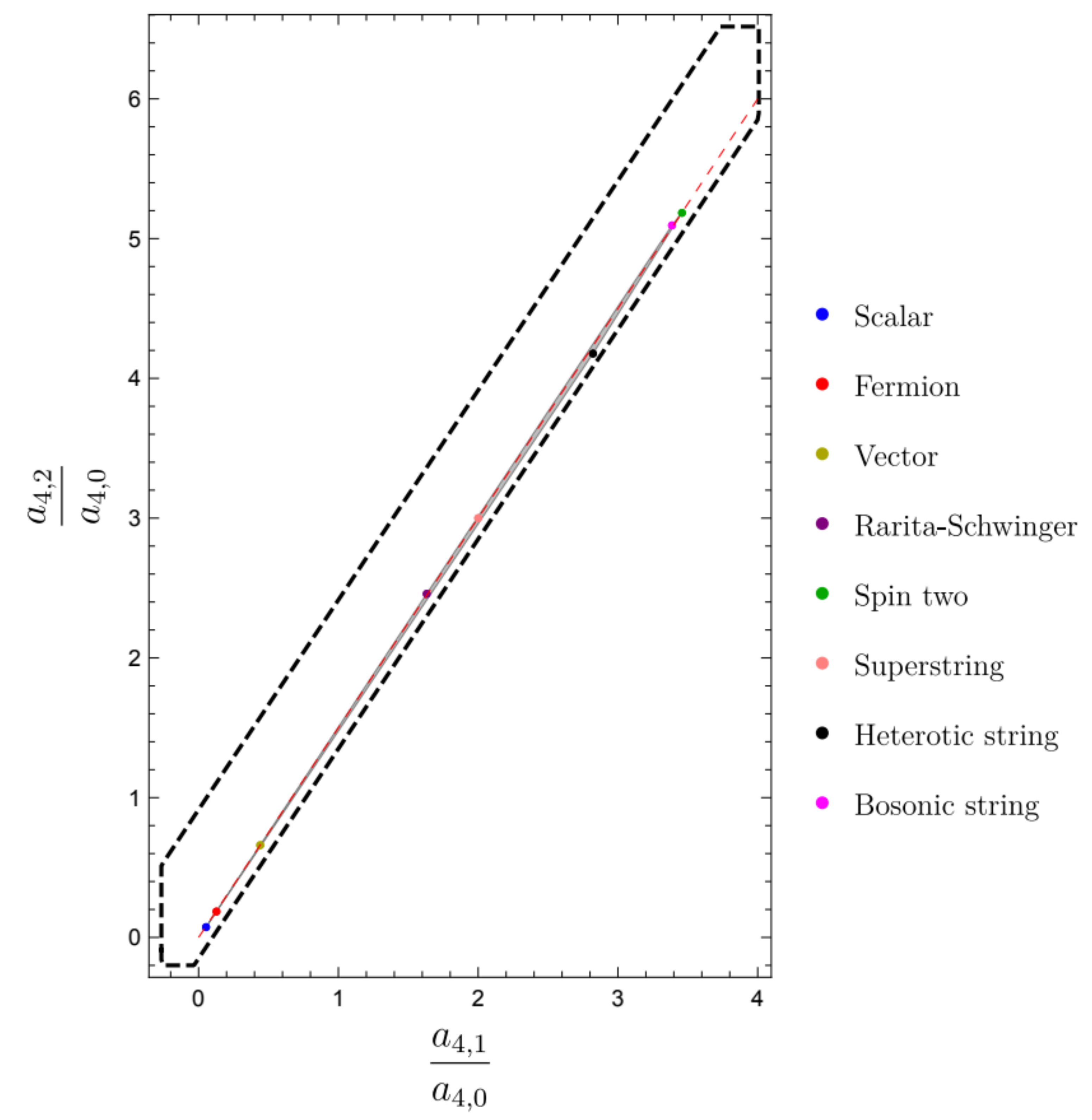
We can generalize to spinning external states $M(+h,+h,-h,-h)$

$$[12]^{2h}\langle 34\rangle^{2h}\Big(\sum_{k,q} a_{k,q}s^{k-q}t^q\Big) = -[12]^{2h}\langle 34\rangle^{2h}\left(\sum_{\ell_a\geq 0} p_i \frac{d^{\ell_i=even}_{0,0}(\theta)}{s-m_i^2} + \sum_{\ell_j\geq 2h} p_j \frac{\tilde{d}^{\ell_j}_{2h,2h}(\theta)}{-t-s-m_j^2}\right)$$

Z. Bern D. Kosmopoulos, A Zhiboedov 2103.12729

D^8R^4



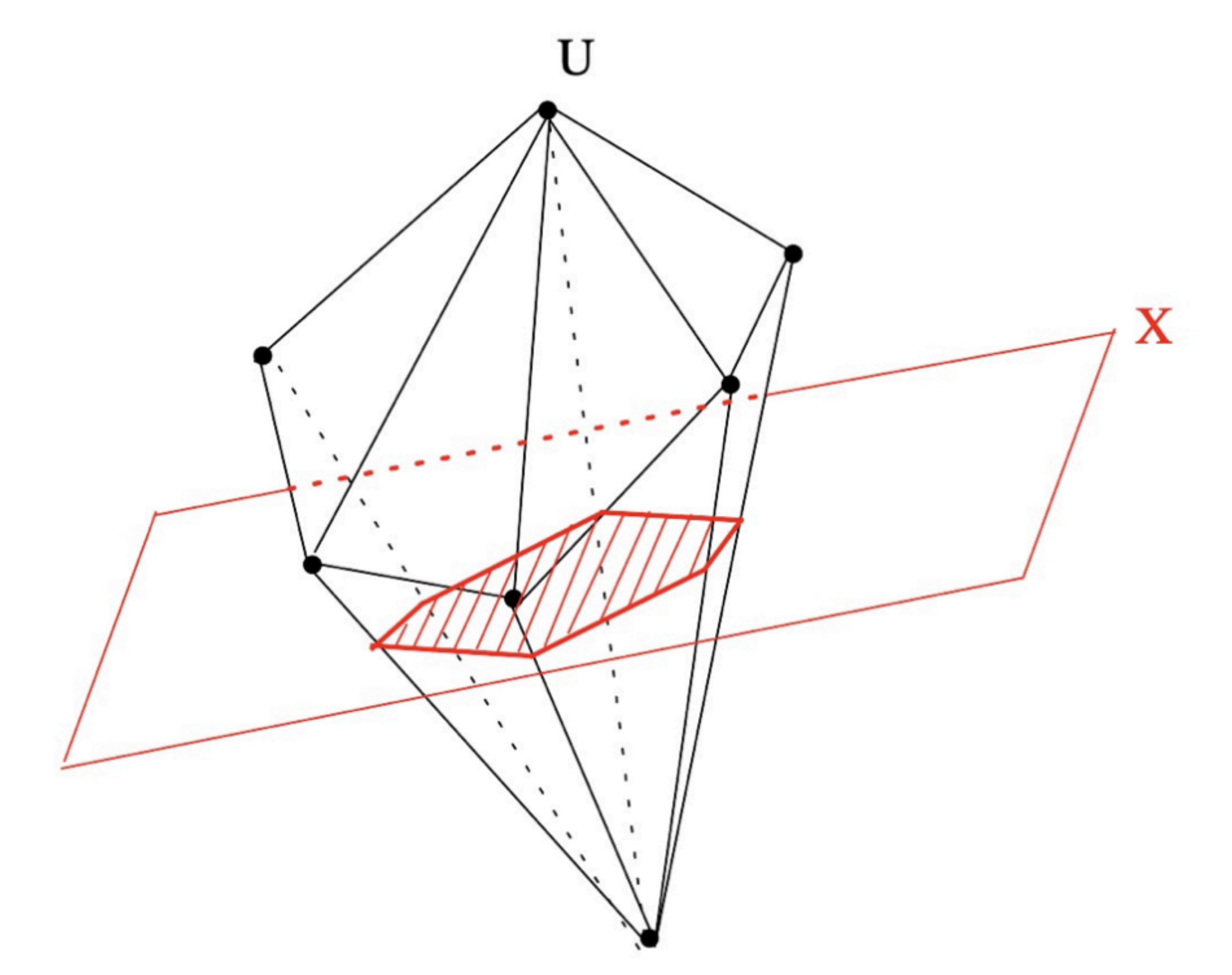


Does large spin suppression emerge from the geometry of the EFThedron ? If so to which extent ?

What constraints on the UV spectrum does the geometry imposes ?

The geometry by nature is an intersection geometry:

A physical spectrum must live on the symmetry plane. A convenient way of formulating this condition is the statement that the hull must have zero image under the projection of the symmetry plane, i.e. the hull must have zero components perpendicular to the symmetry plane.



This leads to "null constraints"

$$n_8: \quad g_{8,4}-\frac{21}{8}g_{8,0}+\frac{1}{4}g_{8,2}=0, \quad g_{8,6}-\frac{21}{8}g_{8,0}+\frac{5}{16}g_{8,2}=0.$$

When using the dispersive representation

$$n_k=\sum_i \frac{p_{\ell_i}}{(m_i^2)^{k+1}}\omega_k(\ell_i)=0\,,$$

where w(l) is a polynomial. The roots impose non-trivial constraint on the spectrum

$$n_k = \sum_i \frac{p_{\ell_i}}{(m_i^2)^{k+1}} \omega_k(\ell_i) = 0 \, ,$$

For example for k=4

$$n_4 = \sum_i \frac{p_i}{m^4} \ell_i(\ell_i+1)(\ell_i^2+\ell_i-8) = 0$$

for most of the spins w(l) is positive

ℓ	0	2	4	6	$\dots$
$\omega_4(\ell)$	0	$-$	$+$	$+$	$+$

Spin-2 must be part of the spectrum

For k=7 we have two null constraints

$$n_7 \equiv g_{7,3} - \frac{4}{5} g_{7,1} = 0, \quad n_7' \equiv g_{7,5} - \frac{16}{3} g_{7,1} = 0$$

→

$$n_7 + n_7' = \sum_i \frac{p_i}{m_i^7} (\ell_i - 2)\ell_i(\ell_i+1)(\ell_i+3) \left(\ell_i^2 + \ell_i - \frac{49}{2} \right) = 0$$

ℓ	0	2	4	6	$\dots$
$\omega_7(\ell)$	0	0	-	$+$	$+$

Spin-4 must be part of the spectrum

$$n_k = \sum_i \frac{p_{\ell_i}}{(m_i^2)^{k+1}} \omega_k(\ell_i) = 0\,,$$

For k=3a+1 we can always arrange for

$$\omega_{3a+1}(\ell) = (\ell^2 + \ell - f_p(a)) \prod_{i=0}^{a-1} (\ell - 2i) \prod_{j=0}^{a-1} (\ell + 2j + 1)$$

Up to a<15 the sign pattern looks

ℓ	0	2	$\dots$	$2a-2$	$2a$	$2a+2$	$\dots$
$\omega_{3a+1}(\ell)$	0	0	0	0	-	$+$	$+$

All spins below 28 must be present !

For higher spins more work is needed

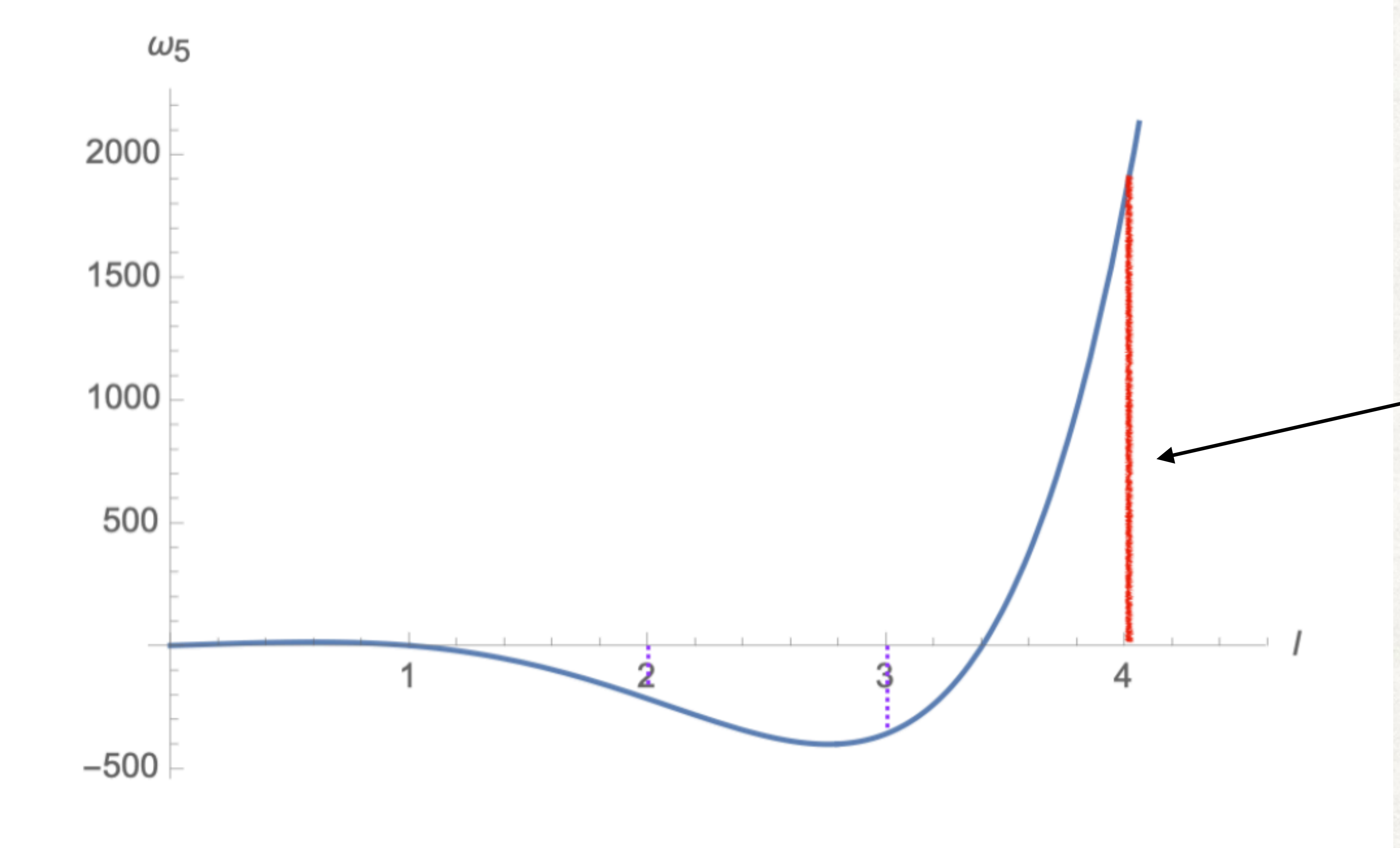
The geometry further imposes constraint on the magnitude for contributions of each spin!

Define the average spinning spectral function

$$\langle p_{k,\ell}\rangle \equiv \sum_{\{i,\ell_i=\ell\}} \frac{p_{\ell_i}}{m_i^{2(k+1)}}$$

The they are reflected in the null constraints as

$$n_k = \sum_i \frac{p_{\ell_i}}{(m_i^2)^{k+1}}\omega_k(\ell_i) = \sum_\ell \langle p_{k,\ell}\rangle \omega_k(\ell) = 0$$



The maximal allowed value for any spin is bounded by the spins with negative contribution, i.e. there's an Upper bound on the ratio

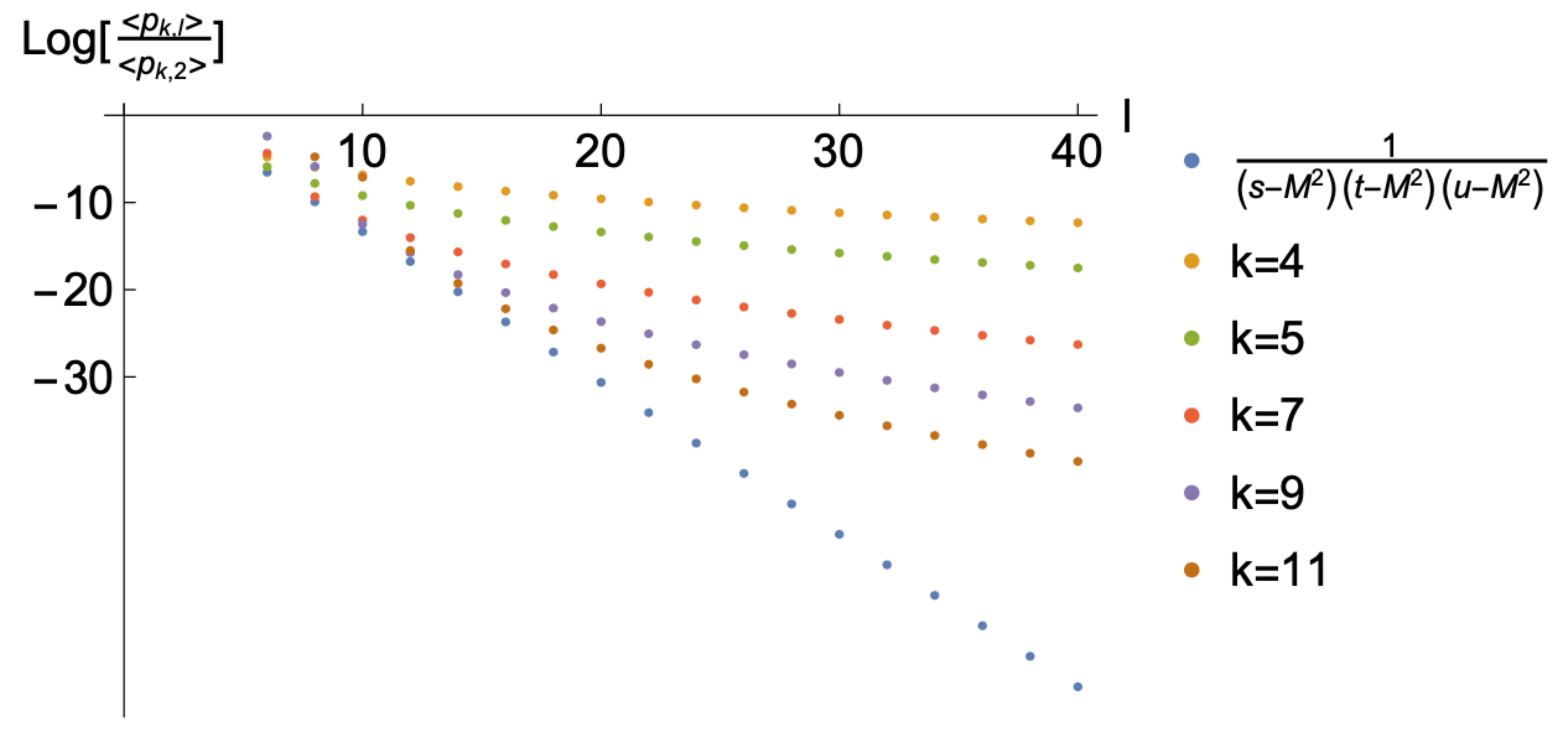
The geometry further imposes constraint on the magnitude for contributions of each spin!

Since we know that spin-2 states must exists, we can bound the ratio of spin-l/spin-2

$$\langle p_{k,\ell}\rangle \equiv \sum_{\{i,\ell_i=\ell\}} \frac{p_{\ell_i}}{m_i^{2(k+1)}}$$

$$\frac{\langle p_{4,\ell}\rangle}{\langle p_{4,2}\rangle} \leq \frac{12}{\ell(\ell+1)\left(\ell^2+\ell-8\right)}, \quad (\ell \geq 4)$$

$$\frac{\langle p_{5,\ell}\rangle}{\langle p_{5,2}\rangle} \leq \frac{216}{\ell(\ell+1)(\ell(\ell+1)(2\ell(\ell+1)-43)+150)}, \quad (\ell \geq 4)$$



We can apply similar analysis to spinning external states

$$[12]^{2h}\langle 34\rangle^{2h}\Big(\sum_{k,q} a_{k,q} s^{k-q} t^{q}\Big) = -[12]^{2h}\langle 34\rangle^{2h}\left(\sum_{\ell_a\geq 0} p_i \frac{d^{\ell_i=even}_{0,0}(\theta)}{s-m_i^2} + \sum_{\ell_j\geq 2h} p_j \frac{\tilde{d}^{\ell_j}_{2h,2h}(\theta)}{-t-s-m_j^2}\right)$$

1<->2 symmetry also leads to null constraints

$$\sum_{l\in\text{even}} \langle p^s_{l,1}\rangle \ell(\ell+1) + \sum_{l\geq 2h} \langle p^u_{\ell,1}\rangle(-4h^2-2h+\ell^2+\ell-1) = 0$$

$$\sum_{l\in\text{even}} \langle p^s_{l,2}\rangle \ell(\ell+1)(\ell+\ell^2-6) + \sum_{l\geq 2h} \langle p^u_{\ell,2}\rangle 4-(2h-\ell)(2h-\ell+1)(2h+\ell+1)(2h+\ell+2) = 0$$

ℓ	2	3	4	5	6	7	...
$h=1, \tilde{\omega}^u_1(\ell)$	-	+	+	+	+	+	...
$h=2, \tilde{\omega}^u_1(\ell)$			-	+	+	+	...
$h=1, \tilde{\omega}^u_2(\ell)$	-	-	+	+	+	+	...
$h=2, \tilde{\omega}^u_1(\ell)$			-	-	+	+	...

We must have l=2 for E&M and l=4 for gravity

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$$h=1: \quad \frac{\langle p^s_{\ell,1}\rangle}{\langle p^u_{2,1}\rangle} \leq \frac{1}{\ell(\ell+1)} \; (\ell\geq 2), \quad \frac{\langle p^u_{\ell,1}\rangle}{\langle p^u_{2,1}\rangle} \leq \frac{1}{\ell^2+\ell-7} \; (\ell \geq 3)$$

$$h=2: \quad \frac{\langle p^s_{\ell,1}\rangle}{\langle p^u_{4,1}\rangle} \leq \frac{1}{\ell(\ell+1)} \; (\ell\geq 2), \quad \frac{\langle p^u_{\ell,1}\rangle}{\langle p^u_{4,1}\rangle} \leq \frac{1}{\ell^2+\ell-21} \; (\ell \geq 5)$$

Deprojecting the EFThedron

So far we've only used $0 \leq \mathsf{p}_\ell(s)$ so the bounds are all projective

$$g_{k,q} = \frac{1}{q!}\frac{d^q}{dt^q}\left(\sum_a \frac{\mathsf{p}_a G_{\ell_a}(1+2\frac{t}{m_a^2})}{(m_a^2)^{k-q+1}} + \sum_b \int ds' \mathsf{p}_{b,\ell}(s')\frac{G_\ell(1+2\frac{t}{s'})}{(s')^{k-q+1}} + \{u\}\right)\Bigg|_{t=0}$$

positive

however unitarity requires $0 \leq \mathsf{p}_\ell(s) \leq 2(2\ell+1)$

In terms of moments, we are interested in the "L" moment problem

$$a_k = \int dz\, p(z) z^k, \quad 0 \leq p(z) \leq L$$

For bounded z, we can easily derive necessary conditions by considering

$$b_k = \int_0^1 dz\, (L - p(z)) z^k = \frac{L}{k+1} - a_k$$

positive simply impose Hankel constraints on b !

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$$exp\left[\frac{1}{L}(\frac{a_0}{x}+\frac{a_1}{x^2}+\frac{a_2}{x^3}\cdots)\right] = 1+\frac{b_1}{x}+\frac{b_2}{x^2}+\cdots$$

A solution for p(z) can be constructed if b satisfies the corresponding Hankel constraints

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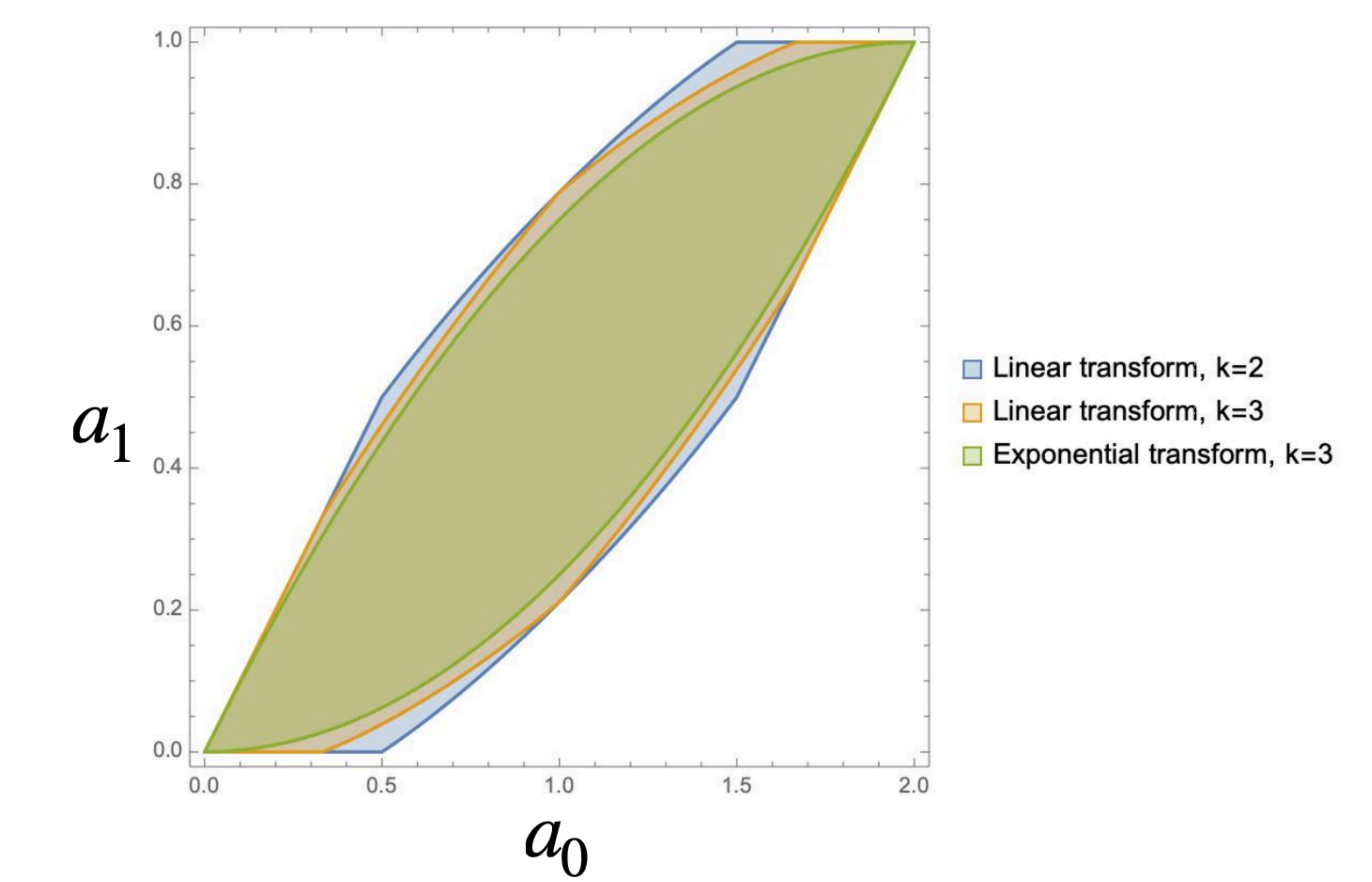

$$p(x) = \begin{cases} L & x \in [0,m] \\ 0 & x \notin [m,1] \end{cases}$$
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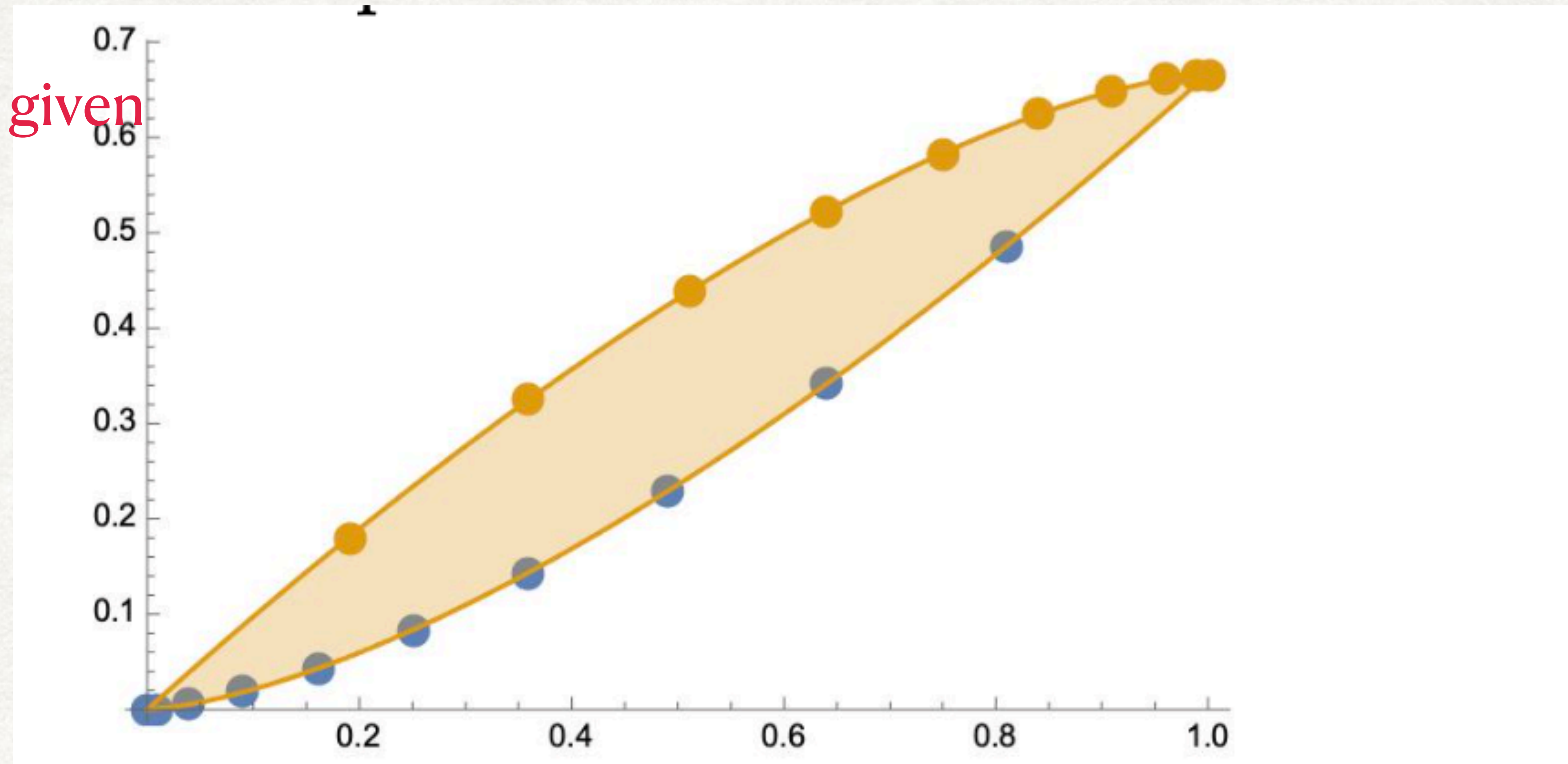
$$a_1 = \int_0^m Lx dx = Lm^2/2$$
$$a_2 = \int_0^m Lx^2 dx = Lm^3/3$$

The boundaries are given
By finite segments



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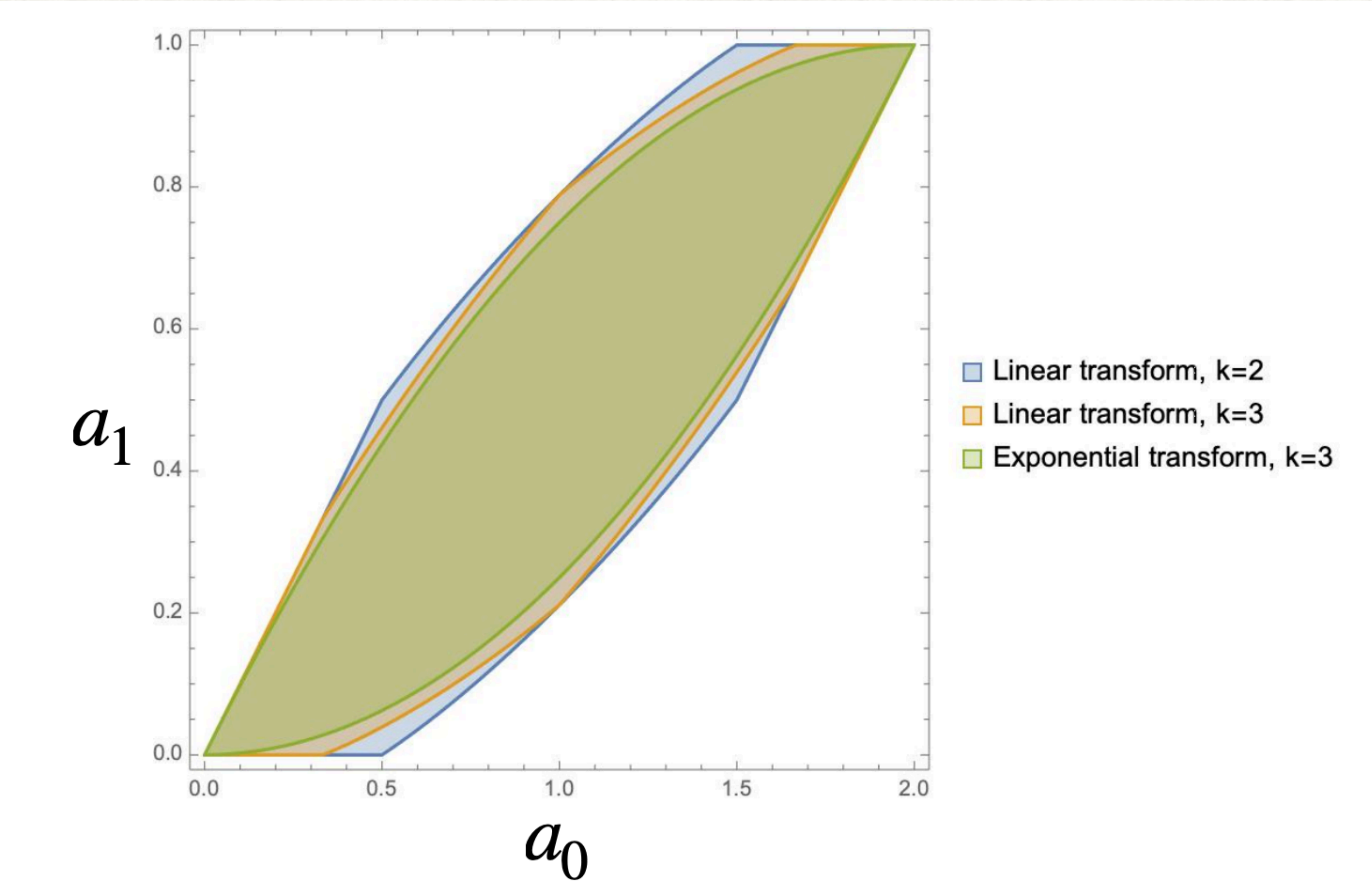
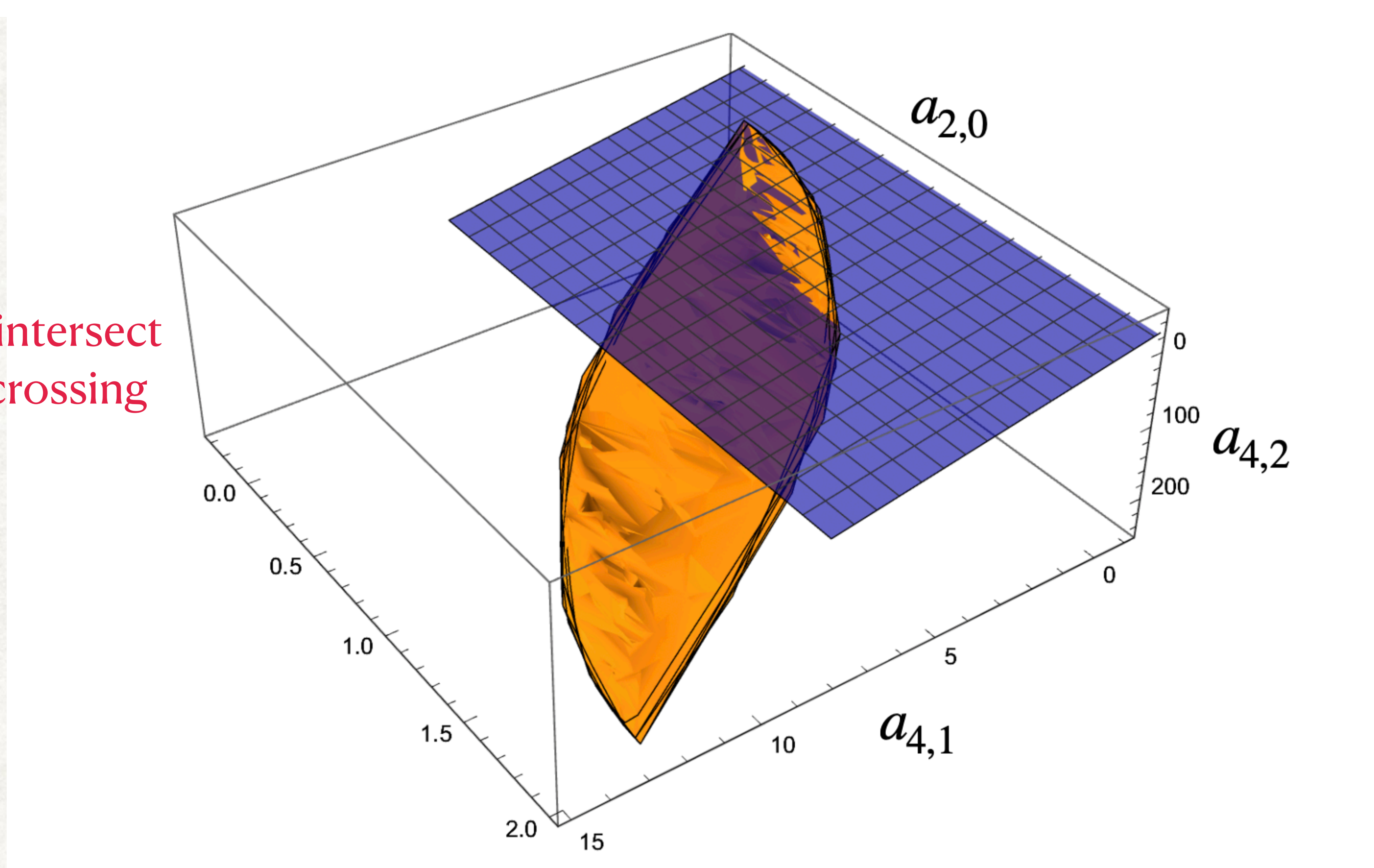


We can further intersect the space with crossing Plane



- The convex hull of product moments is the linchpin of the geometry behind the fixed t EFT bootstrap
- This gives us exact analytic bounds on EFT coefficients, and constrains the UV spectra
- The spectral parameter have a natural upper bound from unitarity alone (p<2), can be straight forwardly incorporated into the geometry
- Generalization to massive external states is straight-forward

$$M^{\mathrm{IR}}(s,t) = -\sum_a p_a \frac{P_l\left(1+\frac{2t}{M_a^2-4m^2}\right)}{s-M_a^2} \rightarrow \sum_{k,q} g_{k,q}(s-4m^2)^{k-q}t^q \qquad g_{k,q} = \sum_a p_a \frac{v_{l_a,q}}{(M_a^2-4m^2)^{k+1}}$$

- Allowing for external states to form irreps under global symmetry (spectrahedron)

Cen Zhang, Shuang-Yong Zhou, PRL. 125, 201601 (2020)

$$M(s,t) = \{massless/massive\ poles\} + \sum_{k,q} a^{\Lambda}_{k,q} s^{k-q} t^{q}$$

$$\sum_{k,q} a^{(n)}_{k,q} z^{k-q} t^{q} = \frac{\Gamma[-s]\Gamma[-t]\Gamma[-u]}{\Gamma[1+s]\Gamma[1+u]\Gamma[+t]} - \left[\sum_{a=1}^{n} R_a(t)\left(\frac{1}{s-a} + \frac{1}{u-a}\right)\right]$$