The Hypersimplex VS the Amplituhedron: Signs, Triangulations, and Eulerian Numbers

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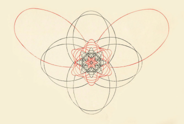

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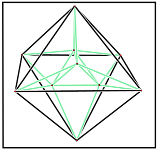
Introduction



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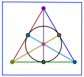
[Gelfand, Goresky, MacPherson, Serganova '87]

moment map, torus orbits
theory of matroids and positroids



tropical geometry

cluster algebras

tropical (positive) Grassmannian

[Kapranov '93; Speyer '08; Ardila, Fink, Rincon '10; Herrmann, Joswig, Speyer '14; Ardila, Rincon, Williams '16]

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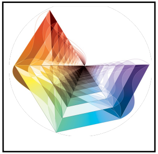

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generalizes polytopes and pos. Grassmannian
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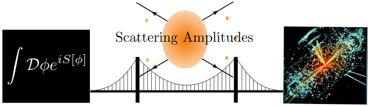

in $\mathcal{N}=4$ super Yang-Mills theory

Positive Geometries [A.Hamed, Bai, Lam '17]

combinatorics/geometry $\leftrightarrow$ physical properties

boundaries $\leftrightarrow$ locality, unitarity

triangulations $\leftrightarrow$ different/novel repr.
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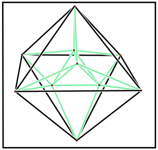

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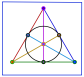

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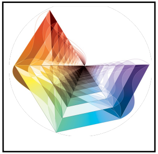

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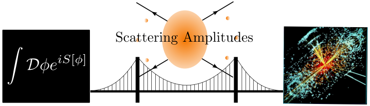

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volume of Hypersimplex ←Eulerian numbers→ # Amplituhedron chambers

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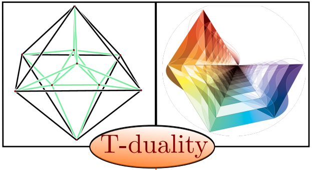

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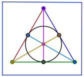

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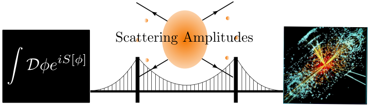

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volume of Hypersimplex $\xleftrightarrow{\text{Eulerian numbers}}$ # Amplituhedron chambers

joint work with *L. K. Williams* and *M. Sherman-Bennett* [Preprint, arXiv:2104.08254]

Outline

The Combinatorics of the Positive Grassmannian

- **Grassmannian**

  $\mathrm{Gr}_{k,n} := \{V : V \subset \mathbb{R}^n, \dim(V) = k\}$

  Represent V by a full-rank $k \times n$ matrix C. Can think of $\mathrm{Gr}_{k,n}$ as $\mathrm{Mat}_{k,n}/\sim$

  Plücker coordinates: $p_I(C) = k \times k$ minor of C with $I \in \binom{[n]}{k}$ as column set

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- **Positive Grassmannian** [Postnikov, Lusztig, Rietsch]

  $\mathrm{Gr}^+_{k,n} \subset \mathrm{Gr}_{k,n}$ where $p_I \geq 0$, $\forall I \in \binom{[n]}{k}$

  partition $\mathrm{Gr}^+_{k,n}$ into pieces: $S_{\mathcal{M}} := \{C \in \mathrm{Gr}^+_{k,n} : p_I(C) > 0 \text{ iff } I \in \mathcal{M}\}$, $\mathcal{M} \subseteq \binom{[n]}{k}$

  Theorem [Postnikov - '06]

  If $S_{\mathcal{M}}$ is non-empty, it is a cell, i.e. homeomorphic to an open ball. So we have:

  $\mathrm{Gr}^+_{k,n} = \bigsqcup S_{\mathcal{M}}$ **positroid cell** decomposition

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  positroid cells of $\mathrm{Gr}^+_{k,n}$ $\leftrightarrow$ *decorated permutations* π on $[n]$ with k anti-excedances

  permutation in which each fixed point is designated either *loop* or *coloop*
  i is *anti-excedence* iff $\pi(i) < i$ or $\pi(i) = i$ is a coloop

$\pi = \{2, 4, 1, 3\} = 2413$
$S_\pi \subset \mathrm{Gr}^+_{2,4}$

nice combinatorics!

planar bicolored (plabic) graphs / $\sim$
Le-diagrams
$\vdots$

The Hypersimplex and the Amplituhedron

- **Moment Map** [Gelfand, Goresky, MacPherson, Serganova - '87]

  $\mu : \mathrm{Gr}^{+}_{k+1,n} \to \mathbb{R}^n$

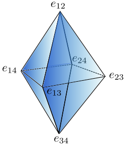
  $$A \mapsto \mu(A) := \frac{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2}$$

  $\{e_1, \ldots, e_n\}$ std basis of $\mathbb{R}^n$, $e_I := \sum_{i \in I} e_i$

- **the Hypersimplex**

  $\Delta_{k+1,n} := \mu(\mathrm{Gr}^{+}_{k+1,n}) = \mathrm{Conv}\{e_I\}_{I \in \binom{[n]}{k+1}}$

  polytope of $\dim(\Delta_{k+1,n}) = n - 1$ in $\mathbb{R}^n$

  $\Delta_{2,4} =$ 

- **Positroid Polytopes**

  $\Gamma_\pi := \mu(\bar{S}_\pi) \subseteq \Delta_{k+1,n}$

  $S_\pi \subseteq \mathrm{Gr}^{+}_{k+1,n}$ pos. cell

  $x_1, \ldots, x_n$ coordinates in $\mathbb{R}^n$

  $x_{[i,j]} := x_i + x_{i+1} + \ldots + x_j$

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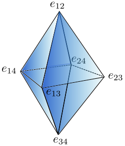
  $\Delta_{2,4} =$ 

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- **Amplituhedron Map** [A.Hamed, Trnka '13]

  $\tilde{Z} : \mathrm{Gr}^+_{k,n} \to \mathrm{Gr}_{k,k+2}$

  $$C \mapsto \tilde{Z}(C) := C \cdot Z$$

  $Z \in \mathrm{Mat}^+_{n,k+2}$ (max minors > 0)

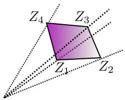
- **the Amplituhedron**

  $\mathcal{A}_{n,k,2}(Z) := \tilde{Z}(\mathrm{Gr}^+_{k,n})$

  not a polytope, $\dim(\mathcal{A}_{n,k,2}) = 2k$ in $\mathrm{Gr}_{k,k+2}$

  $\to (k = 1) : n-$gon

  $\to (k = n-2) : \mathrm{Gr}^+_{k,k+2}$

  $\mathcal{A}_{4,1,2} =$ 

- **Grasstopes**

  $Z_\nu := \tilde{Z}(\bar{S}_\nu) \subseteq \mathcal{A}_{n,k,2}$

  $S_\nu \subseteq \mathrm{Gr}^+_{k,n}$ pos. cell

  $Y \in \mathrm{Gr}_{k,k+2}$, *Twistor coordinates:*

  $\langle Y\, i\, j \rangle := \det_{k+2}(Y_{k \times (k+2)} | Z_i | Z_j)$

Generalized Triangles and T-Duality

- A **generalized triangle** (GT) is a positroid polytope Γ_π / Grasstope Z_ν:
  i) it is full-dimensional, i.e. $n-1$ / $2k$
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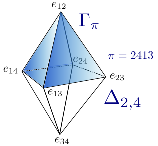

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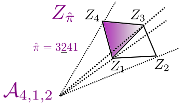
$$\hat{\pi}_i = \pi_{i-1}$$



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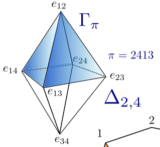

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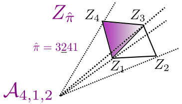

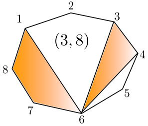

$(k,n)-$unpunctured Plabic Tilings [Oh, Postnikov, Speyer - '15] [Lukowski, MP, Spradlin, Volovich '19]

collection of noncrossing shaded polygons in n-gon triangulated by k triangles

arc $i \to j$ *compatible* if does not cross arcs of shaded polygons
area($i \to j$) = # shaded triangles at the left of $i \to j$

comp. arcs	area
$1 \to 8$	3
$1 \to 7$	2
$1 \to 6$	1
$1 \to 3$	0

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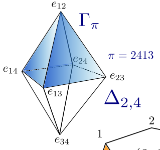

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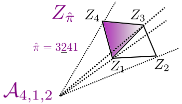

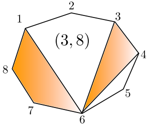

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- **Theorem (Characterization of GTs via Plabic Tilings)**
  GTs of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}$ are in bijection with (k,n)-unpunctured plabic tilings $\mathcal{T}$

$$\Gamma_{\mathcal{T}} \subset \mathbb{R}^n \qquad\qquad Z_{\mathcal{T}} \subset \mathrm{Gr}_{k,k+2}$$

$$x_{[i,j-1]} \geq \mathrm{area}_{\mathcal{T}}(i \to j) \qquad\qquad \mathrm{sign}\langle Yij\rangle = (-1)^{\mathrm{area}_{\mathcal{T}}(i\to j)}$$

$i \to j$ *compatible* arcs of $\mathcal{T}$

Signs and Eulerian Numbers

- **Sign Stratification**

$$\mathcal{A}_{n,k,2} = \bigcup \overline{\mathcal{A}}^{\sigma}_{n,k,2} \qquad \sigma \in \{+,-\}^{\binom{n}{2}}$$

$$\mathcal{A}^{\sigma}_{n,k,2} := \{Y \in \mathrm{Gr}_{k,k+2} : \mathrm{sign}\langle Yij\rangle = \sigma_{ij}\}$$

amplituhedron chambers Q: which is realizable?

Signs and Eulerian Numbers

Eulerian Numbers

$w = w_1 \dots w_{n-1} n \in S_n$
with $k+1$ cyclic descents

$\# w = E_{n-1,k}$

$I_a :=$ {cyclic descents of the rotation of w ending at $a-1$}

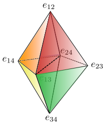
w-Simplices

$\Delta_w := \mathrm{Conv}\{e_{I_1}, \dots, e_{I_n}\}$

Theorem

$$\Delta_{k+1,n} = \bigcup_w \Delta_w$$

[Stanley - '77] [Sturmfels - '96]
[Lam, Postnikov - '07]



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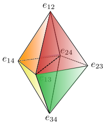

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w-Chambers

$$\hat{\Delta}_w := \{Y \in \mathrm{Gr}_{k,k+2} : \mathrm{sign}\langle Yaj \rangle = (-1)^{|I_a \cap [a+1,j]|-1}\}$$

Theorem

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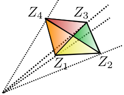


The amplituhedron is the union of w-chambers
- counted by *Eulerian numbers*

Signs and Eulerian Numbers

Eulerian Numbers

$w = w_1 \dots w_{n-1} n \in S_n$ with $k+1$ cyclic descents

$\# w = E_{n-1,k}$

$I_a :=$ {cyclic descents of the rotation of w ending at $a-1$}

Sign Stratification

$$\mathcal{A}_{n,k,2} = \bigcup \overline{\mathcal{A}}^{\sigma}_{n,k,2} \qquad \sigma \in \{+,-\}^{\binom{n}{2}}$$

$$\mathcal{A}^{\sigma}_{n,k,2} := \{Y \in \mathrm{Gr}_{k,k+2} : \mathrm{sign}\langle Yij \rangle = \sigma_{ij}\}$$

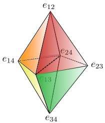
amplituhedron chambers Q: which is realizable?

w-Simplices

$$\Delta_w := \mathrm{Conv}\{e_{I_1}, \dots, e_{I_n}\}$$

Theorem

$$\Delta_{k+1,n} = \bigcup_w \Delta_w$$



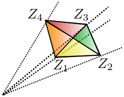
[Stanley - '77] [Sturmfels - '96]
[Lam, Postnikov - '07]

w-Chambers

$$\hat{\Delta}_w := \{Y \in \mathrm{Gr}_{k,k+2} : \mathrm{sign}\langle Yaj \rangle = (-1)^{|I_a \cap [a+1,j]|-1}\}$$

Theorem

$$\mathcal{A}_{n,k,2} = \bigcup_w \hat{\Delta}_w$$



The amplituhedron is the union of w-chambers - counted by *Eulerian numbers*

Theorem (w-simplices, w-chambers, and T-duality)

Let Γ_π and $Z_{\hat{\pi}}$ be T-dual GTs of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}$ resp.:

$$\Delta_w \subset \Gamma_\pi \quad \Leftrightarrow \quad \hat{\Delta}_w \subset Z_{\hat{\pi}}$$

Triangulations and T-Duality

- A **positroid triangulation** is a collection of GTs $\{\Gamma_\pi\}$ / $\{Z_\nu\}$:

  i) have disjoint interiors

  ii) cover $\Delta_{k+1,n}/\mathcal{A}_{n,k,2}$

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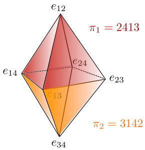
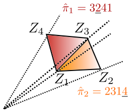
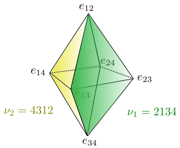
  $\{\Gamma_\pi\}$ is a pos. triangulation of $\Delta_{k+1,n}$ $\Leftrightarrow$ $\{Z_{\hat{\pi}}\}$ is a pos. triangulation of $\mathcal{A}_{n,k,2}(Z)$, for all Z

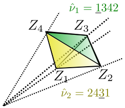
$\Delta_{2,4}$

$\mathcal{A}_{4,1,2}$









Triangulations and T-Duality

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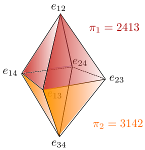
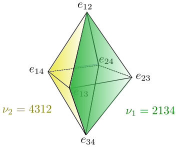
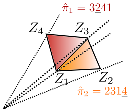
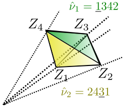




(k,n)	$\sharp$ Triangulations
$(1,n)$	C_{n-2}
$(2,5)$	5
$(2,6)$	120
$(2,7)$	3073
$(2,8)$	6443460
$(3,6)$	14
$(3,7)$	3073

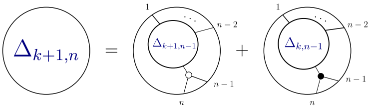
$\mathcal{A}_{4,1,2}$



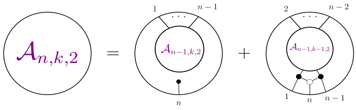


Triangulations from BCFW and Pos Trop Grassmannian

BCFW Recursions



⇔



[Lukowski, MP, Williams '20]

[Bao, He '19]

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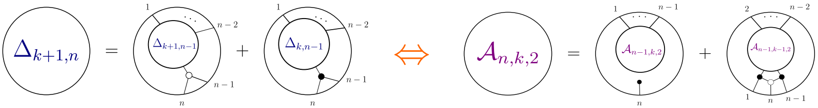

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Positive Tropical Grassmannian [Speyer, Williams '05]

$\mathrm{Trop}^+\mathrm{Gr}_{k+1,n}$ = set of *positive tropical Plücker vectors* in $\mathbb{R}^{\binom{n}{k+1}}$

$$P = \{P_I\} \in \mathbb{R}^{\binom{n}{k+1}} : P_{acS} + P_{bdS} = \min\{P_{abS} + P_{cdS}, P_{adS} + P_{bcS}\}$$

[Lukowski, MP, Williams '20]
[A.Hamed, Lam, Spradlin '20]

Theorem ⇔ $P \in$ max cone of $\mathrm{Trop}^+\mathrm{Gr}_{k+1,n}$

$\mathcal{D}_P = \{\Gamma_\pi\}$ is a pos. triangulation of $\Delta_{k+1,n}$

$\mathcal{D}_P$: *regular* triangulation induced by *height vector* P

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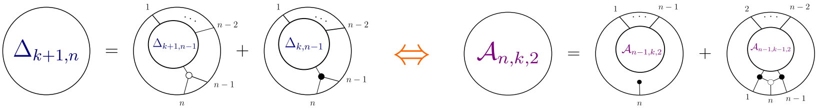

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Theorem ⇔ $P \in$ max cone of $\mathrm{Trop}^+\mathrm{Gr}_{k+1,n}$ ⇒ **Corollary**

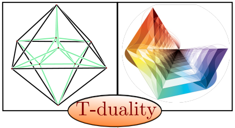
$\mathcal{D}_P = \{\Gamma_\pi\}$ is a pos. triangulation of $\Delta_{k+1,n}$ $\{Z_{\hat{\pi}}\}$ is a pos. triangulation of $\mathcal{A}_{n,k,2}$

$\mathcal{D}_P$: *regular* triangulation induced by *height vector* P

(k,n)	$(1,n)$	$(2,5)$	$(2,6)$	$(2,7)$	$(2,8)$	$(2,9)$	$(3,6)$	$(3,7)$	$(3,8)$	$(3,9)$
# max cones	C_{n-2}	5	120	693	13 612	346 710	14	693	90 609	30 659 424

Summary and Outlook

the Hypersimplex



the Amplituhedron

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$\dim(\Delta_{k+1,n}) = n-1$ in $\mathbb{R}^n$		$\dim(\mathcal{A}_{n,k,2}) = 2k$ in $\mathrm{Gr}_{k,k+2}$
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$\Gamma_{\mathcal{T}}$ (Positroid Polytope)	unpunctured plabic tiling $\mathcal{T}$	(Grasstope) $Z_{\mathcal{T}}$
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Summary and Outlook

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T-duality

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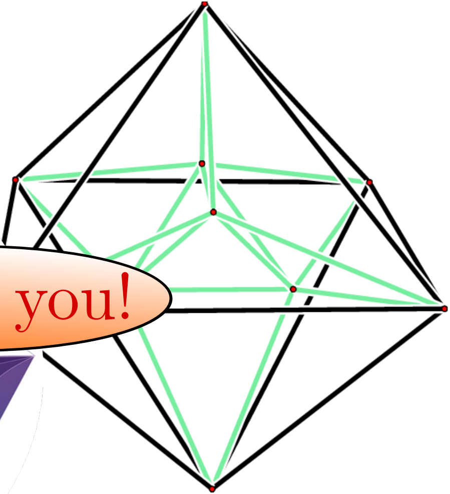
- Eulerian numbers - connection with *Scattering Equations*? [Spradlin, Volovich '05]

Pure mathematics and physics are becoming ever more closely connected, though their methods remain different. One may describe the situation by saying that mathematicians play a game in which they themselves invent the rules, while the physicists play a game in which the rules are provided by Nature. However, as time goes on, it becomes increasingly evident that the rules which mathematicians find interesting are the same as those which Nature has chosen.

['The Relation between Mathematics and Physics' Paul A.M. Dirac, 1939]

Thank you!

[*Artistic depiction of the Amplituhedron* – Gilmore]



[*Schlegel diagram of the Hypersimplex* – Ziegler '05]

Questions?