PennState

A Prescriptive Basis at 2-loops and 6-points

Cameron Langer
In collaboration with: J. Bourjaily, Y. Zhang (to appear)

Based on earlier work with: E. Herrmann, A. McLeod, J. Trnka
[1909.09131] [1911.09106] [2007.13905]

@ Amplitudes 2021

Motivation

Choosing bases… wisely

- Generalized and prescriptive unitarity
- *Graph-theoretic* power-counting at two loops
- Counting the size of the basis
- Constructing a *good* basis

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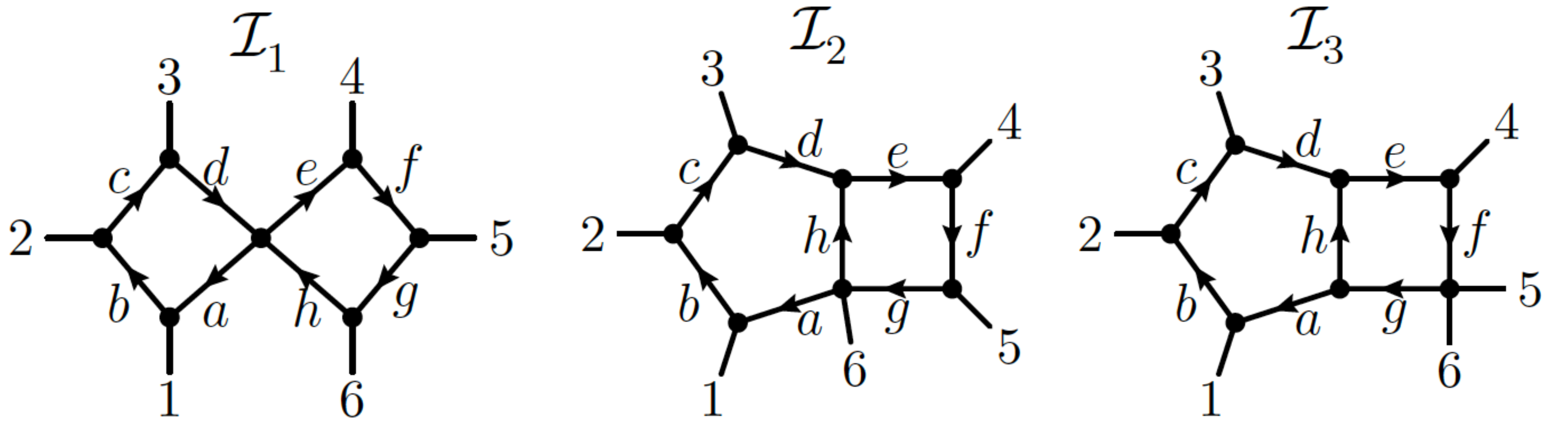
Result: 3-gon basis

- Fully diagonalized 388-dim. space
- $d\,\log$ and pure (when applicable)
- Stratified according to IR divergences



+… (94 total integrand topologies)

Generalized Unitarity

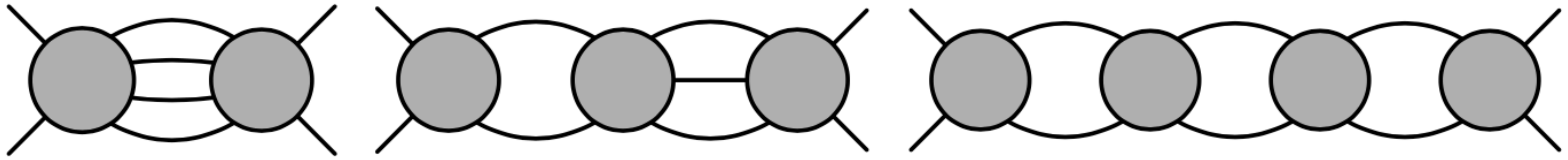
[Bern, Dixon, Dunbar, Kosower: 9403226, 9409265]
[Britto, Cachazo, Feng: 0412103]

- Amplitude integrands are rational functions—so, they may be expanded in a basis $\mathfrak{B}$

$$\mathcal{A} = \sum_{\mathfrak{b}^i \in \mathfrak{B}} a_i \, \mathfrak{b}^i$$

(in any sufficiently well-behaved QFT…)

- The coefficients are fixed by matching a spanning set of field theory cuts e.g.,



$$f_\Gamma \equiv \prod_i \Big(\sum_{\text{states}} \int d^{d-1}\text{LIPS}_i \Big) \prod_v \mathcal{A}_v$$

Computable from first principles in *any* QFT as **on-shell functions**

- Reduces the computation to linear algebra

[Long history of refining this approach from many perspectives:

Passarino, Veltman '79; Ossola, Papadopoulos, Pittau: 0609007; Mastrolia,Ossola,Reiter,Tramontano: 1006.0710; Ellis, Giele, Kunszt: 0708.2398; Badger, Frellesvig, Zhang 1202.2019; Mastrolia, Peraro, Primo: 1605.03157; Ita: 1510.05626, Feng, Huang: 1209.3747…]

Generalized Unitarity

Bourjaily, Herrmann, Trnka: 1704.05460;
Bourjaily, Herrmann, CL, McLeod, Trnka: 1909.09131, 1911.09106

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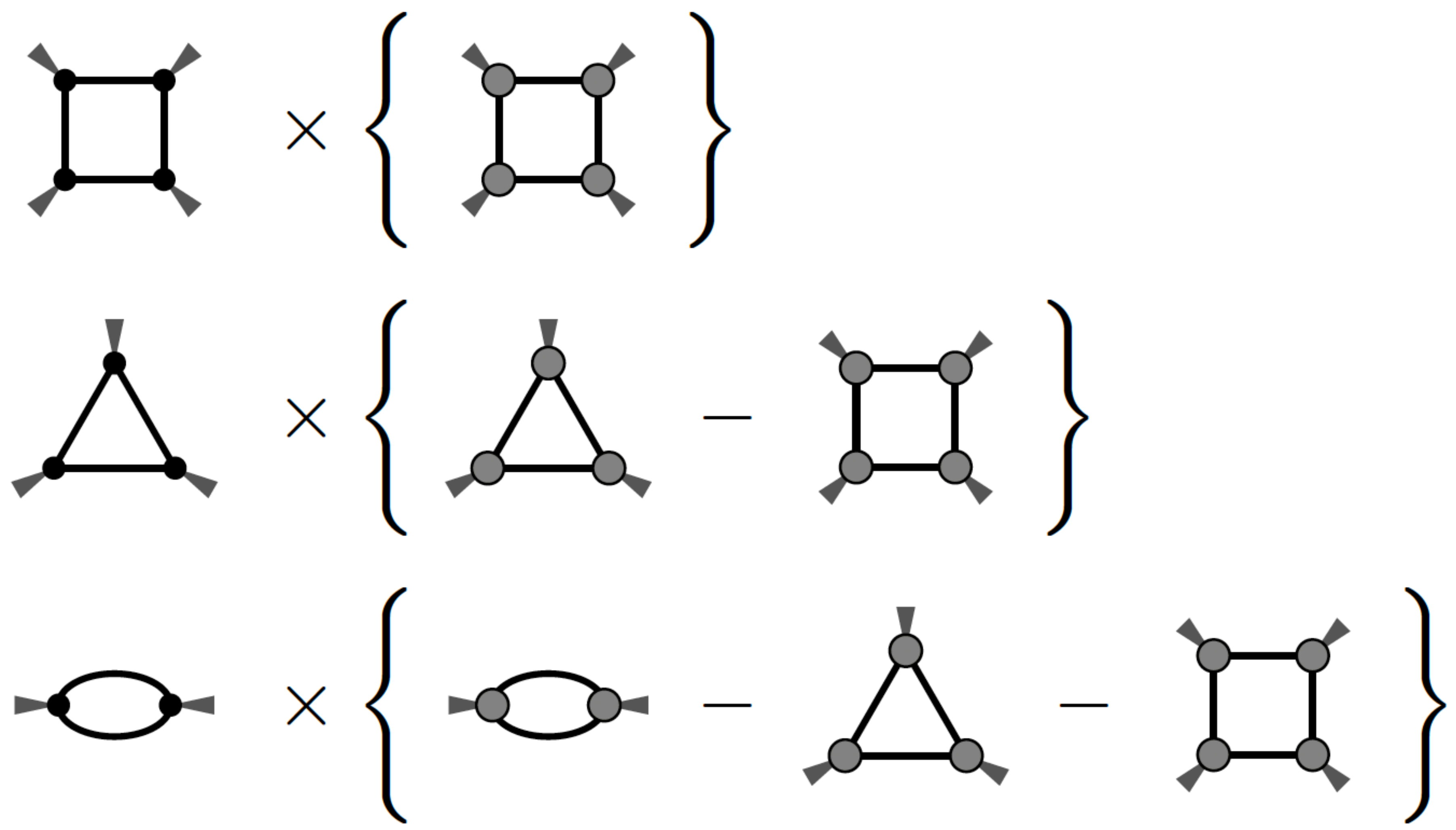
- For an *arbitrary* choice of basis, the coefficients are *arbitrary* linear combinations of field theory residues

- E.g., in terms of *scalar* integrands, amplitudes with bubble power-counting are



- **Boxes**: fixed by quadruple cuts
- **Triangles**: triangle cut (evaluated at a point), *minus* the pollution of the scalar box

…

Prescriptive Unitarity

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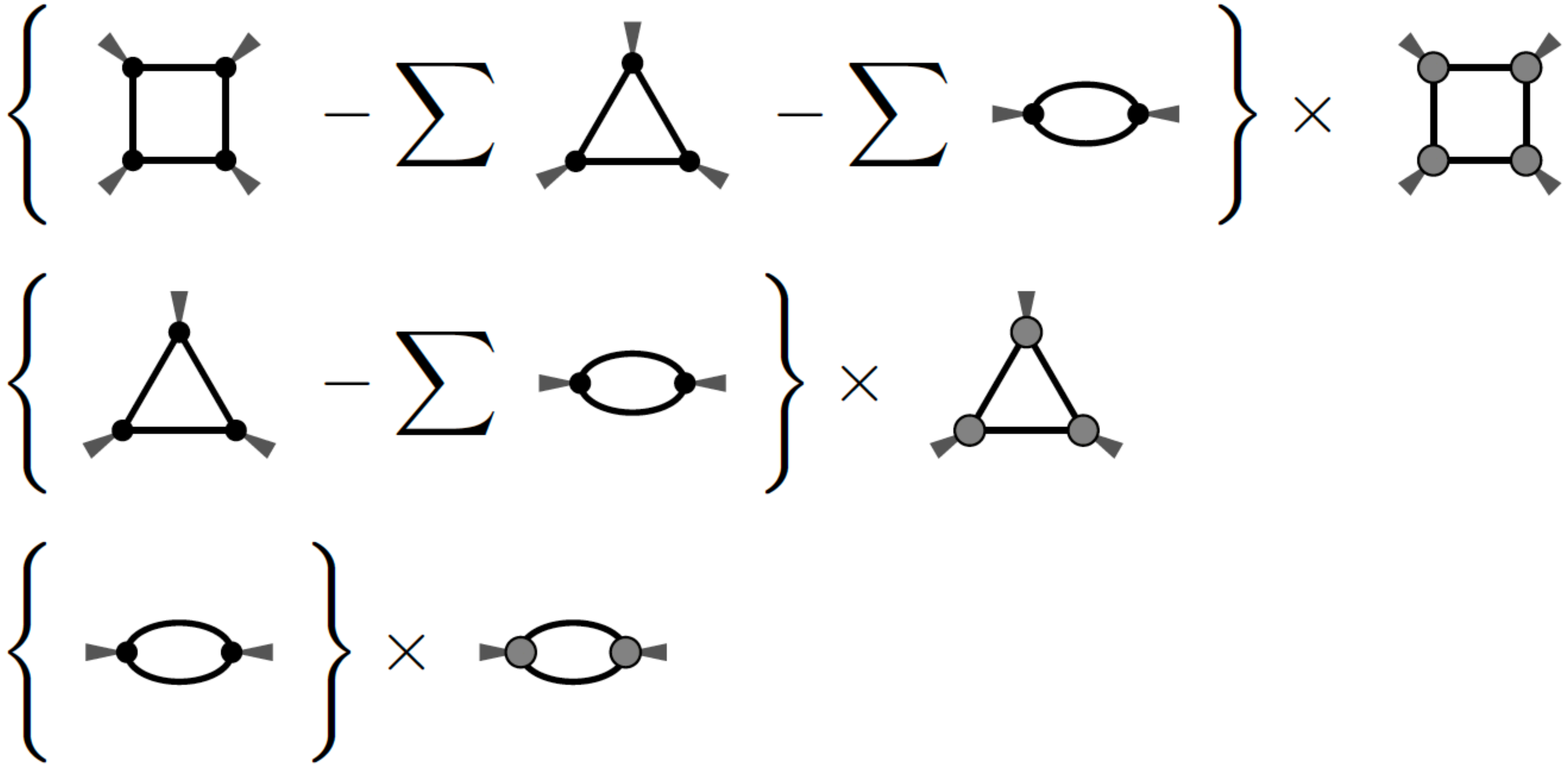
- Exploit the fact that residues of field theory are *easy to compute:* diagonalize the basis with respect to a spanning set of cuts



- **Every** integrand is tailored to match a single field theory cut manifestly, and vanish on all other defining cuts
- All other cuts matched by completeness of the basis (via residue theorems)

Bourjaily, Trnka: 1505.05886
Bourjaily, Herrmann, Trnka: 1704.05460;
Bourjaily, Herrmann, CL, McLeod, Trnka:
1909.09131, 1911.09106

Prescriptive Unitarity at Two Loops

Workflow for finding ‘good’ bases

1. Write down an (arbitrary) basis of integrands (the size of which is dictated by power-counting)

2. Enumerate a spanning set of cuts/contours in field theory

3. Diagonalize the basis with respect to your choice in 2.

For six particles and triangle power-counting, the basis can be stratified according to:

- Polylogarithmicity
- Purity
- Infrared Divergence/Finiteness

[Bourjaily, Herrmann, CL, Trnka 2007.13905]

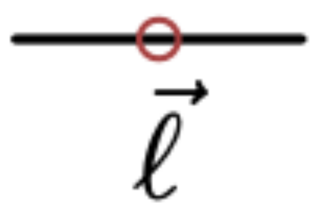
Building Bases of Loop Integrands

Before we discuss 'nice' integrands, we need to know *how many* there are in the first place!

- Size of the basis depends on space-time dimension and choice of power-counting
- At one loop, power-counting is 'obvious': e.g., the space of integrands with triangle PC scale as:

$$\lim_{\ell \to \infty} \mathcal{I} \sim \frac{1}{(\ell^2)^3}$$

"scales like a scalar triangle at infinity"

- Using a graphical notation for loop-dependent numerator insertions 

$\coloneqq \frac{[\ell]}{\ell^2}$

where the vector space of numerators is the span of generalized inverse propagators

$$[\ell] = \text{span}\left(\ell^2, \ell \cdot k_1, \ell \cdot k_2, \ell \cdot k_3, \ell \cdot k_4, 1\right)$$

$$\text{rank}([\ell]) = 6 = 2 + 4$$

[Bourjaily, Herrmann, CL, Trnka 2007.13905]

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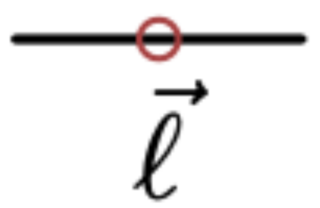
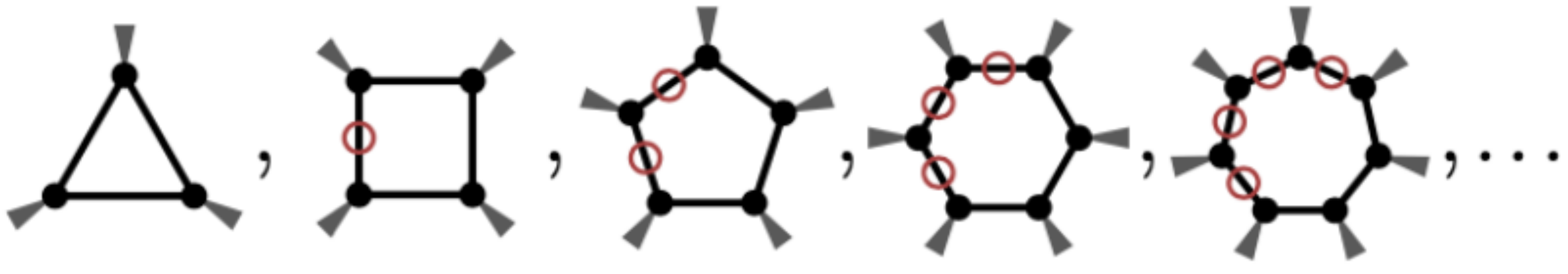
- Using a graphical notation for loop-dependent numerator insertions  $\coloneqq \frac{[\ell]}{\ell^2}$

An over-complete description of the space of integrands with 3-gon PC is

$\mathfrak{B}_3 \coloneqq \text{span}\Big\{$  $\Big\}$

[Bourjaily, Herrmann, CL, Trnka 2007.13905]

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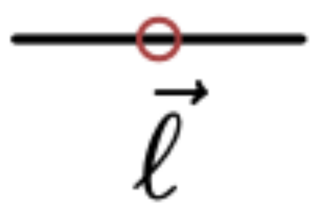
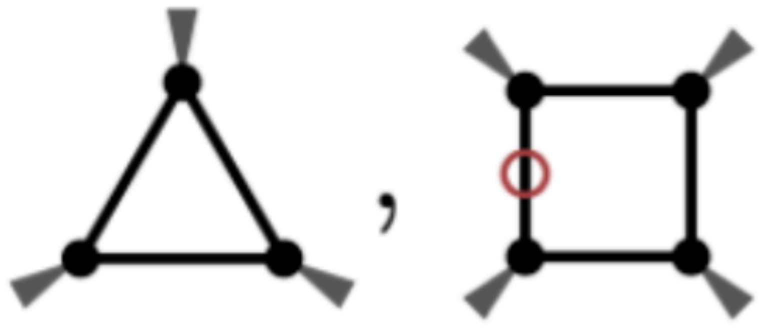
- Using a graphical notation for loop-dependent numerator insertions  $\coloneqq \frac{[\ell]}{\ell^2}$

Since pentagons and higher are *reducible* (in four dimensions)

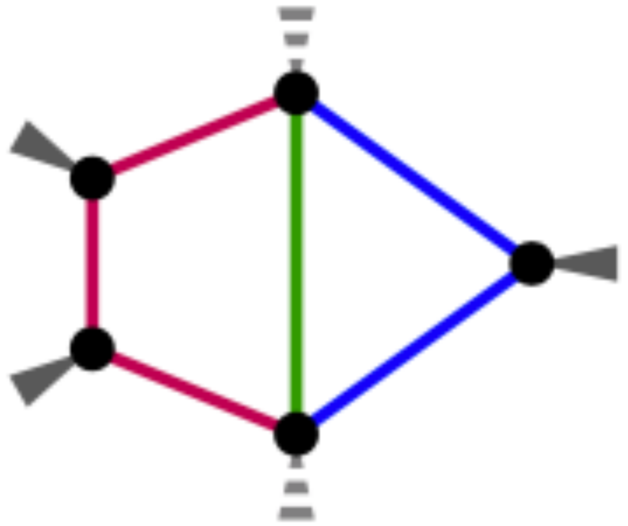
$\mathfrak{B}_3 \coloneqq \text{span}\{$  $\}$

Convenient choice of basis: *chiral boxes* and scalar triangles
[Bourjaily, Caron-Huot, Trnka: 1303.4734]

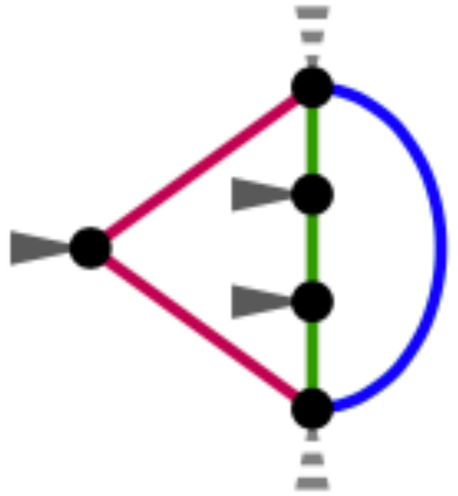
Triangle Power-Counting at Two Loops

Problem: naive scaling according to some loop-momentum routing is not *canonical*

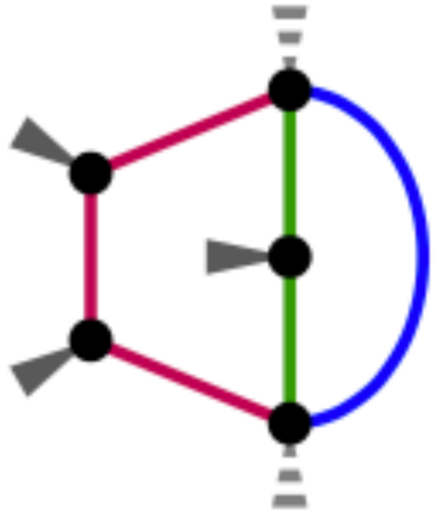
- How many (ℓ_1, ℓ_2) propagators per loop?



$(4, 3)$



$(5, 4)$



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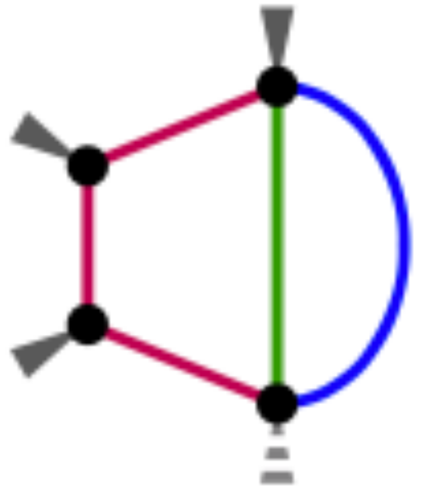
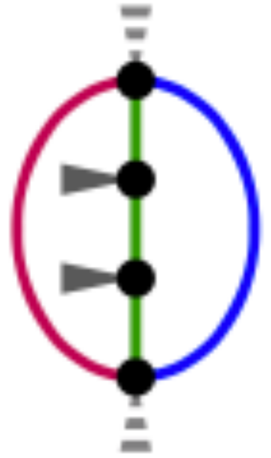


$(4, 2)$



$(4, 4)$

Resolution:

<u>Define power-counting for a graph relative to its contact terms</u>

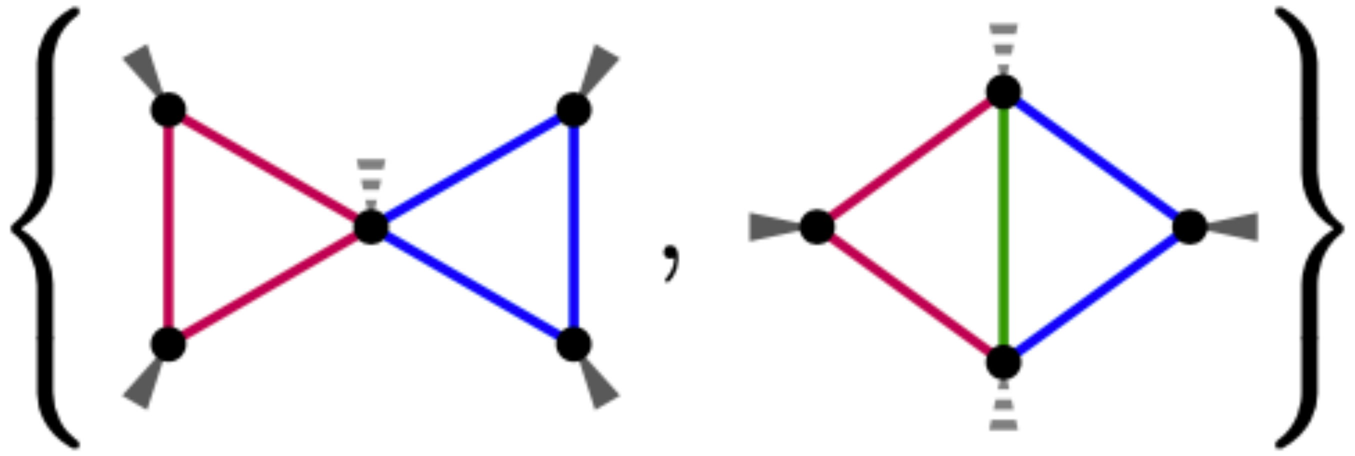
Triangle Power-Counting at Two Loops

<u>Define power-counting for a graph relative to its contact terms</u>

- Definition: a scalar *p*-gon is an integrand whose graph has girth *p,* such that all daughters have girth strictly less than *p*

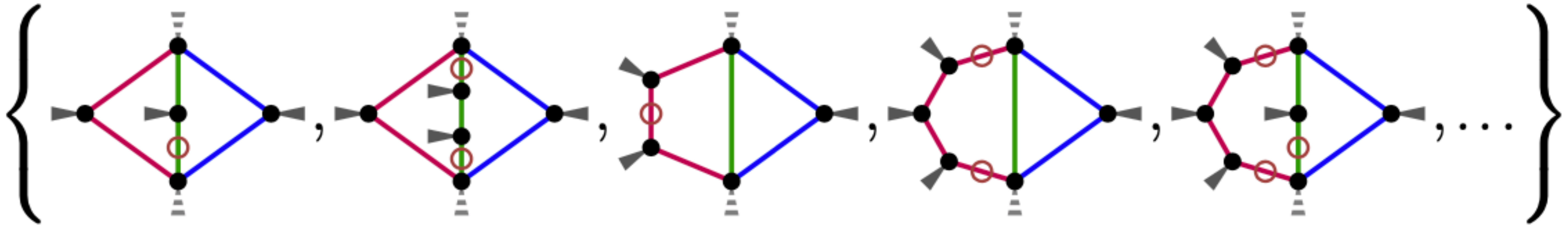
- The set of scalar *p*-gons is easy to enumerate, e.g.,

3-gon power-counting scalars $\mathfrak{S}_3^2 \coloneqq$



Girth: length of the shortest cycle of a graph
Daughters: graphs obtained by single-edge contractions

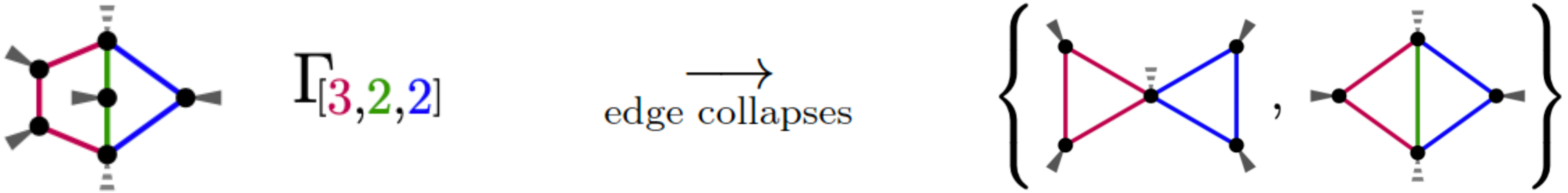
An integrand with *p*-gon PC
'scales like a scalar *p*-gon'



Triangle Power-Counting at Two Loops

Assigning vector spaces of numerators

- The numerator space is defined as the (sum of the) products of translated inverse propagators for all sets of edges that—upon collapsing—lead to a scalar 3-gon



- Three sets of edge collapses:

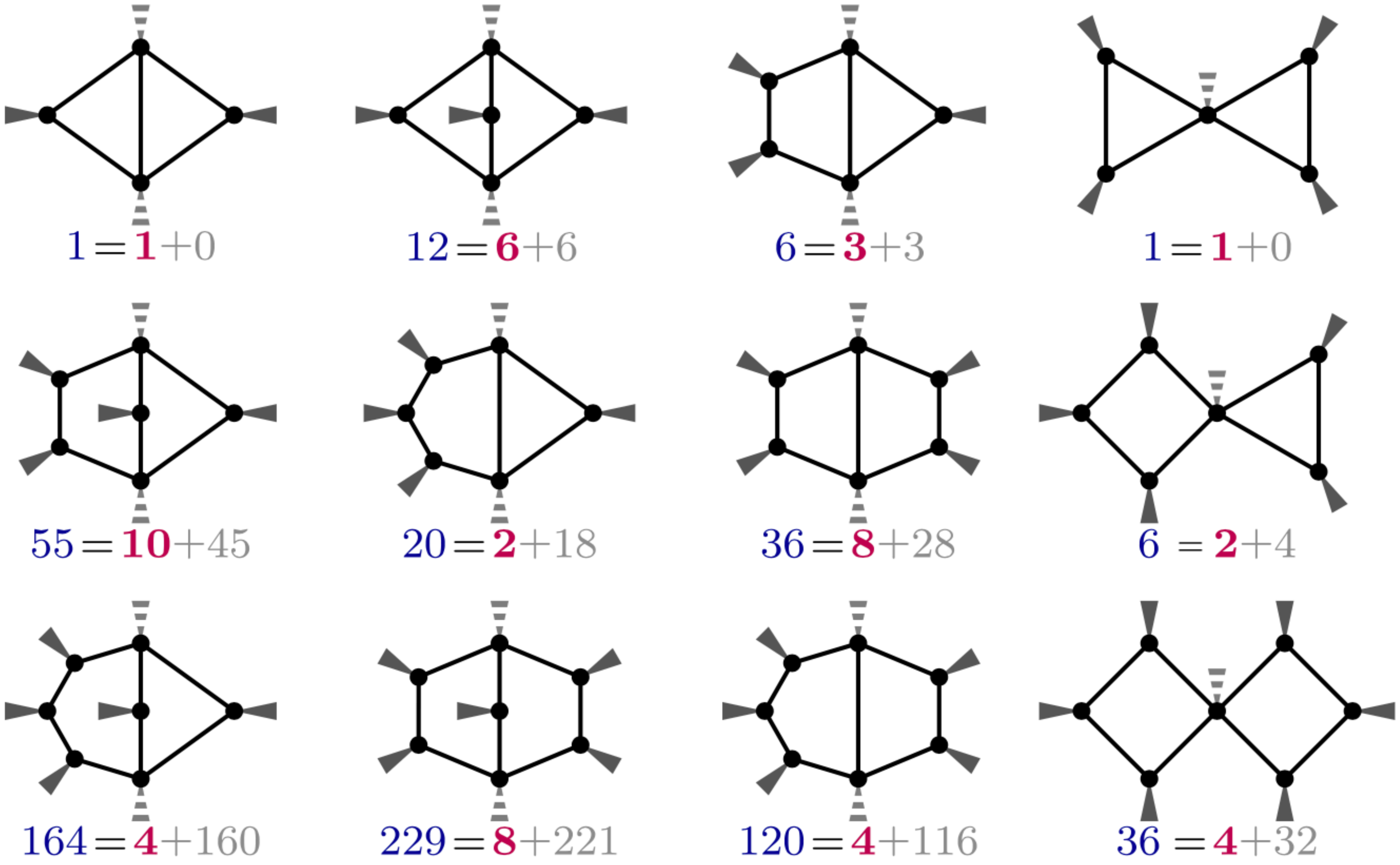
$$\mathfrak{N}_3(\Gamma_{[3,2,2]}) = [\ell_1]^2 \oplus [\ell_1][\ell_2] \oplus [\ell_1][\ell_1 - \ell_2] \qquad \text{rank}[\mathfrak{N}_3(\Gamma_{[3,2,2]})] = 55 = 10 + 45$$

- Convenient separation of this 55-dimensional space into top level and contact term d.o.f.

Two-loop triangle power-counting basis

- Complete list of irreducible integrand topologies, together with the dimension of the numerator spaces

- Red: rank of the numerator space *modulo contact-term degrees of freedom*

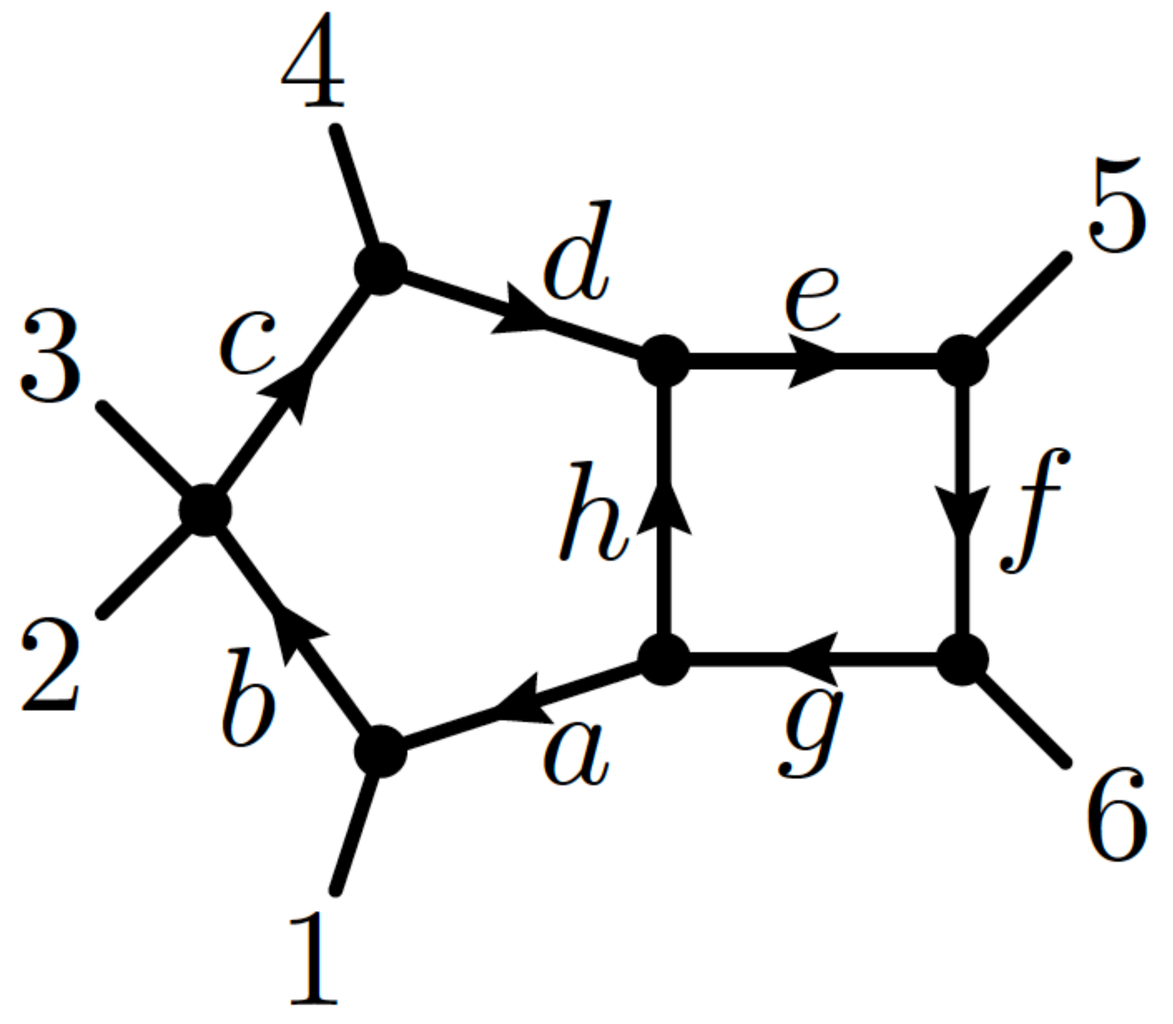


A Good Start: Block Diagonality

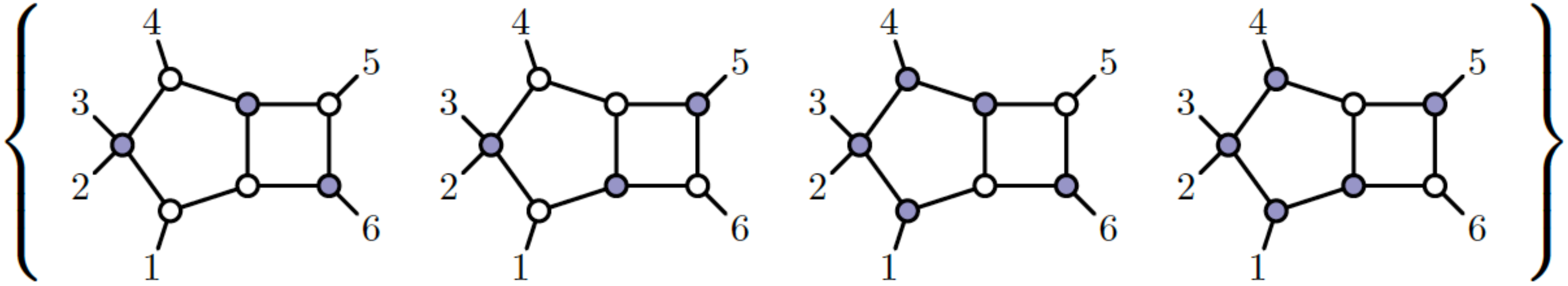
The top level d.o.f. for each topology are normalized on codimension-8 contours

- For each topology, choose a spanning set of cuts with which to match field theory *manifestly*
- Guiding principle: choose as many d.o.f. to match *leading* singularities—including those responsible for infrared divergences—as possible
- The contours on which we normalize/diagonalize can be represented by *on-shell functions:*

$$\mathfrak{N}_3(\Gamma_{[4,1,3]}) = [\ell_1]^2[\ell_2] \oplus [\ell_1][\ell_1 - \ell_2]$$



$120 = 4 + 116$



For 8-propagator integrands,
top-level d.o.f. ↔ solutions to cut equations

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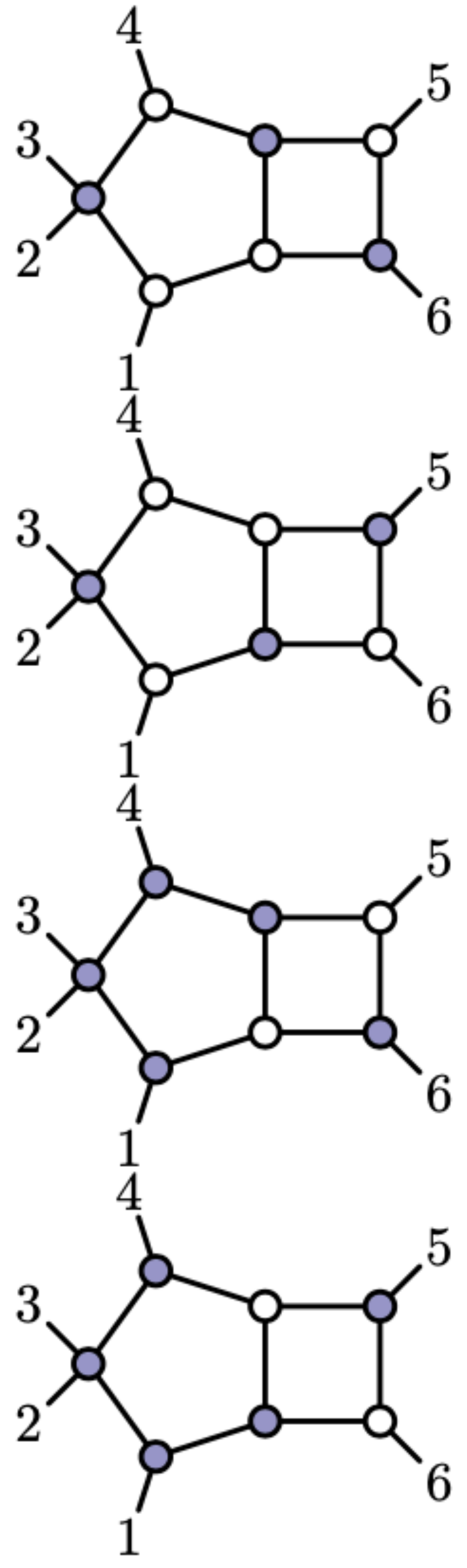
- Constructing block-diagonal numerators:

$$\mathfrak{N}_3(\Gamma_{[4,1,3]}) = [\ell_1]^2[\ell_2] \oplus [\ell_1][\ell_1-\ell_2]$$



$$-\,[\![p_1, b, c, p_4]\!]\,[\![e, f, p_6, d]\!]$$

$$[\![p_1, b, c, p_4]\!][\![d, e, f, p_6]\!]$$

$$-\,[\![b, c, p_4, p_1]\!]\,[\![e, f, p_6, d]\!]$$

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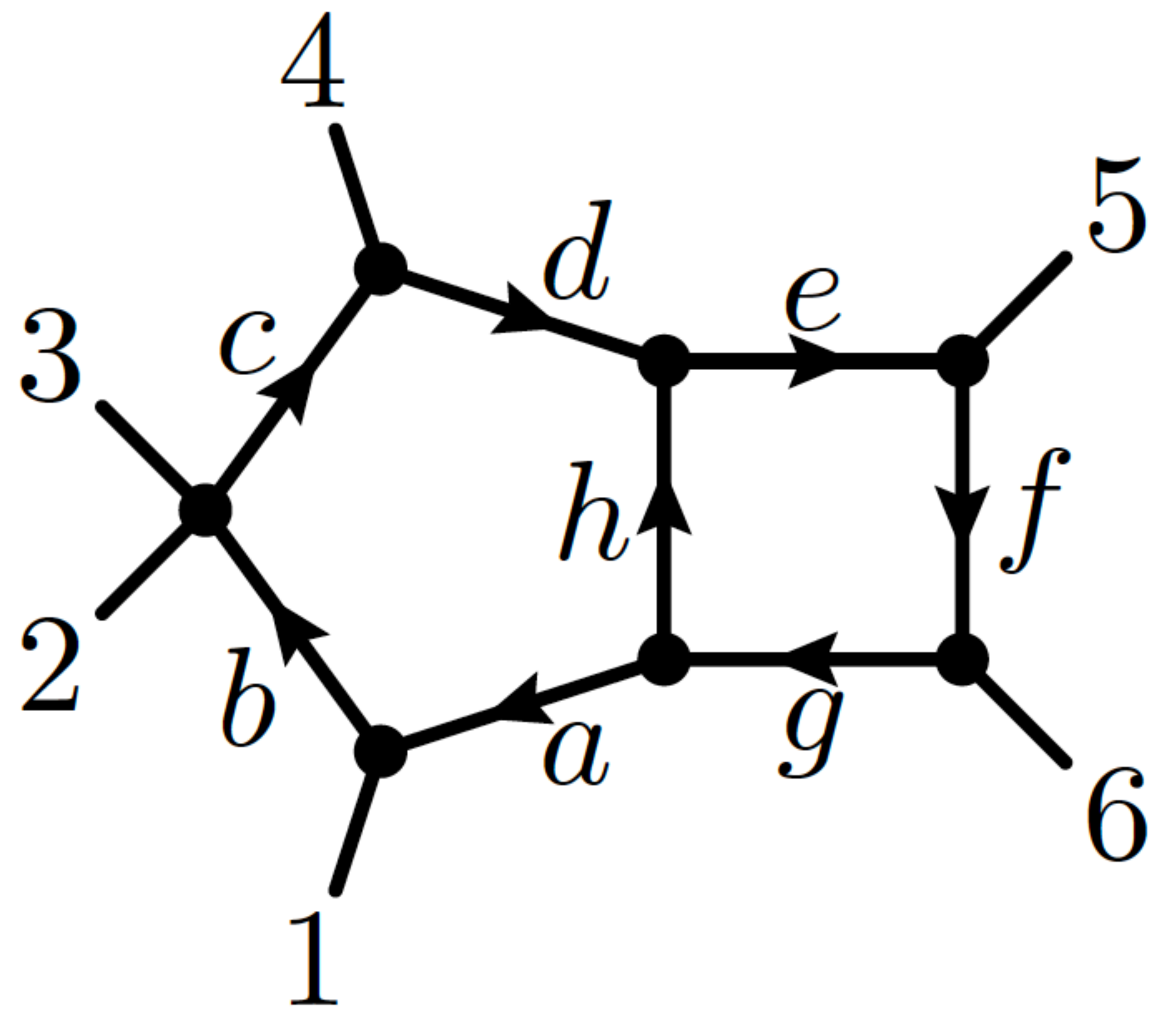

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Natural objects: chiral traces

$$[\![a_1, a_2, b_1, b_2, \cdot, \cdot, c_1, c_2]\!] \coloneqq \Big[(a_1\cdot a_2)^{\alpha}{}_{\beta}(b_1\cdot b_2)^{\beta}{}_{\gamma}\cdots(c_1\cdot c_2)^{\delta}{}_{\alpha}\Big]$$

$$= \mathrm{tr}_+(a_1, a_2, b_1, b_2, \ldots, c_1, c_2)$$

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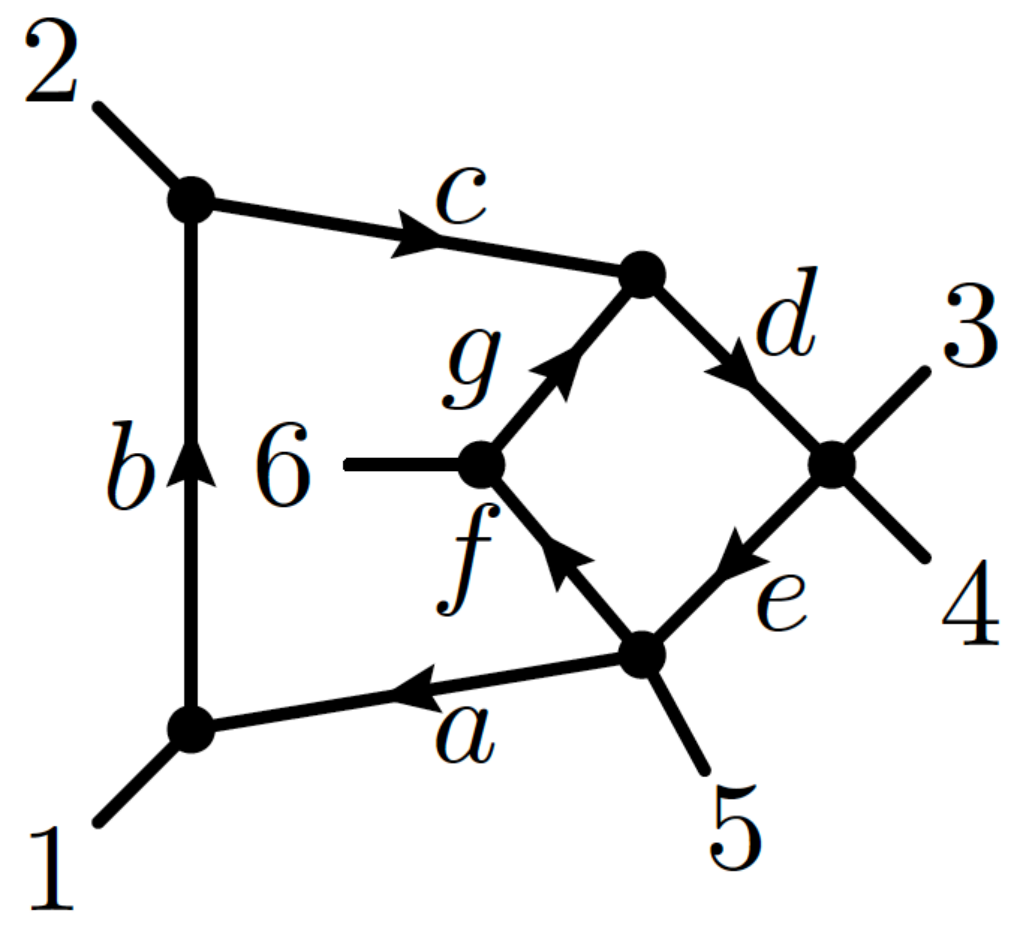

$55 = \mathbf{10} + 45$

- For each topology, choose a spanning set of cuts with which to match field theory *manifestly*

- Choose as many d.o.f. to match *physical* singularities—e.g., those responsible for infrared divergences—as possible

- Fill out the rest of the basis with contours "at infinity"

- For graphs with <8 propagators, need *composite* leading singularities where momenta are *collinear* and/or *soft*

A Good Start: Block Diagonality

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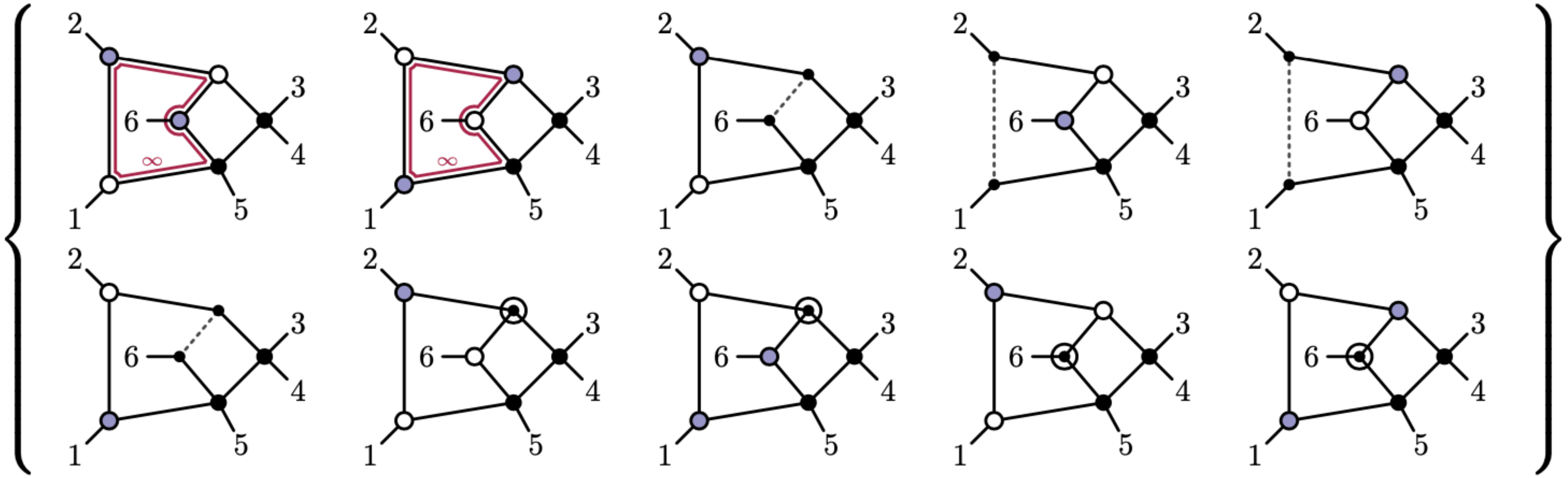
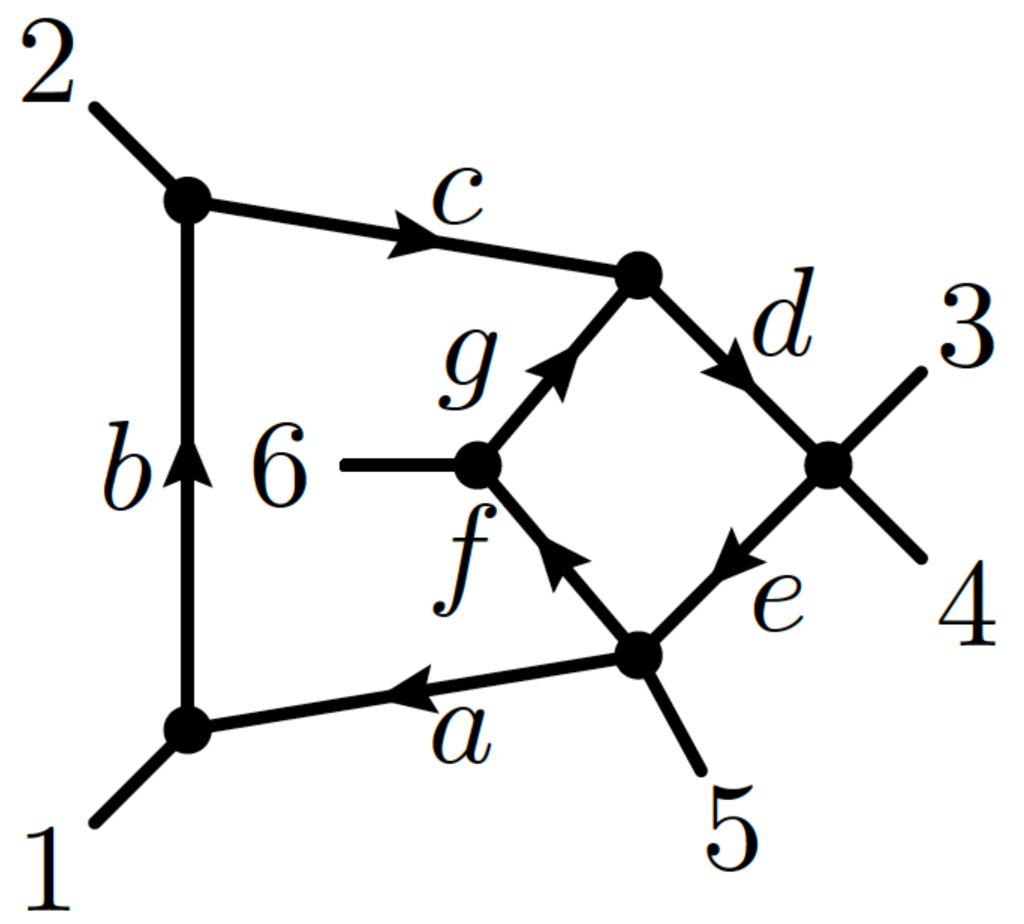


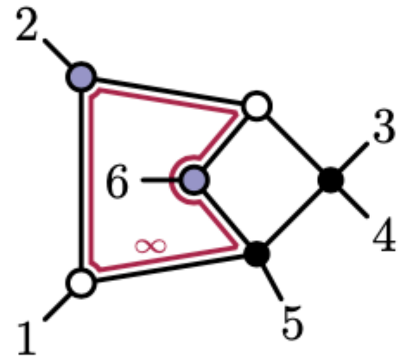


$55 = 10 + 45$

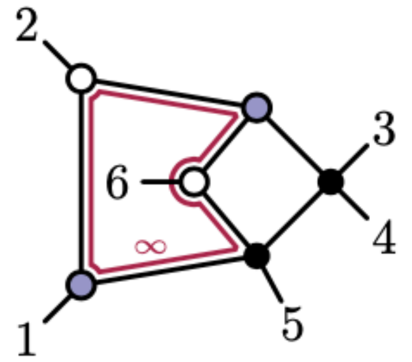
- **Dashed edge:** momentum flow is zero (*soft*)
- **Encircled vertex:** momenta are *collinear*
- **Red loop:** infinite loop momentum, $\ell \to \infty$

Nota bene: maximally SYM amplitudes vanish at infinity
$\Longrightarrow$ basis elements normalized here have coefficient zero!

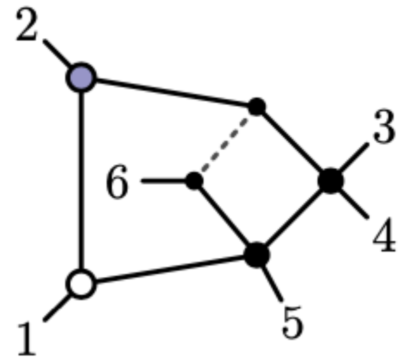
A Good Start: Block Diagonality



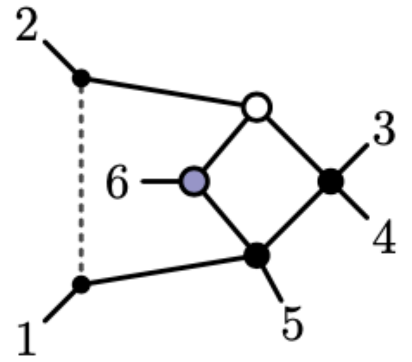
$$\tfrac{1}{2}\llbracket p_1, b, c, g, f, p_{34}\rrbracket$$



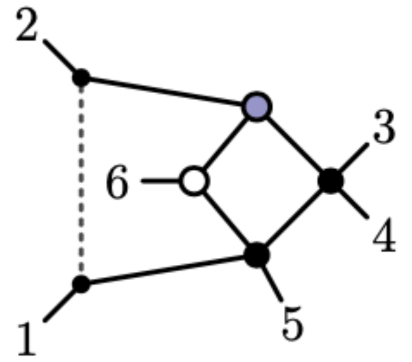
$$\tfrac{1}{2}\llbracket b, c, g, f, p_{34}, p_1\rrbracket$$



$$-\llbracket b, a, e, d, p_6, p_2\rrbracket - \tfrac{1}{2}\llbracket p_1, b, c, g, f, p_{34}\rrbracket$$



$$-s_{12}\,\llbracket d, g, p_6, p_{34}\rrbracket + \tfrac{1}{2}\llbracket p_1, b, c, g, f, p_{34}\rrbracket$$



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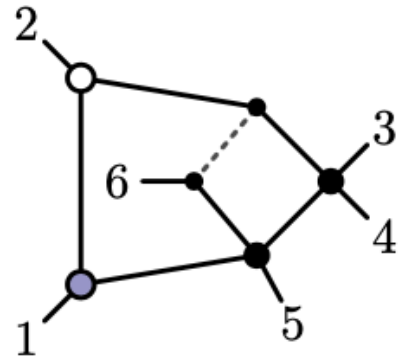
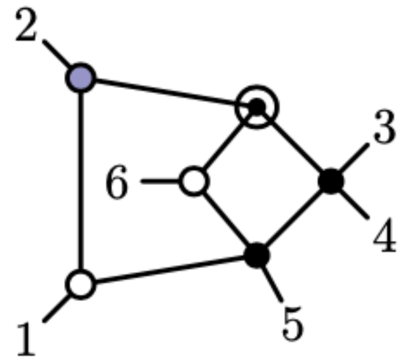
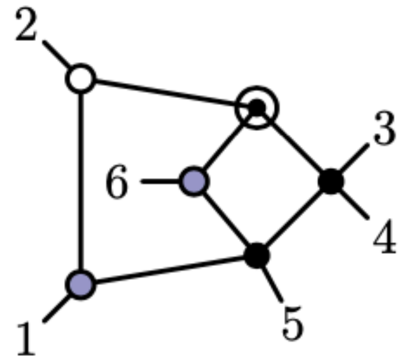


$$\llbracket p_2, b, a, e, d, p_6\rrbracket + \tfrac{1}{2}\llbracket b, c, g, f, p_{34}, p_1\rrbracket$$



$$-\llbracket p_1, b, c, p_{34}, g, f\rrbracket$$



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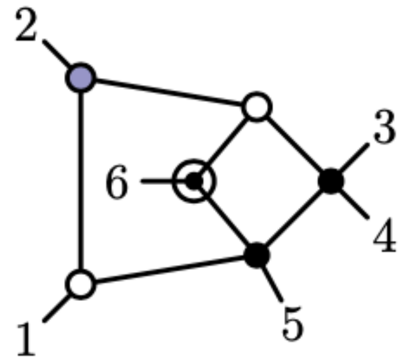

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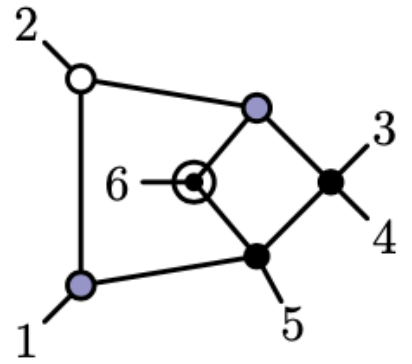
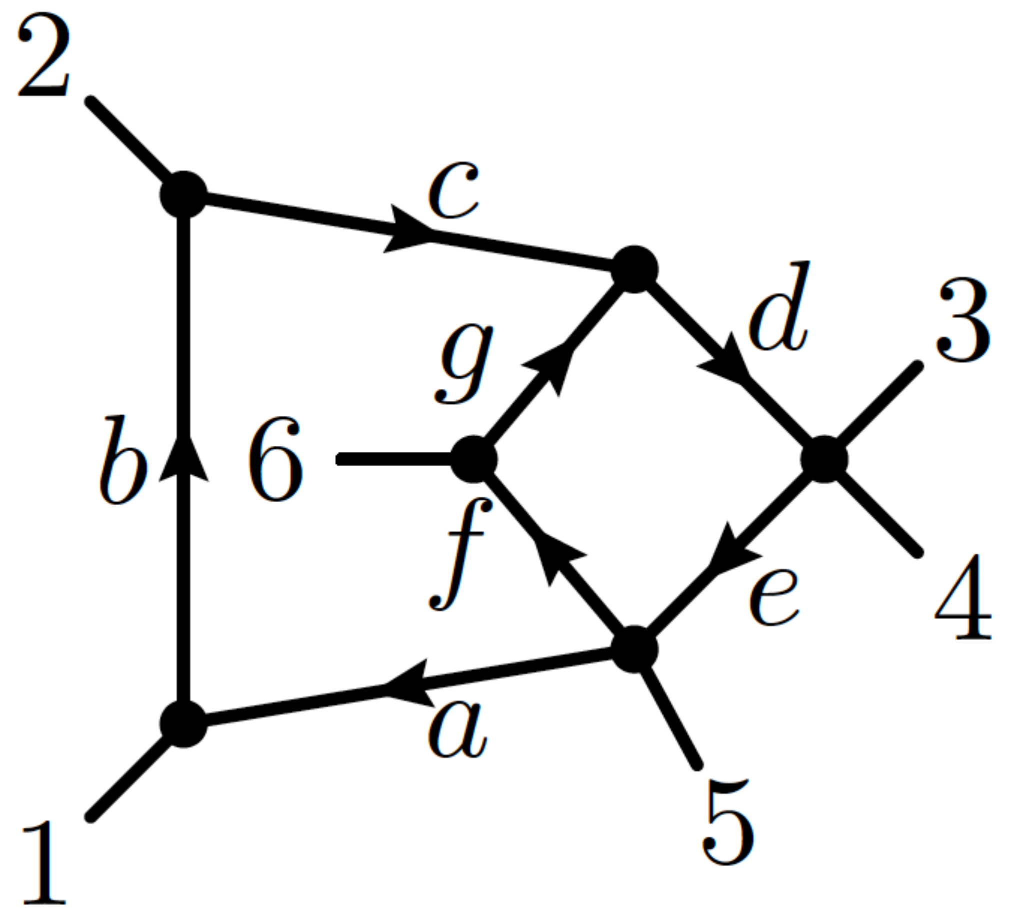


$$\llbracket p_2, b, a, p_6, e, d\rrbracket + \tfrac{1}{2}\llbracket b, c, g, f, p_{34}, p_1\rrbracket$$



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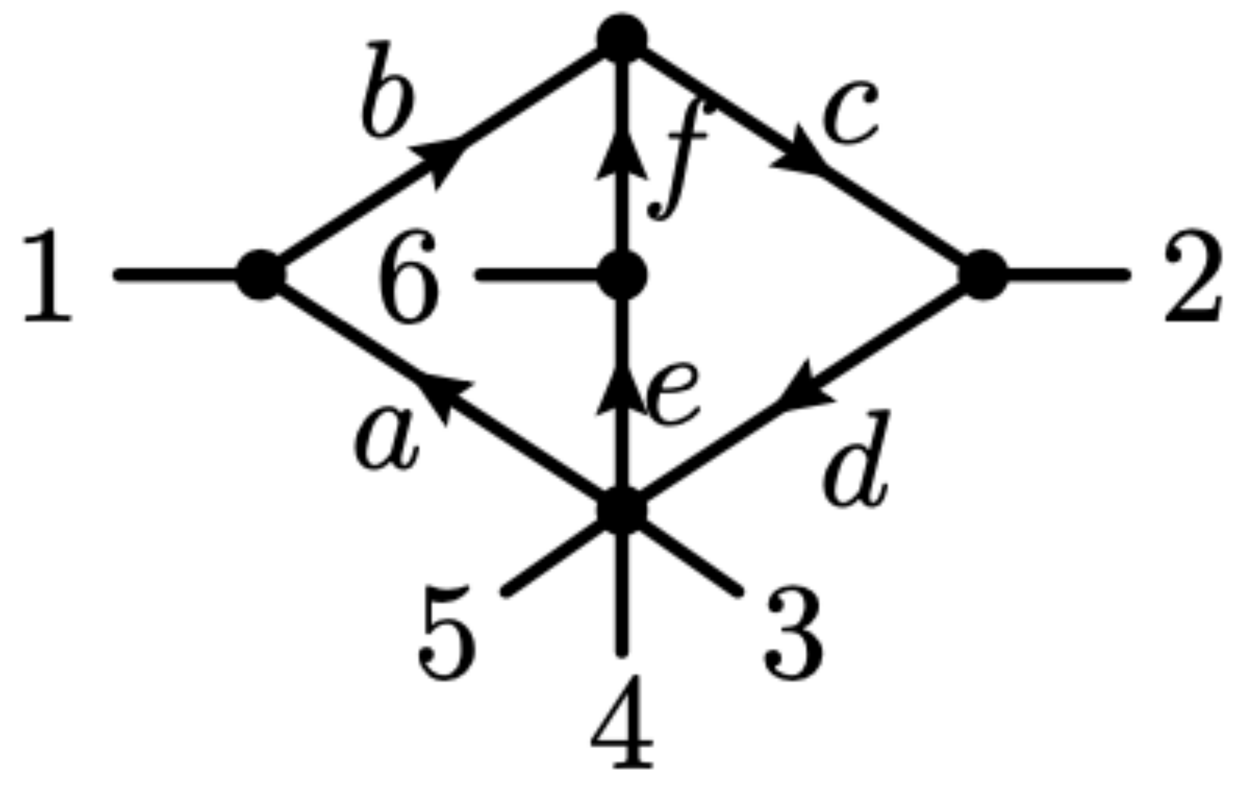
$\Longrightarrow$ basis elements normalized here have coefficient zero!

Dealing with Double Poles

- Part of the basis is inescapably not logarithmic, e.g.,
- Setting $a = \alpha p_1, \quad c = \beta p_2$



The scalar integral has a double pole on any further residue:

$$\mathrm{Res}(\mathcal{I}^{\mathrm{scalar}}) = \frac{d\alpha\, d\beta}{\alpha\beta(\alpha s_{13} + \beta s_{23} + \alpha\beta s_{12})}$$

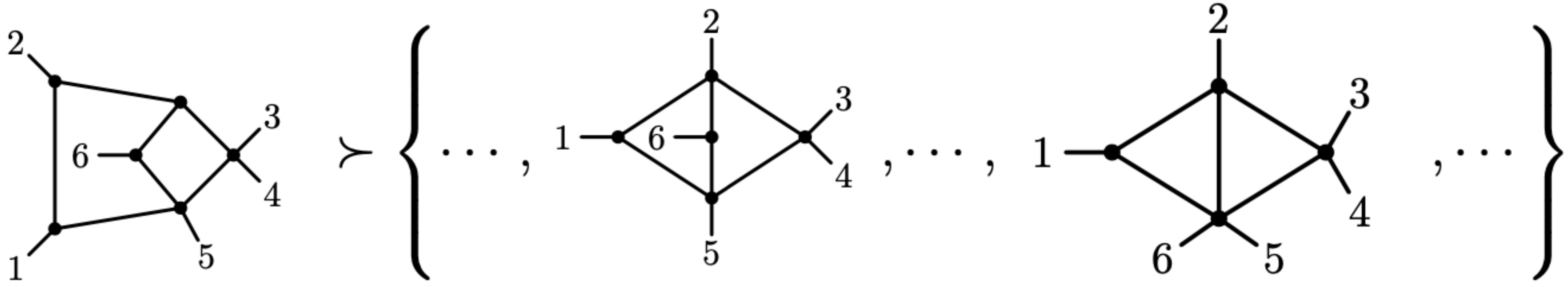
Every other basis element *vanishes* on all such defining points (as do amplitudes)

Global Diagonalization

How do we fix the 'contact term' degrees of freedom?

- Any initially block-diagonal basis of integrands normalized on a spanning set of cuts is automatically *triangular in cuts*

- To diagonalize the entire basis amounts to iterative subtractions e.g.,



- Parent numerators must vanish on *all* defining contours of every daughter

($\sim$ 3200 -dimensional linear system)

Final Result and Discussion

A fully diagonalized basis of integrands with 3-gon power-counting

- Partitioned according to transcendental weight i.e.,

$$\mathfrak{B} = \{d\log \ \text{integrands}\} \sqcup \{\text{integrands with double poles}\}$$

- Cleanly separated into IR finite and divergent integrands
    - By construction, only those integrands *normalized* on collinear and soft-collinear contours (are expected to) generate IR divergences upon integration
- To represent amplitudes in this basis requires only the list of non-vanishing leading singularities in the spanning set:

$$\mathcal{A}^{\text{2-loops}}_{\text{MHV,non-planar}} = \sum_{\substack{\text{inequivalent MHV} \\ \text{leading singularities } \mathfrak{f}}} \mathfrak{f}^{i}_{\text{MHV}} \mathfrak{b}^{i}$$

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Outlook

- Our basis ideally suited to direct integration
- All-multiplicity generalization (requires *elliptic* LS and beyond)
- Upgrade basis with μ-terms for dimensional regularization
- Manifest IR divergence—term-wise finite representation of the ratio function?

Thanks for your time!