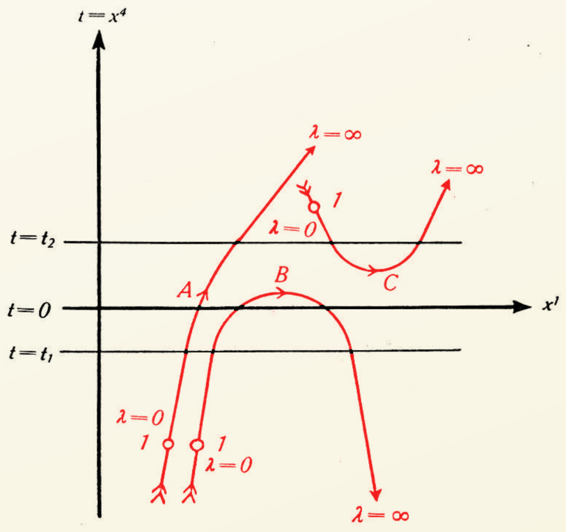
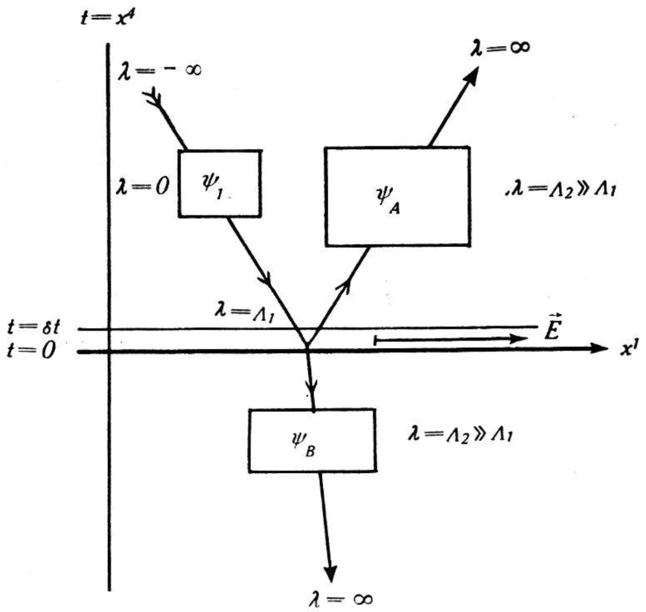


CROSSING

Sebastian Mizera (IAS)

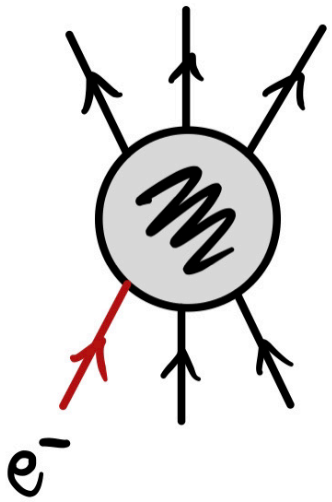
Amplitudes 1941: one of the very first Feynman diagrams!



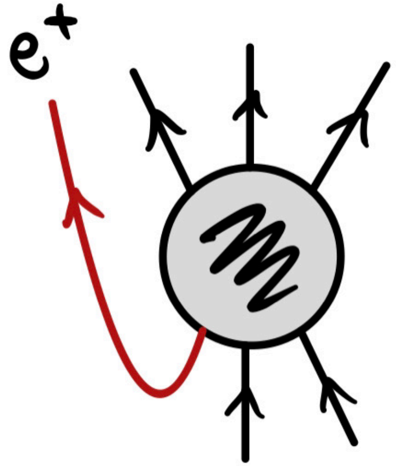
[Stueckelberg, Helv. Phys. Acta **14**, 588 (1941)]

In the same paper: particles indistinguishable from antiparticles with the opposite energy-momentum

At the level of observables in QFT: *crossing symmetry* for the full S-matrix



$$\overset{?}{\simeq}$$



$p_i^2 = M_i^2$

(on-shell)

$\sum_i p_i^\mu = 0$

(mom. cons.)

Are the two scattering processes related by analytic continuation?

(not to be confused with permutation invariance)

[Gell-Mann, Goldberger, Thirring ‘54]

Unfinished business in the S-matrix theory:

What *analytic* properties of the S-matrix guarantee that it corresponds to a *causal* scattering process in space-time?

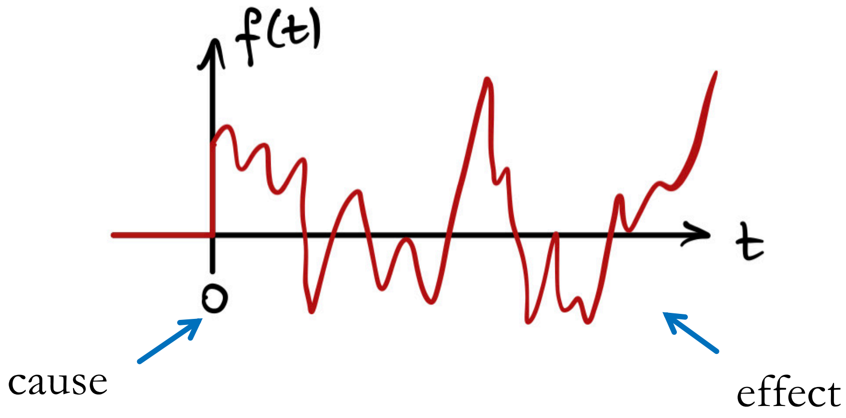
Previous attempts at proving crossing symmetry failed because within the LSZ formalism the notions of *"S-matrix"* and *"space-time causality"* are intrinsically incompatible

(a sign we need a better formulation; cf. work on the flat-space limit of AdS/CFT)

[long literature; talk by Caron-Huot]

Toy model:

Fourier transform a causal signal to the energy space



$$\tilde{f}(E) = \int_0^\infty \mathrm{d}t\, e^{iEt} f(t)$$

causality

Exists only when $\mathrm{Im}\, E > 0$: exponential suppression as $t \to \infty$

Incompatible with dispersion relations ("on-shell conditions")

$$E^2 = \omega^2 \geqslant 0$$

because

$$\operatorname{Im} E^2 = 2\underbrace{(\operatorname{Im} E)}_{>0}(\operatorname{Re} E) = 0$$

$$\operatorname{Re} E^2 = \underbrace{(\operatorname{Re} E)}_{=0}{}^2 - \underbrace{(\operatorname{Im} E)}_{>0}{}^2 < 0$$

contradiction

We are *forced* to define the physical $\tilde{f}(E)$ by analytic continuation

$$\tilde{f}(E) = \underbrace{\lim_{E^2 \to \omega^2}}_{\text{in this case a simple limit}} \int_0^\infty \mathrm{d}t\, e^{iEt} f(t)$$

In QFT we use microcausality:

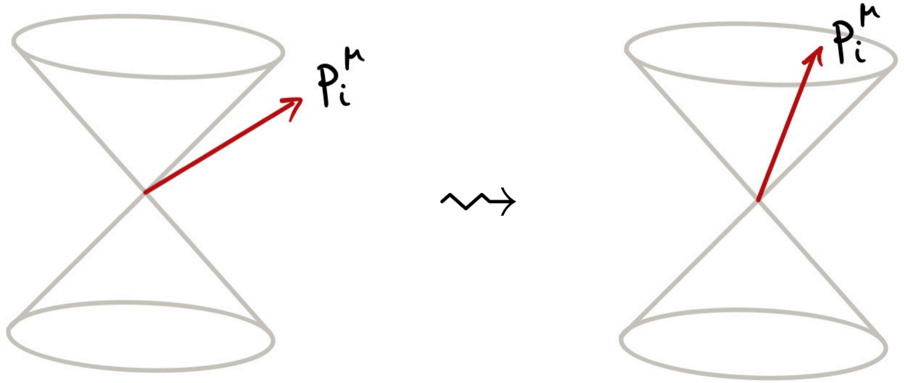
$$[\mathcal{O}_1(x_1), \mathcal{O}_2(x_2)] = 0 \qquad \text{when} \qquad (x_1 - x_2)^2 < 0$$

together with locality and unitarity

The problem becomes extremely severe:

$$S = \underbrace{\cancel{\lim_{p_i^2 \to M_i^2}}}_{\text{not a limit!}} \underbrace{\int \prod_i \mathrm{d}^4 x_i \, e^{i p_i \cdot x_i} \langle \mathrm{out} | \cdots \mathcal{O}_j(x_j) \cdots | \mathrm{in} \rangle}_{\text{doesn't exist on-shell (for } p_i^2 \geqslant 0)}$$

All the physics has to be understood by a supposed analytic continuation across the lightcone:

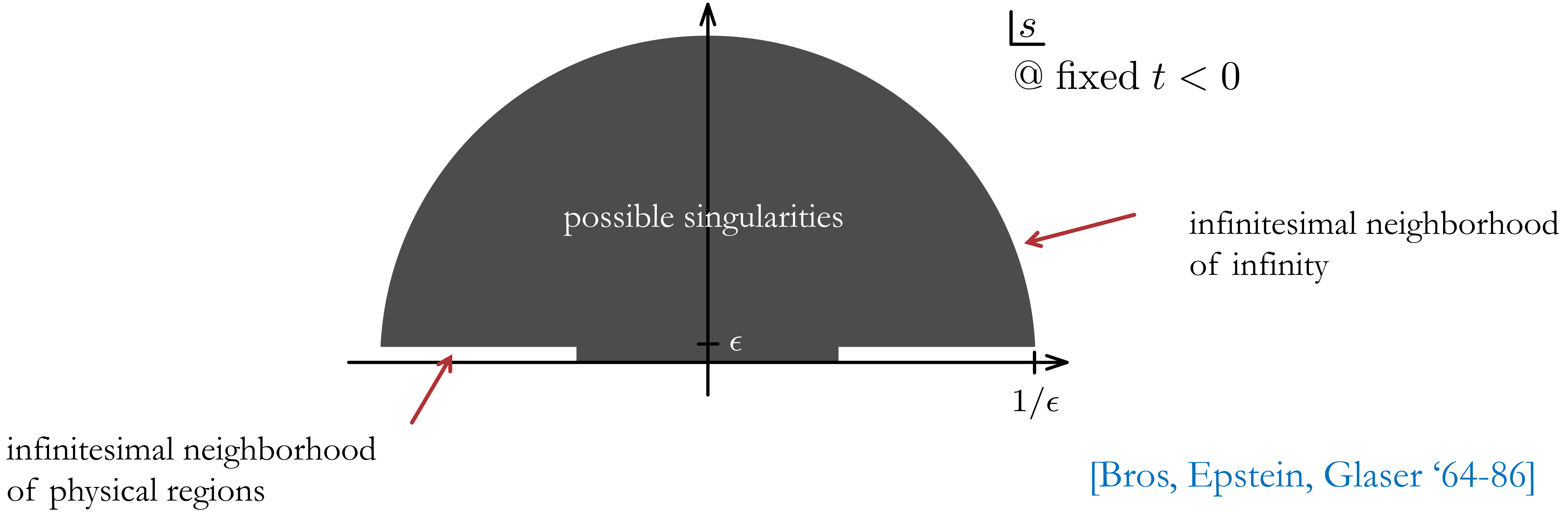


$p_i^2 < 0$ or $\operatorname{Im} p_i^2 \neq 0$ $\qquad$ $p_i^2 \geqslant 0$

Does it always exist? Is it unique? Why/why not?

Progress in the last century:

Exclude massless particles, higher-point processes, crossing of a single particle



[Bros, Epstein, Glaser '64-86]

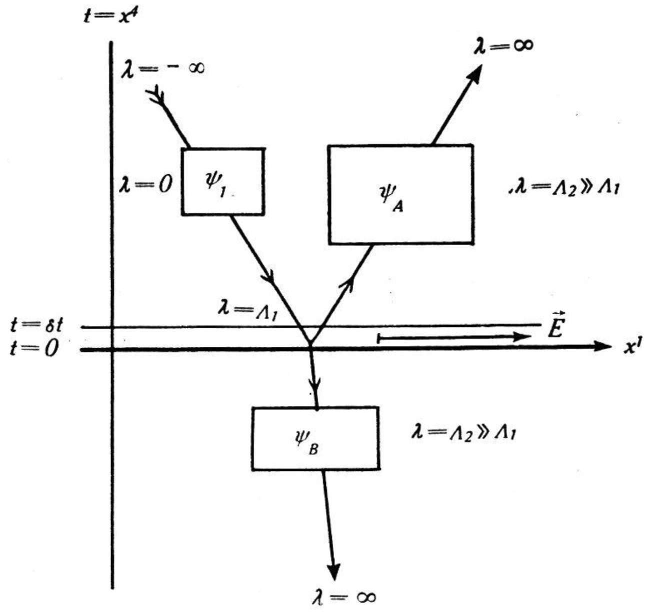
("Euclidean" region doesn't exist for a generic S-matrix element)

Leaves many unanswered questions:

- Can we identify what kind of singularities are absent? Why?
- Is the connection to asymptotic kinematics accidental? What about finite energy?
- Can we separate causality from locality assumptions?
- How does it generalize to higher multiplicity, massless particles, etc.?
- Can a single particle be exchanged for a single antiparticle?

Clearly, a new strategy is needed…

We reconsider this problem in *perturbation theory*
(in the worldline formalism)



- Work to all loop orders, any multiplicity, spins, masses, …

- Can separate analyticity questions from UV/IR divergences

- Any theory in $\mathrm{D} > 2$ satisfying CPT:
  Feynman *rules* are crossing-invariant

Simplification coming with perturbation theory:

singularities $\Leftrightarrow$ wordline saddle points

algebraic problem

Contribution from a single worldline Feynman diagram:

Schwinger parameters $\alpha_{e=1,2,\ldots,\mathrm{E}}$

polynomial numerator

$$\int \frac{\mathrm{d}^{\mathrm{E}}\alpha}{\mathcal{U}^{\mathrm{D}/2}}\,\mathrm{N}\,e^{i\mathcal{V}/\hbar}$$

worldline action

Symanzik polynomials

The action is simply: $\mathcal{V} = \mathcal{F}/\mathcal{U}$,

Feynman $i\epsilon$/convergence of the integral: $\mathrm{Im}\,\mathcal{V} > 0$

Singularities in the classical limit, $\hbar \to 0$, at the saddle points:

$$\underbrace{\alpha_e}_{\text{boundary saddles}} \underbrace{\frac{\partial \mathcal{V}}{\partial \alpha_e}}_{\text{bulk saddles}} = 0, \qquad e = 1, 2, \ldots, \mathrm{E}$$

Equivalent to *Landau equations*, imposing

internal propagators on-shell $\nearrow$ $q_e^2 - m_e^2 = 0$

[Bjorken, Landau, Nakanishi ‘59]

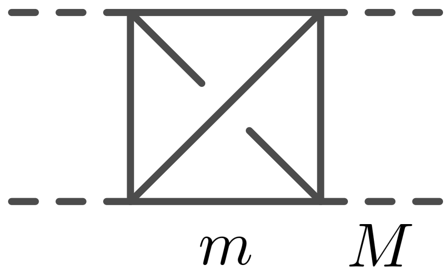
These are known as *anomalous* (or normal) *thresholds*:
intrinsically *Lorentzian* phenomena,
at least partially encoding causality in perturbation theory

How complicated can scattering amplitudes be?

$$S = \sum(\cdots) + \sum \mathrm{Li}(\cdots) + \sum \mathrm{ELi}(\cdots) + \ldots$$

Can the arguments be always written in closed form?
Conjecture: No

Simple example at three loops:



All particles massless, $m = M = 0$:

$$stu = 0$$

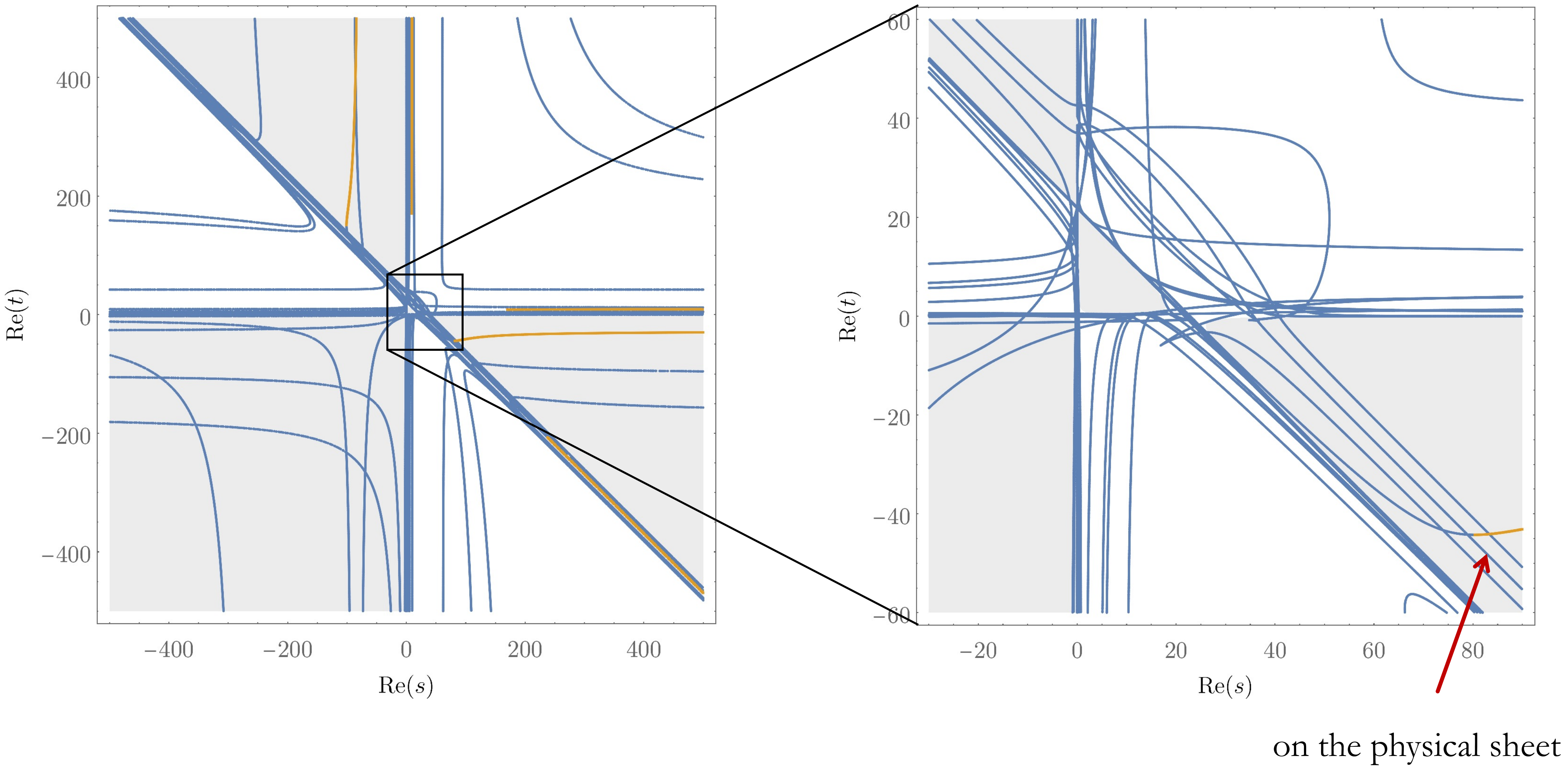
“alphabet” of singularities

More generally, for $m, M > 0$:

degree-128 reducible curve

$$
\begin{aligned}
s(s-4m^2)(s-16m^2)t(t-4m^2)(t-16m^2)u(u-4m^2)(u-16m^2)& \\
\left(40m^2+4M^2-11s\right)\left(9m^4-10m^2M^2+m^2s+M^4\right)& \\
\left(40m^2+4M^2-11t\right)\left(9m^4-10m^2M^2+m^2t+M^4\right)& \\
\left(40m^2+4M^2-11u\right)\left(9m^4-10m^2M^2+m^2u+M^4\right)& \\
\left(64m^{12}M^6+216m^{10}M^4st-19m^4M^8st^2+\ 29\text{ terms}\right)& \\
\left(1728m^{10}M^6+1080m^{10}M^2st+576m^8M^2st^2+\ 80\text{ terms}\right)& \\
\left(1728m^{10}M^6+1080m^{10}M^2st+576m^8M^2ts^2+\ 80\text{ terms}\right)& \\
\left(1728m^{10}M^6+1080m^{10}M^2su+576m^8M^2su^2+\ 80\text{ terms}\right)& \\
\left(32142336m^{12}M^2s^2t^3+24592384m^{10}M^6s^3t+475136m^4M^{20}+\ 344\text{ terms}\right)&=0
\end{aligned}
$$

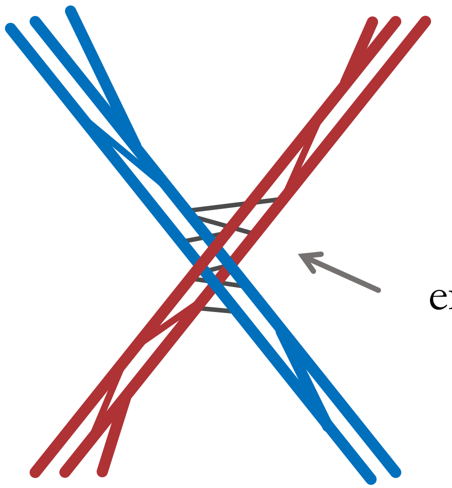
Obtained with the package `Landau.jl` [arXiv:21so.oooon with Simon Telen]



400
200
0
−200
−400
Re(t)
Re(s)
60
40
20
−20
−40
−60
80
on the physical sheet

Luckily, for the question of crossing symmetry
we only need to know where singularities *cannot* appear

Only a certain class of anomalous thresholds can pose a potential obstruction to crossing symmetry!



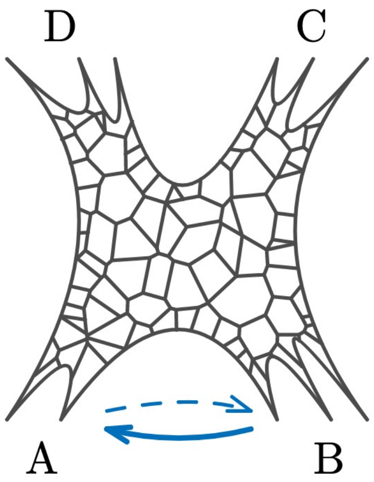
exchanging massless states

aligned along two beams
(at finite energy)

[Witten]

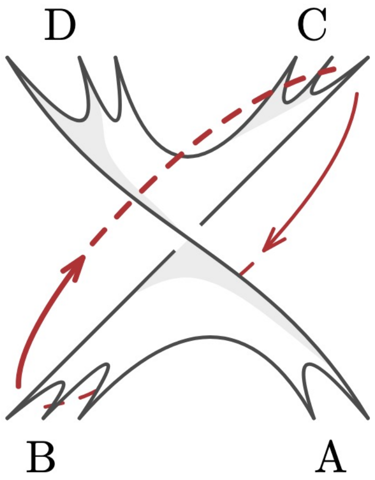
To understand why, we need to figure out how to avoid

Real singularities



(within a physical region)

Complex singularities



(across different physical regions)

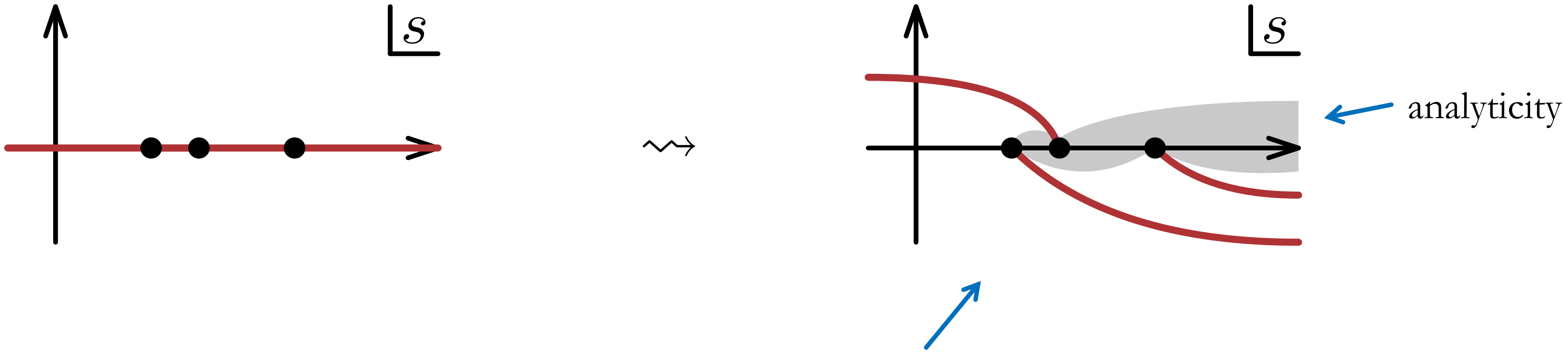
$\operatorname{Im}\mathcal{V} > 0$ imposed by giving worldlines infinitesimal phases:

$$\alpha_e \to \alpha_e \exp\left(i\varepsilon \frac{\partial \mathcal{V}}{\partial \alpha_e}\right)$$

The action acquires a small non-negative imaginary part

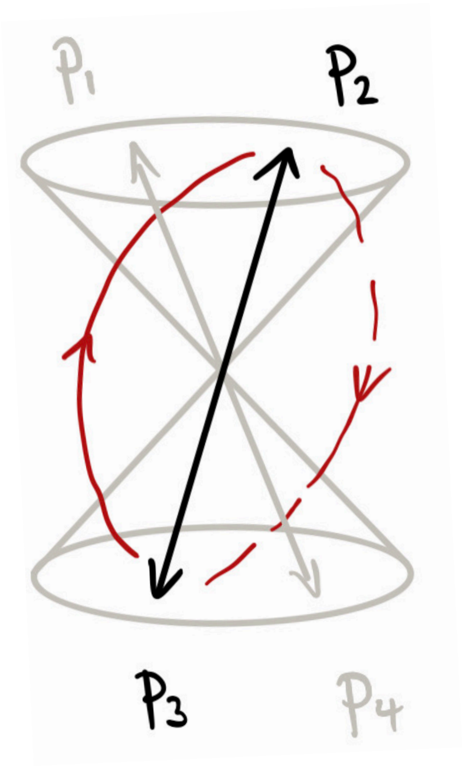
$$\mathcal{V} \to \mathcal{V} + i\varepsilon \underbrace{\sum_e \alpha_e \left(\frac{\partial \mathcal{V}}{\partial \alpha_e}\right)^2}_{>0 \quad \text{except at saddle points}} + \dots$$

Resolves branch cuts in the kinematic space



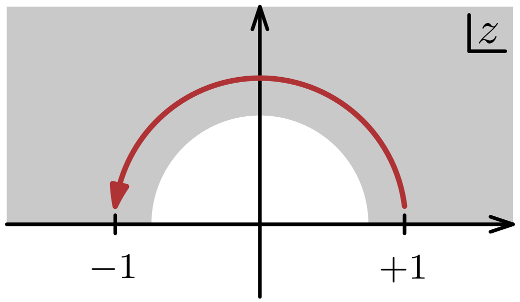
doesn’t imply anything about analyticity away from the physical kinematics

Analytic continuation of *external* energies within the complexified lightcone (say at 4-pt):



lightcone coordinates

$$
\begin{aligned}
p_1^\mu &= (\;\; p_1^+,\;\; p_1^-,\; \vec{p}_1) \\
p_2^\mu &= (\; z p_2^+,\; \tfrac{1}{z} p_2^-,\; \vec{p}_2) \\
p_3^\mu &= (-z p_2^+, -\tfrac{1}{z} p_2^-,\; \vec{p}_3) \\
p_4^\mu &= (\,-\,p_1^+,\; -\,p_1^-,\; \vec{p}_4)
\end{aligned}
$$

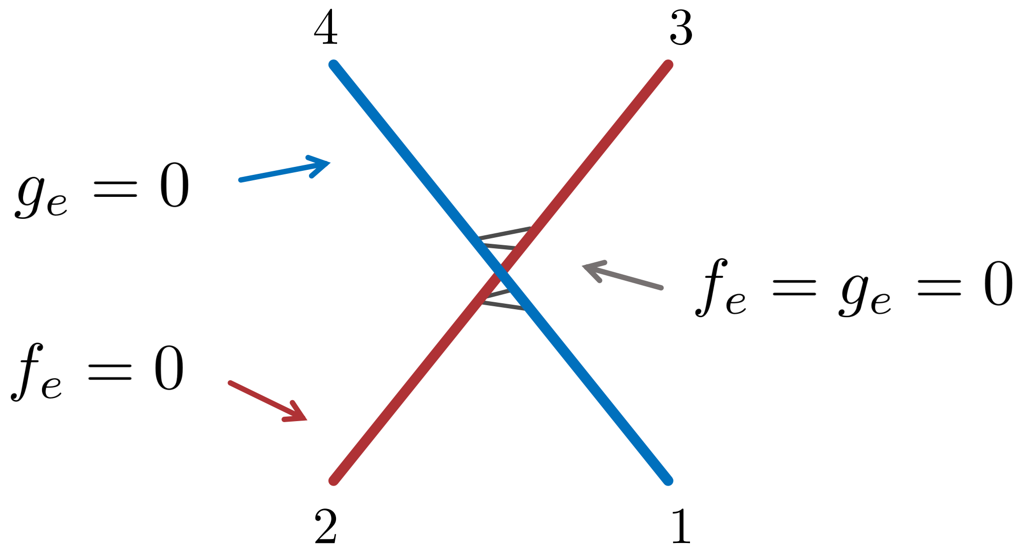


(preserves on-shell conditions and mom. cons.)

Every *internal* momentum q_e^μ can be decomposed as

$$q_e^\pm = p_1^\pm f_e + z^{\pm 1} p_2^\pm g_e$$

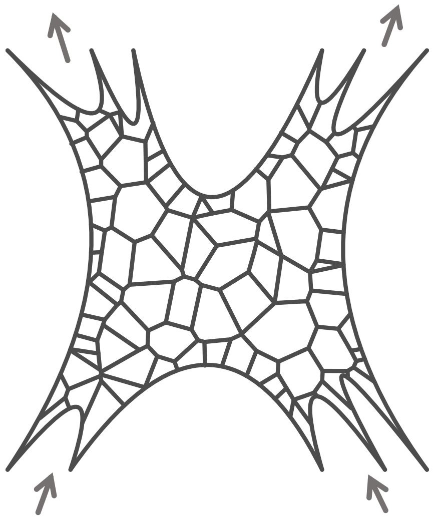
Putting them on-shell implies $0 = \mathrm{Im}(q_e^2 - m_e^2) \propto f_e g_e$



Unknown if such anomalous thresholds exist in an arbitrary theory

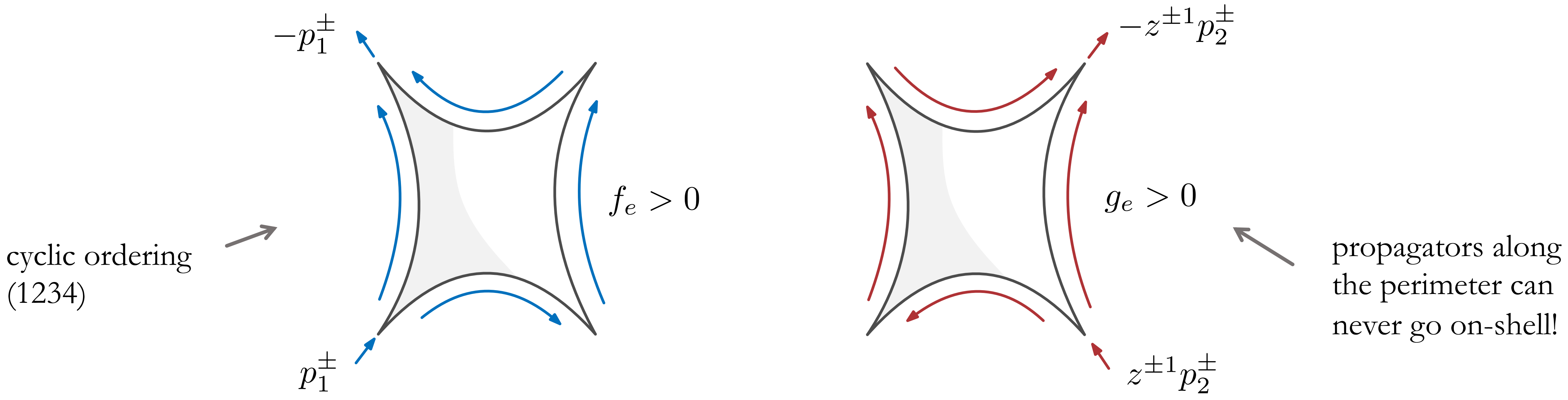
Specialize to planar amplitudes, e.g., large-N QCD

all outgoing consecutive



all incoming consecutive

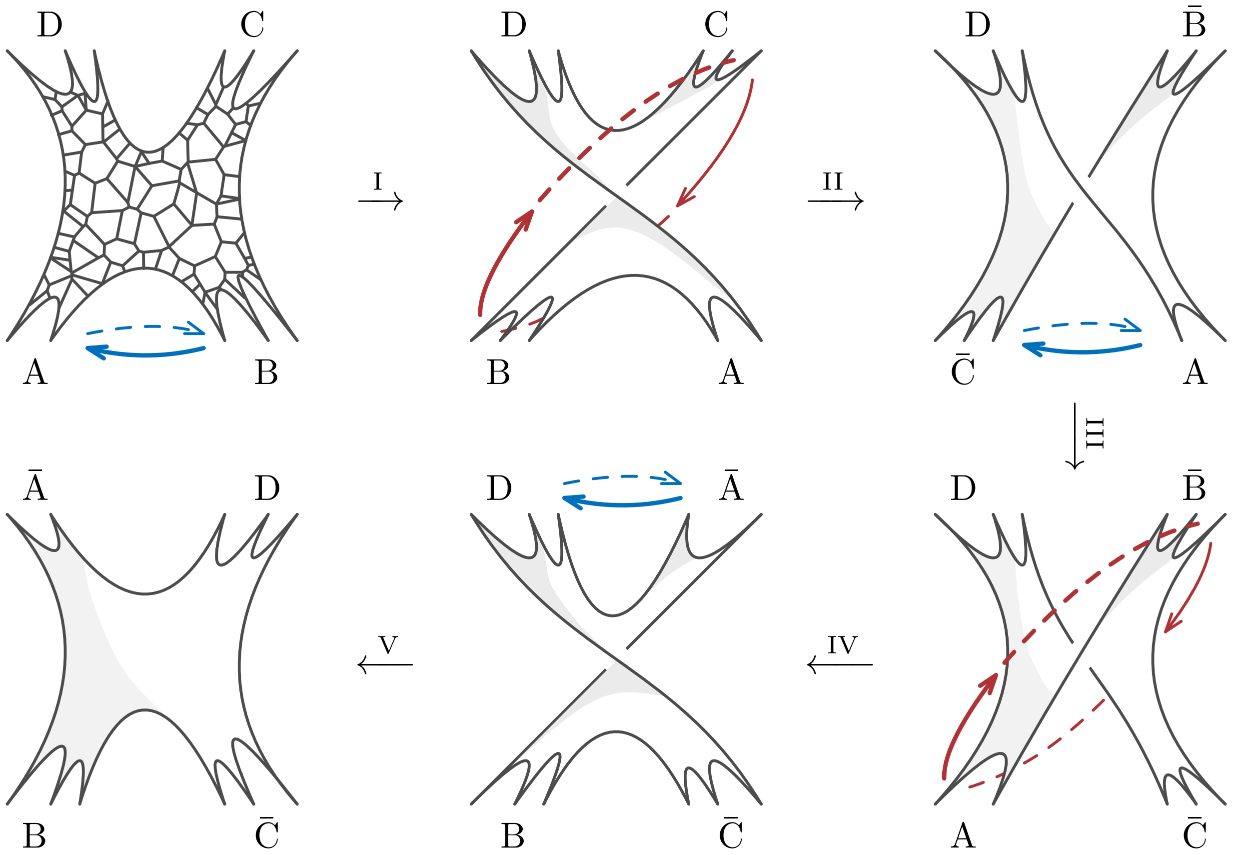
The dangerous singularities never appear for planar amplitudes:



$$\mathrm{Im}(q_e^2 - m_e^2) \propto f_e g_e \neq 0 \quad \Rightarrow$$

analyticity when rotating from the past to the future lightcones and vice versa

Sequence of rotating the energies between crossing channels:



[details in hep-th/2104.12776]

This gives us analytic continuation between physical channels:

$$S_{\mathrm{AB}\to\mathrm{CD}} \;=\; S_{\mathrm{B}\bar{\mathrm{C}}\to\mathrm{D}\bar{\mathrm{A}}}$$

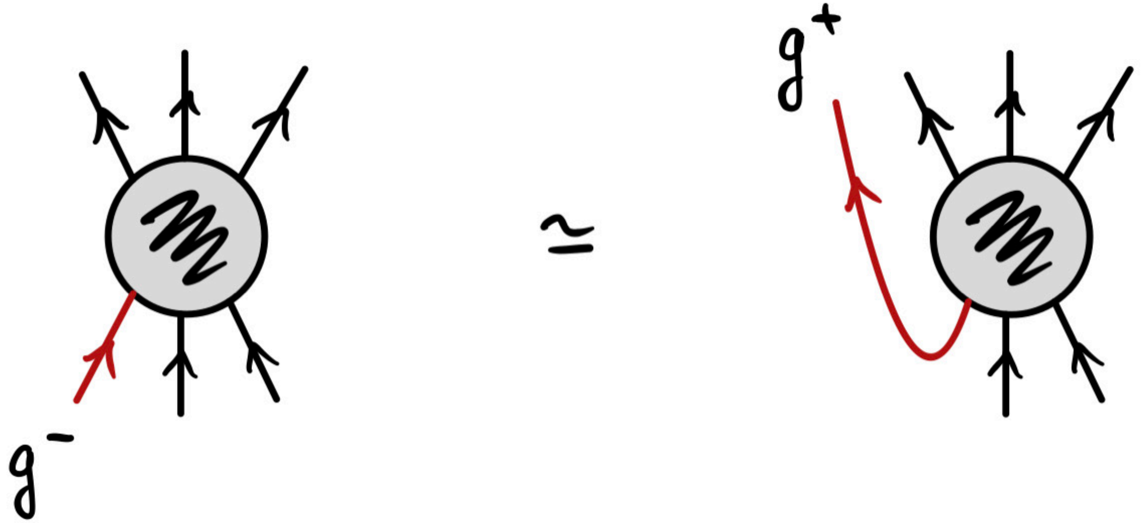
Crossing symmetry follows from:

$$S_{\mathrm{IJ}\to\mathrm{KL}n} \;=\; S_{\mathrm{J}\bar{\mathrm{K}}\bar{\mathrm{L}}\to n\bar{\mathrm{I}}} \;=\; S_{\bar{\mathrm{L}}\bar{n}\to\bar{\mathrm{I}}\bar{\mathrm{J}}\mathrm{K}} \;=\; S_{\bar{n}\mathrm{IJ}\to\mathrm{KL}}$$

particle n antiparticle $\bar{n}$

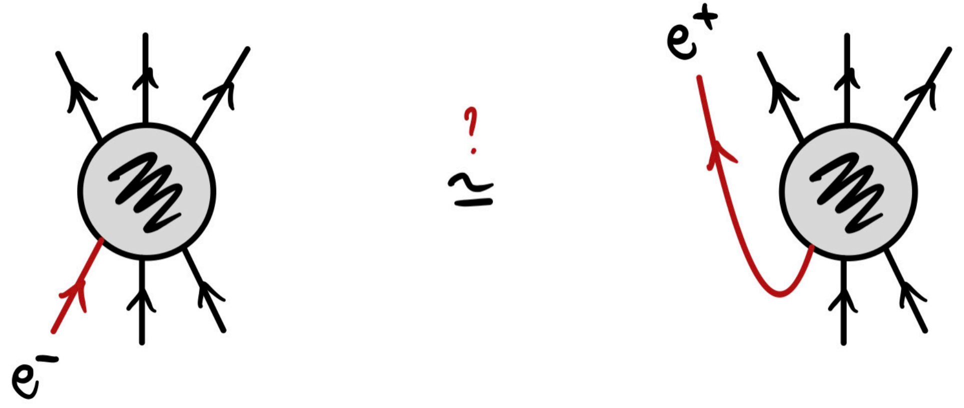
[details in hep-th/2104.12776]

First realization of crossing symmetry, for planar amplitudes at every order in perturbation theory with any masses, spins, multiplicity, …



(for $n \geqslant 5$, processes with consecutive in/out states in CPT-invariant theories)

Summary



- Difficulties with non-perturbative approaches
    - Singularities as worldline saddle points
- Crossing symmetry for planar amplitudes

Thank you!