Negative Amplituhedron geometry and amplitudes at strong coupling

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Motivation

- Use Amplituhedron picture for planar N=4 SYM amplitudes, perform all-loop order calculation, go to strong coupling, compare with integrability and eventually study the “strong coupling geometry”

Outline

- Amplituhedron: geometric picture for scattering amplitudes in planar N=4 SYM theory
- Define "negative geometries" which naturally give the logarithm of the amplitude to all loop orders
- Define IR finite object by freezing one of the loops and integrated over the others, relation to Wilson loop picture
- Approximation: special class of negative geometries, evaluate to all loops, resummation, strong coupling limit

Amplituhedron

(Arkani-Hamed, JT 2013)

(Arkani-Hamed, Thomas, JT 2017)

Four point amplitudes

- In this talk we are interested in 4pt amplitudes in planar N=4 SYM theory

- Amplituhedron picture for the all-loop integrand

$$M_4 = \int \Omega_4$$

dlog form on the Amplituhedron geometry

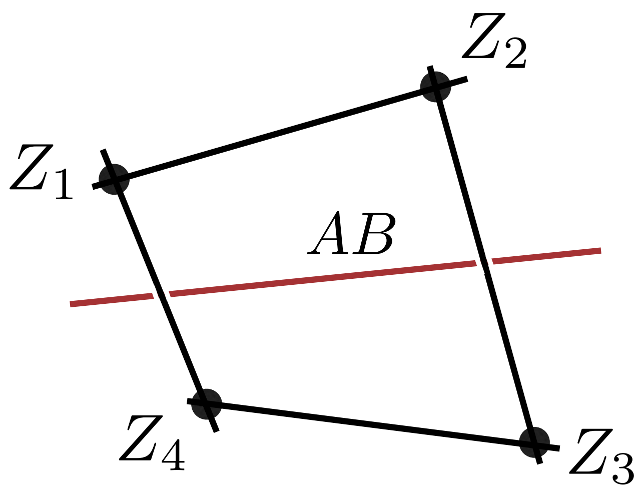
- Amplitude M_4 is divergent and needs to be regulated

- The integrand form Ω_4 is rational and finite

Amplituhedron geometry

- Convenient kinematical variables: momentum twistors

(Hodges, 2009)



external Z_1, Z_2, Z_3, Z_4

points in $\mathbb{P}^3$

loops $(AB)_j$ $\longrightarrow$ lines

- Fixed convex external data: $\langle 1234 \rangle = \epsilon_{abcd} Z_1^a Z_2^b Z_3^c Z_4^d > 0$

- Amplituhedron: configuration space of all lines $(AB)_j$

One-loop Amplituhedron

- One-loop Amplituhedron: configuration of all lines (AB) which satisfy following conditions

$$\langle AB12\rangle, \langle AB23\rangle, \langle AB34\rangle, \langle AB14\rangle > 0, \qquad \langle AB13\rangle, \langle AB24\rangle < 0$$

where $\langle AB12\rangle = \epsilon_{abcd} Z_A^a Z_B^b Z_1^c Z_2^d$

- Convenient parametrization

$$Z_A = Z_1 + xZ_2 + yZ_4 \qquad Z_B = Z_3 - zZ_2 + wZ_4$$

the space reduces to

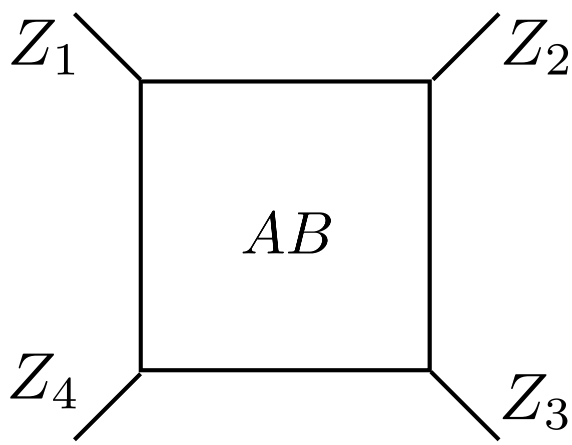
$$x, y, z, w > 0$$

One-loop Amplituhedron

- Logarithmic form on this space

$$\Omega = \frac{dx}{x}\frac{dy}{y}\frac{dz}{z}\frac{dw}{w} = \frac{d\mu_{AB}\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle}$$

- This corresponds to the one-loop box integral



measure

$$d\mu_{AB} = \langle AB\, d^2 A\rangle\langle AB\, d^2 B\rangle$$

Two-loop Amplituhedron

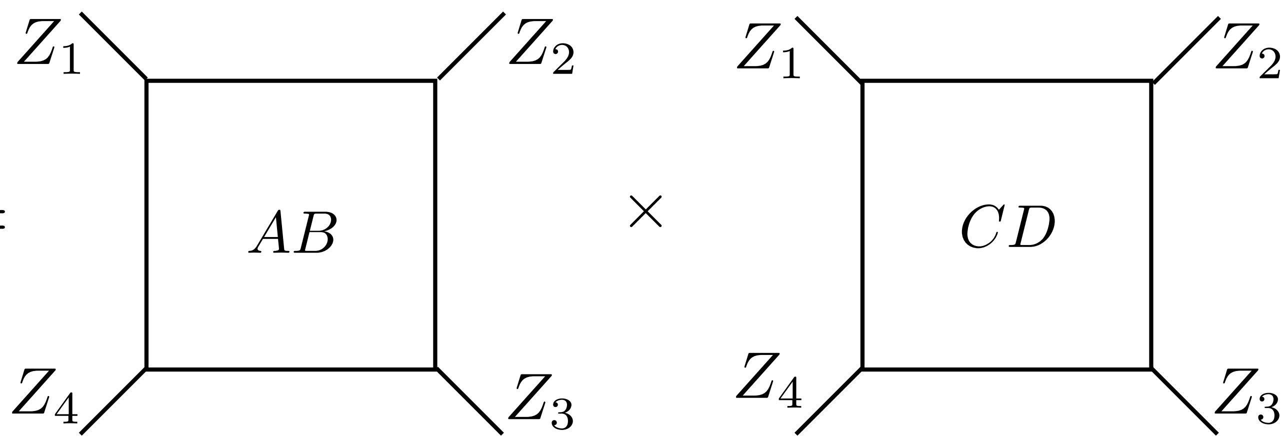
- Configuration of two lines (AB) and (CD)

each line lives in the one-loop Amplituhedron

$$\langle AB12\rangle, \langle AB23\rangle, \langle AB34\rangle, \langle AB14\rangle > 0, \qquad \langle AB13\rangle, \langle AB24\rangle < 0$$
$$\langle CD12\rangle, \langle CD23\rangle, \langle CD34\rangle, \langle CD14\rangle > 0, \qquad \langle CD13\rangle, \langle CD24\rangle < 0$$

if nothing else is imposed: square of one-loop problem

$\Omega =$



Two-loop Amplituhedron

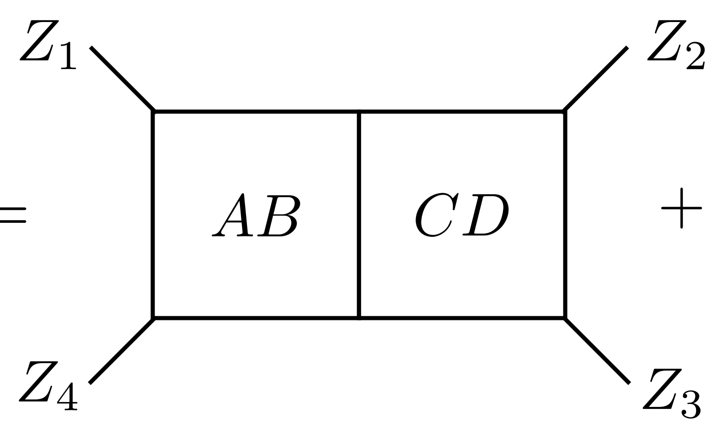
- Impose mutual positivity condition $\langle ABCD \rangle > 0$

$$x_1, y_1, z_1, w_1 > 0 \qquad x_2, y_2, z_2, w_2 > 0$$

$$D_{12} = -(x_1 - x_2)(w_1 - w_2) - (y_1 - y_2)(z_1 - z_2) > 0$$

- Logarithmic form: two-loop integrand

$$\Omega = \frac{x_1 w_2 + x_2 w_1 + y_1 z_2 + y_2 z_1}{x_1 y_1 z_1 w_1 x_2 y_2 z_2 w_2 D_{12}} =$$



$+\,\mathrm{cycl}$

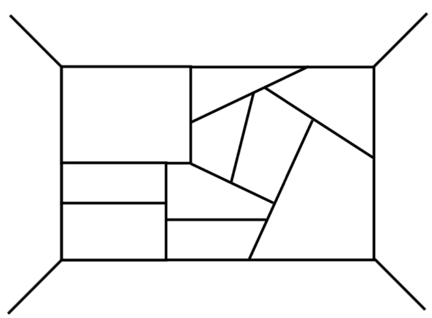
L-loop Amplituhedron

- At L-loops we have configuration of L lines $(AB)_i$
  - each line in the one-loop Amplituhedron
  - for any two lines we impose $\langle (AB)_i (AB)_j \rangle > 0$
- In our usual parametrization

$$x_i, y_i, z_i, w_i > 0 \qquad D_{ij} = -(x_i - x_j)(w_i - w_j) - (y_i - y_j)(z_i - z_j) > 0$$

This is a very complicated space and captures the complexity of L-loop integrand

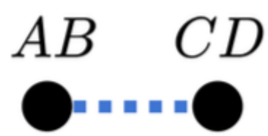
$\Omega =$



$+$ many others

Graphical notation

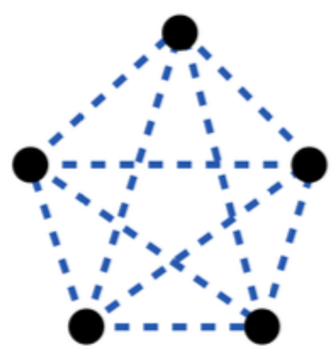
- We introduced a graphic notation:
  - vertex: loop line $(AB)_i$
  - blue dashed link: mutual positivity condition $\langle (AB)_i (AB)_j \rangle > 0$
- We denote the dlog form on the two-loop space



$\equiv \Omega(\mathcal{A}_2)$

- The L-loop dlog form: complete graph

$\Omega(\mathcal{A}_L) =$



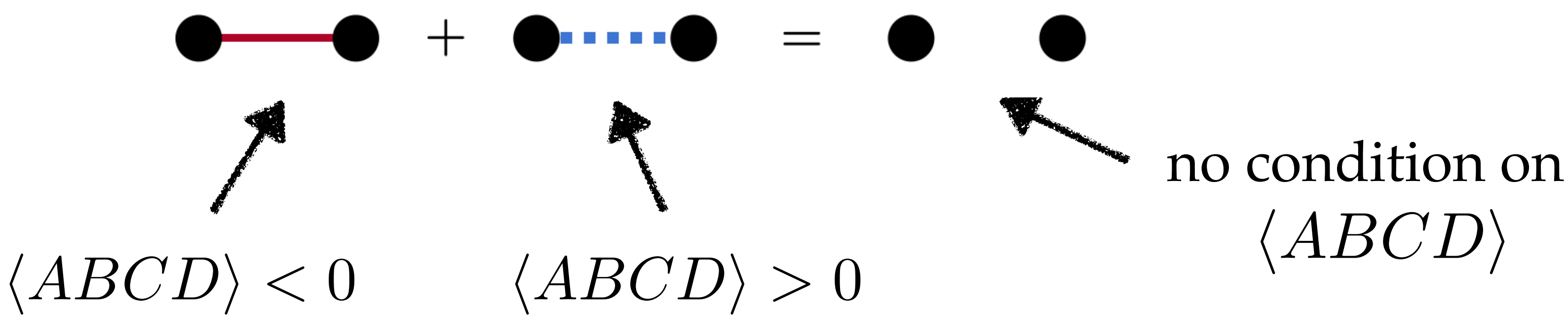
Negative geometry

Mutual negativity

- Complement to mutual positivity:

$$\langle ABCD \rangle < 0$$

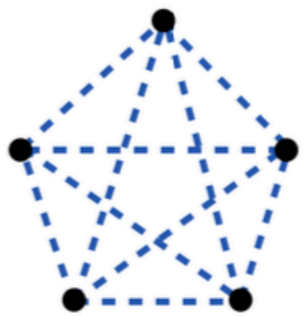
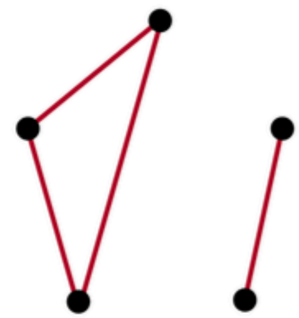
- Graphical notation:



- Express all positive links using negative and empty

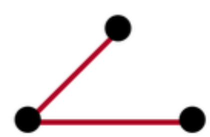
New formula for form

- New formula for L-loop Amplituhedron dlog form:

$\Omega(\mathcal{A}_L) =$  $= \sum_{\text{all } G} (-1)^{E(G)}$ 

sum over all graphs

- Example: L=3

$\Omega(\mathcal{A}_3) =$  $+$  $+$  $+$ 

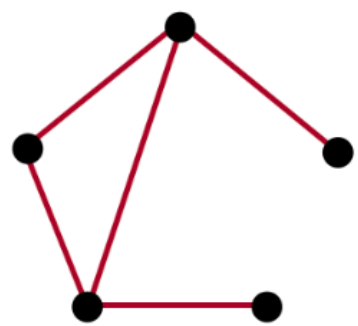
Exponentiation

- We can now write a formal sum over all loops

$$\Omega(g) = \sum_{L=0}^{\infty} (-g^2)^L \Omega(\mathcal{A}_L) \qquad \text{where } \Omega(\mathcal{A}_0) = 1$$

- The formula for $\Omega(g)$ exponentiates

$$\Omega(g) = \exp\left\{ \sum_{\substack{\text{all connected} \\ \text{graphs } G}} (-1)^{E(G)} (-g^2)^L \right.$$



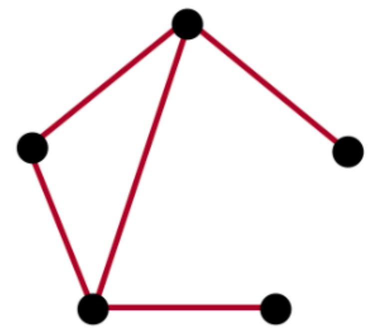
$$\left. \right\}$$

- We take the logarithm of both sides and expand in g

Expansion of the log

- The L-loop logarithm is then

$$\log \Omega(g)\Big|_{(-g^2)^L} = \widetilde{\Omega}_L = \sum_{\substack{\text{all connected} \\ \text{graphs } G \\ \text{with } L \text{ vertices}}} (-1)^{E(G)}$$

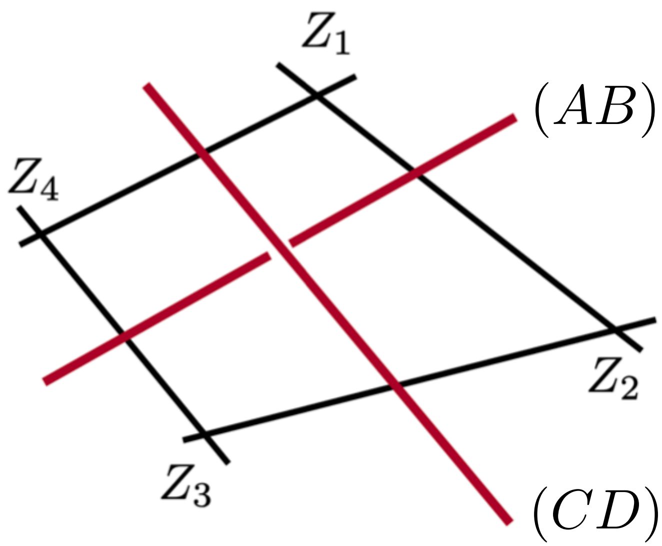


- We are left with the collection of all connected graphs with negative links

Collinear safety

Positivity and planarity

- Mutual positivity ensures the object is planar



(AB) cuts lines (12) and (34)
(CD) cuts lines (23) and (14)

"non-planar cut"

here forced $\langle ABCD\rangle < 0$

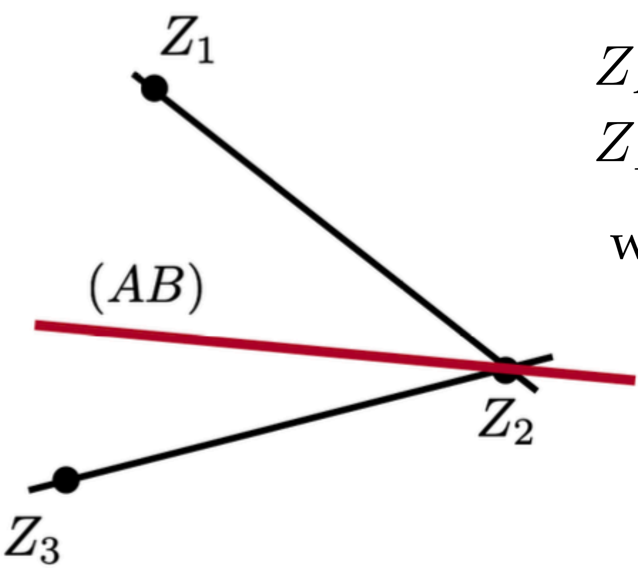
Our connected graphs with negative links are not planar

- Products of amplitudes are also not planar

Example: one-loop square space — no constraint on $\langle ABCD\rangle$

Implications of negativity

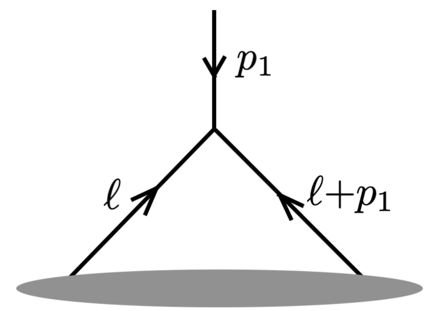
- Collinear configuration of line (AB)



$$Z_A = Z_2$$
$$Z_B = Z_3 - \alpha Z_1$$

with $\alpha > 0$

in momentum space



$$\ell = \alpha p_1$$

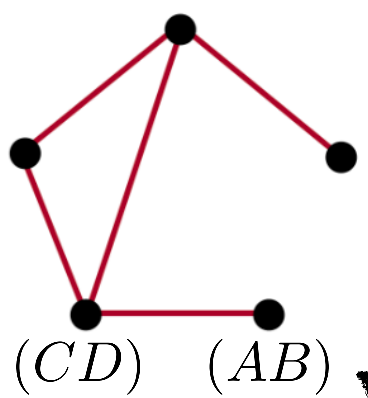
- This is forbidden if we have $\langle ABCD \rangle < 0$

because $\langle ABCD \rangle = \langle CD23 \rangle + \alpha \langle CD12 \rangle > 0$

all positive

Implications of negativity

- Connected graph: each vertex connected to at least one other vertex — that link is the mutual negativity condition



no loop line can access any collinear region

$\langle ABCD \rangle < 0$ is violated if the line (AB) accesses the collinear region

- Only all lines simultaneously access collinear region

all $\langle (AB)_i (AB)_j \rangle = 0$

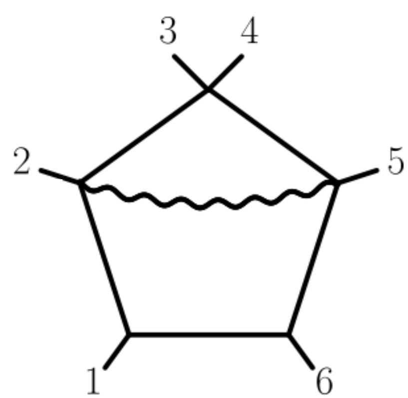
IR finite object

IR divergences

- IR divergences: caused by singularities of the integrand in the collinear (and soft) regions
- For N=4 SYM: no collinear support = no IR divergences
- IR finite integrals: either no massless corners or special numerators

(Arkani-Hamed, Bourjaily, Cachazo, Trnka 2010)

Example:
chiral pentagons



numerator cancels all collinear regions

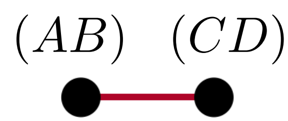
$$= \frac{d\mu_{AB}\langle AB(123)\cap(456)\rangle\langle 1256\rangle}{\langle AB45\rangle\langle AB56\rangle\langle AB16\rangle\langle AB12\rangle\langle AB23\rangle}$$

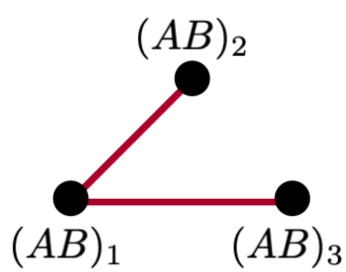
IR divergences

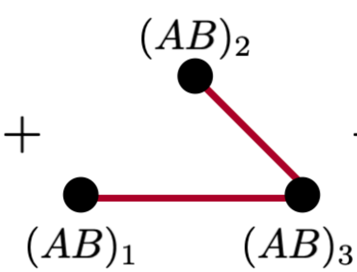
- Connected graph with negative links: integrate the form — only very mild IR divergence

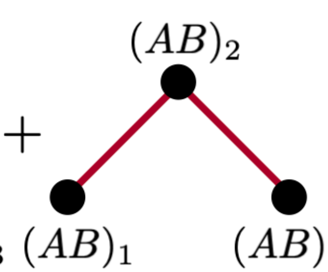
$$\int \Omega_\Gamma = \frac{1}{\epsilon^2}(\dots) + \mathcal{O}\left(\frac{1}{\epsilon}\right)$$

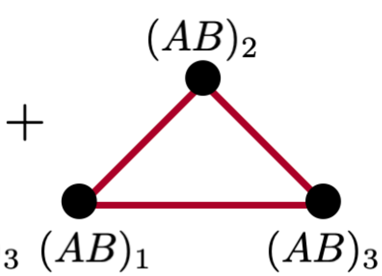
- Simplest examples:

$\widetilde{\Omega}_2 =$ 

$\widetilde{\Omega}_3 =$ 

$+$ 

$+$ 

$+$ 

each graph generates only $\dfrac{1}{\epsilon^2}$ divergence

Logarithm of the amplitude

- From amplitudes point of view:

$$M = M^{(0)} + gM^{(1)} + g^2 M^{(2)} + g^3 M^{(3)} + g^4 M^{(4)} + \ldots$$

expand the logarithm

$$\log M = gM^{(1)} + g^2 \left[M^{(2)} - \frac{1}{2}(M^{(1)})^2 \right] + g^3 \left[M^{(3)} - M^{(2)} M^{(1)} + \frac{1}{3}(M^{(1)})^3 \right]$$
$$+ g^4 \left[M^{(4)} - M^{(3)} M^{(1)} - \frac{1}{2}(M^{(2)})^2 + M^{(2)} (M^{(1)})^2 - \frac{1}{4}(M^{(1)})^4 \right] + \ldots$$

the IR divergence has simple structure to all loops

$$\log M = - \sum_{L \geq 1} g^{2L} \frac{\Gamma^{(L)}_{\text{cusp}}}{(L\epsilon)^2} + \mathcal{O}\left(1/\epsilon\right)$$

Frozen loop

- IR divergence in connected graphs, and hence in $\log M$
  from all lines simultaneously in collinear region
  which is a part of the integration region

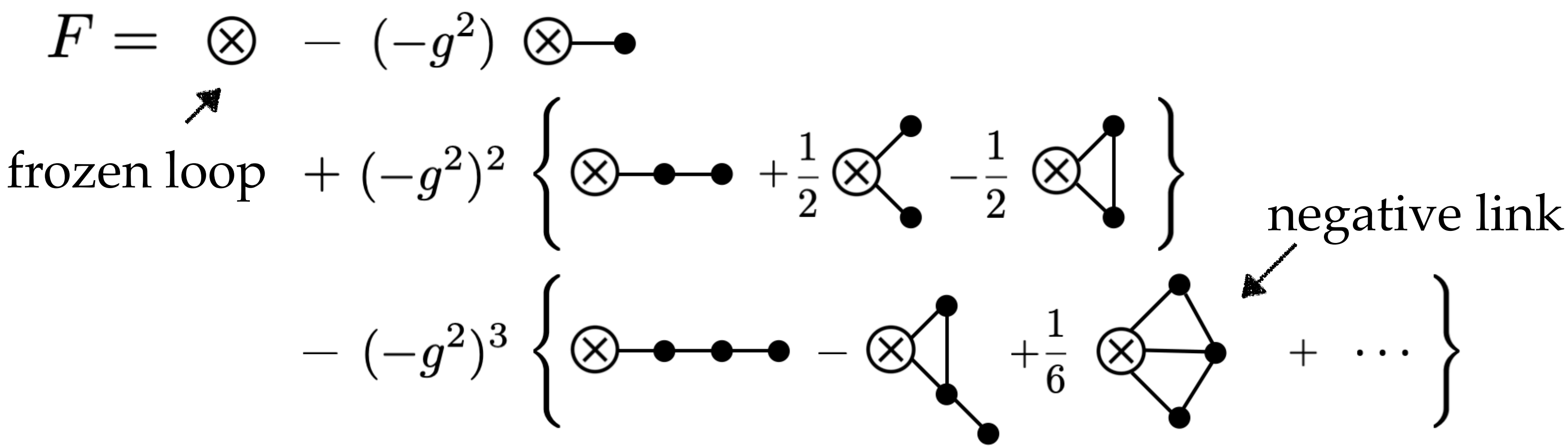
- Freeze one of the lines, and integrate over all others:
  no collinear support — IR finite object

$$\mathcal{F}_\Gamma(AB_0) = \int d\mu_{AB_1} \dots d\mu_{AB_{L-1}} \Omega_\Gamma$$

for each connected graph

Definition of IR finite object

- Define the same object for the logarithm of the amplitude (= sum over all connected graphs)



Definition of IR finite object

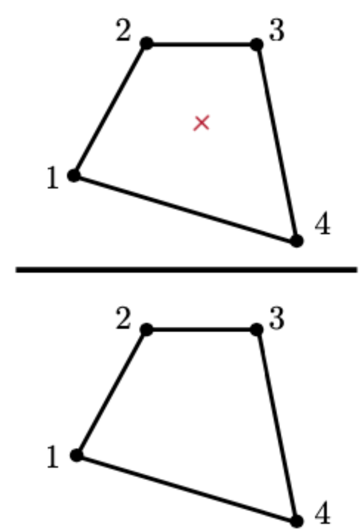
- This is IR finite function of one cross ratio

$$\mathcal{F}(g, z) \qquad\qquad z = \frac{\langle AB_0 12\rangle\langle AB_0 34\rangle}{\langle AB_0 14\rangle\langle AB_0 23\rangle}$$

- This object is also natural from the dual Wilson loop picture

(Alday, Buchbinder, Tseytlin, 2011) (Englund, Roiban, 2011)

$$\frac{\langle W_F(x_1,x_2,x_3,x_4)\mathcal{L}(x_0)\rangle}{\langle W_F(x_1,x_2,x_3,x_4)\rangle} = \frac{1}{\pi^2}\frac{x_{13}^2 x_{24}^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} F(g;z)$$



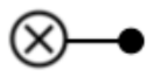
Wilson loop with Lagrangian insertion at point x_0

$$F = -g^2\mathcal{F}$$

Low loop order results

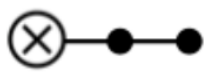
We can find the dlog forms for all graphs Ω_Γ and integrate

$L = 2$



$$= \left[\pi^2 + \log^2(z)\right]$$

$L = 3$



$$= -\frac{1}{12}\left[\pi^2 + \log^2(z)\right] \times \left[5\pi^2 + \log^2 z\right]$$

$$= -\frac{1}{2}\left[\pi^2 + \log^2 z\right]^2$$

$$= -\frac{1}{6}\log^4 z + \log^2 z\left[-\frac{2}{3}\mathrm{Li}_2\left(\frac{1}{z+1}\right) - \frac{2}{3}\mathrm{Li}_2\left(\frac{z}{z+1}\right) + \frac{\pi^2}{9}\right]$$
$$+ \log z\left[4\mathrm{Li}_3\left(\frac{z}{z+1}\right) - 4\mathrm{Li}_3\left(\frac{1}{z+1}\right)\right] - \frac{2}{3}\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right]^2$$
$$- \frac{8}{3}\pi^2\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right] - 8\mathrm{Li}_4\left(\frac{1}{z+1}\right) - 8\mathrm{Li}_4\left(\frac{z}{z+1}\right) - \frac{\pi^4}{18}.$$

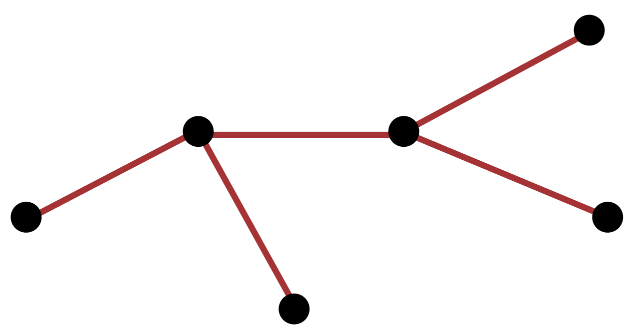
Trees and ladders

Trees vs loops in the loop space

- We see in L=3 that "tree" graphs are much simpler than "loop" graph

- In fact, we can find the dlog form for all tree graphs

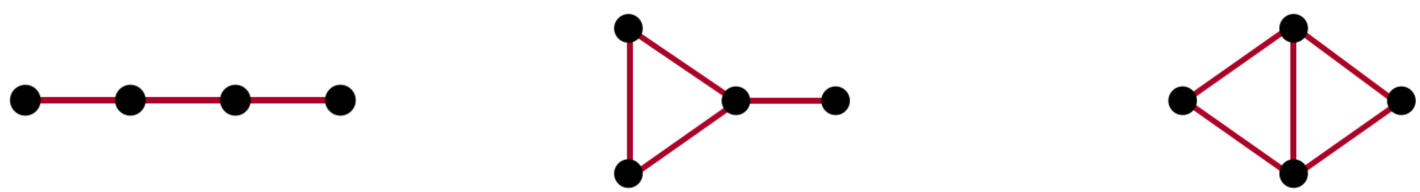
$$\widetilde{\Omega} = \underbrace{\prod_{k=1}^{L}}_{\text{vertices}} \frac{d\mu_k}{D_k} \times \underbrace{\prod_{\Gamma}}_{\text{links}} n^{(ij)}$$



$$D_k = \langle AB_k 12\rangle\langle AB_k 23\rangle\langle AB_k 34\rangle\langle AB_k 14\rangle$$
$$n^{(ij)} = \langle AB_i 13\rangle\langle AB_j 24\rangle + \langle AB_i 24\rangle\langle AB_j 13\rangle$$

Trees vs loops in the loop space

- We also have closed formula for all one-loop graphs but they are more complicated
- Note that both tree and loop graphs have the same number of standard loops

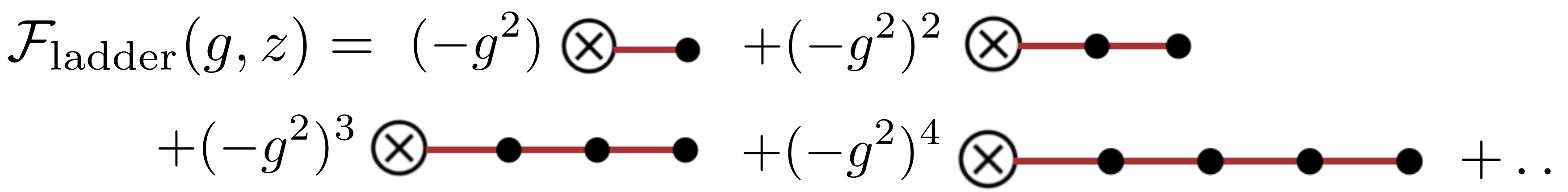


These are all L=4 graphs, but the forms and integrated formulas get more complicated with more graph loops

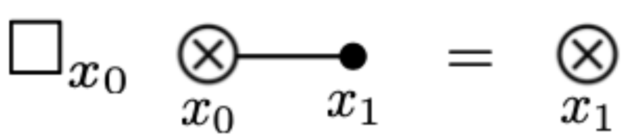
Ladders

- Tree approximation: only tree graphs are considered

- There is even more special class of tree graphs: "ladders"



- Because of the simple structure of the dlog form, we can find the Laplace operator which acts on graphs as



Closed formula for ladders

- We have a function of single cross ratio, rewrite the differential operator as

$$\frac{1}{2}(z\partial z)^2\mathcal{F}_{\mathrm{ladder}}(g,z)+g^2\mathcal{F}_{\mathrm{ladder}}(g,z)=0$$

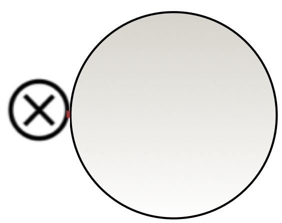
which is solved by

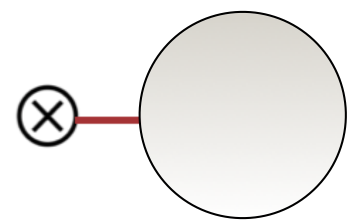
$$\mathcal{F}_{\mathrm{ladder}}(g,z)=\frac{\cos(\sqrt{2}g\log z)}{\cosh(\sqrt{2}g\pi)}$$

(satisfying certain boundary conditions)

All trees

- Next, we consider the sum over all trees
- For that it is useful to define the generating function for all trees with special link

$\mathcal{F}_{\text{tree}}(g, z) =$ 

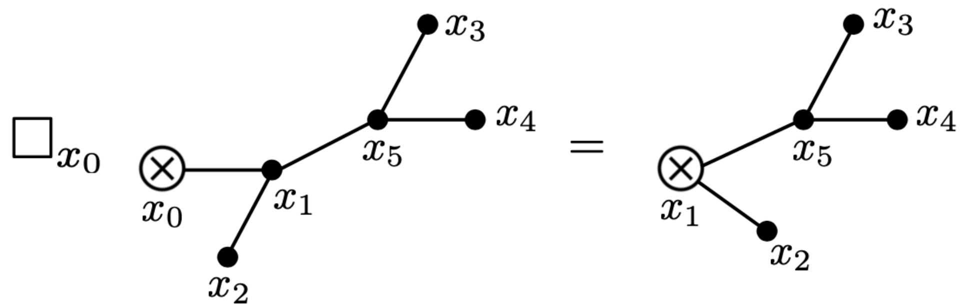
$\mathcal{H}_{\text{tree}}(g, z) =$ 

We can check that both functions are related

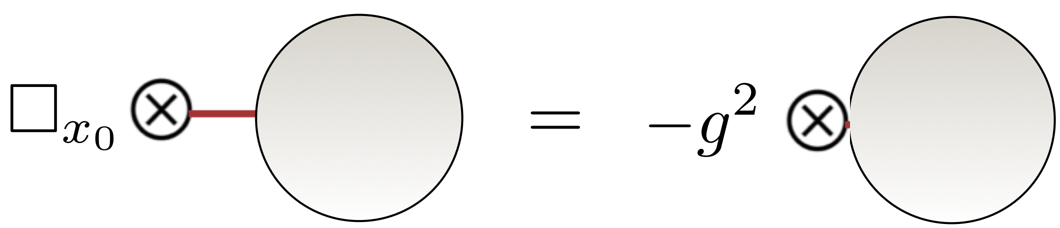
$$\mathcal{F}_{\text{tree}}(g, z) = e^{\mathcal{H}_{\text{tree}}(g, z)}$$

All trees

- We can apply the same differential operator on any tree graph



- From that we can read off an equation for $\mathcal{H}_{\text{tree}}(g, z)$



$$\frac{1}{2}(z\partial_z)^2\mathcal{H}_{\text{tree}}(g, z) + g^2 e^{\mathcal{H}_{\text{tree}}(g,z)} = 0$$

All trees

- With proper boundary conditions we can solve this equation

$$\mathcal{F}_{\text{tree}}(g,z) = \frac{A^2}{g^2}\frac{z^A}{(z^A+1)^2}\,, \quad \text{where} \quad \frac{A}{2g\cos\frac{\pi A}{2}} = 1$$

- We can use our formulas for $\mathcal{F}_{\text{ladder}}(g,z)$ and $\mathcal{F}_{\text{tree}}(g,z)$ to go to strong coupling and also find contributions to cusp anomalous dimension

Strong coupling

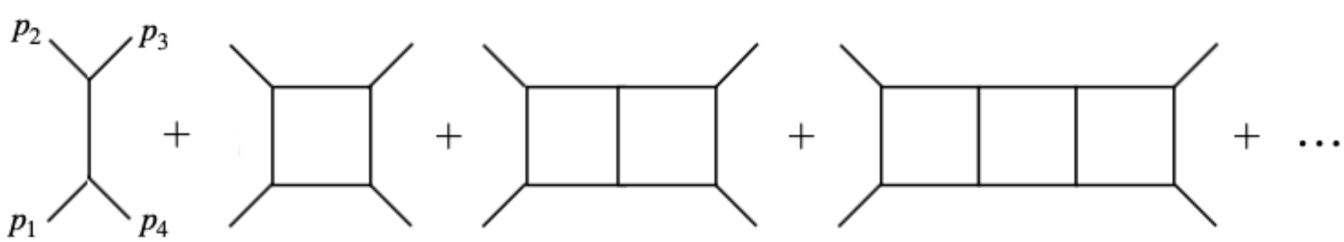
Strong coupling of ladders

- At strong coupling the ladder contribution is exponentially suppressed

$$\mathcal{F}_{\text{ladder}}(g;z) \leq \frac{1}{\cosh(\sqrt{2}g\pi)} \leq 2e^{-\sqrt{2}g\pi} \overset{g\gg 1}{\to} 0$$

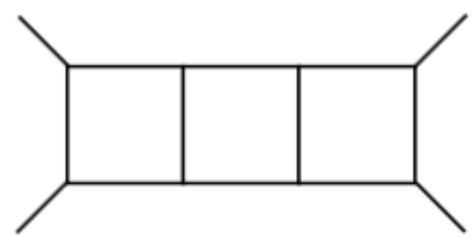
This is similar as what we would get from summing actual ϕ^3 ladder diagrams

(Broadhurst, Davydychev, 2010)



Strong coupling of ladders

- Our ladders are very different, but the only thing they have in common is the number of internal $\langle (AB)_i (AB)_j \rangle$ propagators in the integrand form



both have (L-1) propagators but very different numerators

Strong coupling of ladders

- The exponential suppression is in contrast with the full gamma cusp result

$$F(z) \overset{g \gg 1}{=} g \frac{z}{(z-1)^3} \left[2(1-z) + (z+1)\log z\right] + \dots .$$

which is linear at strong coupling

- In comparison the sum over trees is

$$F_{\text{tree}} = -\frac{z}{(1+z)^2} + \mathcal{O}\left(\frac{1}{g}\right)$$

Missing g term

But still has 1/g expansion unlike ladders!

Gamma cusp

From F to gamma cusp

- Recall the expression for the logarithm of the amplitude

$$\log M = -\sum_{L\geq 1} g^{2L} \frac{\Gamma_{\text{cusp}}^{(L)}}{(L\epsilon)^2} + \mathcal{O}(1/\epsilon)$$

- The cusp anomalous dimension is

$$\Gamma_{\text{cusp}}(g) = \sum_{L\geq 1} g^{2L} \Gamma_{\text{cusp}}^{(L)}$$

and can be obtained from function $F(g,z)$

(Alday, Henn, Sikorowski, 2013) (Henn, Korchemsky, Mistlberger 2019)

$$g \frac{\partial}{\partial_g} \Gamma_{\text{cusp}}(g) = -2\mathcal{I}[F(g,z)] \quad \text{where} \quad \mathcal{I}[z^p] = \frac{\sin(\pi p)}{\pi p}$$

From F to gamma cusp

- For the sum of ladders we can finish the calculation

$$\Gamma_{\text{ladder}}(g) = \frac{4}{\pi}\log\cosh(\sqrt{2}\pi g)$$

which is very close to $\Gamma_{\text{octagon}}(g)$ which controls the six-point remainder function in particular limit

$$\Gamma_{\text{ladder}}(g) = 2\,\Gamma_{\text{octagon}}(\frac{g}{\sqrt{2}})$$

(Kostov, Petkova, Serban, 2019) (Basso, Dixon, Papathanasiou, 2020) (Caron-Huot, Coronado, 2021)

From F to gamma cusp

- We just need to expand $F_{\text{tree}}(g,z)$ in powers of z

$$F_{\text{tree}}(g,z) = A\sum_{m=0}^{\infty}(mA)(-1)^m z^{mA}$$

Plugging back into the formula for gamma cusp

$$g\partial_g\Gamma_{\text{tree}}(g) = -\frac{2A}{\pi}\sum_{m=0}^{\infty}(-1)^m\sin(\pi m A)$$

$$= \frac{4A}{\pi}\tan\left(\frac{\pi A}{2}\right) \qquad \text{where} \qquad \frac{A}{2g\cos\left(\frac{\pi A}{2}\right)} = 1$$

Can not solve it analytically but we will extract strong coupling data

Asymptotics of gamma cusp

- Now we look at the weak and strong coupling asymptotic of gamma cusp

radius of convergence

$$\Gamma_{\rm cusp}(g) \to \begin{cases} 4g^2 - 8\zeta_2 g^4 + \cdots & g \ll 1 \\ 2g - \frac{6\log 2}{4\pi}\frac{1}{g} + \cdots & g \gg 1 \end{cases}$$

$g_* = 0.25$

(Beisert, Eden, Staudacher, 2005)

for ladders

$$\Gamma_{\rm ladder}(g) \to \begin{cases} 4g^2 - 8\zeta_2 g^4 + \cdots & g \ll 1 \\ 4\sqrt{2}g - 4\frac{\log 2}{\pi} + \frac{4}{\pi}e^{-2\sqrt{2}g\pi} + \cdots & g \gg 1 \end{cases}$$

$g_* = 0.35$

no 1/g terms

less non-perturbative

Asymptotics of gamma cusp

- Now we look at the weak and strong coupling asymptotic of gamma cusp

radius of convergence

$$\Gamma_{\text{cusp}}(g) \to \begin{cases} 4g^2 - 8\zeta_2 g^4 + \cdots & g \ll 1 \\ 2g - \frac{6\log 2}{4\pi}\frac{1}{g} + \cdots & g \gg 1 \end{cases}$$

$g_* = 0.25$

(Beisert, Eden, Staudacher, 2005)

while for trees

$$\Gamma_{\text{tree}}(g) \to \begin{cases} 4g^2 - 8\zeta_2 g^4 + \cdots & g \ll 1 \\ \frac{8}{\pi}g + \frac{1}{\pi}\frac{1}{g} + \cdots & g \gg 1 \end{cases}$$

$g_* = 0.21$

All the correct qualitative behavior

Comparison at weak coupling

- Compare numerically weak coupling coefficients

$$r_{i/j} = \sum_{k\geq 1} g^{2k} \frac{c_{i,k}}{c_{j,k}}$$

$$r_{\text{ladder/cusp}} = g^2 + g^4 + 0.73g^6 + 0.44g^8 + 0.25g^{10} + 0.14g^{12} + 0.07g^{14} + \dots .$$

$$r_{\text{cusp/tree}} = g^2 + g^4 + 0.92g^6 + 0.83g^8 + 0.74g^{10} + 0.63g^{12} + 0.53g^{14} + \dots .$$

better numerical approximation

Summary/Outlook

Summary

- Logarithm of the amplitude: expansion in terms of dlog forms on negative geometries, manifest IR
- Freeze one loop: IR finite object
- Summing ladders and all trees, strong coupling, gamma cusp — trees give correct qualitative behavior
- The function of z — no g term: need "loops of loops"

Outlook

- Calculate geometric "loop" corrections: forms and integrals, new differential operators needed
- Strong coupling: see how the correct behavior for F(z) emerges
- Extension to five points - more cross ratios

(Arkani-Hamed, Chicherin, Henn, JT, in progress)

Thank you!