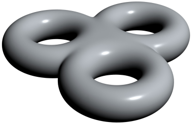
Superstring loop amplitudes from the field theory limit

Yvonne Geyer



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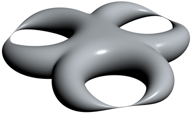
Bangkok



Amplitudes 2021

arXiv:2106.03968

with Ricardo Monteiro and Ricardo Stark-Muchão

Status of loop amplitudes: Superstring vs Supergravity

- Superstring

  4-pt amplitude, massless external states

  - tree-level and 1-loop: [Green, Schwarz '82]
  - 2-loops: [D'Hoker, Phong; Berkovits '05]
  - 3-loops: partial work [D'Hoker, Phong; Cacciatori, d.Piazza, v.Greemen] [Gomez, Mafra '13]

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  [BCJ et.al. '17-'18]

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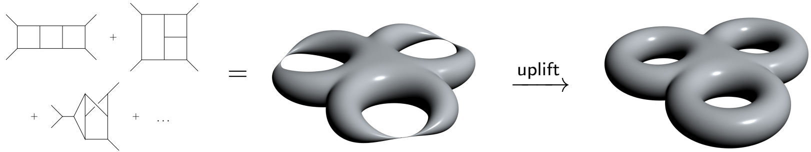
GOAL: sugra advances $\longrightarrow$ superstring

Tools: modern amplitudes techniques

- colour-kinematics duality
- ambitwistor string

Outline

Superstring loop amplitudes from field theory limit:



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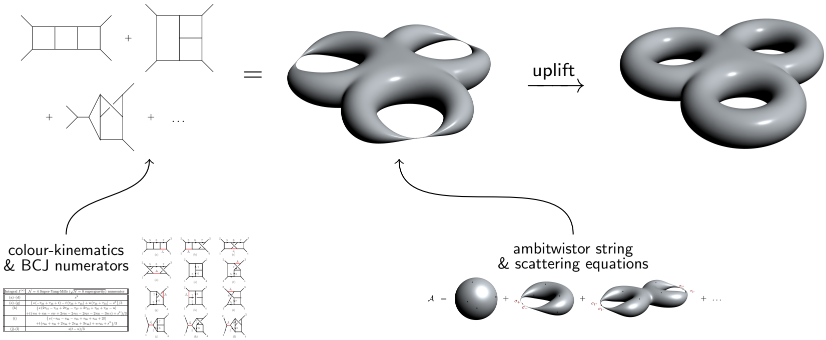

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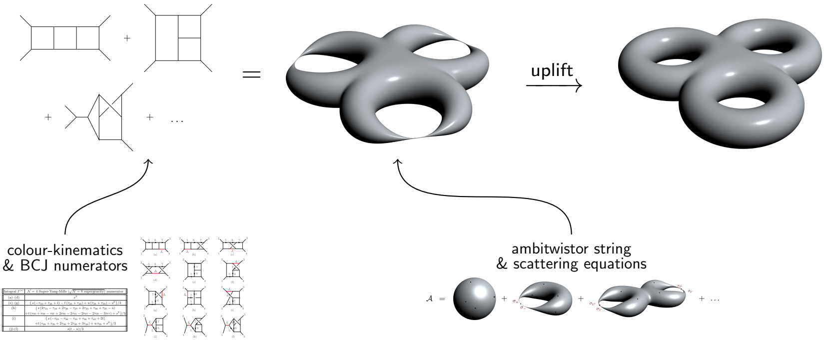


$\Rightarrow$ conjecture for 3-loop 4-pt superstring amplitude $\mathcal{A}_{\mathbb{S}}^{(3)}$

Amplitudes toolkit

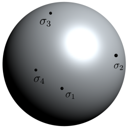
Worldsheet models for Field Theory

Ambitwistor string [Mason,Skinner '13; c.f. Berkovits]

$\overline{D} = \bar{\partial} + e\partial$

no α'!

$$S_{\mathbb{A}} = \frac{1}{2\pi}\int_{\Sigma} P \cdot \bar{D}X - \frac{\tilde{e}}{2}P^2 + S_M$$



- *chiral* worldsheet theory: $X^\mu \in \Omega^0(\Sigma)$, $P_\mu \in \Omega^0(K_\Sigma)$
- 'RNS' model: $S_M = S_{\psi_1} + S_{\psi_2}$ (others possible)
  - action: $S_\psi = \int \psi \cdot \overline{D}\psi + \chi\, P \cdot \psi$ with $\psi^\mu_{r=1,2} \in \Pi\Omega^0(K_\Sigma^{1/2})$
  - BRST: free, linear CFTs with $d_{\text{crit}} = 10$

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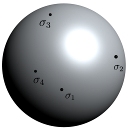

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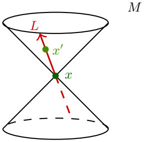
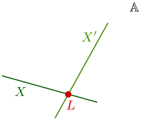
- target space: $\mathbb{A}$ = phase space of complexified null geodesics





Spectrum and correlators

- <u>Spectrum: type II supergravity</u>

$$V_{\mathsf{NS}} = c\tilde{c}\,\delta(\gamma_1)\delta(\gamma_2)\,\epsilon_{\mu\nu}\psi_1^\mu\psi_2^\nu\, e^{ik\cdot X} \qquad \text{with} \quad k^2 = \epsilon_{\mu\nu}k^\nu = \epsilon_{\mu\nu}k^\mu = 0$$

$\Rightarrow$ worldsheet theory for QFT amplitudes

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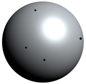
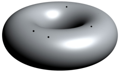
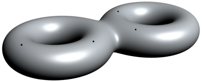
$\mathcal{A}$ =  +  +  + ...

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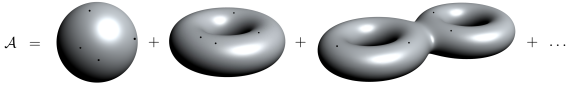

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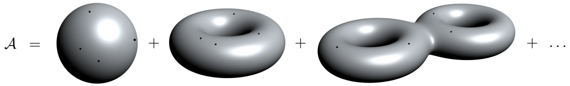

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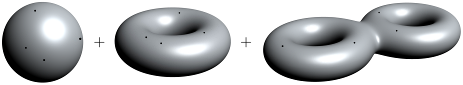
$\mathcal{A} =$  $+ \dots$

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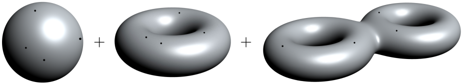
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  - $P^2 = 0 \quad \leftrightarrow$ scattering equations $\mathcal{E}_i = \text{Res}_{\sigma_i} P^2 = 2k_i \cdot P(\sigma_i)$

CHY amplitudes [Cachazo, He, Yuan '13]

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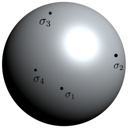

- Measure
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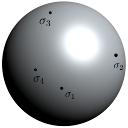
$\bar{\delta}(z) = \bar{\partial}\left(\frac{1}{2\pi i\, z}\right)$



- **Measure**
  - integral over $\mathfrak{M}_{0,n}$
  - fully localized on scattering equations

momenta $k_i \in \mathbb{R}^d$
$k_i^2 = 0$

$$\mathcal{E}_i = \text{Res}_{\sigma_i} P^2 = \sum_{j\neq i} \frac{2k_i \cdot k_j}{\sigma_i - \sigma_j} \qquad \text{with} \quad P_\mu(\sigma) = \sum_i \frac{k_{i\,\mu}}{\sigma - \sigma_i} d\sigma$$

$\sigma_i \in \mathbb{CP}^1$

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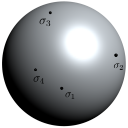

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  - specifies theory

Yang-Mills, $\epsilon_\mu\, t^{\mathfrak{a}}\, e^{ik\cdot X}$: $\mathcal{I}_{\text{YM}} = \mathcal{I}_{\text{kin}}(\sigma_i,k_i,\epsilon_i) \times \mathcal{C}(\sigma_i,\mathfrak{a}_i)$

Gravity, $\epsilon_\mu\, \tilde{\epsilon}_\nu\, e^{ik\cdot X}$: $\mathcal{I}_{\text{grav}} = \mathcal{I}_{\text{kin}}(\sigma_i,k_i,\epsilon_i) \times \mathcal{I}_{\text{kin}}(\sigma_i,k_i,\tilde{\epsilon}_i)$

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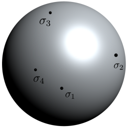

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  - 'woldsheet double copy' c.f [Kawai,Lewellen,Tye '86; Bern,Carrasco,Johansson '08]

Gravity $\sim$ YM^2

Integrand and the Colour-kinematics duality

- Colour-kinematics duality

[Bern, Carrasco, Johansson '08]

$f^{\mathfrak{a}_1\mathfrak{a}_2\cdot} f^{\cdot\mathfrak{a}_3\cdot} \dots$

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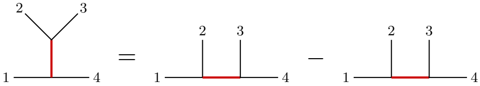
Kinematic numerators N_α satisfying same Jacobi's as C_α:



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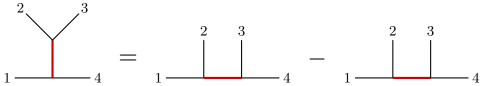

- CHY integrands

  [CHY '13; Bjerrum-Bohr et.al. '16, ...]

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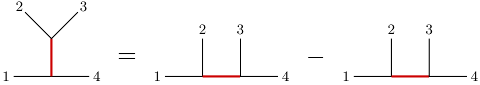

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$$\mathcal{C}(\mathfrak{a}) = \sum_{\alpha\in S_{n-2}} \frac{C_\alpha(\mathfrak{a})}{(1\,\alpha\,n)} \qquad\qquad \mathcal{I}_{\text{kin}}(\epsilon) \overset{\text{SE}}{=} \sum_{\alpha\in S_{n-2}} \frac{N_\alpha(\epsilon)}{(1\,\alpha\,n)}$$

Parke-Taylor factor

$(12\ldots n) := \sigma_{12}\sigma_{23}\ldots\sigma_{n1}$

Integrand and the Colour-kinematics duality

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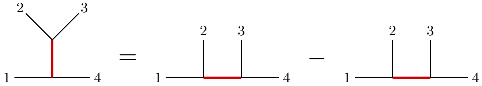

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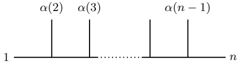
  - Colour C_α and BCJ numerators N_α for 'half-ladder' master diagrams



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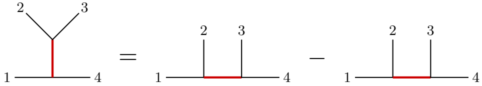

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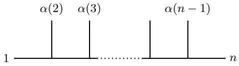
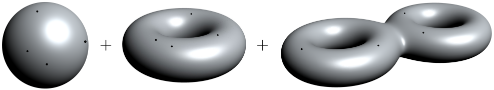


genus-g correlators = loop integrands

$\mathcal{A} =$  $+ \dots$

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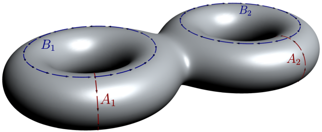
- Moduli space $\mathfrak{M}_{g,n}$
  - homology basis: $\#(A_I, B_J) = \delta_{IJ}$
    modular group: $\mathrm{Sp}(4, \mathbb{Z})_{\#}$
  - holomorphic differentials ω_I

$$\delta_{IJ} = \oint_{A_I} \omega_J \qquad \qquad \Omega_{IJ} = \oint_{B_I} \omega_J$$



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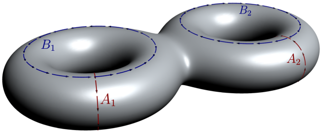

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  - P determined by $\bar{\partial} P = \sum_i k_i\, \bar{\delta}(z - z_i)\, dz$

$$P_\mu(z) = 2\pi i\, \ell_\mu^I\, \omega_I(z) + \sum_i k_{i\,\mu}\, \omega_{i,*}(z)$$

hom. solution
loop momenta

merom. diff's $\omega_{[ij]}$
$\mathrm{Res}_{z_i} \omega_{ij} = 1$

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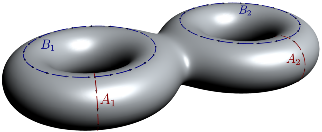

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$$\boxed{\mathcal{E}_i = \mathrm{Res}_{z_i} P^2} \qquad \boxed{P^2\Big|_{\mathcal{E}_i = 0} = u^{IJ}\, \omega_I \omega_J}$$

Higher genus amplitude formulae

$$\mathcal{A}_n^{(g)} = \int d^{10}\ell^I \int_{\mathfrak{M}_{g,n}} \prod_{I \leq J} d\Omega_{IJ}\, \bar\delta(u^{IJ}) \prod_i \bar\delta(\mathcal{E}_i)\, \mathcal{I}_n^{(g)}$$

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  - localization on scattering equations

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- Questions
  - loop integration UV divergent in $d = 10$
  - calculation of loop integrand?

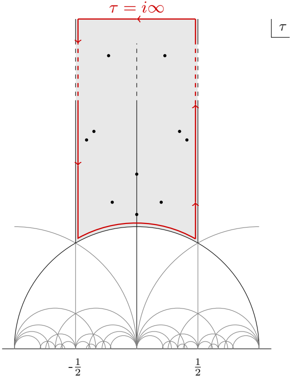
Field theory! How can we see that the integrand is rational?

Residue theorem to the nodal sphere

[YG, Mason, Monteiro, Tourkine '15-'18]

- <u>Residue theorem on fundamental domain</u>

  Look at $g = 1$:



solutions to $u = 0$
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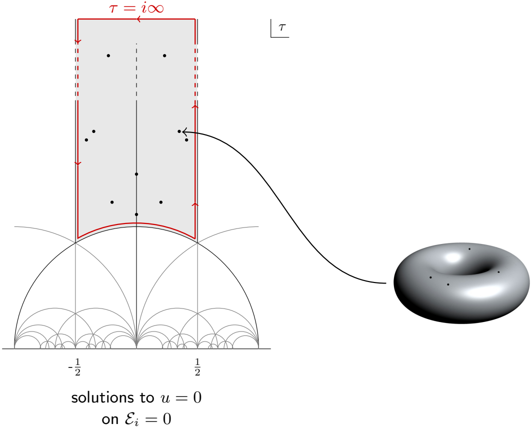

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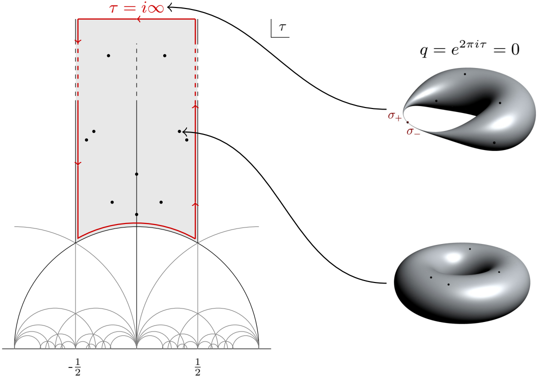

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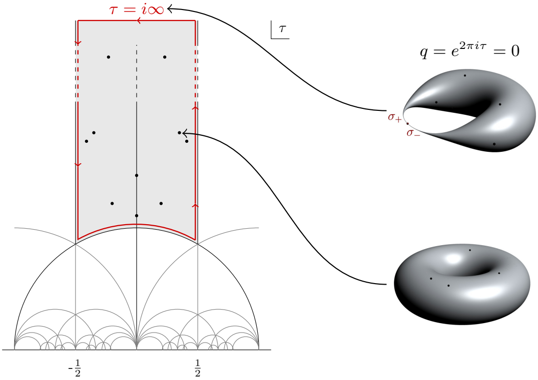

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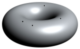
- Integrand localizes on nodal sphere

$$\mathfrak{I}_n^{(1)} = \int_{\mathfrak{M}_{1,n}} \frac{dq}{q}\, \bar{\delta}(u)\, \mathcal{I}^{(1)}(q)$$

$=$ 

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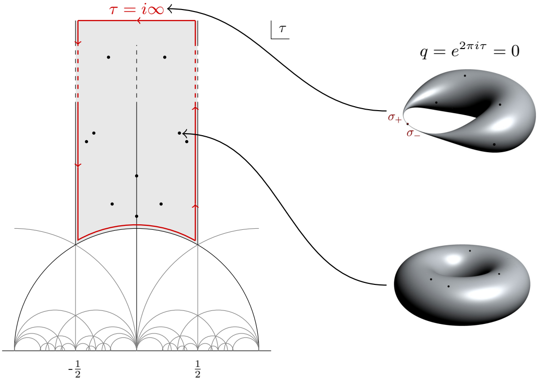

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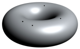
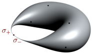
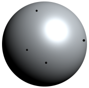
$=$  $\overset{\text{res}}{=}$ 

loop expansion = nodal expansion

$\mathcal{A}$ =



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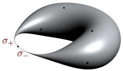

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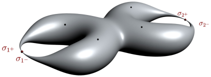

+ $\ldots$

Loop amplitudes from the nodal sphere

$$\mathcal{A}_n^{(g)} = \int \frac{d^{10}\ell^I}{\prod(\ell^I)^2} \int_{\mathfrak{M}_{0,n+2g}} c^{(g)} \left(\mathcal{J}^{(g)}\, \mathcal{I}_L^{(g)}\right) \left(\mathcal{J}^{(g)}\, \mathcal{I}_R^{(g)}\right) \prod_{A=1}^{n+2g}{}' \bar{\delta}\left(\mathcal{E}_A\right)$$

- From residue theorem
  - traded localization on $P^2 = 0$ for $q_{II} = e^{i\pi\Omega_{II}} = 0$

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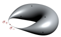

$$J^{(1)} = \left(\sigma_{+-}\right)^{-2} \quad **$$

$$J^{(2)} = \left(\sigma_{1+2+}\sigma_{1+2-}\sigma_{1-2+}\sigma_{1-2-}\right)^{-1}$$

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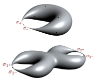

$$c^{(1)} = 1$$

$$c^{(2)} = \frac{\sigma_{1+2-}\,\sigma_{1-2+}}{\sigma_{1+1-}\,\sigma_{2+2-}}$$

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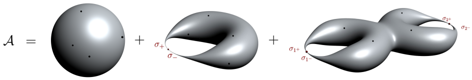
$$\boxed{\mathcal{E}_A = \text{Res}_{\sigma_A} \mathfrak{P}^{(g)}} \qquad \mathfrak{P}^{(g)} = P^2 - (\ell^I \omega_{I^+I^-})^2 + L_{(g)}^{IJ}\, \omega_{I^+I^-}\omega_{J^+J^-}$$

$L_{(1)}^{IJ} = 0$

$L_{(2)}^{12} = \ell_1^2 + \ell_2^2$

Comments

- **Different theories possible**
  - dim. red. to $d \leq 10$
  - sugra and sYM (next slide)



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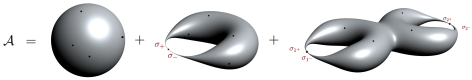

- **Unorthodox integrand representation**
  - 'linear' propagator factors of form $2\ell_I \cdot K + K^2$
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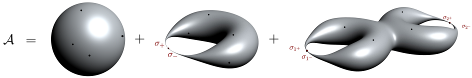

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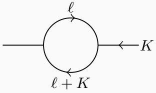

$$\frac{1}{\ell^2(\ell+K)^2} = \frac{1}{\ell^2(2\ell\cdot K + K^2)} + \frac{1}{(\ell+K)^2(-2\ell\cdot K - K^2)}$$

$$\xrightarrow{\text{shift}} \frac{1}{\ell^2}\left(\frac{1}{2\ell\cdot K + K^2} + \frac{1}{-2\ell\cdot K + K^2}\right)$$

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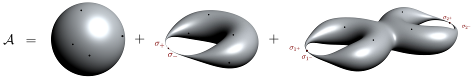

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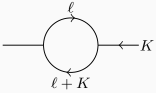

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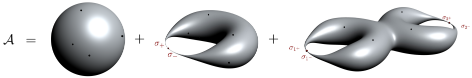

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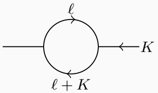

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- Physical interpretation of $\mathfrak{P}^{(g)}$ and $c^{(g)}$
  - $\mathfrak{P}^{(g)}$: correct poles in 'linear' representation
  - $c^{(g)}$: no unphysical poles

Colour-kinematics at loop level

[Bern,Carrasco,Johansson '10]

- BCJ double copy at g loops

State-of-the-art: 5 loops

[Bern,Carrasco,Chen,Edison,Johansson,Parra-Martinez,Roiban,Zeng '17-18]

$$\mathcal{A}^{(g)}_{\text{YM}} = \sum_{\alpha\in\Gamma^{(g)}_n} \int \prod_{I=1}^{g} d^D\ell^I \, \frac{N_{\alpha}(\epsilon)\, C_{\alpha}(\mathfrak{a})}{S_\alpha\, D_\alpha} \qquad \mathcal{A}^{(g)}_{\text{grav}} = \sum_{\alpha\in\Gamma^{(g)}_n} \int \prod_{I=1}^{g} d^D\ell^I \, \frac{N_{\alpha}(\epsilon)\, N_{\alpha}(\tilde{\epsilon})}{S_\alpha\, D_\alpha}$$

symmetry factor

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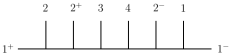
- **Nodal sphere**
  [He,Schlotterer,Zhang '16-'17; YG,Monteiro '17-19'; ...]
  - sYM from single copy

$$\mathcal{I}^{(g)}_{\text{YM}} = \mathcal{C}^{(g)} \left(\mathcal{J}^{(g)}\mathcal{I}^{(g)}_{\text{kin}}(\epsilon)\right) \qquad \mathcal{I}^{(g)}_{\text{grav}} = \left(\mathcal{J}^{(g)}\mathcal{I}^{(g)}_{\text{kin}}(\epsilon)\right)\left(\mathcal{J}^{(g)}\mathcal{I}^{(g)}_{\text{kin}}(\tilde\epsilon)\right)$$

  - Half-integrands in BCJ representation:

$$\mathcal{C}^{(g)} = \sum_{\alpha\,\in\, S_{n+2g-2}} \frac{C^{(g)}(1^+\alpha\, 1^-)}{(1^+\,\alpha\, 1^-)} \qquad \mathcal{J}^{(g)}\mathcal{I}^{(g)}_{\text{kin}} = \sum_{\alpha\,\in\, S_{n+2g-2}} \frac{N^{(g)}(1^+\alpha, 1^-)}{(1^+\,\alpha\, 1^-)}$$

  - 'half-ladder' master diagrams



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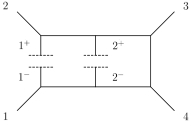

From field theory to superstring amplitudes

4-pt amplitudes for $g \leq 2$

- Supergravity
  from ambitwistor string, higher genus and nodal sphere

$$\mathcal{A}_{\mathbb{A}}^{(g)} = \mathcal{R}^4 \int d^{10}\ell^I \quad \int_{\mathfrak{M}_{g,4}} \prod_{I \leq J} d\Omega_{IJ} \; \left(\mathcal{Y}_{\mathbb{A}}^{(g)}\right)^2 \prod_{i=1}^{4} \bar{\delta}(\mathcal{E}_i) \prod_{I \leq J} \bar{\delta}(u^{IJ})$$

$$= \mathcal{R}^4 \int \frac{d^{10}\ell^I}{\prod_I (\ell^I)^2} \int_{\mathfrak{M}_{0,4+2g}} c^{(g)} \left(\mathcal{J}^{(g)}\, \mathcal{Y}^{(g)}\right)^2 \prod_{A=1}^{4+2g}{}' \bar{\delta}(\mathcal{E}_A)$$

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$$\mathcal{A}^{(g)}_{\mathbb{A}} = \mathcal{R}^4 \int d^{10}\ell^I \quad \int_{\mathfrak{M}_{g,4}} \prod_{I\leq J} d\Omega_{IJ} \, \left(\mathcal{Y}^{(g)}_{\mathbb{A}}\right)^2 \prod_{i=1}^{4} \bar{\delta}(\mathcal{E}_i) \prod_{I\leq J} \bar{\delta}(u^{IJ})$$

$$= \mathcal{R}^4 \int \frac{d^{10}\ell^I}{\prod_I (\ell^I)^2} \int_{\mathfrak{M}_{0,4+2g}} c^{(g)} \left(\mathcal{J}^{(g)}\, \mathcal{Y}^{(g)}\right)^2 \prod_{A=1}^{4+2g}{}' \bar{\delta}(\mathcal{E}_A)$$

$$\mathcal{A}^{(0)} = \frac{\mathcal{R}^4}{s_{12}s_{13}s_{14}}$$

4-pt amplitudes for $g \leq 2$

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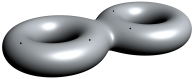
- **Type II superstring**
  chiral splitting form [D'Hoker,Phong '88, '05]



$$\mathcal{A}^{(g)}_{\mathbb{S}} = \mathcal{R}^4 \int_{\mathfrak{M}_{g,4}} \left| \prod_{I \leq J} d\Omega_{IJ} \right|^2 \int d^{10}\ell^I \; \left|\mathcal{Y}^{(g)}_{\mathbb{S}}\right|^2$$

$$\times \prod_{i<j} \left|E(z_i, z_j)\right|^{\alpha' s_{ij}/2} \left| e^{\frac{\alpha'}{2}\left(i\pi\, \Omega_{IJ}\, \ell^I \ell^J + 2\pi i \sum_j \ell^I k_j \int_{z_0}^{z_j} \omega_I\right)} \right|^2$$

Chiral integrands

- Observation 1:
  $\exists$ representations s.t.

$$\mathcal{Y}_{\mathbb{S}}^{(g)} \cong \mathcal{Y}_{\mathbb{A}}^{(g)} \qquad \text{mod} \ \ (d\text{-exact}, (\mathcal{E}, u))$$

  - superstring: mod d-exact terms, $\mathcal{Y}_{\mathbb{S}}^{(g)}$ independent of α'
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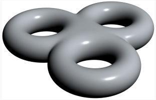
Assumptions

- straightforward extension of $\mathcal{A}_{\mathbb{S}}^{(g)}$ and $\mathcal{A}_{\mathbb{A}}^{(g)}$ to $g = 3$
  - Schottky problem for $g \geq 4$
  - non-projectedness of supermoduli space for $g \geq 5$ [Donagi,Witten '13; Witten '15]
  - scattering equations on nodal sphere for $g \geq 4$?



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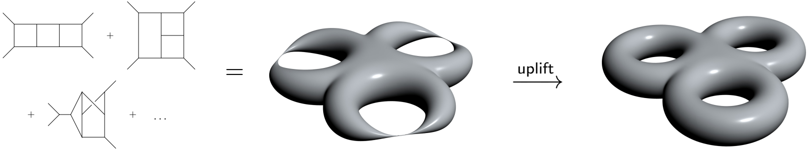
Strategy



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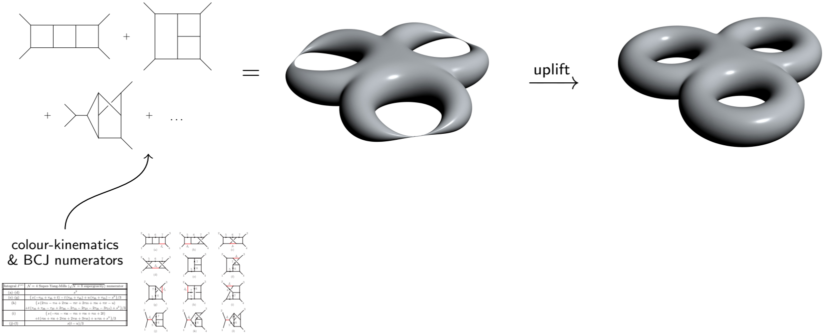

(i) start with supergravity loop integrand in a BCJ representation, $N^{(g)}$

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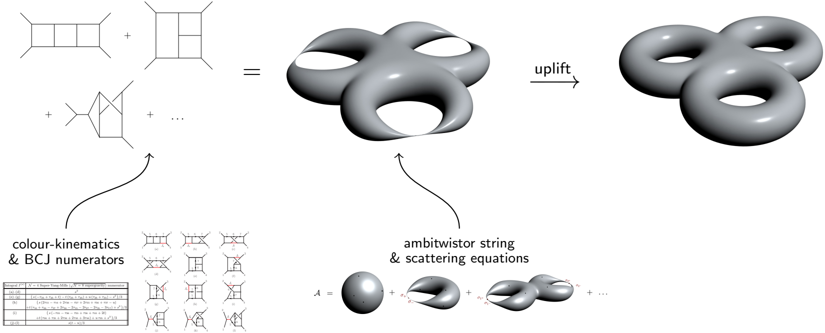

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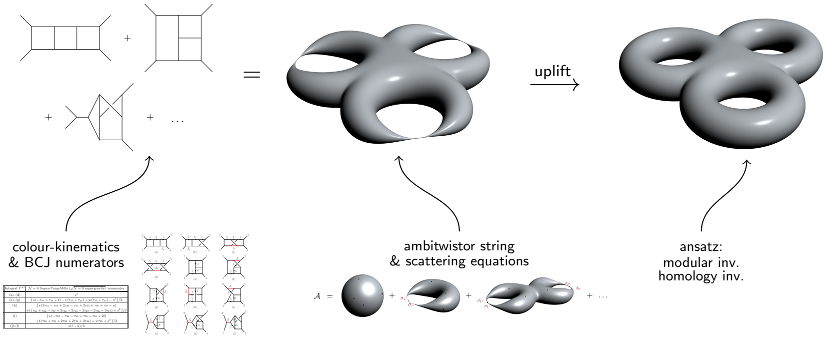

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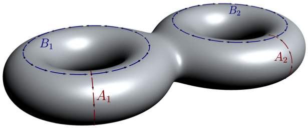
- $\mathcal{Y}^{(g)}_{\mathbb{A}}\big|_{\text{nodal}} = \mathcal{Y}^{(g)}$
- modular invariance

Proof of concept:
2-loop integrand



Upshot: reproduces known $\mathcal{Y}_{\mathbb{S}}^{(2)}$
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2-loop superstring amplitude from field theory

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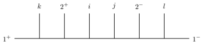
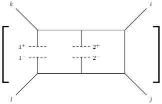
Numerators: $N^{(2)}\Big[$ 

$\Big] = N^{(2)}\Big[$ 

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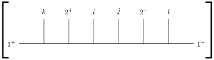
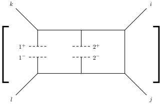
Numerators: $N^{(2)}\Big[$ 

$\Big] = N^{(2)}\Big[$ 

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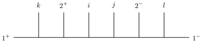
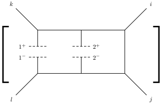
Numerators: $N^{(2)}\Big[$ 

$\Big] = N^{(2)}\Big[$ 

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Ansatz with correct modular weight:

- modular weight -2
- construct from $\Delta_{ij}^{(2)} = \varepsilon^{IJ}\omega_I(z_i)\omega_J(z_j)$

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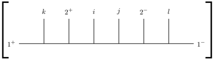
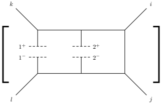
Numerators: $N^{(2)}\Big[$ 

$\Big] = N^{(2)}\Big[$ 

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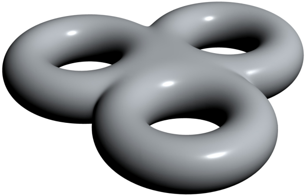
  - RNS superstring: $\Xi_8[\delta]/\Psi_{10}$ chiral measure

    $S^\delta(z_i, z_j)$ Szegő kernels

$n \leq 3$pt: $$\sum_\delta \Xi_8[\delta]\,(S^\delta_{..})^n_{\text{cyc}} = 0$$

4pt: $$\sum_\delta \frac{\Xi_8[\delta]}{\Psi_{10}}(S^\delta_{..})^4_{\text{cyc}} \cong \pi^4\left(\Delta^{(2)}_{12}\Delta^{(2)}_{34} - \Delta^{(2)}_{14}\Delta^{(2)}_{23}\right)$$

New results:
3-loop integrand



3 loops (i): BCJ representation

(i) Supergravity integrand in BCJ representation

[Bern, Carrasco, Johansson '10]

Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\left(-\tau_{35}+\tau_{45}+t\right)-t\left(\tau_{25}+\tau_{45}\right)+u\left(\tau_{25}+\tau_{35}\right)-s^2\right)/3$
(h)	$\left(s\left(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u\right)\right.$ $\left.+t\left(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}\right)+s^2\right)/3$
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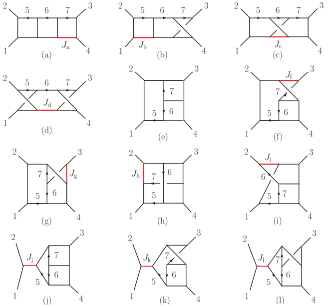
(j)-(l)	$s(t-u)/3$

$(\tau_{ir}=2k_i\cdot\ell_{r-4})$



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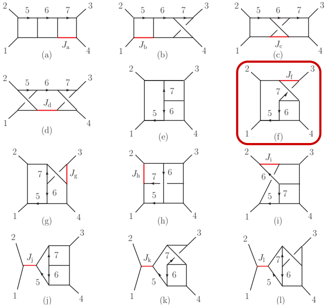
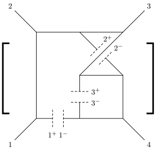


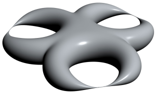


$$N^{(3)}\left[\;\cdot\;\right] = \frac{1}{3}\, s_{12}(s_{12}-s_{14}) + \frac{2}{3}\,\ell^1 \cdot k_2\,(s_{13}-s_{14})$$
$$+\frac{2}{3}\,\ell^1 \cdot \Big(k_3\,(s_{13}-s_{12}) + k_4\,(s_{12}-s_{14})\Big)$$

3 loops (ii): nodal sphere

(ii) Translate to nodal sphere
Use colour-kinematics on the worldsheet

$$(2\pi i)^4 \mathcal{J}^{(3)} \mathcal{Y}^{(3)} = \sum_{\alpha \in S_{6+2}} \frac{N^{(3)}(1^+ \alpha\, 1^-)}{(1^+ \alpha\, 1^-)}$$



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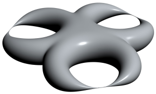

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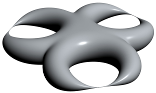

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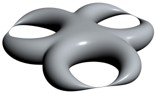

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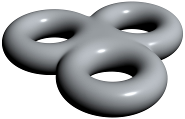
- Hyperelliptic locus $y^2 = \prod_{a=1}^{2g+2}(x - x_a)$: $\Psi_9 = 0$

$$\Psi_9 = \sqrt{-\prod_{\delta} \vartheta_\delta(0)} \quad \text{and} \quad \Psi_9\Big|_{\text{nodal}} = J_{\text{hyp}}\, J^{(2)}_{12} J^{(2)}_{13} J^{(2)}_{23} \prod \sigma^3_{I^+I^+}\, q^2_{II}$$

Take-away:
- $\mathcal{J}^{(3)}\mathcal{Y}^{(3)} \neq 0$ on hyperelliptic $J_{\text{hyp}} = 0$
- $\mathcal{Y}^{(3)}_{\mathbb{S}} \sim \frac{\chi_8(z_i)}{\Psi_9} + \ldots$

3 loops (iii): higher genus

$$\mathcal{Y}_{\mathbb{S}}^{(g)} = \ell_\mu^I \, \mathcal{Y}_I^\mu + \frac{\mathcal{Y}_0}{2\pi i}$$



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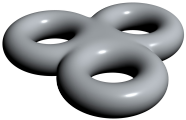

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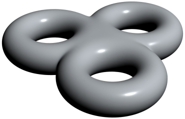

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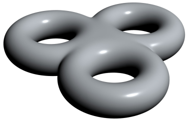

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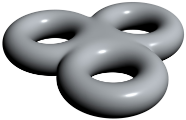

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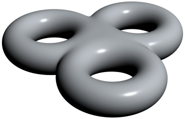

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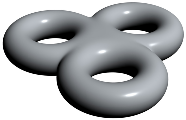

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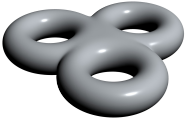

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  - sum over 36 even spin structures δ
  - chiral measure Ξ_8/Ψ_9 [C,DP,vG '08]

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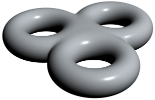

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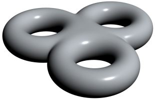

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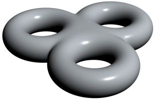

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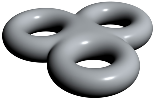

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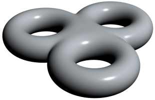
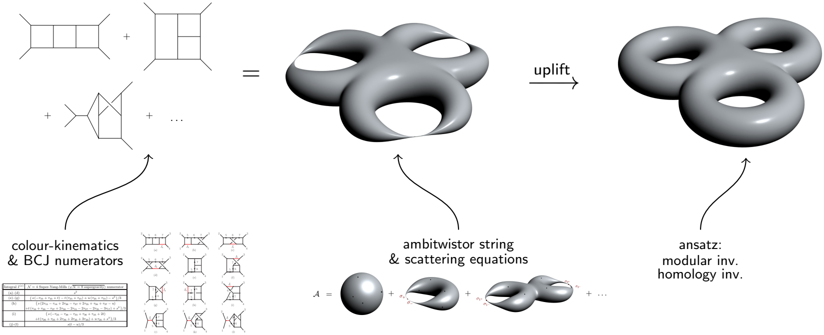
- Questions
  - simplification of $\mathcal{S}_{12,34}$
  - RNS origin of measure unclear [Witten '15]
  - Functional basis? $\leftrightarrow$ Uniqueness?



Outlook

- Strategy for importing field theory results to superstring



$\Rightarrow$ proposal for 4-pt 3-loop superstring amplitude

- Outlook
  - uniqueness?
  - stronger evidence / proof?
  - higher loops?

Thank you!