QCD scattering amplitudes at the precision frontier

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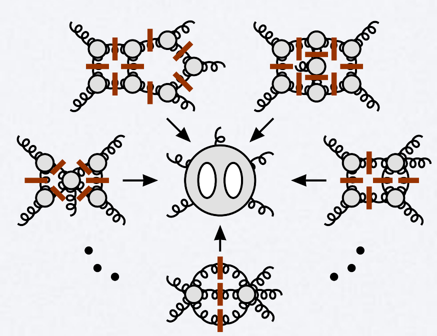
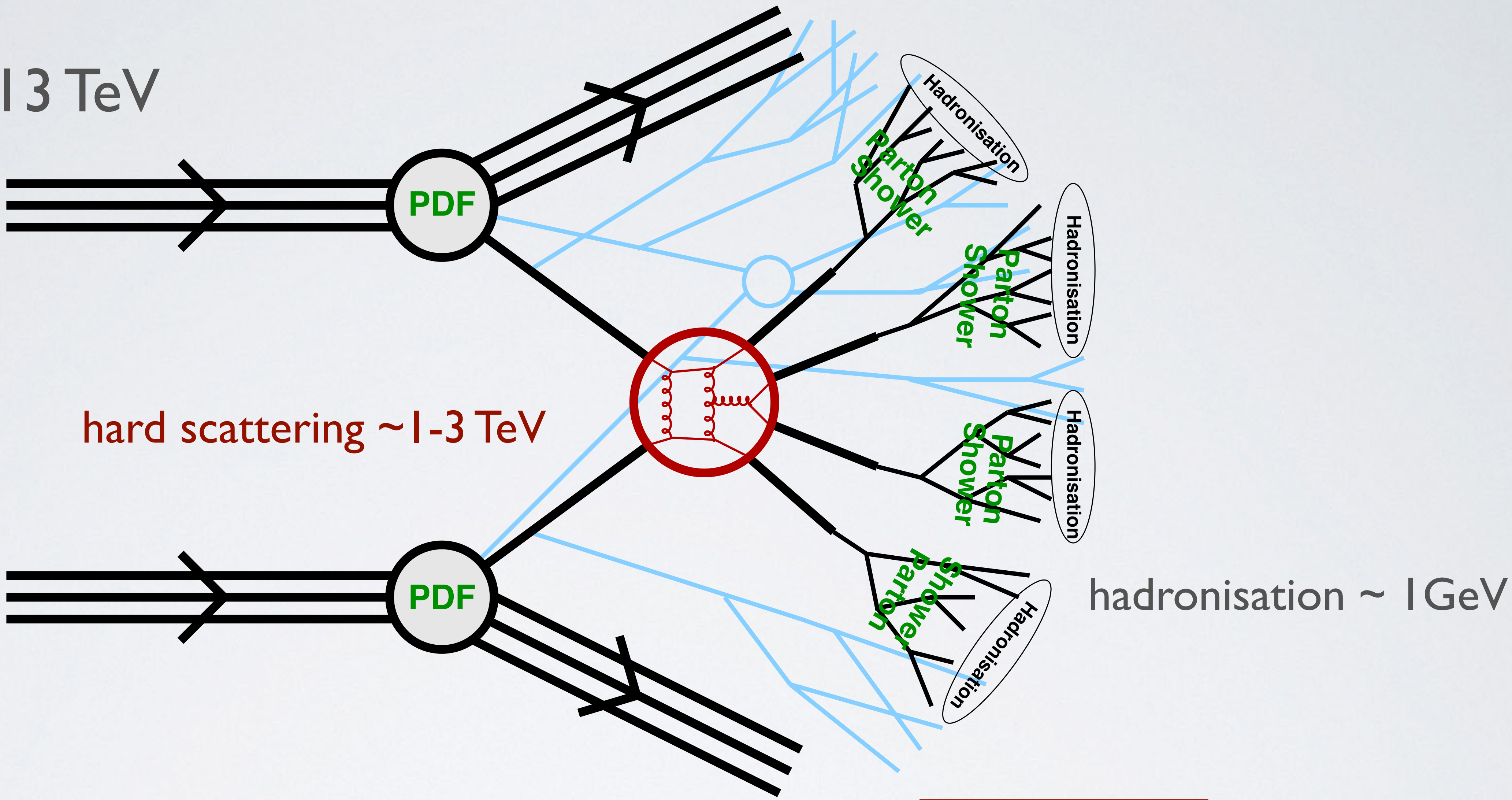
based on work with Brønnum-Hansen, Chaubey, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Marzucca, Moodie, Peraro, Kryś, Zoia

17th August 2021
Amplitudes Conference



hadron collider predictions

LHC : pp 13 TeV



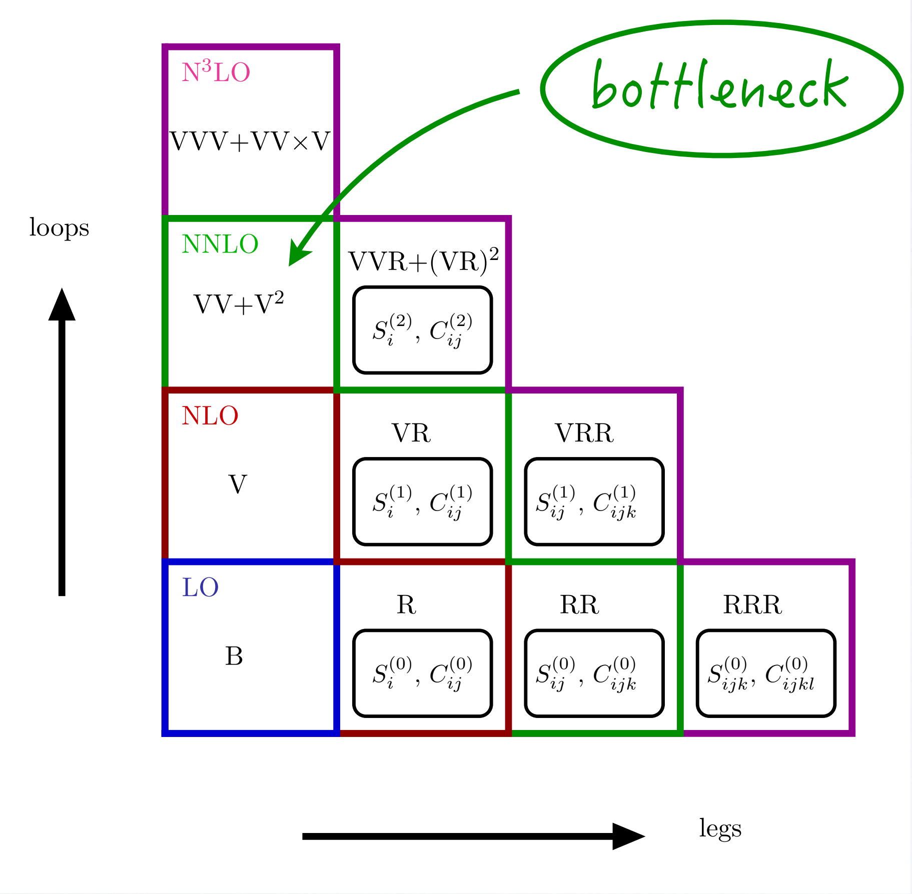
$$d\sigma(pp \to X) \sim \sum_{i,j \in \{g,u,d,\ldots\}} \int dx_1 dx_2 f_i(x_1, Q^2) f_j(x_2, Q^2) d\widehat{\sigma}(ij \to X)$$

$$d\widehat{\sigma} \sim d\Phi |\mathcal{A}|^2$$

precision QCD predictions

$$d\sigma = d\sigma^{\mathrm{LO}} + \alpha_s d\sigma^{\mathrm{NLO}} + \alpha_s^2 d\sigma^{\mathrm{NNLO}}$$

~10-30 % **~1-10 %**



complicated integrals

complicated phase-space

(d)σ N3LO (2→1)

[Anasastiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger]
[Berhing, Melnikov, Rietkerk, Tancredi, Wever]
[Dulat, Mistlberger, Pelloni] [Duhr, Dulat, Mistlberger]

Bernhard Mistlberger's talk

towards dσ N3LO 2→2

3-loop 4-point amplitudes
[Ahmed, Henn, Mistlberger]
[Jin, Luo] [Caola, von Manteuffel, Tancredi]

dσ NNLO (fully differential 2→3)

qq→3γ [Chawdhry, Czakon, Mitov, Poncelet]
[Kallweit, Sotnikov, Wiesemann]
qq→γγj [Chawdhry, Czakon, Mitov, Poncelet]
pp→3j [Czakon, Mitov, Poncelet]

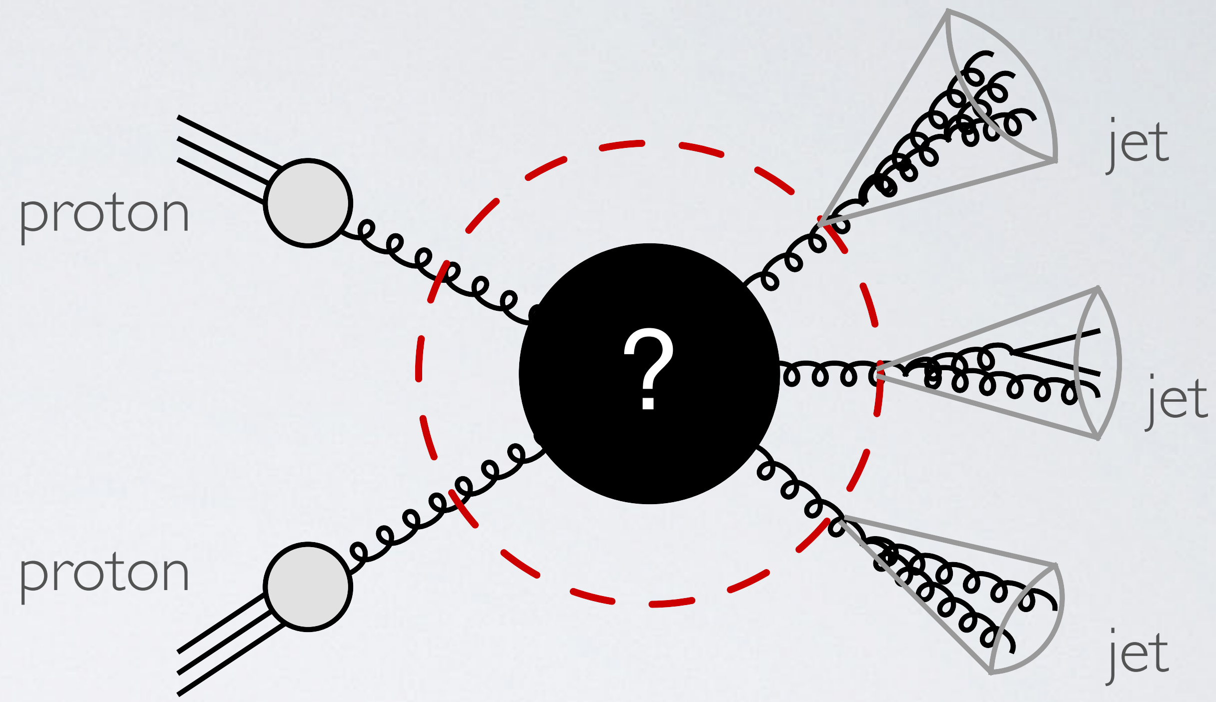
don't forget! (N)NLO EW, mass effects, resummation, showers…

precision frontier: 2 → 3 scattering

(2 → 3)/(2 → 2) ratio quantities become accessible

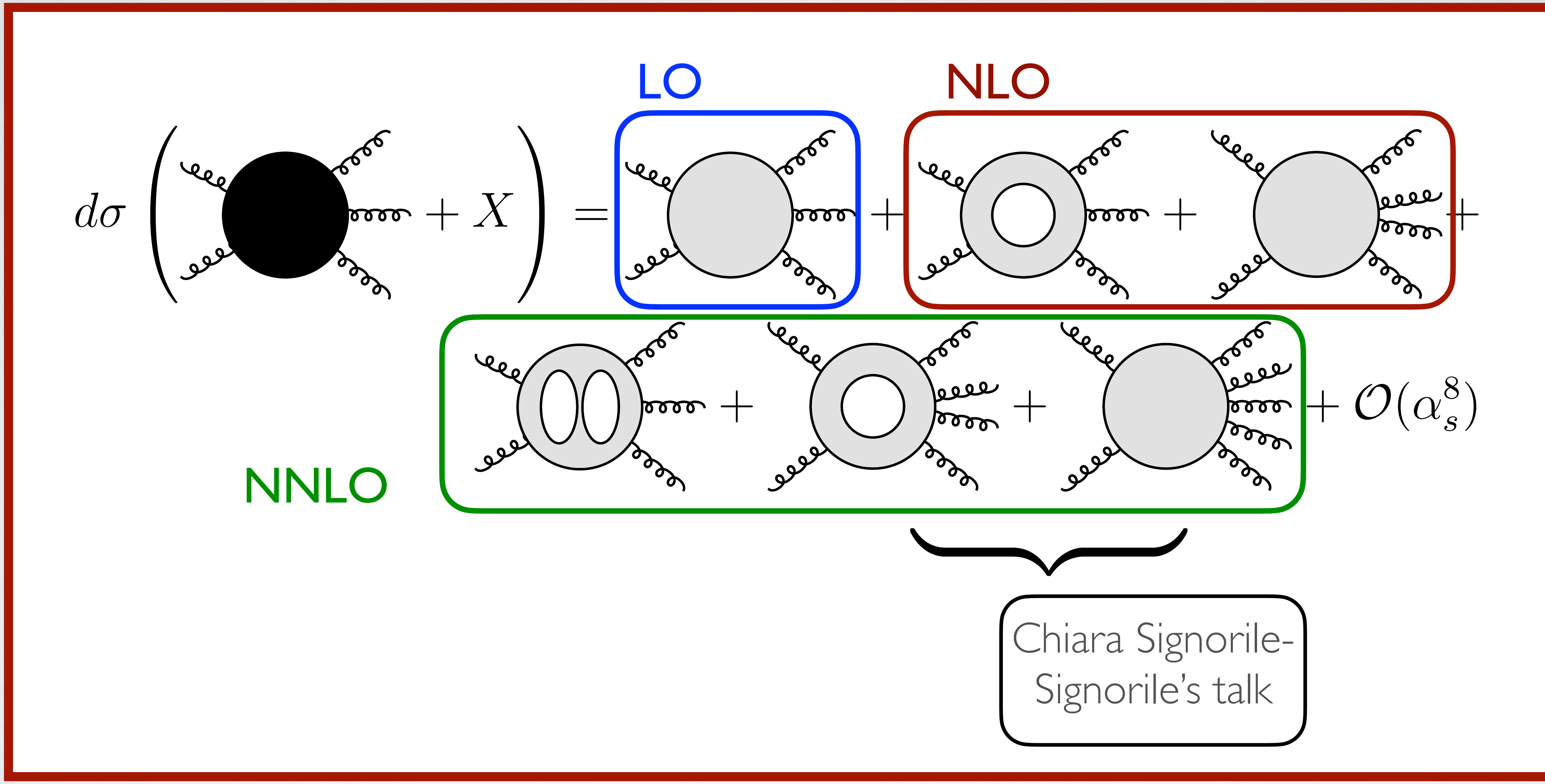
systematic errors cancel

high precision observables



process	precision observables
$pp \to 3j$	jet multiplicity ratios, α_s at high energies, 3-jet mass
$pp \to \gamma\gamma + j$	background to Higgs p_T, signal/background interference effects
$pp \to H + 2j$	Higgs p_T, Higgs coupling through vector boson fusion (VBF)
$pp \to V + 2j$	Vector boson p_T, W^+/W^- ratios and multiplicity scaling
$pp \to VV + j$	backgrounds to p_T spectra for new physics decaying via vector boson

precision frontier: 2 → 3 scattering



infrared subtraction problem is highly non-trivial,
plenty of new ideas needed here too of course!

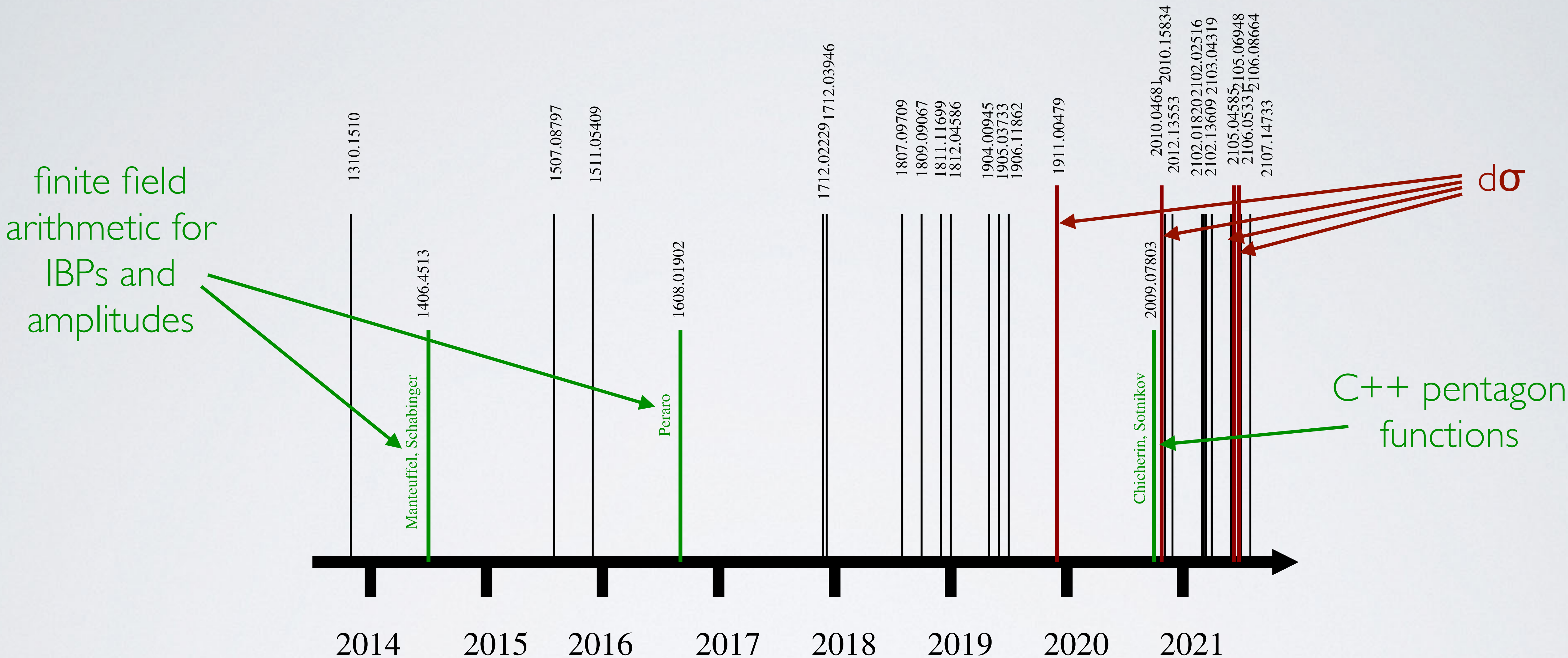
bottlenecks

many channels, colour, helicity configurations:
high level of automation required!

large intermediate expressions:
new methods needed to overcome
algebraic complexity

complicated
function spaces:
efficient and stable
numerical evaluation
required. need to
overcome analytic
complexity

precision frontier: 2 → 3 scattering



Abreu, Agarwal, SB, Brønnum-Hansen, Buccioni, Chawdhry, Chicherin, Czakon, Dixon, Dormans, Febres Cordero, Gehrmann, Hartanto, Heinrich, Henn, Herrmann, Ita, Kraus, Kryś, Lo Presti, Mitev, Mitov, Mogull, Ochirov, O'Connell, Page, Papadopoulos, Pascual, Peraro, Poncelet, Ruf, Sotnikov, Tancredi, Tommasini, von Manteuffel, Wasser, Wever, Zeng, Zhang, Zoia, ...

new results!

massless 5-particle scattering

a basis of pentagon functions identified!

Gehrmann, Henn, Lo Presti (2018)

Chicherin, Henn, Mitev (2018)

Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia (2020)

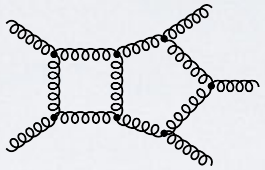
Abreu, Dixon, Herrmann, Page, Zeng (2020)

efficient numerical evaluations for all master integrals

Sotnikov, Chicherin (2020)

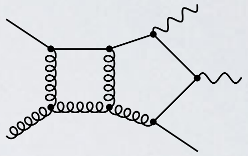
fast numerical codes for evaluation in physical region

pp→3j



Abreu, Febres-Cordero, Ita Page, Sotnikov [2102.13609]

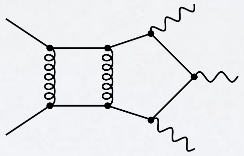
pp→γγ +j



Agarwal, Buccioni, von Manteuffel, Tancredi [2102.01820]

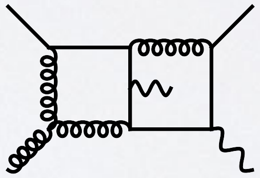
Chawdhry, Czkaon, Mitov Poncelet [2103.04319]

pp→γγγ

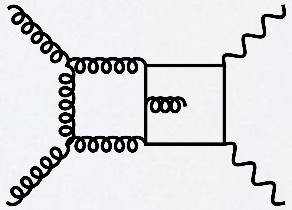


Chawdhry, Czkaon, Mitov Poncelet [2103.04319]

Abreu, Page, Pascual, Sotnikov [2010.15834]



Agarwal, Buccioni, von Manteuffel, Tancredi [2105.04585]



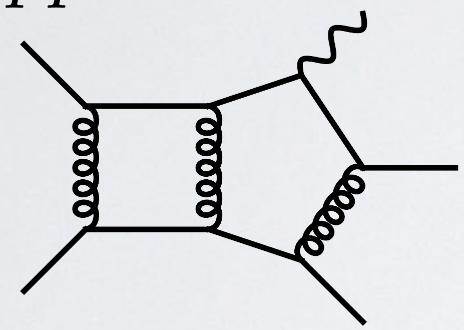
SB, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Marzucca, Moodie, Peraro, Krys, Zoia [2106.08664]

new results!

5-particle scattering with an off-shell leg

analytic finite remainders.
numerical evaluation with generalised series expansions

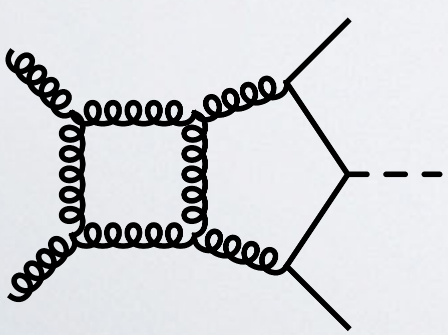
$pp \to W b\bar{b}$



leading colour, on-shell W, massless b

SB, Hartanto, Zoia
[2102.02516]

$pp \to H b\bar{b}$



leading colour, massless b

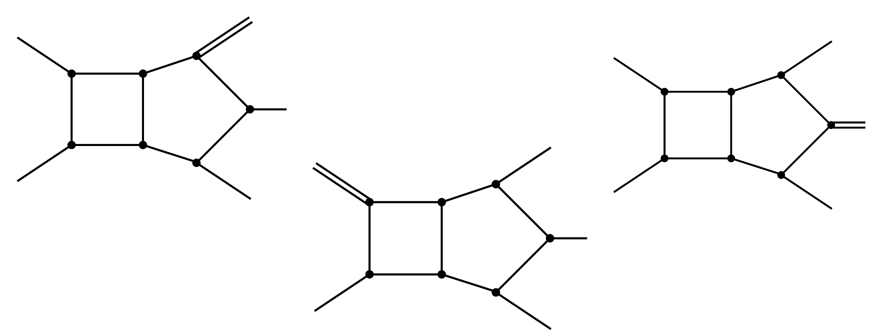
SB, Hartanto, Kryś, Zoia
[2107.14733]

all planar integrals known!

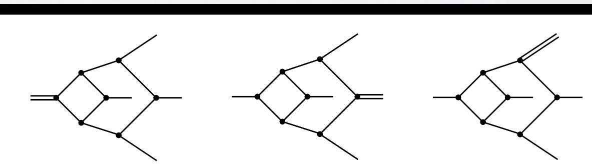
[Papadopoulos Tommasini, Wever (2019)]
[Padadopoulos, Wever (2019)]

[Abreu, Ita, Moriello, Page, Tschernow, Zeng (2020)]

[Canko, Padadopoulos, Syrrakos (2020)]
[Syrrakos (2020)]



non-planar hexa-box



[Abreu, Ita, Page, Tschernow 2107.14180]

new results!

analytic scattering amplitudes with massive propagators

well studied process - fully analytic form still challenging

numerical solutions very successful

[Baernreuther, Czakon, Chen, Fiedler, Poncelet (2008-2018)]

analytic solutions for qq→tt and all non-elliptic sectors of gg→tt known

[Bonciani, Ferroglia, Gehrmann, Studerus, von Manteuffel, Di Vita, Laporta, Mastrolia, Primo, Schubert, Becchetti, Casconi, Lavacca (2009-2019)]

$$gg \to t\bar{t}$$

leading colour helicity amplitudes with top-quark loops

SB, Chaubey, Hartanto, Marzucca [2102.13450]

$$e^+e^- \to \mu^+\mu^-$$

complete analytic form

Bonciani, Broggio, Di Vita, Ferroglia, Mandal, Mastrolia, Mattiazzi, Primo, Ronca, Schubert, Torres Bobadilla, Tramontano [2106.13179] bare amplitudes

$$A^{(L),4-2\epsilon} = \sum_i c_i(\epsilon, \{p\}) \mathcal{F}_i(\epsilon, \{p\})$$

rational functions

integrals/special functions

finite remainders

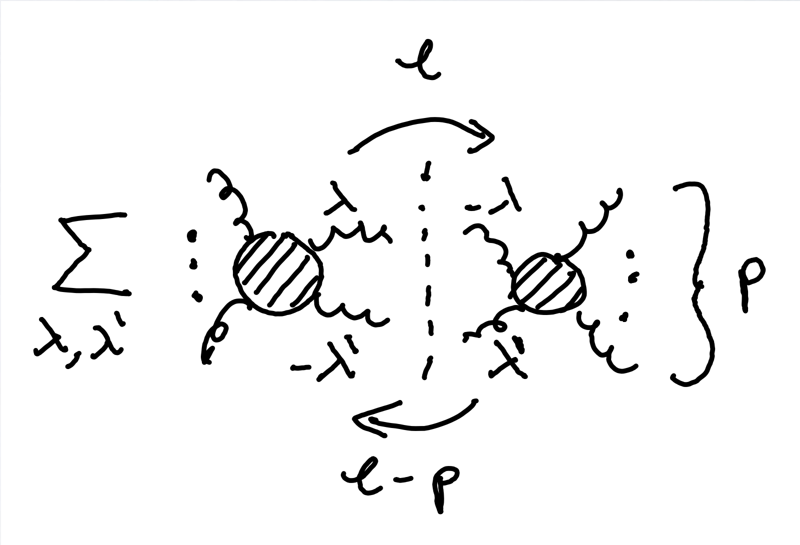
$$F^{(L)} = A^{(L),4-2\epsilon} - \sum_{k=1}^{L} I^{(k),4-2\epsilon} A^{(L-k),4-2\epsilon}$$

universal IR/UV poles

[Catani (1998)][Becher, Neubert (2009)]
[Magnea, Gardi (2009)]

integrands

unitarity cuts [Bern, Dixon, Dunbar, Kosower (1994)]



(also generalised unitarity, numerical unitarity, prescriptive unitarity…)

Feynman diagrams
not very elegant but…

helicity amplitudes
expose **on-shell** simplicity

(physical) projectors
[e.g. Binoth, Glover, Marquard, van der Bij (2002)]
[Peraro, Tancredi (2019,2020)][Chen (2020)]

off-shell recursion etc.
[Berends-Giele, OpenLoops]

reduction

integration-by-parts identities [Chetyrkin, Tkachov (1981)]

Algorithmic solution: large linear algebra problem [Laporta (2000)]

avoid large intermediate expressions with
finite field arithmetic

[von Mantueffel, Schabinger (2014)]
[Peraro (2016, 2018)]

FiniteFlow, FinRed, Fire6, Kira+FireFly, …

efficient solutions using computational algebraic geometry:
[Gluza, Kajda, Kosower, Schabinger, Larsen, Johansson, Ita, Abreu, Febres Cordero, Jaquier, Page, Zeng, Bosma, Boehm, Georgoudis, Schulze, Schoenemann, Zhang,…]

integrals

differential equations

Kotikov (1991), Bern, Dixon, Kosower (1993), Remiddi (1997), Gehrmann, Remiddi (2000), Henn (2013)

'canonical form'

$$\frac{d}{ds}\vec{MI}(s,\epsilon) = \epsilon A(s).\vec{MI}(s,\epsilon)$$

↙ iterated integrals

↘ poly-logarithms?

computational framework

QGRAF + FORM/MATHEMATICA + rational phase-space (Momentum Twistors) → colour ordered helicity amplitudes

$$M^{(2)}(\{p\},\epsilon) = \sum_i c_i(\{p\},\epsilon)\mathcal{F}_i(\{p\},\epsilon)$$

IBPs

$$M^{(2)}(\{p\},\epsilon) = \sum_i d_i(\{p\},\epsilon)\mathrm{MI}_i(\{p\},\epsilon)$$

IR/UV sub + expansion to function basis

$$F^{(2)}(\{p\}) = \sum_i e_i(\{p\})\mathrm{mon}_i(f_j^{(w)})$$

linear relations, univariate apart, polynomial reconstruction

complete reduction setup implemented in FINITEFLOW

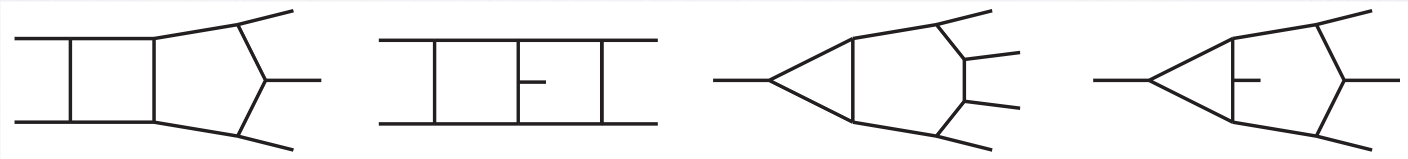
IBPs generated with help from LITERED/ FINITEFLOW

$gg \to \gamma\gamma g$

SB, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto,
Henn, Marcoli, Moodie, Peraro, Zoia [2106.08664]

$$\mathcal{A}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma) = g_s g_e^2 \left(Q_u^2 N_u + Q_d^2 N_d\right) f^{a_1 a_2 a_3} \sum_{\ell=1}^{\infty} \left(n_\epsilon \frac{\alpha_s}{4\pi}\right)^\ell A^{(\ell)}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma)$$

$$\begin{aligned} A^{(1)}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma) &= A_1^{(1)}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma)\,, \\ A^{(2)}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma) &= N_c A_1^{(2)}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma) \\ &\quad + \frac{1}{N_c} A_2^{(2)}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma) + n_f A_3^{(2)}(1_g, 2_g, 3_g, 4_\gamma, 5_\gamma) \end{aligned}$$



# permutations:	84	18	120	21

IBPs with LiteRed/FiniteFlow. Syzygy relations for planar sectors

analytic reconstruction

polynomial degrees in momentum twistor parametrisation

finite remainder	original	stage 1	stage 2	stage 3*	stage 4*
$F^{(2)}_{1;1}(1^-,2^-,3^+,4^+,5^+)$	69/60	28/20	24/0	19/10	11/5
$F^{(2)}_{1;0}(1^-,2^-,3^+,4^+,5^+)$	78/69	44/35	43/0	21/10	16/9
$F^{(2)}_{1;1}(1^-,2^+,3^+,4^-,5^+)$	59/55	30/27	29/0	18/15	17/4
$F^{(2)}_{1;0}(1^-,2^+,3^+,4^-,5^+)$	89/86	38/36	38/0	20/16	17/3
$F^{(2)}_{1;1}(1^+,2^+,3^+,4^-,5^-)$	40/42	25/27	25/0	15/18	15/0
$F^{(2)}_{1;0}(1^+,2^+,3^+,4^-,5^-)$	66/66	32/33	32/0	13/13	12/3

stage 1: linear relations

stage 2: factor matching

stage 3*: univariate apart

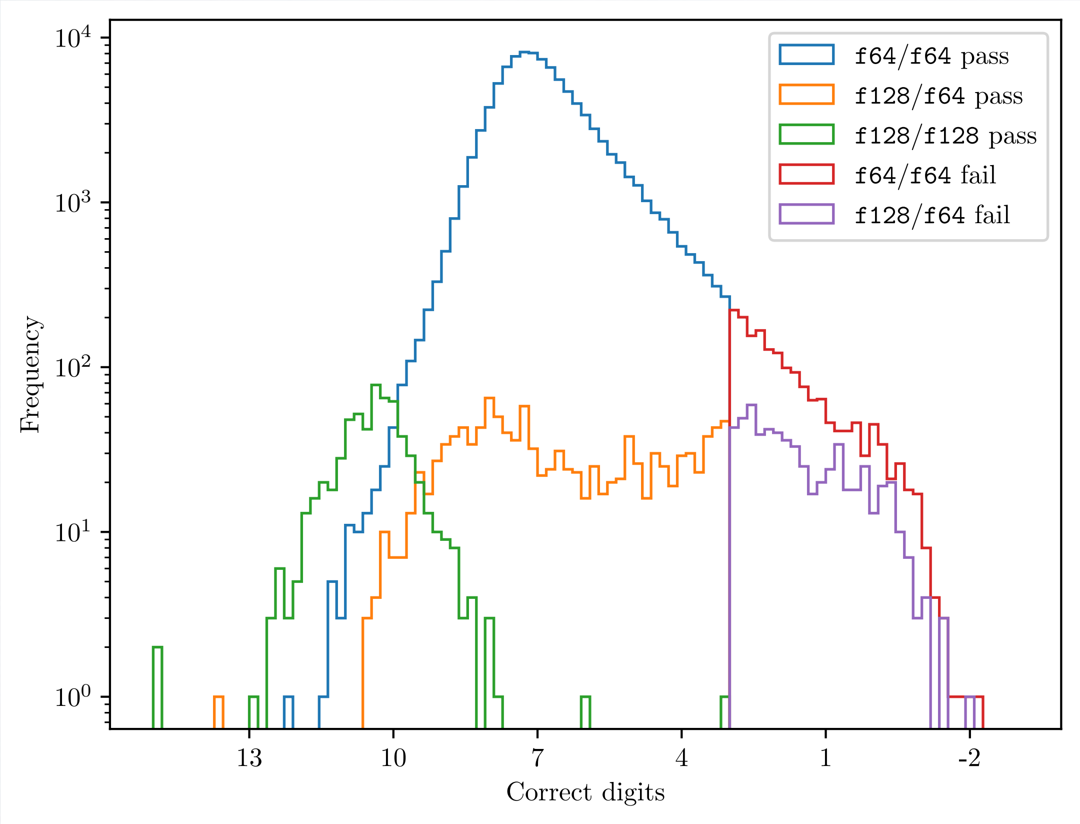
stage 4*: univariate apart +factor matching

numerical performance

complete colour and
helicity summed hard functions

C++ code available at
https://bitbucket.org/njet/njet

100,000 points over
physical phase-space



average evaluation time including stability tests and higher precision corrections ~ **26s per point**

Aside: framework also used to compute leading colour $pp \to 3j$

full agreement with 2102.13609v2

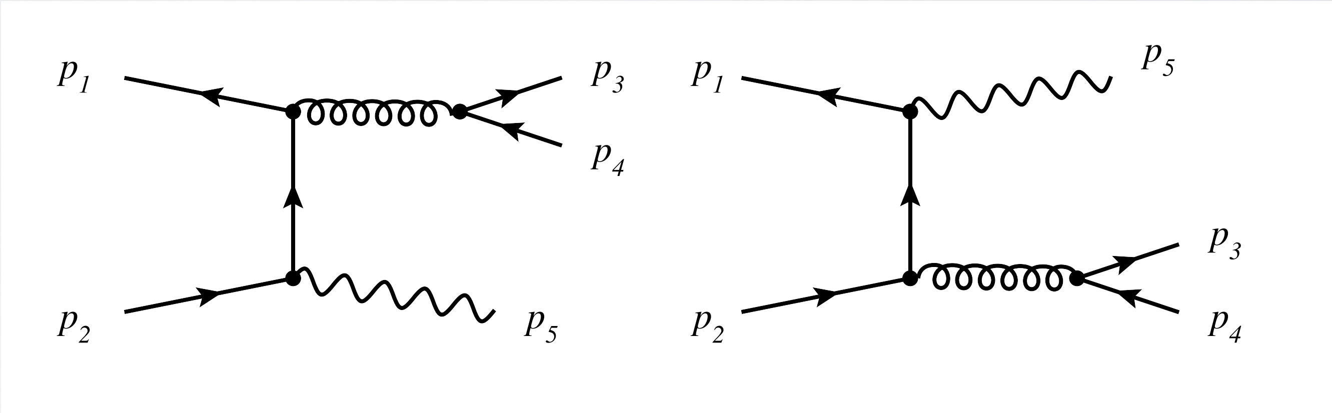
will be included in next NJET version

$pp \to Wb\bar{b}$

SB, Hartanto, Zoia [2102.02516]

two leading order diagrams

background to associated HW production



$$\bar{d}(p_1) + u(p_2) \to b(p_3) + \bar{b}(p_4) + W^+(p_5)$$

6 scalar invariants

$$s_{12} = (p_1 + p_2)^2\,, \quad s_{23} = (p_2 - p_3)^2\,, \quad s_{34} = (p_3 + p_4)^2\,,$$
$$s_{45} = (p_4 + p_5)^2\,, \quad s_{15} = (p_1 - p_5)^2\,, \quad s_5 = p_5^2\,. \qquad (4)$$

- leading colour approximation
- on-shell W boson

$$i\sum_\lambda \varepsilon_W^{\mu *}(p_5,\lambda)\varepsilon_W^{\nu}(p_5,\lambda) = -g^{\mu\nu} + \frac{p_5^\mu p_5^\nu}{m_W^2}.$$

special functions

58 letters including 3 square roots

thanks to work of [Abreu et al (2020)]:

↓

also in this work: numerical evaluation with generalised series expansions [Moriello (2019)]

$$d\overrightarrow{\mathrm{MI}} = \epsilon \sum_{i=1}^{58} a_i d\log w_i \overrightarrow{\mathrm{MI}}$$

this form allows easy expansion to Chen iterated integrals

[Chen (1977)]

[Canko, Papadopoulos, Syrakkos (2020)]

obey shuffle relations ⇒ basis

of independent function

(lengthy) GPL expressions from simplified differential equations

function basis

- use **master integral components as function basis**
  - $\Rightarrow$ $MI_i^{(k)}$ for the $\mathcal{O}(\epsilon^k)$ component of the i^{th} master integral
- high precision evaluation of GPL form (~1000 digits)
  - $\Rightarrow$ **analytic boundaries via PSLQ**
- **determine relations** between integral components by solving linear system
  - $\Rightarrow$ $f_i^{(k)}$ for the function at weight k
- derive **new differential equation** for independent integral components
- find **analytic cancellation of IR poles**

numerical evaluation

gen. series exp. only with $f_i^{(k)}$ in finite remainder

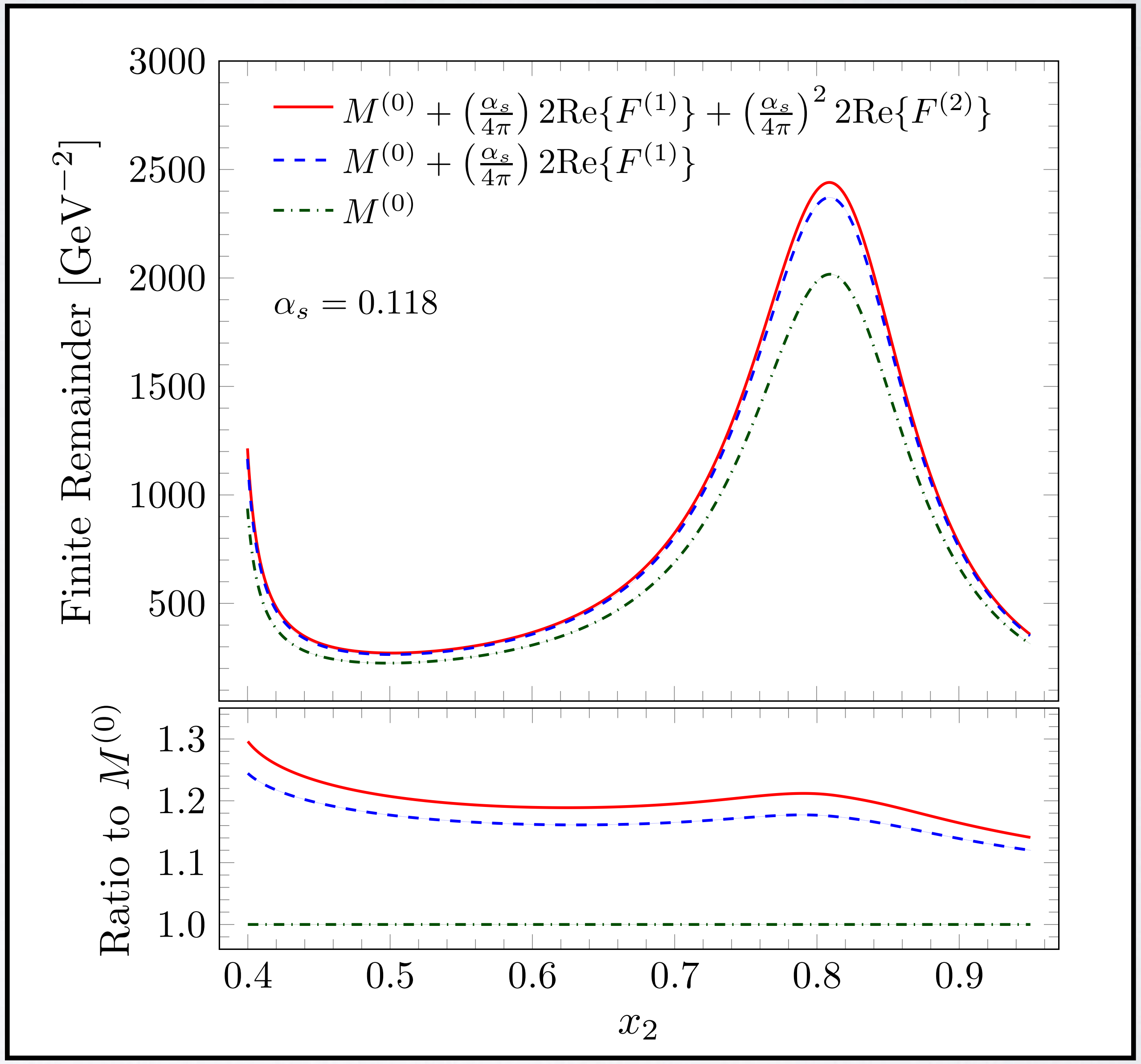
evaluate with DiffExp [Hidding (2020)]

$$p_3 = \frac{x_1\sqrt{s}}{2}\,(1,1,0,0)\ ,$$
$$p_4 = \frac{x_2\sqrt{s}}{2}\,(1,\cos\theta,-\sin\phi\sin\theta,-\cos\phi\sin\theta)$$
$$p_5 = \sqrt{s}\,(1,0,0,0) - p_3 - p_4\,,$$

$$\cos\theta = 1 + \frac{2}{x_1x_2}\left(1 - x_1 - x_2 - \frac{m_W^2}{s}\right)$$

$$s = 1, m_W^2 = 0.1, \phi = 0.1, x_1 = 0.6$$

evaluation time ~ 260s per point



$pp \to Hb\bar{b}$

SB, Hartanto, Kryś, Zoia [2107.14733]

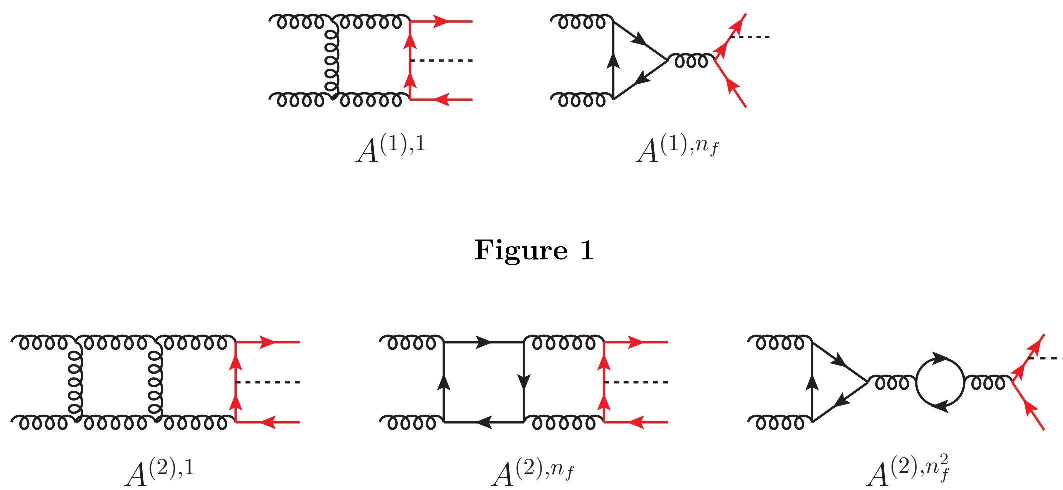


Figure 1

$$A^{(0)}(1^+_{\bar{b}}, 2^+_b, 3^+_g, 4^+_g, 5_H) = \frac{s_5}{\langle 23\rangle\langle 34\rangle\langle 41\rangle},$$

$$A^{(0)}(1^+_{\bar{b}}, 2^+_b, 3^-_g, 4^-_g, 5_H) = -\frac{[12]^2}{[23][34][41]},$$

$$A^{(0)}(1^+_{\bar{b}}, 2^+_b, 3^+_g, 4^-_g, 5_H) = \frac{\langle 24\rangle\langle 4|5|1]^2}{s_{234}\langle 23\rangle\langle 34\rangle\langle 2|5|1]} - \frac{s_5[13]^3}{s_{134}[14][34]\langle 2|5|1]},$$

complete set of leading colour two-loop helicity amplitudes (incl n_f terms)

$$0 \to \bar{b}(p_1) + b(p_2) + g(p_3) + g(p_4) + H(p_5)\,,$$
$$0 \to \bar{b}(p_1) + b(p_2) + \bar{q}(p_3) + q(p_4) + H(p_5)\,,$$
$$0 \to \bar{b}(p_1) + b(p_2) + \bar{b}(p_3) + b(p_4) + H(p_5)\,,$$

$\bar{b}bggH$	helicity configurations	$r_i(x)$	independent $r_i(x)$	partial fraction in x_5	number of points
$F^{(2),1}$	$++++$	63/57	52/46	20/6	3361
	$+++-$	135/134	119/120	28/22	24901
	$++--$	105/111	105/111	22/12	4797
$F^{(2),n_f}$	$++++$	45/41	45/41	16/6	1381
	$+++-$	94/95	94/95	17/6	1853
	$++--$	89/95	62/69	18/3	2492
$F^{(2),n_f^2}$	$++++$	12/8	9/7	0/0	3
	$+++-$	11/16	11/16	3/0	22
	$++--$	12/20	8/16	8/0	242

$pp \to Hb\bar{b}$

finite remainder basis functions

$$f_i^{(k)} \to h_i^{(k)}$$

7 letters drop out in finite remainders

188 → 23 weight 4 functions

(same basis for $pp \to Wb\bar{b}$)

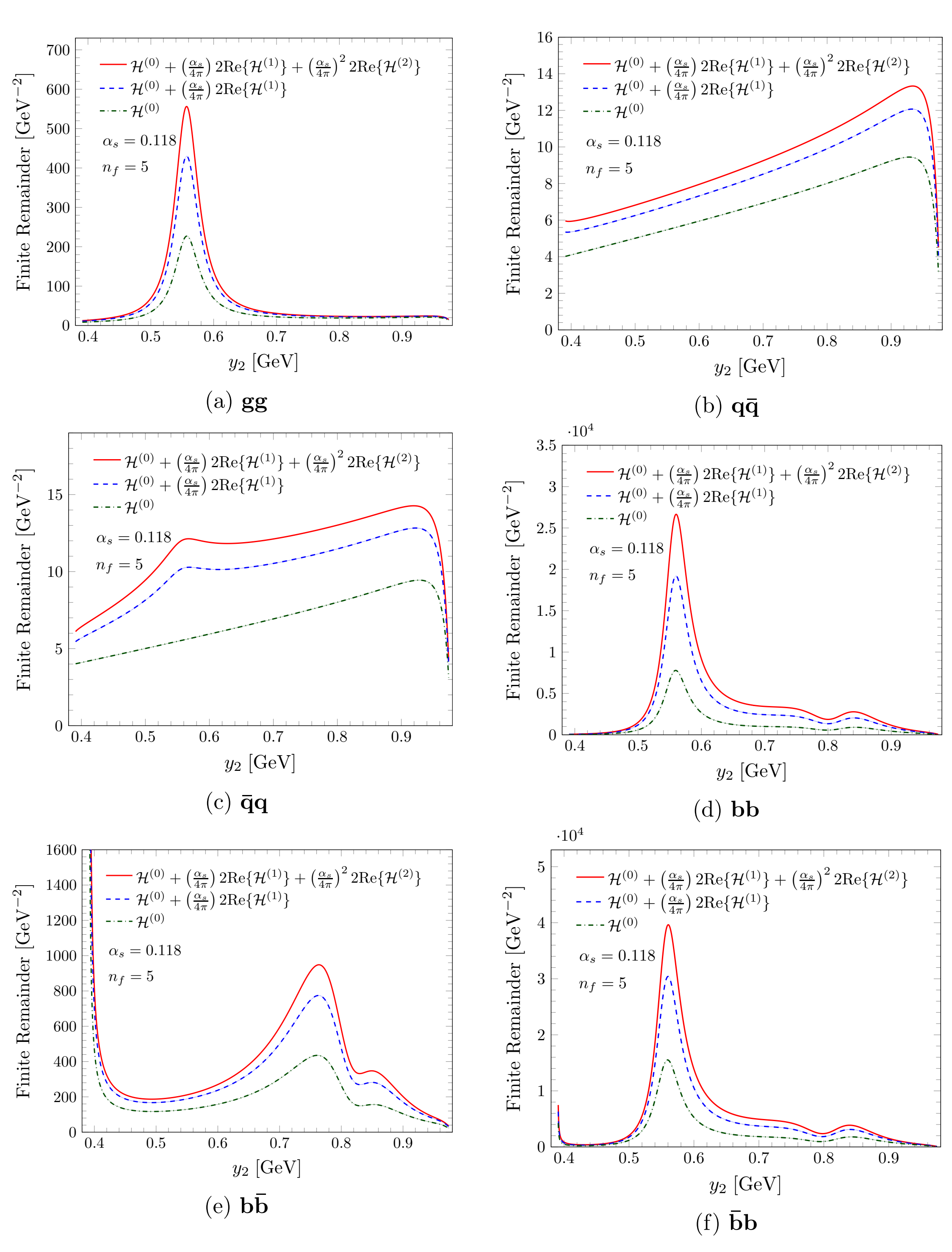
analytic subtraction of IR/UV poles

UV counter-terms required to renormalise Yukawa coupling

$pp \to Hb\bar{b}$

channel	Re $\mathcal{H}^{(2),1}$	Re $\mathcal{H}^{(2),n_f}$	Re $\mathcal{H}^{(2),n_f^2}$
gg	156680.6267	−41215.80337	405.9379563
$\mathbf{q\bar{q}}$	0.09391314268	−0.02045942258	−0.004225713438
$\mathbf{\bar{q}q}$	0.3494872243	−0.08069122736	−0.004225713438
$\mathbf{b\bar{b}}$	48640.80398	−26530.01855	2458.442153
$\mathbf{\bar{b}b}$	−141130.5373	42183.03094	3711.445449
$\mathbf{bb/\bar{b}\bar{b}}$	−53679.25708	1988.662899	894.7895467

$$p_1 = \frac{y_1\sqrt{s}}{2}\,(1\,,1\,,0\,,0)\ ,$$
$$p_2 = \frac{y_2\sqrt{s}}{2}\,(1\,,\cos\theta\,,-\sin\theta\sin\phi\,,-\sin\theta\cos\phi)\ ,$$
$$p_3 = \frac{\sqrt{s}}{2}\,(-1\,,0\,,0\,,-1)\ ,$$
$$p_4 = \frac{\sqrt{s}}{2}\,(-1\,,0\,,0\,,1)\ ,$$

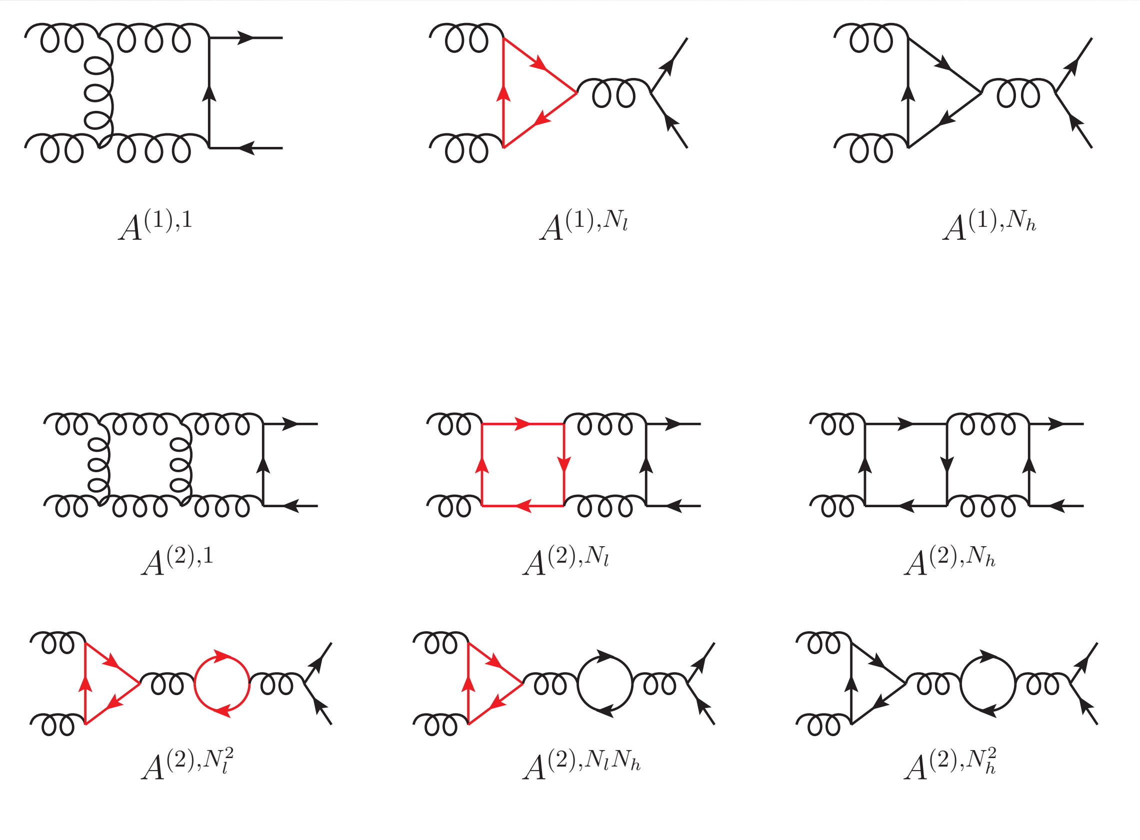


(a) **gg** (b) $\mathbf{q\bar{q}}$ (c) $\mathbf{\bar{q}q}$ (d) **bb** (e) $\mathbf{b\bar{b}}$ (f) $\mathbf{\bar{b}b}$

$gg \to t\bar{t}$

SB, Chaubey, Hartanto, Marzucca [2102.02516]

massive internal propagators have always challenged analytic methods



numerical solutions very successful even if computationally intensive
[Baernreuther, Czakon, Chen, Fiedler, Poncelet (2008-2018)]

analytic solutions for qq→tt and all non-elliptic sectors of gg→tt known
[Bonciani, Ferroglia, Gehrmann, Studerus, von Manteuffel, Di Vita, Laporta, Mastrolia, Primo, Schubert, Becchetti, Casconi, Lavacca (2009-2019)]

helicity amplitudes

include full top-quark decays efficiently

e.g. at one-loop [Melnikov, Schulze (2008)]

$$p^{\flat,\mu} = p^\mu - \frac{m^2}{2p.n} n^\mu$$

$$u_+(p,m) = \frac{(\not{p}+m)|n\rangle}{\langle p^\flat n\rangle}, \qquad u_-(p,m) = \frac{(\not{p}+m)|n]}{[p^\flat n]},$$
$$v_-(p,m) = \frac{(\not{p}-m)|n\rangle}{\langle p^\flat n\rangle}, \qquad v_+(p,m) = \frac{(\not{p}-m)|n]}{[p^\flat n]}.$$

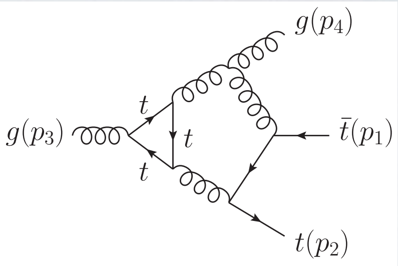
$$\begin{aligned} A^{(L)}(1_{\bar{t}}^+, 2_t^+, 3^{h_3}, 4^{h_4}; n_1, n_2) = m \frac{\Phi^{h_3 h_4}}{\langle 1^\flat n_1\rangle \langle 2^\flat n_2\rangle} \Bigg(& \langle n_1 n_2\rangle A^{(L),[1]}(1_{\bar{t}}^+, 2_t^+, 3^{h_3}, 4^{h_4}) \\ & + \frac{\langle n_1 3\rangle\langle n_2 4\rangle}{\langle 34\rangle} A^{(L),[2]}(1_{\bar{t}}^+, 2_t^+, 3^{h_3}, 4^{h_4}) \\ & + \frac{s_{34}\langle n_1 3\rangle\langle n_2 3\rangle}{\langle 3|14|3\rangle} A^{(L),[3]}(1_{\bar{t}}^+, 2_t^+, 3^{h_3}, 4^{h_4}) \\ & + \frac{s_{34}\langle n_1 4\rangle\langle n_2 4\rangle}{\langle 4|13|4\rangle} A^{(L),[4]}(1_{\bar{t}}^+, 2_t^+, 3^{h_3}, 4^{h_4}) \Bigg) \end{aligned}$$

obtain rational parametrisation in terms of only 2 variables (via Momentum twistors, m_t=1)

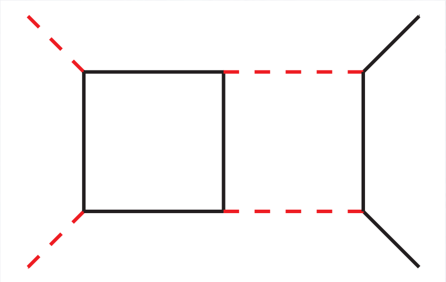
$$-\frac{s}{m_t^2} = \frac{(1-x)^2}{x}, \qquad \frac{t}{m_t^2} = y.$$

master integrals

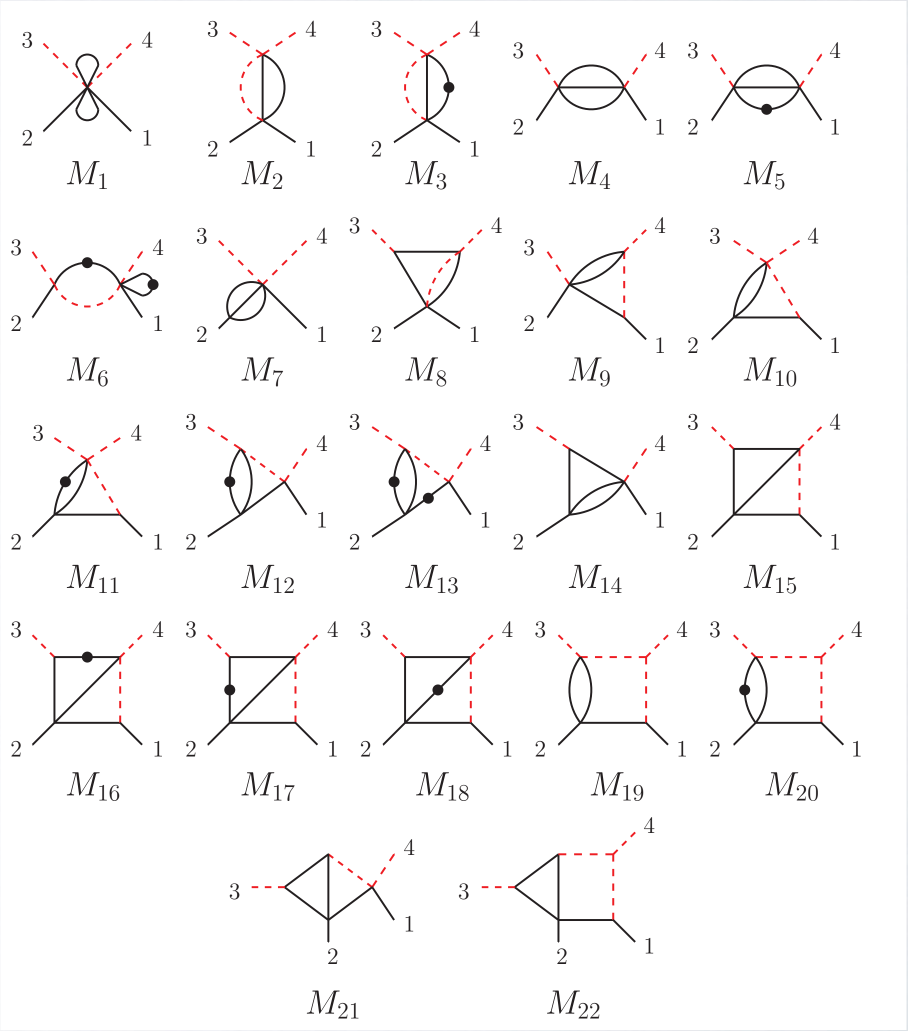
top-box integral recently computed analytically in terms of iterated integrals over **3 elliptic curves**



[Adams, Chaubey, Weinzierl (2018)]



one missing integral in the amplitude



derived canonical form DE obtained **iterated integral form** and boundary constants

analytic finite remainders

direct reconstruction of finite remainders

	monomials	monos. with rels.
amplitude	12025	11791
finite remainder	3586	3158

$$\begin{aligned} I(a_{3,3}^{(b)}, f, \ldots) &= \int a_{3,3}^{(b)} I(f, \ldots) \\ &= \int d\left(\psi_1^{(b)} \frac{x(-1+y)}{\pi(-1+x)^2}\right) \cdot I(f, \ldots) \\ &= \left[\psi_1^{(b)} \frac{x(-1+y)}{\pi(-1+x)^2} I(f, \ldots)\right]_{(0,1)}^{(x,y)} - I\left(\psi_1^{(b)} \frac{x(-1+y)}{\pi(-1+x)^2} f, \ldots\right) \end{aligned}$$

additional function relations necessary to cancel IR poles - beyond shuffle relations

- test evaluations match with previous numerical results
- analytic continuation of iterated integrals needs further investigation

lots more to understand! → talks by Klemm, Weinzierl and Duhr

outlook

- techniques for **multi-scale two-loop amplitudes** reaching maturity
- **modular arithmetic** has played a key role in reducing complexity
- **fast and reliable analytic expressions** for many processes
- looking forward to phenomenological applications

a lot of progress since LH SMWG 2019! [2003.01700]

$pp \to 3\,\text{jets}$	$\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$	NNLO_{QCD}

Czakon et al. [2106.05331]

$pp \to \gamma\gamma + j$	NLO_{QCD} NLO_{EW}	$\text{NNLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$
$pp \to \gamma\gamma\gamma$	NNLO_{QCD}	

Kallweit et al. [2010.04681]
Chawdhry et al. [1911.00479]

$pp \to V + 2j$	$\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ NLO_{EW}	NNLO_{QCD}
$pp \to V + b\bar{b}$	NLO_{QCD}	$\text{NNLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$

$pp \to H + 2j$	$\text{NLO}_{\text{HTL}} \otimes \text{LO}_{\text{QCD}}$ $\text{N}^3\text{LO}_{\text{QCD}}^{(\text{VBF}^*)}$ (incl.) $\text{NNLO}_{\text{QCD}}^{(\text{VBF}^*)}$ $\text{NLO}_{\text{EW}}^{(\text{VBF})}$	$\text{NNLO}_{\text{HTL}} \otimes \text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ $\text{NNLO}_{\text{QCD}}^{(\text{VBF})} + \text{NLO}_{\text{EW}}^{(\text{VBF})}$

first (leading colour) amplitudes

$pp \to t\bar{t} + j$	NLO_{QCD} (w/ decays) NLO_{EW}	$\text{NNLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ (w/ decays)

Chawdhry et al. [2105.05331]