Non-relativistic physics in AdS and its CFT dual

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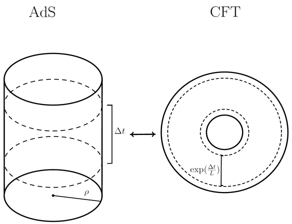
Motivation

The study of the correspondence between AdS spacetime and CFT has started more than 20 years ago with Maldacena 98' and Witten 98':



Theory of quantum gravity in d+1-dim AdS spacetime $\Longleftrightarrow$ d-dim CFT

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Even though this correspondence has been studied for 20 years, there are much more physical insights and intuition left to be understood. In this direction we aim to study the duality in the well-known Non-relativistic limit.

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In particular some questions we focus on are as follows:

- Can we understand the CFT scaling dimension and OPE spectrum (Regge trajectories) in the NR limit?
- Can we understand non-relativistic Flat space scattering from the CFT correlator?
  Expected when we take the AdS radius to infinity.

The full relativistic version of these questions have been of interest for many years and have been massively studied in different limit:

Polchinski, Giddings, Gary, Penedones, Kaplan, Fitzpatrick, Katz, Poland, Simmons-Duffin, Raju, van Rees, Caron-huot,...

Non-relativistic two-body problem in AdS

- The metric of global AdS is

$$ds^2 = -c^2 f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2$$

where $f(r) = \frac{r^2}{L^2} + 1$.

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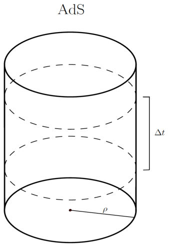
where $f(r) = \frac{r^2}{L^2} + 1$.
In the global coordinate which will be used often during the talk we have: $\tan\rho \equiv \frac{r}{L}$ so pictorially we get:



We are interested in non-relativistic dynamics so if τ is the proper time of a geodesic in this background we get $\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}$. This in turn means we can identify the time component of the metric as follows:

$$g_{tt} = -(c^2 + 2\Phi)$$

Where Φ is the gravitational potential given as:

$$\Phi = \frac{1}{2}\omega^2 r^2$$

with $\omega = \frac{c}{L}$. This potential confines the dynamics of non-relativistic particles to $r \ll L$. We can now write the Hamiltonian of the NR particles in AdS:

$$H = \sum_{i=1}^{2} \left(\frac{p_i^2}{2m_i} + \frac{1}{2} m_i \omega^2 r_i^2 \right) + V(r)$$

Outline

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Conformal Symmetry

We start by studying the symmetry algebra in order to find out which state representation trivializes them and hence are simpler.

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The symmetry algebra of AdS_{d+1} is SO(d,2) which is spanned by the dilatation operator D, momentum P_i, special conformal transformation K_i and rotation $M_{ij} = -M_{ji}$. We have:

$$D^\dagger = D, \quad M_{ij}^\dagger = M_{ij}, \quad K_i^\dagger = P_i$$

The algebra is given by:

$$[D, P_i] = P_i, \quad [D, K_i] = -K_i, \quad [K_i, P_j] = 2\delta_{ij} D - 2iM_{ij}$$

Along with the $SO(d)$ algebra of rotation for M_{ij}. Also K_i and P_i transform like vectors under rotation.

Taking Non-relativistic limit of the Symmetry Group

The dilation operator is the time translate operator in global coordinates in AdS so, $D = i\partial_t$. So it is our Hamiltonian. In the NR limit for particles of total mass M, this will become the rest-energy Mc^2 plus the non-relativistic Hamiltonian:

$$D \sim M + H_{NR}$$

The conformal algebra then reduces to:

$$[H_{NR}, P_i] = P_i, \quad [H_{NR}, K_i] = -K_i, \quad [K_i, P_j] = 2\delta_{ij}M.$$

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Expressing the action of P_i in global coordinate we have:

$$P_i \sim (mx_i - ip_i)$$

P_i acts linearly in mx_i and p_i thus it acts on the center of mass wavefunction

Non-relativistic Symmetry Group of AdS

P_i looks familiar from quantum mechanics! It is creation operator forthe harmonic oscillator, $P_i \sim A_i^\dagger$. We can then rewrite the algebra:

$$[H_{NR}, A_i^\dagger] = A_i^\dagger, \quad [H_{NR}, A_i] = -A_i, \quad [A_i, A_j^\dagger] = \delta_{ij}$$

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From this we we can rewrite H_{NR}:

$$H_{NR} = (A_i^\dagger \cdot A_i + \frac{d}{2}) + H_{\text{Relative}}.$$

We focus on the primary states by considering states annihilated by K_i (or equivalently A_i). This trivialized the symmetry and correspond to the ground state of the harmonic oscillator.

The energy shift of the primary two-particle states $[OO]_{E,\ell}$ comes from the H_{relative}

Outline

CFT correlation Function

The correlation function of 4 scalar primary operators is given as

$$\langle \phi_1^\dagger(x_1)\phi_2^\dagger(x_2)\phi_1(x_3)\phi_2(x_4)\rangle = \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_1+\Delta_2}{2}}}\mathcal{G}(z,\bar{z}),$$

where $x_{ij} = x_i - x_j$ and the conformal cross-ratios z, $\bar{z}$ are defined as

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\ , \qquad (1-z)(1-\bar{z}) = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}\ .$$

In the NR limit we have $\Delta_1, \Delta_2 \gg 1$ with $z\Delta_i$ and $\bar{z}\Delta_i$ fixed.
The operator dimension is related to the masses of the dual particles as $\Delta \sim m + O(1)$.
We now aim to build a flat space scattering amplitude which is equivalent to this CFT correlator in the relevant regime.

Preparing Single-Particle state

- Primary state $|O\rangle$ or ground state of HO: operator inserted at distant Euclidean past $t_E = -\infty$ on the cylinder (or at origin in the y-plane)
- Apply the translation operator $e^{-i\vec{y}\cdot\vec{P}}$ to the ground state to create a single state operator at any other point at Euclidean time and angle:

$$
\begin{aligned}
|O\rangle &\propto e^{-iy\cdot\vec{P}}|O\rangle \\
&\sim e^{\sqrt{2m}\vec{y}\cdot\vec{A}^\dagger}|0\rangle
\end{aligned}
$$

We see that we get a coherent state: a Gaussian wavefunction with the same width as the ground state but center offset from the origin

Preparing Two-Particles state

- Non-interacting two particle state: tensor product of two coherent states defined above. Splitting this into center of mass and separation of the particles is simple:
  - Split the center of mass special conformal operator as a sum:

    $$\vec{K}_1 + \vec{K}_2 = -i\sqrt{2m_1}\vec{a}_1 - i\sqrt{2m_2}\vec{a}_2$$

    The tensor product state is then eigenstate of $\vec{K}$ with eigenvalues $-2i(m_1\vec{y}_1 + m_2\vec{y}_2)$
  - We then choose $m_1\vec{y}_1 + m_2\vec{y}_2 = 0$ so the product state is annihilated by $\vec{K}$ and is a primary state.

Preparing Two-Particles state

- This motivates us to consider the 'Euclidean scattering states $|\psi(\tau_{in}, \Omega_{in})\rangle$ defined by:

$$|O_1 O_2\rangle \propto |\psi\rangle \otimes |0\rangle_{\text{COM}}$$

where

$$|\psi\rangle \sim e^{\vec{\alpha}_{in} \cdot \vec{a}^\dagger} |0\rangle$$

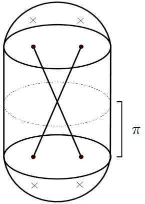
Here $\vec{a}^\dagger$ are creation operator built from the separation variables.

Flat Space Amplitude

We build the flat space scattering amplitude by inserting a real time $\frac{\pi}{\omega}$ evolution operator between the prepared initial and final states:

$$G^{\mathsf{scat}}(E,\theta) = \langle \psi_{\mathsf{in}} | e^{-i\frac{\pi}{\omega}H} | \psi_{\mathsf{out}} \rangle$$

where from the Euclidean insertion point we can find out the dependence of the correlator on z and $\bar{z}$. Pictorially we have:

$\mathcal{G}(z,\bar{z}) =$ 

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CFT data from quantum theory in the bulk

As argued any information about the energy shifts of a primary two particle-state can be read form the relative Hamiltonian. The non-relativistic potential for the two body problem is given as:

$$H_{\text{relative}} = \frac{p^2}{2\mu} + \frac{1}{2}\mu\omega r^2 + V(r)$$

We use effective quantum theory to obtain the shift in the energy of the two particles in AdS due to a potential V(r) when $|V(r)| \ll \frac{1}{2}\mu\omega r^2$:

$$\delta E_{\ell,n} = \langle n\ell|V|n\ell\rangle + \sum_{m\neq n}^{\infty} \frac{|\langle m\ell|V|n\ell\rangle|^2}{E_{n,\ell} - E_{m,\ell}} + \dots$$

Since the dilation operator is the Hamiltonian in AdS we have :

$$\delta E_{\ell,n} = \gamma_{n,\ell}$$

Kepler Potential in d=4

Focusing on the Kepler potential we can separate the energy shifts of different order in G_N as follows:

$$\gamma^{(1)}_{n,\ell} = \delta E^{(1)}_{\ell,n} = \int_{\text{AdS}} \phi_{n\ell}(r)\phi^*_{n\ell}(r)V(r)d\text{Vol}$$

$$\gamma^{(2)}_{n,\ell} = \delta E^{(2)}_{\ell,n} = \sum_{m=0}^{\infty} \frac{1}{E_{\ell,n} - E_{\ell,m}} |\int_{\text{AdS}} \phi_{n\ell}(r)\phi^*_{n\ell}(r)V(r)d\text{Vol}|^2$$

where $\langle r|n\ell\rangle = \phi_{n,\ell}(r)$, $V(r) = -\frac{G_N m_1 m_2}{r}$ and dVol is the volume element of AdS at large L_{AdS}.

Kepler Potential in d=4

We can then calculate the integral to obtain:

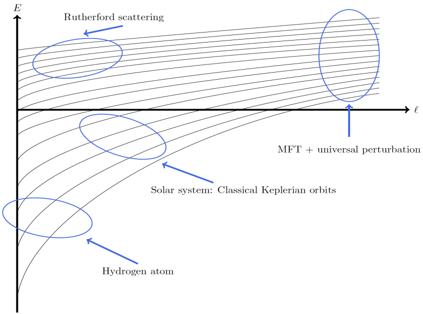
$$\gamma_{n,\ell}^{(1)} = G_N m_1 m_2 \sqrt{\mu} \frac{\left(\frac{1}{2}\right)_n \Gamma(\ell+1)}{n!\Gamma(\ell+\frac{3}{2})} {}_3F_2(-n, \ell+1, \frac{1}{2}; \ell+\frac{3}{2}, -n+\frac{1}{2}; 1)$$

It would be illuminating to look at different limit of this formula

- Flat space: large n, large ℓ, $n \gg \ell$ we get $\gamma_{n,\ell}^{(1)} \sim -\frac{G_N m_1 m_2}{\pi}\sqrt{\frac{\mu}{n}}\log(\frac{n}{\ell})$.
- Large ℓ limit: $\gamma_{n,\ell}^{(1)} \sim -G_N m_1 m_2 \sqrt{\frac{\mu}{\ell}}$.

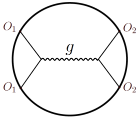
Regge Trajectory

With the above analysis we can then understand the Regge Trajectory:



Diagrammatic Interpretation in the cross-channel

The perturbation discussed above has a nice interpretation in terms of the number of graviton exchanged in the cross-channel:



$+ \dots$

In the AdS/CFT correspondence this then maps to exchange of stress-tensor and its multi-twist operators in the CFT.
We can then use Lorentzian inversion formula, a formula developed by Caron-huot, 2017', to extract the scaling dimension and OPE coefficient of the operators in CFT which is encoded in the poles and residues of the formula.
Note that this analysis is fully relativistic.

Lorentzian Inversion Formula: Graviton Exchange

Explicitly Lorentzian Inversion Formula can be written:

$$c^t(\ell,\Delta) = \int_0^1 \int_0^1 dz d\bar{z} \mu(z,\bar{z},\beta) G_{\ell+d-1,\Delta+1-d}(z,\bar{z}) \, \mathsf{dDisc}_t \left[\mathcal{G}(z,\bar{z})\right].$$

where c^t contains the CFT data through its poles and residues.

$$c(\ell,\Delta) \sim \sum_k \frac{f^2_{\phi\phi O_k}}{\Delta - \Delta_k}$$

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We calculate the dDisc perutrbatively by OPE expanding the correlator and focusing on single twist and higher twist-states of T. The result we get for the anomalous dimension in the large Δ limit is:

$$\gamma^{(1)}_{n=0,\ell} = -2 f_T^2 \frac{128}{3\sqrt{\Delta_1 + \Delta_2}} \frac{\sqrt{\Delta_1 \Delta_2}\Gamma(\ell+1)}{\Gamma(\ell + \frac{3}{2})}$$

We check that this is exactly what we had from the bulk calculation $({}_3F_2)$ for the leading trajectory:

Note that Similar to Froissart-Gribov formula, the inversion formula builds the data from the discontinuity of the correlator.

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Summary & Conclusion

In this talk we analyzed the non-relativistic two-body problem in AdS. Specifically in this limit:

- We have introduced a physical way to extract the flat space bulk s-matrix from the AdS/CFT correspondence. This has a few advantages compared to the constructions available in the literature:
    - Since we used Euclidean time evolution in order to prepare the states we did not need to smear in and out states over real time (compared to Fitzpatrick, Kaplan 2011)
    - We did not had to treat $V(r)$ as a small correction. We split (mass) + (COM) + (relative), where mass is large but the interaction can still be strong (compared to Fitzpatrick, Katz et al 2011)

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  - We did not had to treat $V(r)$ as a small correction. We split (mass) + (COM) + (relative), where mass is large but the interaction can still be strong (compared to Fitzpatrick, Katz et al 2011)
- As a first step we have exploited perturbation theory in order to extract data about the CFT spectrum.
  - We find that we can get leading anomalous dimension of double twist operators $[O_1 O_2]_{n,\ell}$ when $\Delta_i \gg 1$.
  - We then used CFT tools to obtain the anomalous dimension for $n = 0$ and $n = 1$ and finding perfect agreeement after taking $\Delta_i \gg 1$

Future Direction

- NR dynamics turns out to be an easy toy model in which we can prepare the states do the partial wave decomposition and much more. IT is thus the perfect playground to study the strongly coupled bulk physics from the CFT

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- Higher order in perturbation theory. This is interesting from the CFT perspective as it gives the contribution of the resummation of all multi twist operator at a single order.
- Of course getting the OPE coefficient perturbatively using this method would be of interest as well.
- One can also try other central potentials to read CFT data in other regime or exchange of other particles. In the upcoming paper we work out the Yukawa potential which corresponds to the NR exchange of massive particle in the bulk with $\Delta_{\text{Exchanged}} \sim \Delta_{\text{External}}$.

Review: Froissart Gribov Formula

In amplitude, our questions can be answered by using analyticity in spin and Froissart Gribov Formula:

$$A^t(\ell,s) = \frac{-i}{32\pi^2}\int_1^{\infty} dz_s \mathsf{Disc}_t[\mathcal{A}(s,t(s,z_s))]Q_\ell(z_s)$$

The integral is convergent for sufficiently large ℓ.
This needs to be added to u-channel contribution to give the full contribution:

$$A_\ell(s) = A^t(\ell,s) + (-1)^\ell A^u(\ell,s)$$

Here $A^t(\ell,s)$ and $A^u(\ell,s)$ are analytic functions of spin with isolated singularities. We can then start with the representation for the value of ℓ that it converges and analytically continue it to the regions where the representation does not converge.

Lorentzian Inversion Formula

In CFT, we take a similar approach: we use Lorentzian inversion formula which is the CFT Froissart Gribov to obtain the CFT data in terms of the imaginary part of the CFT "Amplitude".
To define the CFT "Amplitude" we use the following inequlaity:

$$|\mathcal{G}(z,\bar{z})| \leq \mathcal{G}_{\text{Eucl}}(z,\bar{z})$$

Then we can have:

$$i\mathcal{M} \equiv \mathcal{G}(z,\bar{z}) - \mathcal{G}_{\text{Eucl}} \quad \text{intuition:} \quad i\mathcal{M} = S - 1$$

The imaginary part of the defined amplitude is then:

$$\text{Im}\mathcal{M} \equiv \text{dDisc}[\mathcal{G}(z,\bar{z})] = \mathcal{G}_{\text{Eucl}}(z,\bar{z}) - \frac{1}{2}(\mathcal{G}^{\circlearrowleft}(z,\bar{z}) + \mathcal{G}^{\circlearrowright}(z,\bar{z}))$$

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Explicitly Lorentzian Inversion Formula can be written:

$$c^t(J,\Delta) = \int_0^1 \int_0^1 dz d\bar{z} \frac{|z-\bar{z}|^{d-2}}{(z\bar{z})^d} \kappa_\beta G_{J+d-1,\Delta+1-d}(z,\bar{z}) \, \mathrm{dDisc}_t \left[\mathcal{G}(z,\bar{z})\right].$$

where this needs to be added to u-channel contribution to give the full contribution:

$$c(J,\Delta) = c^t(J,\Delta) + (-1)^J c^u(J,\Delta)$$

where c contains the CFT data through its poles and residues.

$$c(J,\Delta) \sim \sum_k \frac{f^2_{\phi\phi O_k}}{\Delta - \Delta_k}$$