Covariant Color-Kinematics Duality

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Amplitudes 2021

Double copy is a proven fact
about scattering amplitudes.

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* at tree level, like this talk

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Why is it true?

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about scattering amplitudes.

Why is it true?
Why does it sometimes fail?

* at tree level, like this talk

Upshot of this Talk

- progress towards a QFT double copy
- kinematic algebras + currents
- YM is itself a covariant double copy
- closed formula for BCJ numerators

Outline

1. Biadjoint Scalar Theory
2. Nonlinear Sigma Model
3. Yang-Mills Theory
4. Applications
5. Conclusions

Biadjoint Scalar Theory

Biadjoint scalar (BAS) theory is the template for all future analyses. The Lagrangian is

$$\mathscr{L}^{\mathrm{BAS}} = \frac{1}{2}\partial_\mu \phi^{a\bar{a}} \partial^\mu \phi^{a\bar{a}} - \frac{1}{3!} f^{abc} f^{\bar{a}\bar{b}\bar{c}} \phi^{a\bar{a}} \phi^{b\bar{b}} \phi^{c\bar{c}} + \phi^{a\bar{a}} J^{a\bar{a}}$$

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while the equations of motion (EOM) are

$$\Box\,\phi^{a\bar{a}} + \frac{1}{2}f^{abc}f^{\bar{a}\bar{b}\bar{c}}\phi^{b\bar{b}}\phi^{c\bar{c}} = J^{a\bar{a}}$$

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This sourced solution encodes all tree amplitudes (see Berends-Giele recursion, perturbiner).

The n-point correlator is derived from one-point,

$$\langle \phi^{a_1\bar{a}_1}(p_1)\phi^{a_2\bar{a}_2}(p_2)\cdots\phi^{a_n\bar{a}_n}(p_n)\rangle_{J=0} = \left[\left(\prod_{i=1}^{n-1}\frac{1}{i}\frac{\delta}{\delta J^{a_i\bar{a}_i}(p_i)}\right)\langle\phi^{a_n\bar{a}_n}(p_n)\rangle_J\right]_{J=0}$$

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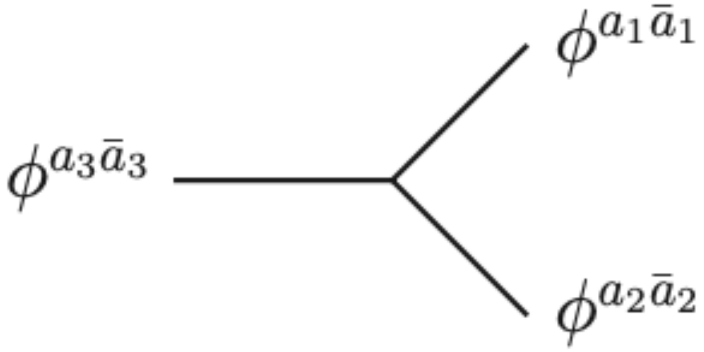
where the associated Feynman rules are

$$= \frac{i\delta^{a_1a_2}\delta^{\bar{a}_1\bar{a}_2}}{p^2}$$



$$= -if^{a_1a_2a_3}f^{\bar{a}_1\bar{a}_2\bar{a}_3}$$

BAS theory has a color and dual color symmetry, as well as the corresponding currents,

$$\phi^{a\bar{a}} \to \phi^{a\bar{a}} + f^{abc}\theta^{b}\phi^{c\bar{a}} \qquad \phi^{a\bar{a}} \to \phi^{a\bar{a}} + f^{\bar{a}\bar{b}\bar{c}}\theta^{\bar{b}}\phi^{a\bar{c}}$$

$$\mathscr{J}^{a}_{\alpha} = f^{abc}\phi^{b\bar{a}}\overleftrightarrow{\partial}_{\alpha}\phi^{c\bar{a}} \qquad \mathscr{K}^{\bar{a}}_{\alpha} = f^{\bar{a}\bar{b}\bar{c}}\phi^{a\bar{b}}\overleftrightarrow{\partial}_{\alpha}\phi^{a\bar{c}}$$

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which are conserved by the EOM and Jacobi, so

$$\partial^{\alpha}\mathcal{J}^{a}_{\alpha} = f^{abc}\phi^{b\bar{a}}\overleftrightarrow{\Box}\phi^{c\bar{a}}$$

$$\sim (f^{abe}f^{ecd} + f^{ace}f^{edb} + f^{ade}f^{ebc})f^{\bar{a}\bar{b}\bar{c}}\phi^{d\bar{a}}\phi^{b\bar{b}}\phi^{c\bar{c}} = 0$$

Gauged biadjoint scalar (GBAS) theory is BAS with color gauged. The scalar sector is

dual color not gauged

$$\mathscr{L}^{\mathrm{GBAS}} = \frac{1}{2} D_\mu \phi^{a\bar{a}} D^\mu \phi^{a\bar{a}} - \frac{1}{3!} f^{abc} f^{\bar{a}\bar{b}\bar{c}} \phi^{a\bar{a}} \phi^{b\bar{b}} \phi^{c\bar{c}} + \phi^{a\bar{a}} J^{a\bar{a}}$$

with covariant derivatives $D_\mu \phi^{a\bar{a}} = \partial_\mu \phi^{a\bar{a}} + f^{abc} A^b_\mu \phi^{c\bar{a}}$.

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$$D^2 \phi^{a\bar{a}} + \frac{1}{2} f^{abc} f^{\bar{a}\bar{b}\bar{c}} \phi^{b\bar{b}} \phi^{c\bar{c}} = J^{a\bar{a}}$$

$$\mathscr{J}^{a}_{\alpha} = f^{abc} \phi^{b\bar{a}} \overleftrightarrow{D}_{\alpha} \phi^{c\bar{a}} \qquad \mathscr{K}^{\bar{a}}_{\alpha} = f^{\bar{a}\bar{b}\bar{c}} \phi^{a\bar{b}} \overleftrightarrow{D}_{\alpha} \phi^{a\bar{c}}$$

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Nonlinear Sigma Model

The usual textbook Lagrangian for the NLSM is

$$\mathscr{L}^{\mathrm{NLSM}} = \frac{1}{2} j^a_\mu j^{a\mu}$$

where the chiral current is

$$j^a_\mu = i\,\mathrm{tr}[g^{-1}\partial_\mu g\, T^a]$$

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on-shell
external
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and the EOM says the chiral current is conserved

$$\partial^\mu j^a_\mu = J^a \quad \longleftarrow$$

conserved modulo
external sources

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Feynman propagator

cubic self-interaction

chiral current sourced by derivative of scalar source

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The chiral current is agnostic about field basis redundancy (also see Freedman-Townsend, 1981).

We want to scatter scalars, not chiral currents,

$$j^a_\mu = -\,\partial_\mu \pi^a + \cdots \qquad \longrightarrow \qquad \pi^a = -\,\frac{q^\mu j^a_\mu}{q\partial} + \cdots$$

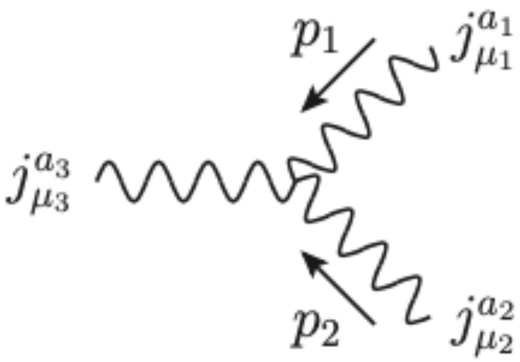
for reference q. All nonlinear field ambiguities vanish on-shell, so π^a = exotically polarized j^a_μ.

$$\langle \pi^a(p)\rangle_J = \tilde{\varepsilon}^\mu(p)\langle j^a_\mu(p)\rangle_J \qquad \text{where} \qquad \tilde{\varepsilon}_\mu(p) = \frac{iq_\mu}{pq}$$

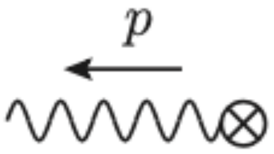
$$\langle \pi^{a_1}(p_1)\pi^{a_2}(p_2)\cdots\pi^{a_n}(p_n)\rangle_{J=0} = \left[\left(\prod_{i=1}^{n-1}\frac{1}{i}\frac{\delta}{\delta J^{a_i}(p_i)}\right)\tilde{\varepsilon}^\mu(p_n)\langle j^{a_n}_\mu(p_n)\rangle_J\right]_{J=0}$$

NLSM Feynman Rules

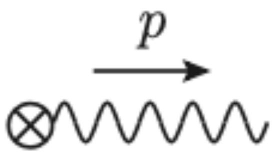
$$= \qquad \frac{i\delta^{a_1a_2}\eta^{\mu_1\mu_2}}{p^2}$$



$$= \qquad -if^{a_1a_2a_3}(ip_2^{\mu_1}\eta^{\mu_2\mu_3} - ip_1^{\mu_2}\eta^{\mu_1\mu_3})$$

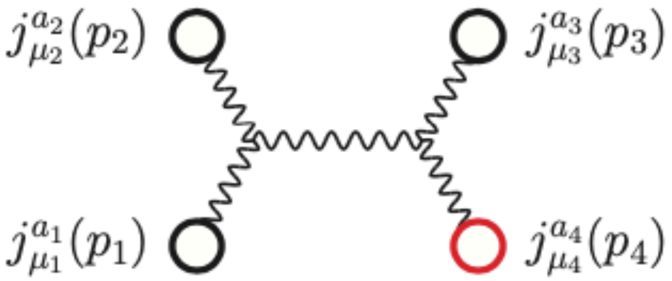


$$= \qquad \varepsilon_\mu = ip_\mu$$

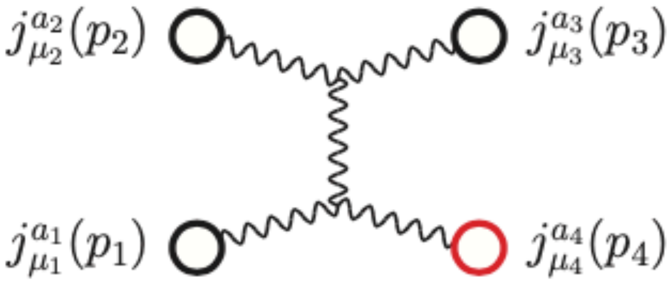


$$= \qquad \tilde{\varepsilon}_\mu = \frac{iq_\mu}{pq}$$

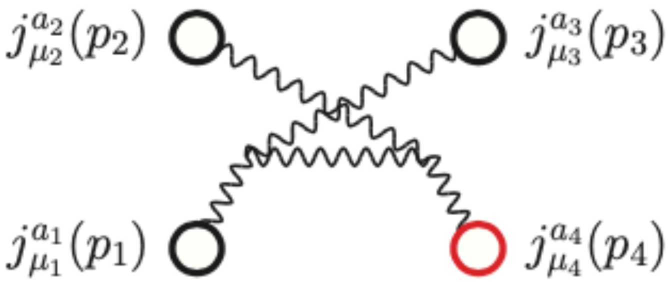
Kinematic Jacobi identity holds off-shell!



$$n_s = p_1^{\mu_2}(p_1 + p_2)^{\mu_3}\eta^{\mu_1\mu_4} + p_2^{\mu_1}p_3^{\mu_2}\eta^{\mu_3\mu_4} - \{1 \leftrightarrow 2\}$$



$$n_t = p_2^{\mu_3}(p_2 + p_3)^{\mu_1}\eta^{\mu_2\mu_4} + p_3^{\mu_2}p_1^{\mu_3}\eta^{\mu_1\mu_4} - \{2 \leftrightarrow 3\}$$



$$n_u = p_3^{\mu_1}(p_3 + p_1)^{\mu_2}\eta^{\mu_3\mu_4} + p_1^{\mu_3}p_2^{\mu_1}\eta^{\mu_2\mu_4} - \{3 \leftrightarrow 1\}$$

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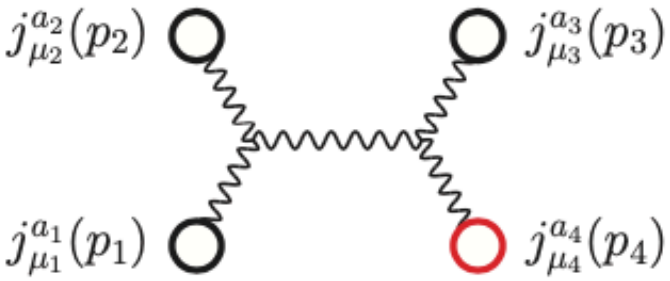

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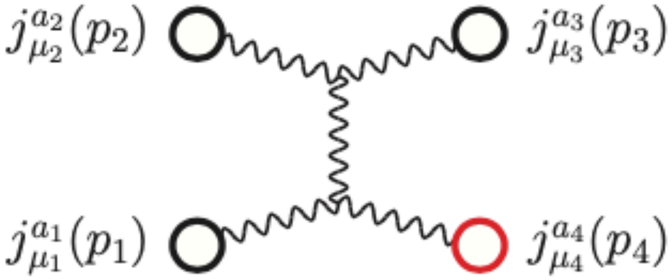

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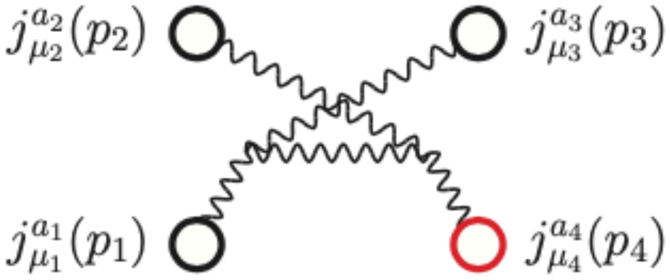

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$$\longrightarrow \qquad n_s + n_t + n_u = 0$$ Why?

BAS

$$\Box\,\phi^{a\bar{a}}+\frac{1}{2}f^{abc}f^{\bar{a}\bar{b}\bar{c}}\phi^{b\bar{b}}\phi^{c\bar{c}}=J^{a\bar{a}}$$

↕ isomorphic

NLSM

$$\Box\,j^{a}_{\mu}+f^{abc}j^{b\nu}\partial_{\nu}j^{c}_{\mu}=\partial_{\mu}J^{a}$$

Define “ $\otimes$ NLSM ” double copy acting on fields,

$$V^a \quad \overset{\text{NLSM}}{\rightarrow} \quad V_\mu$$

$$f^{abc} V^b W^c \quad \overset{\text{NLSM}}{\rightarrow} \quad V^\nu \partial_\nu W_\mu - W^\nu \partial_\nu V_\mu$$

$$J^a \quad \overset{\text{NLSM}}{\rightarrow} \quad \partial_\mu J$$

By inspection, kinematic algebra = diff algebra!

$$[\, V_\mu \partial^\mu, W_\nu \partial^\nu \,] = (V^\nu \partial_\nu W_\mu - W^\nu \partial_\nu V_\mu) \partial^\mu$$

BAS $\otimes$ NLSM = NLSM

	fields	EOM
BAS	$\phi^{a\bar{a}}$	$\Box\phi^{a\bar{a}}+\frac{1}{2}f^{abc}f^{\bar{a}\bar{b}\bar{c}}\phi^{b\bar{b}}\phi^{c\bar{c}}=J^{a\bar{a}}$
NLSM		

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NLSM	j_μ^a	

BAS $\otimes$ NLSM = NLSM

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BAS	$\phi^{a\bar{a}}$	$\Box\phi^{a\bar{a}}+\frac{1}{2}f^{abc}f^{\bar{a}\bar{b}\bar{c}}\phi^{b\bar{b}}\phi^{c\bar{c}} = J^{a\bar{a}}$
NLSM	j^a_μ	$\Box j^a_\mu + f^{abc}j^{b\nu}\partial_\nu j^c_\mu = \partial_\mu J^a$

NLSM $\otimes$ NLSM = SG

	fields	EOM
NLSM	j_μ^a	$\partial_{[\mu} j_{\nu]}^a + f^{abc} j_\mu^b j_\nu^c = 0$ $\partial^\mu j_\mu^a = J^a$
SG new form!		

NLSM $\otimes$ NLSM = SG

	fields	EOM
NLSM	j_μ^a	$\partial_{[\mu} j_{\nu]}^a + f^{abc} j_\mu^b j_\nu^c = 0$ $\partial^\mu j_\mu^a = J^a$
SG new form!	$j_{\mu\bar{\mu}}$	

NLSM ⊗ NLSM = SG

	fields	EOM
NLSM	j_μ^a	$\partial_{[\mu}j_{\nu]}^a + f^{abc}j_\mu^b j_\nu^c = 0$ $\partial^\mu j_\mu^a = J^a$
SG new form!	$j_{\mu\bar{\mu}}$	$\partial_{[\mu}j_{\nu]\bar{\mu}} + j_\mu{}^{\bar{\nu}}\partial_{\bar{\nu}}j_{\nu\bar{\mu}} - j_\nu{}^{\bar{\nu}}\partial_{\bar{\nu}}j_{\mu\bar{\mu}} = 0$ $\partial^\mu j_{\mu\bar{\nu}} = \partial_{\bar{\nu}}J$

YM $\otimes$ NLSM = BI

	fields	EOM
YM	A^a_μ $F^a_{\mu\nu}$	$F^a_{\mu\nu} = \partial_{[\mu}A^a_{\nu]} + f^{abc}A^b_\mu A^c_\nu$ $\partial^\mu F^a_{\mu\nu} + f^{abc}A^{b\mu}F^c_{\mu\nu} = J^a_\nu$
BI new form!		

YM $\otimes$ NLSM = BI

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BI new form!	$A_{\mu\bar{\mu}}$ $F_{\mu\nu\bar{\mu}}$	

YM ⊗ NLSM = BI

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YM	A^a_μ $F^a_{\mu\nu}$	$F^a_{\mu\nu} = \partial_{[\mu}A^a_{\nu]} + f^{abc}A^b_\mu A^c_\nu$ $\partial^\mu F^a_{\mu\nu} + f^{abc}A^{b\mu}F^c_{\mu\nu} = J^a_\nu$
BI new form!	$A_{\mu\bar{\mu}}$ $F_{\mu\nu\bar{\mu}}$	$F_{\mu\nu\bar{\mu}} = \partial_{[\mu}A_{\nu]\bar{\mu}} + A_\mu{}^{\bar{\nu}}\partial_{\bar{\nu}}A_{\nu\bar{\mu}} - A_\nu{}^{\bar{\nu}}\partial_{\bar{\nu}}A_{\mu\bar{\mu}}$ $\partial^\mu F_{\mu\nu\bar{\mu}} + A^{\mu\bar{\nu}}\partial_{\bar{\nu}}F_{\mu\nu\bar{\mu}} - \partial_{\bar{\nu}}A^\mu{}_{\bar{\mu}}F_{\mu\nu}{}^{\bar{\nu}} = \partial_{\bar{\mu}}J_\nu$

The color current enforces color Jacobi identities. What enforces the kinematic Jacobi identities?

$$\mathscr{K}^{\bar{a}}_{\alpha} = f^{\bar{a}\bar{b}\bar{c}} \phi^{a\bar{b}} \overleftrightarrow{\partial}_{\alpha} \phi^{a\bar{c}}$$

$$\overset{\text{NLSM}}{\rightarrow} \qquad \mathscr{K}^{\text{NLSM}}_{\mu\alpha} = j^{a\nu} \partial_{\nu} \overleftrightarrow{\partial}_{\alpha} j^{a}_{\mu}$$

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$$\xrightarrow{\text{NLSM}} \qquad \mathscr{K}^{\text{NLSM}}_{\mu\alpha} = j^{a\nu}\partial_{\nu}\overleftrightarrow{\partial}_{\alpha}j^{a}_{\mu}$$

The kinematic current is conserved on the EOM.

$$\partial^{\alpha}\mathscr{K}^{\text{NLSM}}_{\mu\alpha} = j^{a\nu}\partial_{\nu}\overleftrightarrow{\Box}\, j^{a}_{\mu}$$

$$= -f^{abc}\left[j^{a\nu}\partial_{\nu}(j^{b\rho}\partial_{\rho}j^{c}_{\mu}) - \partial_{\nu}j^{a}_{\mu}j^{b\rho}\partial_{\rho}j^{c\nu}\right] = 0$$

The kinematic current has another name.

$$\mathscr{K}^{\mathrm{NLSM}}_{\mu\alpha} = -\,\Box\, T^{\mathrm{NLSM}}_{\mu\alpha} + \text{improvement terms}$$

And the kinematic symmetry?

The kinematic current has another name.

$$\mathscr{K}_{\mu\alpha}^{\mathrm{NLSM}} = -\Box T_{\mu\alpha}^{\mathrm{NLSM}} + \text{improvement terms}$$

And the kinematic symmetry? It is trivial.

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$$\longrightarrow \quad a^\mu Q_\mu = \lim_{a\to 0} \int d^3x\, \left[J_0(x+a) - J_0(x)\right] = 0$$

Yang-Mills Theory

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field strength sourced by derivative of gauge source

$$D^\rho(a)_{\rho\mu\nu} + D_{[\mu}(b)_{\nu]} \qquad D^2 F^a_{\mu\nu} + f^{abc} F^b_{\rho[\mu} F^{c\rho}_{\nu]} = D_{[\mu} J^a_{\nu]}$$

covariant propagator depends on gauge field A^a_μ

cubic self-interaction

Now define a first-order formulation of YM,

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The field strength evolves like a color charged scalar with a cubic self-interaction.

We aim to scatter gauge fields, not field strengths.

$$F^a_{\mu\nu} = \partial_{[\mu} A^a_{\nu]} + \cdots \qquad \xrightarrow{q^\mu A^a_\mu = \mathcal{O}(A^2)} \qquad A^a_\mu = -\frac{q^\nu F^a_{\mu\nu}}{q\partial} + \cdots$$

for reference q. All nonlinear field ambiguities vanish on-shell, so A^a_μ = exotically polarized $F^a_{\mu\nu}$.

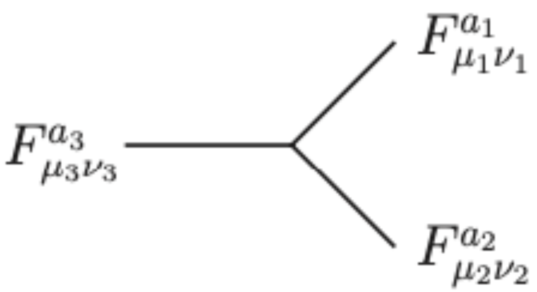
$$\langle A^a_\mu(p)\rangle_J = \tilde{\varepsilon}^\nu(p)\langle F^a_{\mu\nu}(p)\rangle_J \qquad \text{where} \qquad \tilde{\varepsilon}_\mu(p) = \frac{iq_\mu}{pq}$$

$$\langle A^{a_1}_{\mu_1}(p_1)A^{a_2}_{\mu_2}(p_2)\cdots A^{a_n}_{\mu_n}(p_n)\rangle_{J=0} = \left[\left(\prod_{i=1}^{n-1}\frac{1}{i}\frac{\delta}{\delta J^{a_i\mu_i}(p_i)}\right)\tilde{\varepsilon}^{\nu_n}(p_n)\langle F^{a_n}_{\mu_n\nu_n}(p_n)\rangle_J\right]_{J=0}$$

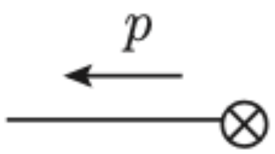
YM Feynman Rules

field strength vertices only !

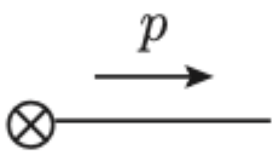
$$= \frac{i\delta^{a_1a_2}\Pi^{\mu_1\nu_1\mu_2\nu_2}}{p^2}$$



$$= 4\, if^{a_1a_2a_3}\Pi^{\mu_1\nu_1\alpha}{}_{\beta}\Pi^{\mu_2\nu_2\beta}{}_{\gamma}\Pi^{\mu_3\nu_3\gamma}{}_{\alpha}$$

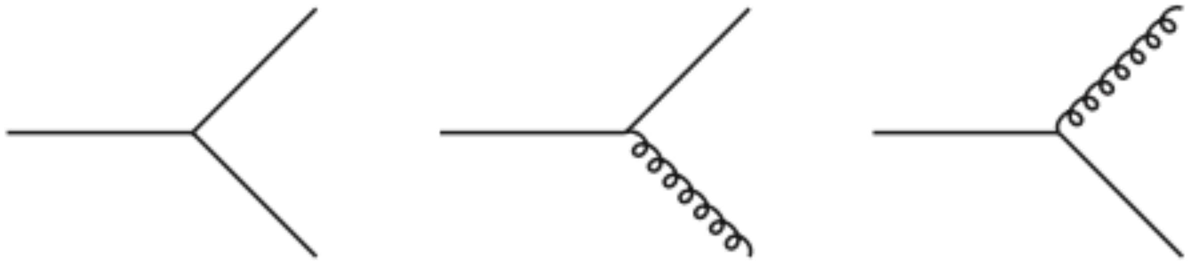


$$= \varepsilon_{\mu\nu} = ip_{[\mu}\varepsilon_{\nu]}$$

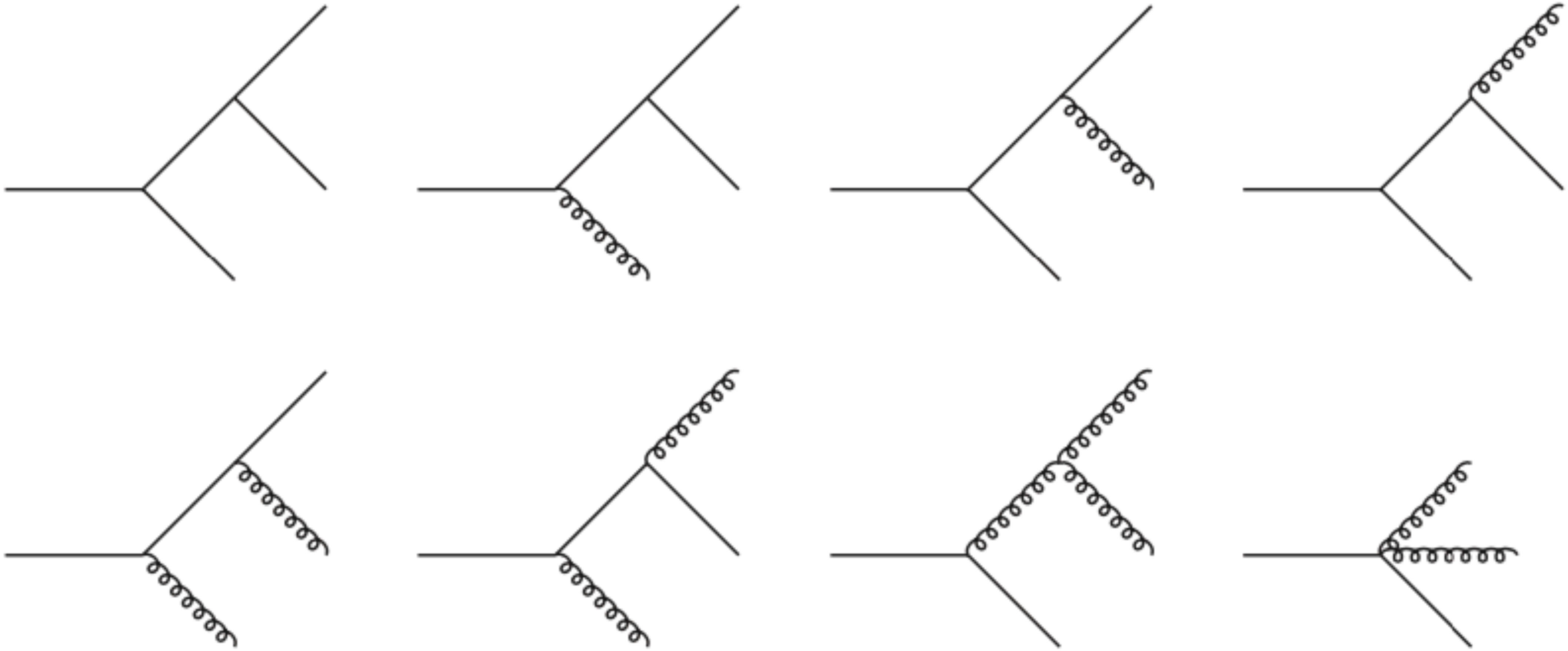


$$= \tilde{\varepsilon}_{\mu\nu} = \frac{iq_{[\mu}\varepsilon_{\nu]}}{pq}$$

$$A_3^{\mathrm{YM}} \quad =$$



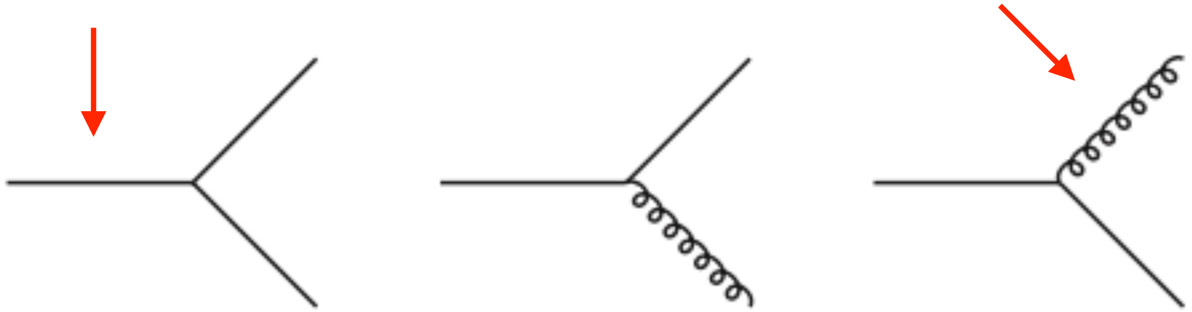
$$A_4^{\mathrm{YM}} \quad =$$



field strength $F^a_{\mu\nu}$

gauge field A^a_μ

$A_3^{\text{YM}} =$



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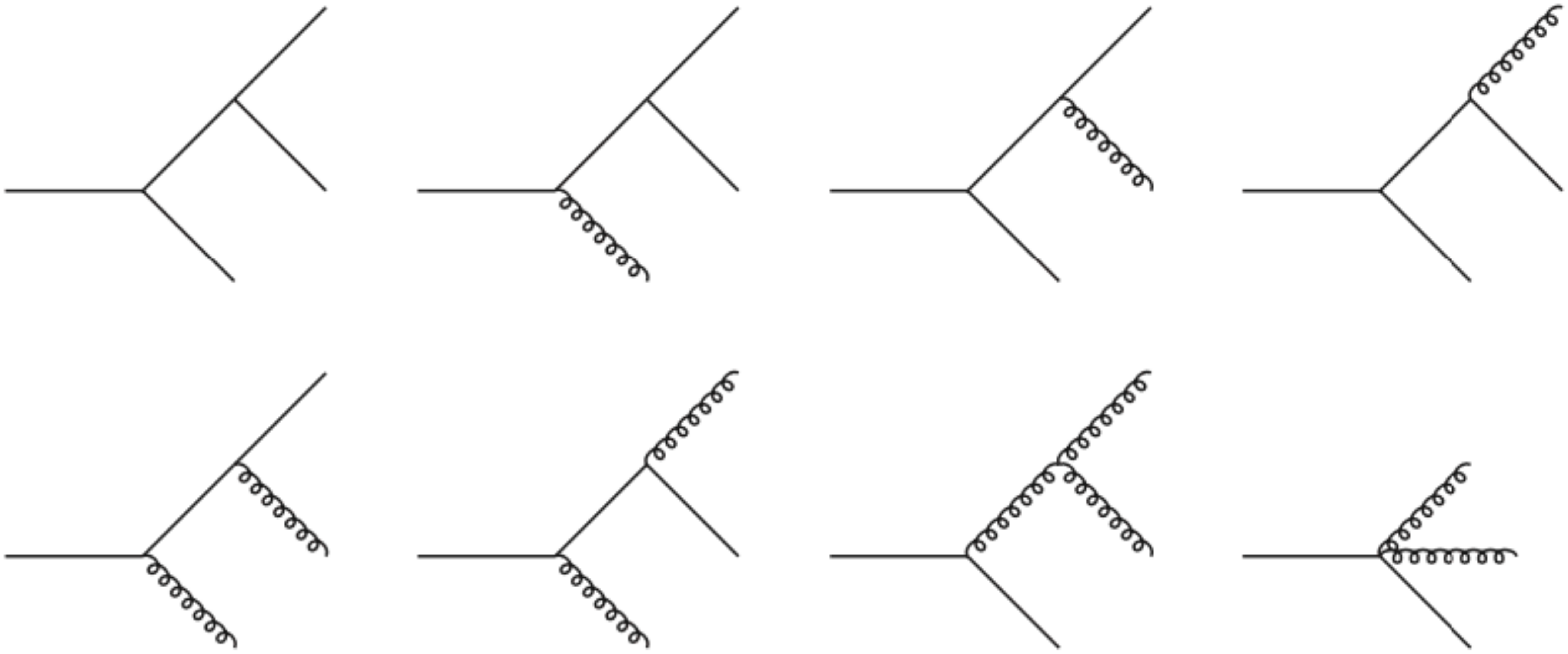

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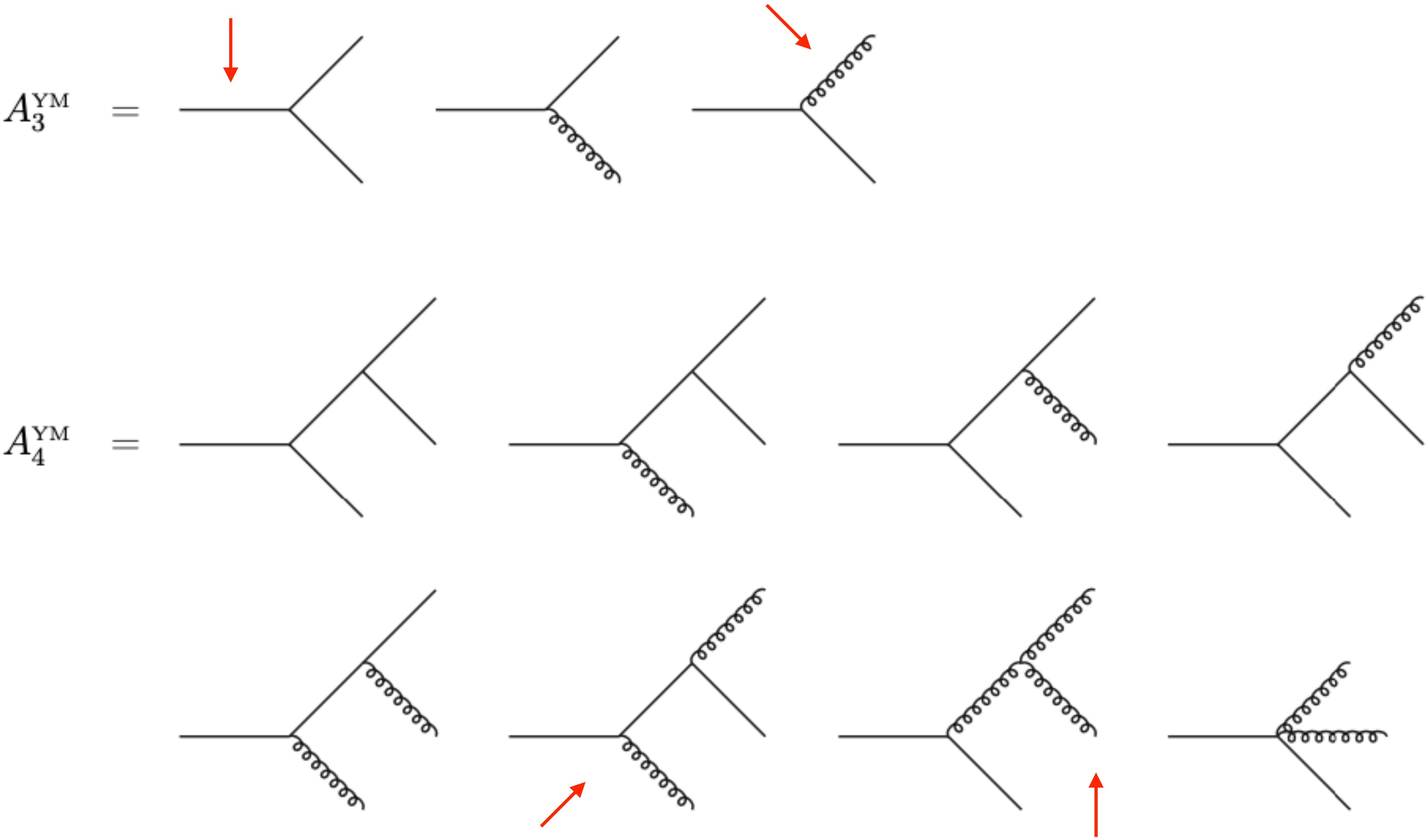

$A_3^{\text{YM}} =$

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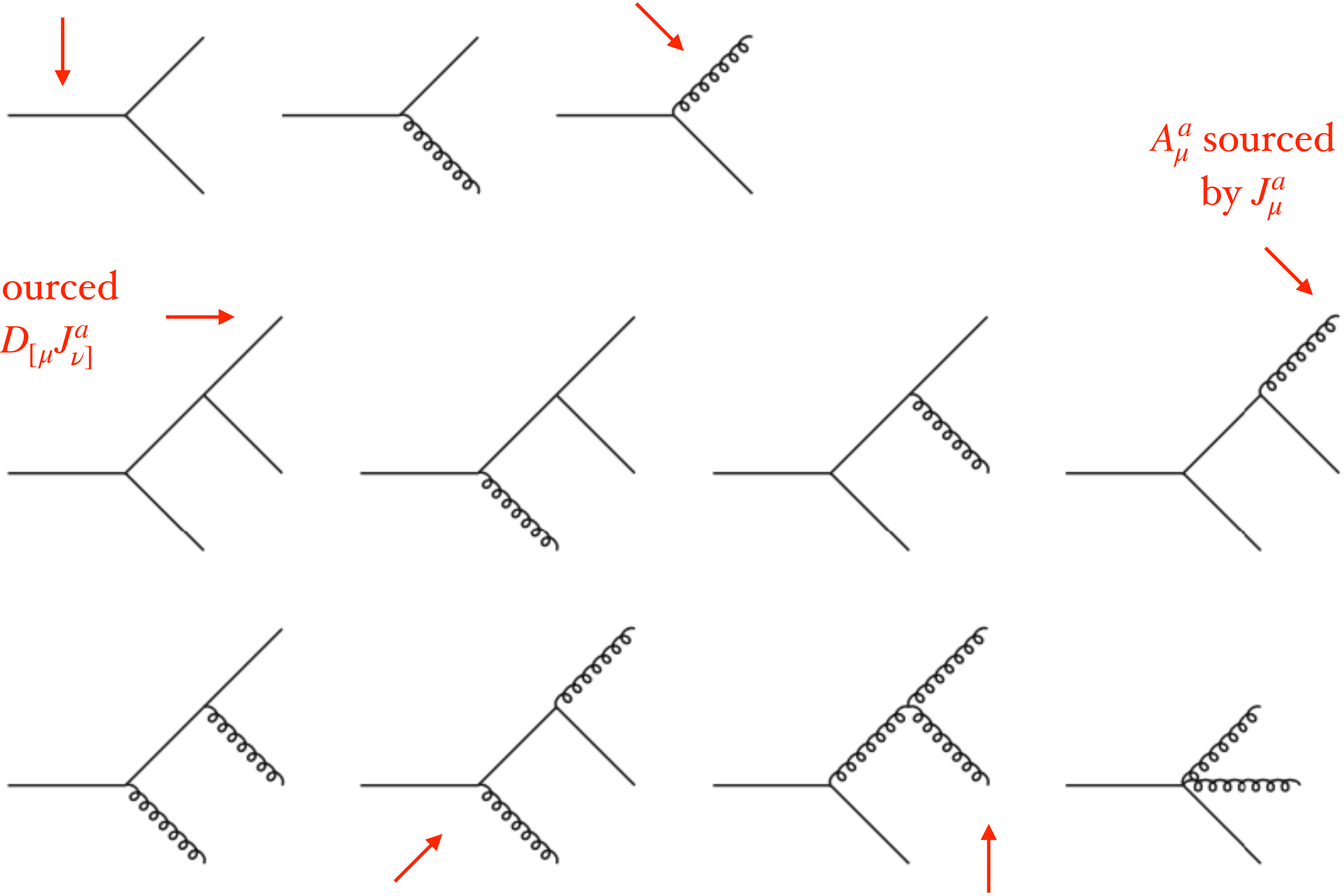
$F^a_{\mu\nu}$ emits A^a_μ via minimal coupling kinetic term $D^2 F^a_{\mu\nu}$

A^a_μ branch to other A^a_μ via standard YM EOM

$A_3^{\text{YM}} =$

$A_4^{\text{YM}} =$



field strength $F_{\mu\nu}^a$

gauge field A_μ^a

$F_{\mu\nu}^a$ sourced
by $D_{[\mu}J_{\nu]}^a$

A_μ^a sourced
by J_μ^a

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A_μ^a branch to other A_μ^a
via standard YM EOM

Covariant color-kinematics duality is identical to the usual story but with $1/\Box$ replaced with $1/D^2$.

Consider an off-shell four-point subdiagram with only field strength vertices included.

$$n_s = 8\,\Pi^{\mu_1\nu_1\alpha}{}_{\beta}\Pi^{\mu_2\nu_2\beta}{}_{\gamma}\Pi^{\mu_3\nu_3\gamma}{}_{\delta}\Pi^{\mu_4\nu_4\delta}{}_{\alpha} - \{3 \leftrightarrow 4\}$$

$$n_t = 8\,\Pi^{\mu_2\nu_2\alpha}{}_{\beta}\Pi^{\mu_3\nu_3\beta}{}_{\gamma}\Pi^{\mu_1\nu_1\gamma}{}_{\delta}\Pi^{\mu_4\nu_4\delta}{}_{\alpha} - \{1 \leftrightarrow 4\}$$

$$n_u = 8\,\Pi^{\mu_3\nu_3\alpha}{}_{\beta}\Pi^{\mu_1\nu_1\beta}{}_{\gamma}\Pi^{\mu_2\nu_2\gamma}{}_{\delta}\Pi^{\mu_4\nu_4\delta}{}_{\alpha} - \{2 \leftrightarrow 4\}$$

$$\longrightarrow \qquad n_s + n_t + n_u = 0$$

GBAS

$$D^2\phi^{a\bar{a}}+\frac{1}{2}f^{abc}f^{\bar{a}\bar{b}\bar{c}}\phi^{b\bar{b}}\phi^{c\bar{c}} = J^{a\bar{a}}$$

$$\updownarrow$$

isomorphic

YM

$$D^2F^a_{\mu\nu}+f^{abc}F^b_{\rho[\mu}F^{c\rho}_{\nu]} = D_{[\mu}J^a_{\nu]}$$

Define “ $\otimes F^3$ ” covariant double copy on fields,

$$V^a \quad \overset{F^3}{\rightarrow} \quad V_{\mu\nu}$$

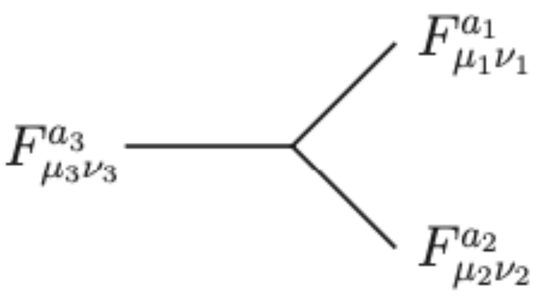
$$f^{abc}V^bW^c \quad \overset{F^3}{\rightarrow} \quad V_{\rho\mu}W_\nu{}^\rho - W_{\rho\mu}V_\nu{}^\rho$$

$$J^a \quad \overset{F^3}{\rightarrow} \quad \partial_{[\mu}J_{\nu]}$$

So, covariant kinematic algebra = Lorentz algebra!

$$[\, V_{\mu\nu}S^{\mu\nu}, W_{\rho\sigma}S^{\rho\sigma}] = (V_{\rho\mu}W_\nu{}^\rho - W_{\rho\mu}V_\nu{}^\rho)S^{\mu\nu}$$

The kinematic structure constants are Feynman vertices of an F^3 theory of antisymmetric tensors.



=

tensorial generalization
of cubic BAS vertex

In this context F^3 is a renormalizable coupling.

dimensionless
couplings
⟶

$$D^2F^a_{\mu\nu} + f^{abc}F^b_{\rho[\mu}F^{c\rho}_{\nu]} = D_{[\mu}J^a_{\nu]}$$

These antisymmetric tensors are simultaneously field strengths and Lorentz generators.

GBAS $\otimes F^3$ = YM

	fields	EOM
GBAS	$\phi^{a\bar{a}}$	$D^2\phi^{a\bar{a}}+\frac{1}{2}f^{abc}f^{\bar{a}\bar{b}\bar{c}}\phi^{b\bar{b}}\phi^{c\bar{c}}=J^{a\bar{a}}$
YM		

GBAS $\otimes F^3 =$ YM

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YM	$F^a_{\mu\nu}$	

GBAS $\otimes F^3$ = YM

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EYM $\otimes F^3 =$ GR

	fields	EOM
EYM	A^a_μ $F^a_{\mu\nu}$	$F^a_{\mu\nu} = \partial_{[\mu} A^a_{\nu]} + f^{abc} A^b_\mu A^c_\nu$ $\nabla^\mu F^a_{\mu\nu} + f^{abc} A^{b\mu} F^c_{\mu\nu} = J^a_\nu$
GR tetrad!		

EYM $\otimes$ F^3 = GR

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EYM	A^a_μ $F^a_{\mu\nu}$	$F^a_{\mu\nu} = \partial_{[\mu}A^a_{\nu]} + f^{abc}A^b_\mu A^c_\nu$ $\nabla^\mu F^a_{\mu\nu} + f^{abc}A^{b\mu}F^c_{\mu\nu} = J^a_\nu$
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EYM $\otimes F^3$ = GR

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GR tetrad!	$\omega_{\mu\bar{\mu}\bar{\nu}}$ $R_{\mu\nu\bar{\mu}\bar{\nu}}$	$R_{\mu\nu\bar{\mu}\bar{\nu}} = \partial_{[\mu}\omega_{\nu]\bar{\mu}\bar{\nu}} + \omega_{\mu\bar{\rho}\bar{\mu}}\omega_{\nu\bar{\nu}}{}^{\bar{\rho}} - \omega_{\nu\bar{\rho}\bar{\mu}}\omega_{\mu\bar{\nu}}{}^{\bar{\rho}}$ $\nabla^\mu R_{\mu\nu\rho\sigma} = \nabla_{[\rho} T_{\sigma]\nu}$

YM

“ field strengths evolve like charged
self-interacting biadjoint scalars ”

GR

“ spacetime curvatures evolve like
field strengths in curved space ”

Covariant color-kinematics duality enforces a trivial “tetradic” variant of classical double copy.

Every GR solution is a YM solution in the same curved background, e.g. Euclidean black hole has

$$A^a_\mu \sim \begin{pmatrix} -\frac{R_S}{2r^2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{1-\frac{R_S}{r}} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{1-\frac{R_S}{r}}\,\sin\theta & \cos\theta & \cdots \end{pmatrix}$$

So a black hole is a “chromo-gravitational” atom where proton/electron = background/gauge field.

We derive the covariant kinematic current of YM from the dual color current of BAS,

$$\mathscr{K}^{\bar{a}}_{\alpha} = f^{\bar{a}\bar{b}\bar{c}}\phi^{a\bar{b}}\overleftrightarrow{D}_{\alpha}\phi^{a\bar{c}}$$

$$\xrightarrow{F^3} \qquad \mathscr{K}^{\mathrm{YM}}_{\mu\nu\alpha} = F^{a}_{\rho\mu}\overleftrightarrow{D}_{\alpha}F^{a\rho}_{\nu}$$

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$$\xrightarrow{F^3} \qquad \mathcal{K}^{\mathrm{YM}}_{\mu\nu\alpha} = F^{a}_{\rho\mu}\overleftrightarrow{D}_{\alpha}F^{a\rho}_{\nu}$$

which is conserved on the EOM,

$$\partial^{\alpha}\mathcal{K}^{\mathrm{YM}}_{\mu\nu\alpha} = F^{a}_{\rho\mu}\overleftrightarrow{D}^{2}F^{a\rho}_{\nu}$$

$$= -f^{abc}(F^{a\rho}{}_{\mu}F^{b}_{\sigma[\nu}F^{c\sigma}_{\rho]} - F^{a\rho}_{\nu}F^{b}_{\sigma[\rho}F^{c\sigma}_{\mu]}) = 0$$

The current is again related to the stress tensor.

$$\mathscr{K}^{\mathrm{YM}}_{\mu\nu\alpha} = -\,\partial_{[\mu} T^{\mathrm{YM}}_{\nu]\alpha} + \text{improvement terms}$$

The charge is the change in energy-momentum.

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trivial charge

↓

no symmetry

↓

no violation of Coleman-Mandula

Applications

Deriving BCJ Numerators

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$$A^{\mathrm{YM}} \quad \sim \quad \sum A^{\mathrm{GBAS}} F \dots F$$

field strength decomposition
(more gluons from fewer gluons)

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$$A^{\text{GBAS}} \sim \mathscr{T} \cdots \mathscr{T} A^{\text{YM}}$$

amplitudes transmutation

(fewer gluons from more gluons)

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$$A^{\mathrm{YM}} \sim \sum A^{\mathrm{BAS}} \mathcal{T} \cdots \mathcal{T} F \cdots F$$

BCJ numerators

To begin, apply " $\otimes F^3$ " covariant double copy to more complicated half-ladder color topologies,

$$f^{a_1a_2a_3} \quad \overset{F^3}{\rightarrow} \quad (-i)\,\mathrm{tr}[[F_1, F_2]\,\widetilde{F}_3]$$

$$f^{a_1a_2a_3a_4} \quad \overset{F^3}{\rightarrow} \quad (-i)^2\,\mathrm{tr}[[[F_1, F_2], F_3]\,\widetilde{F}_4]$$

$$\vdots$$

$$f^{a_1a_2a_3\cdots a_n} \quad \overset{F^3}{\rightarrow} \quad (-i)^{n-2}\,\mathrm{tr}[[\cdots[[F_1, F_2], F_3], \cdots, F_{n-1}]\,\widetilde{F}_n]$$

where the linearized field strengths are

$$\left[F_i\right]_{\mu\nu} = p_{i\mu}\varepsilon_{i\nu} - p_{i\mu}\varepsilon_{i\nu} \qquad \text{for} \qquad \mathrm{i} \neq \mathrm{n}$$

$$[\widetilde{F}_n]_{\mu\nu} = \frac{1}{p_n q}\left(q_\mu\varepsilon_{n\nu} - q_\mu\varepsilon_{n\nu}\right)$$

The field strengths are literally generators of the covariant kinematic algebra in the fundamental.

$$\mathrm{tr}\left[T^{a_1}T^{a_2}T^{a_3}\right] \quad \overset{F^3}{\to} \quad \mathrm{tr}[F_1F_2\widetilde{F}_3]$$

$$\mathrm{tr}\left[T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right] \quad \overset{F^3}{\to} \quad \mathrm{tr}[F_1F_2F_3\widetilde{F}_4]$$

$$\vdots$$

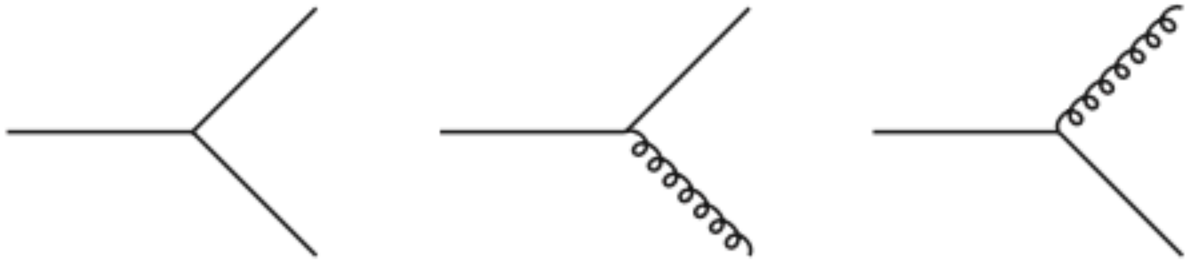
$$\mathrm{tr}\left[T^{a_1}T^{a_2}T^{a_3}\ldots T^{a_{n-1}}T^{a_n}\right] \quad \overset{F^3}{\to} \quad \mathrm{tr}[F_1F_2F_3\cdots F_{n-1}\widetilde{F}_n]$$

For $\sigma = \sigma_1\sigma_2\sigma_3\cdots\sigma_\ell$, define the field strength trace,

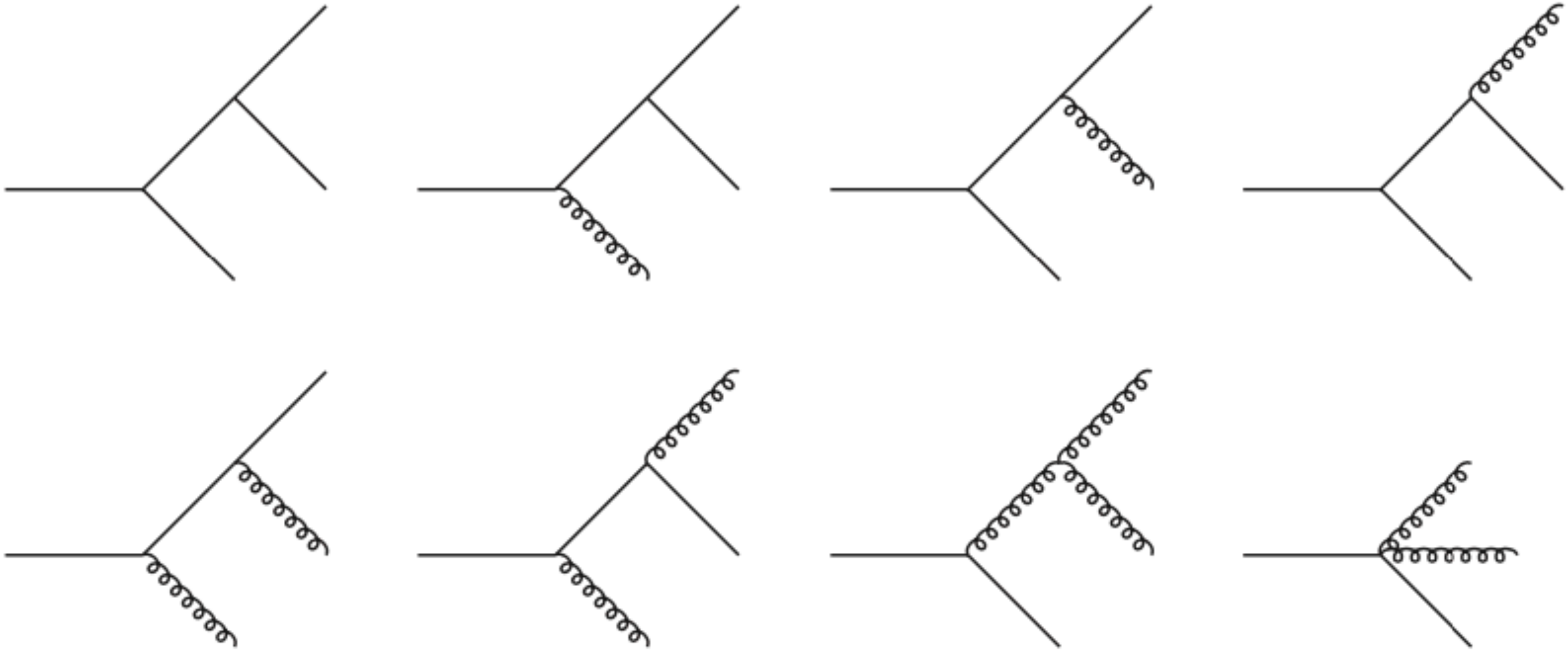
$$F[\sigma n] = \mathrm{tr}[F_\sigma \widetilde{F}_n]$$

$$F_\sigma = \prod_i^{|\sigma|} F_{\sigma_i}$$

$A_3^{\text{YM}} \quad =$



$A_4^{\text{YM}} \quad =$



From the color decomposed GBAS amplitude,

$$A_{\phi n}^{\text{GBAS}} = \sum_{\sigma \in S(\phi)} A[\sigma n]\, \text{tr}\left[T^{a_{\sigma_1}} \cdots T^{a_{\sigma_\ell}} T^{a_n}\right] \longleftarrow$$

dressed with
the non-gauged
biadjoint color

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covariant double copy generates the formulas,

$$\sum_{\phi\in\mathbb{P}^+(1\cdots n-1)} A_{\phi n}^{\mathrm{GBAS}} \quad \xrightarrow{F^3} \quad A_n^{\mathrm{YM}} = \sum_{\phi\in\mathbb{P}^+(1\cdots n-1)}\ \sum_{\sigma\in S(\phi)} A[\sigma n]\,F[\sigma n]$$

From the color decomposed GBAS amplitude,

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dressed with
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covariant double copy generates the formulas,

$$\sum_{\phi \in \mathbb{P}^+(1 \cdots n-1)} A^{\mathrm{GBAS}}_{\phi n} \quad \xrightarrow{F^3} \quad A_n^{\mathrm{YM}} = \sum_{\phi \in \mathbb{P}^+(1 \cdots n-1)} \sum_{\sigma \in S(\phi)} A[\sigma n]\, F[\sigma n]$$

$$A_n^{\mathrm{GR}} = \sum_{\substack{\phi \in \mathbb{P}^+(1 \cdots n-1) \\ \bar{\phi} \in \mathbb{P}^+(1 \cdots n-1)}} \sum_{\substack{\sigma \in S(\phi) \\ \bar{\sigma} \in S(\bar{\phi})}} A[\sigma n \,|\, \bar{\sigma} n]\, F[\sigma n]\, \bar{F}[\bar{\sigma} n]$$

We derive GBAS from YM via transmutation,

fewer gluons more gluons

$$A[123\cdots\ell n] = \mathscr{T}[123\cdots\ell n]\,A_n^{\rm YM} = \left(\mathscr{T}_{1\,n}\prod_{i=1}^{\ell-1}\mathscr{T}_{i\,i+1\,n}\right)A_n^{\rm YM}$$

where $\mathscr{T}_{ij} = \dfrac{\partial}{\partial(\varepsilon_i\varepsilon_j)}$ and $\mathscr{T}_{ijk} = \dfrac{\partial}{\partial(p_k\varepsilon_j)} - \dfrac{\partial}{\partial(p_i\varepsilon_j)}$

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where $\mathscr{T}_{ij} = \dfrac{\partial}{\partial(\varepsilon_i \varepsilon_j)}$ and $\mathscr{T}_{ijk} = \dfrac{\partial}{\partial(p_k \varepsilon_j)} - \dfrac{\partial}{\partial(p_i \varepsilon_j)}$

Applying transmutation to the field strength decomposition, we derive inverse transmutation

$$A[123\cdots\ell n] = \sum_{\theta\in\mathbb{P}^+(\ell+1\cdots n-1)} \sum_{\tau\in S(\theta)} \sum_{i=1}^{\ell} \sum_{\rho\in\tau\wedge(i+1\cdots\ell)} A[1\cdots i\rho n] \left\{ -\frac{p_i F_\tau q}{p_{123\cdots\ell n} q} \right\}$$

more gluons fewer gluons

Eliminating gluons one by one, we express YM purely in terms of BAS, i.e. the BCJ construction!

BAS amplitude

$$A_n^{\text{YM}} = \sum_{\sigma \in S(1 \cdots n-1)} A[\sigma n] \, K[\sigma n] \longleftarrow$$

BCJ numerator in trace basis

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BAS amplitude

$$A_n^{\mathrm{YM}} = \sum_{\sigma\in S(1\cdots n-1)} A[\sigma n]\, K[\sigma n] \quad \longleftarrow \quad \text{BCJ numerator in trace basis}$$

$$K[123\cdots n] = \sum_{\tau\in \mathrm{part}(1\cdots \mathrm{n}-1)} F[\tau_1 n] \prod_{i=2}^{|\tau|} G[(\tau_1\cdots\tau_{i-1})_{<\tau_i}, \tau_i, (\tau_1\cdots\tau_{i-1})_{>\tau_i} n]$$

where $F[\sigma n] = \mathrm{tr}[F_\sigma \widetilde{F}_n]$ and $G[\sigma, \tau, \rho n] = -\dfrac{p_\sigma F_\tau q_{\sigma\rho}}{p_{\sigma\rho n} q_{\sigma\rho}}$

Eliminating gluons one by one, we express YM purely in terms of BAS, i.e. the BCJ construction!

BAS amplitude

$$A_n^{\mathrm{YM}} = \sum_{\sigma\in S(1\cdots n-1)} A[\sigma n]\, K[\sigma n] \quad \longleftarrow \quad \text{BCJ numerator in trace basis}$$

$$K[123\cdots n] = \sum_{\tau\in \mathrm{part}(1\cdots \mathrm{n}-1)} F[\tau_1 n] \prod_{i=2}^{|\tau|} G[(\tau_1\cdots\tau_{i-1})_{<\tau_i}, \tau_i, (\tau_1\cdots\tau_{i-1})_{>\tau_i} n]$$

where $\quad F[\sigma n] = \mathrm{tr}[F_\sigma \widetilde{F}_n] \quad$ and $\quad G[\sigma, \tau, \rho n] = -\dfrac{p_\sigma F_\tau q_{\sigma\rho}}{p_{\sigma\rho n} q_{\sigma\rho}}$

Via standard double copy, we obtain gravity,

$$A_n^{\mathrm{GR}} = \sum_{\sigma\in S(1\cdots n-1)} \sum_{\bar\sigma\in S(1\cdots n-1)} A[\sigma n\,|\,\bar\sigma n]\, K[\sigma n]\, \bar K[\bar\sigma n]$$

Properties of our BCJ Numerators

- no implicit algorithms, diagrammatics, integrals
- permutation invariant on $n - 1$ legs
- gauge invariant on $n - 1$ legs
- reference dependent, but can be made local

The BCJ numerators for a few examples are

$$
\begin{aligned}
K[123] ={}& F[123] + F[13]\, G[1,2,3] \\
K[1234] ={}& F[1234] + F[124]\, G[12,3,4] + F[134]\, G[1,2,34] \\
&+ F[14]\, G[1,23,4] + F[14]\, G[1,2,4]\, G[12,3,4] + F[14]\, G[1,3,4]\, G[1,2,34] \\
K[12345] ={}& F[12345] + F[1345]\, G[1,2,345] + F[1245]\, G[12,3,45] + F[1235]\, G[123,4,5] \\
&+ F[145]\, G[1,2,345]\, G[1,3,45] + F[145]\, G[1,2,45]\, G[12,3,45] + F[145]\, G[1,23,45] \\
&+ F[135]\, G[1,2,345]\, G[13,4,5] + F[135]\, G[1,2,35]\, G[123,4,5] + F[135]\, G[1,24,35] \\
&+ F[125]\, G[12,3,45]\, G[12,4,5] + F[125]\, G[12,3,5]\, G[123,4,5] + F[125]\, G[12,34,5] \\
&+ F[15]\, G[1,2,345]\, G[1,3,45]\, G[1,4,5] + F[15]\, G[1,2,45]\, G[1,4,5]\, G[12,3,45] \\
&+ F[15]\, G[1,2,345]\, G[1,3,5]\, G[13,4,5] + F[15]\, G[1,2,35]\, G[1,3,5]\, G[123,4,5] \\
&+ F[15]\, G[1,2,5]\, G[12,3,45]\, G[12,4,5] + F[15]\, G[1,2,5]\, G[12,3,5]\, G[123,4,5] \\
&+ F[15]\, G[1,2,345]\, G[1,34,5] + F[15]\, G[1,24,5]\, G[12,3,45] \\
&+ F[15]\, G[1,23,5]\, G[123,4,5] + F[15]\, G[1,4,5]\, G[1,23,45] \\
&+ F[15]\, G[1,3,5]\, G[1,24,35] + F[15]\, G[1,2,5]\, G[12,34,5] + F[15]\, G[1,234,5],
\end{aligned}
$$

Our arXiv ancillary files, `BCJ_numerators.m` and `Examples.nb`, implement this formula.

The BCJ numerators for YM evaluate in $d = 4$ to

$$F[\sigma n] \overset{d=4}{\to} \frac{\mathrm{PT}(\sigma^- q)\overline{\mathrm{PT}}(\sigma^+)}{\langle qn\rangle^2}$$

inverse Parke-Taylor
denominator factor

$$G[\sigma, \tau, \rho n] \overset{d=4}{\to} -\frac{1}{\langle r|p_{\sigma\rho n}|r]} \sum_{i=1}^{|\sigma|} \frac{\mathrm{PT}(\sigma_i \tau^- r)\overline{\mathrm{PT}}(\sigma_i \tau^+ r)}{\langle r|p_{\sigma_i}|r]}$$

and map to NLSM by sending $F_{i\mu\nu}$ to $\Pi_{i\mu\nu} = p_{i\mu}p_{i\nu}$

$$F[\sigma n] \overset{\mathrm{NLSM}}{\to} \mathrm{tr}[\Pi_\sigma \widetilde{\Pi}_n]$$

manifest Adler zero

$$G[\sigma, \tau, \rho n] \overset{\mathrm{NLSM}}{\to} -\frac{p_\sigma \Pi_\tau q_{\sigma\rho}}{p_{\sigma\rho n} q_{\sigma\rho}}$$

BCJ numerators
for YM, NLSM

standard
double
copy

tree-level amplitudes for
YM, GR, NLSM, SG, BI

Conclusions

- Double copy on fields and EOM for currents and field strengths. No field basis redundancy.

- BAS → NLSM via color algebra → diff algebra. Implies new formulations of SG, BI.

- GBAS → YM via color algebra → Lorentz algebra. Implies tetrad classical double copy.

- Closed formula for BCJ numerators in YM and NLSM at any point in general dimensions.

Thank You!

Backup Slides

Fundamental BCJ follows from a color factor symmetry of amplitudes (Brown + Naculich).

To implement this directly on EOM, we define

$$f^{abc} \to f^{abc} + \delta f^{abc}$$

$$\delta f^{abc} V^b V^c = \epsilon^a \delta^{bc} V^b \overleftrightarrow{\Box} W^c$$

The EOM of BAS is invariant!

dual color conservation

$$\delta f^{abc} f^{\bar{a}\bar{b}\bar{c}} \phi^{b\bar{b}} \phi^{c\bar{c}} = \epsilon^a \partial^\alpha \mathscr{K}^{\bar{a}}_\alpha \overset{\downarrow}{=} 0$$

YM has a non-covariant first-order formulation,

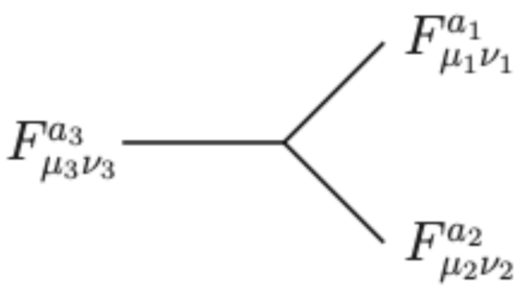
$$\Box F^a_{\mu\nu} + f^{abc}\left[\partial^\rho(A^b_{[\rho}F^c_{\mu\nu]}) - \partial_{[\mu}(A^{b\rho}F^c_{\nu]\rho})\right] = \partial_{[\mu}J^a_{\nu]}$$

Feynman propagator

$$\text{where} \quad A^a_\mu = -\frac{q^\nu F^a_{\mu\nu}}{q\partial}$$

lightcone gauge

which is cubic with no aux fields. Kinematic Jacobi fails but three-point encodes all scattering,



$$= \left[\frac{if^{a_1a_2a_3}q^{\mu_1}\eta^{\mu_2\mu_3}}{4qp_1}\left(\frac{1}{2}p_3^{\nu_1}\eta^{\nu_2\nu_3} - p_3^{\nu_2}\eta^{\nu_3\nu_1} + p_3^{\nu_3}\eta^{\nu_1\nu_2}\right)\right]_{\text{antisym}} + \{1 \leftrightarrow 2\},$$