D-instanton Amplitudes in String Theory

Ashoke Sen

Harish-Chandra Research Institute, Allahabad, India

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In spirit, modern amplitudes program has its root in string theory

Bern, Kosower, 1988

A string amplitude with a given set of external states has one term at every loop order

– avoids having to sum over large number of Feynman diagrams.

Simplest example: Veneziano amplitude

$$\int_0^1 dy\, y^{2p_1.p_2}(1-y)^{2p_1.p_3}$$

– gives the tree level 'color ordered' amplitude of four open strings in $\alpha' = 1$ unit

– no sum over s-channel, t channel and contact diagrams

However many string theory amplitudes written in this form actually diverge.

Example: Veneziano amplitude

$$\int_0^1 dy\, y^{2p_1.p_2}(1-y)^{2p_1.p_3}$$

– diverges for $2p_1.p_2 \le -1$ or $2p_1.p_3 \le -1$.

Conventional wisdom: Define the integral for $2p_1.p_2 > -1$ and $2p_1.p_3 > -1$ and then analytically continue to other regions.

There are more elegant formalisms involving defining the integrals as contour integrals in the complex plane

Berera; Witten; Mizera; $\cdots$

However there are cases where all these prescriptions fail to give finite results.

Strategy:

– first understand the origin of the simple divergences that can be treated via analytic continuation

– then use this insight to address the cases where analytic continuation fails

Main tool: String field theory (SFT)

SFT is a regular quantum field theory (QFT) with infinite number of fields

Perturbative amplitudes: sum of Feynman diagrams

Each diagram covers part of the integration region over the world-sheet variables (moduli of Riemann surfaces and locations of vertex operators)

Sum of the diagrams covers the full integration region.

There are no UV divergences since interaction terms are exponentially suppressed at large Euclidean momenta.

How do we get integral over world-sheet variables from a Feynman diagram?

Use Schwinger parametrization of the internal propagators:

$$(k^2 + m^2)^{-1} = \int_0^\infty dt\, e^{-t\,(k^2+m^2)}$$

The integration over t gives integration over world-sheet variables after a change of variable.

Divergences come from the $t \to \infty$ region for $k^2 + m^2 \leq 0$.

All divergences in string theory are of this kind.

Analytic continuation amounts to defining the integral for $k^2 + m^2 > 0$ and then continuing the result to $k^2 + m^2 < 0$.

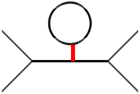
SFT on the other hand just uses the lhs i.e. $(k^2 + m^2)^{-1}$ for computation

$$(k^2 + m^2)^{-1} = \int_0^\infty dt\, e^{-t\,(k^2+m^2)}$$

The problematic cases arise when $k^2 + m^2$ is forced to vanish due to momentum conservation

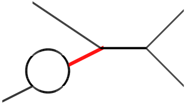
– both sides diverge and no analytic continuation is possible

Massless tadpole:



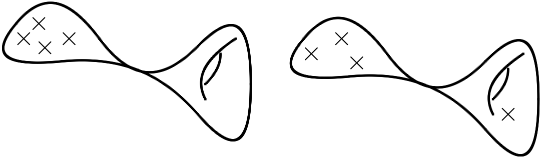
The red line has k=0, m=0 and gives divergent propagator

Mass renormalization:



The red line has $k^2 + m^2 = 0$ and gives divergent propagator.

In the world-sheet description these divergences show up as divergences arising from degenerate Riemann surfaces.



World-sheet theory cannot make sense of these divergences

In SFT, we can address these divergences using standard QFT rules

Massless tadpole problem is addressed by shifting the vacuum

Mass renormalization problem is addressed by shifting the definition of on-shell states to $k^2 + m^2 + \delta m^2 = 0$

In principle SFT gives a completely well defined expression for perturbative string amplitudes to all orders.

Today we are going to apply SFT techniques to address another problem in string theory where similar divergences are present

– D-instanton amplitudes

D-instantons are to D-branes what ordinary instantons are to solitons in QFT

– describe classical solutions in the Euclidean theory that are localized in all non-compact directions including Euclidean time

(uses Dirichlet b.c. on open strings along all non-compact directions)

– give non-perturbative contribution to string amplitudes of order e^{-C/g_s}

C: numerical constant, g_s: string coupling

D-instanton contributions are expected to play important role in many aspects of string phenomenology.

1. They provide contribution to the superpotential needed for moduli stabilizations

– lift flat directions in the scalar field potential (KKLT, LVS)

2. In many cases perturbative contribution to Yukawa type couplings vanish and D-instantons provide the leading contribution.

D-instanton contributions to the amplitudes can in principle be computed systematically using perturbative world-sheet methods

– theory of open strings on the D-instanton and closed strings

Problem: Opens strings on D-instantons carry zero momentum along the non-compact directions

– no variable to carry out analytic continuation

– need to use SFT to extract sensible answers.

We shall illustrate this procedure for a particular class of problems

– D-instanton contribution to IIB string theory amplitudes in 10 dimensions.

However the method is quite general and applies to any euclidean D-brane in any string theory. Alexandrov, A.S., Stefanski

The problem

Consider four graviton scattering amplitude in type IIB string theory.

At tree level, it is given by the supergravity result and an additional R^4 contact interaction: Gross, Witten

$$\frac{i}{4}\,\kappa^2\,K_c\,\left[\frac{64}{stu}+2\zeta(3)\right]\,(2\pi)^{10}\,\delta^{(10)}(p_1+p_2+p_3+p_4)$$

in $\alpha' = 1$ unit.

$$\kappa^2 \equiv 8\pi G = 2^6\pi^7 g_s^2$$

g_s: string coupling, normalized so that the D-string tension is $1/(2\pi g_s)$.

K_c: A kinematic factor that depends on the polarizations and momenta of the external gravitons (carries 8 powers of momentum).

$$\frac{i}{4}\,\kappa^2\,K_c\,\left[\frac{64}{stu}+2\zeta(3)\right]\,(2\pi)^{10}\,\delta^{(10)}(p_1+p_2+p_3+p_4)$$

The supergravity contribution, given by the first term, is S-duality invariant.

The second term is not, but there is an S-duality invariant completion containing one loop and non-perturbative terms.

Green, Gutperle

$$\frac{i}{4}\,\kappa^2\,K_c\,\left[\frac{64}{stu}+2\zeta(3)+\frac{2\pi^2}{3}\,g_s^2+4\,\pi\,g_s^{3/2}\sum_{k=1}^{\infty}\sqrt{k}\left(\sum_{d|k}d^{-2}\right)\left\{e^{2\pi ik\tau}+e^{-2\pi ik\tau^*}\right\}\{1+O(g_s)\}\right]$$

$\tau = a + i/g_s$, a: vev of RR scalar field

$e^{2\pi ik\tau} = e^{-2\pi k/g_s}\times e^{2\pi ia}$: k D-instanton contribution,

$e^{-2\pi ik\tau^*} = e^{-2\pi k/g_s}\times e^{-2\pi ia}$: k anti-D-instanton contribution

$O(g_s)$ contains higher powers of g_s

This gives the following prediction for the leading term in the k D-instanton contribution to the four graviton amplitude:

$$i\, 2^6 \pi^8\, g_s^{7/2}\, e^{2\pi i k\tau}\, K_c\, k^{1/2} \sum_{d|k} d^{-2}$$

Our goal: Reproduce this from direct D-instanton computation.

Once this is done, all the higher order terms in power series expansion in g_s can be obtained using the differential equation implied by supersymmetry

Green, Sethi; Green, Vanhove; $\cdots$, Wang, Yin

– determines the function completely without help of S-duality

Direct calculation should also be possible

Agmon, Balthazar, Cho, Rodriguez, Yin (ABCRY)

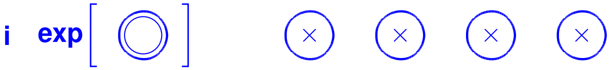
$$i\,2^6\pi^8\,g_s^{7/2}\,e^{2\pi ik\tau}\,K_c\,k^{1/2}\sum_{d|k} d^{-2}$$

The $e^{2\pi ik\tau}$ comes from exponential of the action of k instantons.

Our goal: Reproduce the $2^6\pi^8\,g_s^{7/2}\,K_c\,k^{1/2}\sum_{d|k} d^{-2}$ factor.

Single instanton

The leading contribution comes from the product of four disk one point functions and arbitrary number of annulus zero point functions.



Note: we include disconnected world-sheet since individual world-sheets do not conserve momentum

– restored at the end after integration over zero modes

Define N = i exp[A], A: Annulus zero point function

Our first task will be to calculate N.

At this order, all the subtleties reside in the calculation of N.

$$\mathbf{N} = \mathbf{i}\, \mathbf{e}^{\mathbf{A}}$$

For type IIB D-instantons:

$$\mathbf{A} = \int_0^\infty \frac{\mathbf{dt}}{\mathbf{2t}} \left[\frac{\mathbf{1}}{\mathbf{2}}\, \eta(\mathbf{it})^{-\mathbf{12}} \left\{\vartheta_{\mathbf{3}}(\mathbf{0}, \mathbf{it})^{\mathbf{4}} - \vartheta_{\mathbf{4}}(\mathbf{0}, \mathbf{it})^{\mathbf{4}} - \vartheta_{\mathbf{2}}(\mathbf{0}, \mathbf{it})^{\mathbf{4}} + \vartheta_{\mathbf{1}}(\mathbf{0}, \mathbf{it})^{\mathbf{4}}\right\}\right]$$

ϑ_α are Jacobi theta functions.

The integrand vanishes by theta function identity

– result of cancellation between NS and R sector open string states.

The first two terms come from NS sector and the last two terms come from the R sector.

The general structure of A:

$$A = \int_0^\infty \frac{dt}{2t}\, Tr\left[e^{-2\pi t L_0}(-1)^F\right]$$

The vanishing of the integrand shows that the contribution from the positive L_0 states cancel in Bose-Fermi pair.

For these there are no subtleties and the cancellation is genuine.

However the cencellation cannot be trusted for the zero modes and we need to treat them carefully.

For this it will be useful to regulate the system by introducing a small non-zero L_0 value h for the zero modes at the intermediate steps

– put slightly shifted boundary condition on the two boundaries

– preserves conformal and BRST invariance on the world-sheet

In this regulated system

$$A = \int_0^\infty \frac{dt}{2t}\left(8\,e^{-2\pi t h} - 8\,e^{-2\pi t h}\right) = \int_0^\infty \frac{dt}{2t}\left(10\,e^{-2\pi t h} - 2\,e^{-2\pi t h} - 8\,e^{-2\pi t h}\right)$$

$\downarrow$ **absence of UV divergence as $t \to 0$**

$$\Rightarrow N = i\,e^{A} = i\sqrt{\frac{h^8\,h^2}{h^{10}}} = i\int\left\{\prod_{\mu=0}^{9}\frac{d\xi_\mu}{\sqrt{2\pi}}\right\} dp\,dq\,\exp\left[-\frac{1}{2}h\sum_{\mu=0}^{9}\xi_\mu\xi^\mu - h\,p\,q\right]$$

$$\int\prod_{\alpha=1}^{16} d\chi_\alpha\,\exp\left[\frac{1}{2}g_{\alpha\beta}\chi_\alpha\chi_\beta\right]$$

ξ_μ: **10 grassmann even modes related to D-instanton position**

p,q: 2 grassmann odd modes representing ghosts

χ_α: **16 grassmann odd modes representing fermion zero modes**

$g_{\alpha\beta}$: **an anti-symmetric, 16×16 hermitian matrix satisfying:**

$$g^2 = h\,I_{16}\,, \quad I_{16}\text{: } 16 \times 16 \text{ identity matrix}$$

We now proceed as follows.

1. First we interpret N as the Siegel gauge fixed path integral of open string field theory on the instanton

– fixes the normalization of the integration measure.

2. In this interpretation, the modes p and q represent Faddeev-Popov ghosts.

3. Then we show that the Siegel gauge becomes singular in the $h \to 0$ limit, and this is the reason why the coefficient of the p q term, representing the ghost kinetic operator, vanishes.

4. The remedy is to work with the original gauge invariant path integral before gauge fixing.

This gets rid of the zero modes p and q from the integral.

Gauge invariant path integral

$$\mathbf{N} = \mathbf{i} \int \left\{ \prod_\mu \frac{\mathbf{d}\xi^\mu}{\sqrt{\mathbf{2}\pi}} \right\} \left\{ \prod_\alpha \mathbf{d}\chi_\alpha \right\} \mathbf{d}\phi \, \mathbf{e}^{-\mathbf{S}} \Bigg/ \int \mathbf{d}\theta$$

$$\mathbf{S} = \frac{\mathbf{1}}{\mathbf{4}}(\phi - \sqrt{\mathbf{2h}}\,\xi^{\mathbf{9}})^{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{h} \sum_{\mu=\mathbf{0}}^{\mathbf{8}} \xi_\mu \xi^\mu - \frac{\mathbf{1}}{\mathbf{2}} \mathbf{g}_{\alpha\beta} \chi_\alpha \chi_\beta$$

ϕ: an extra 'field' that is present in the gauge invariant action

θ: a gauge transformation parameter under which

$$\delta\phi = \mathbf{h}\,\theta, \qquad \delta\xi^{\mathbf{9}} = \sqrt{\frac{\mathbf{h}}{\mathbf{2}}}\,\theta$$

9 direction is special because we have taken the shifted b.c. between the two boundaries of the annulus to be along the 9-direction.

Siegel gauge corresponds to setting $\phi = \mathbf{0}$

– gives us back the original expression.

After setting h=0, we get

$$\mathbf{N} = \mathbf{i} \int \left\{ \prod_{\mu} \frac{\mathbf{d}\xi^{\mu}}{\sqrt{\mathbf{2}\pi}} \right\} \left\{ \prod_{\alpha} \mathbf{d}\chi_{\alpha} \right\} \mathbf{d}\phi \, \mathbf{e}^{-\phi^{\mathbf{2}}/\mathbf{4}} \Bigg/ \int \mathbf{d}\theta$$

We can now do the ϕ integral and write:

$$\mathbf{N} = \mathbf{i} \, (\mathbf{2}\pi)^{-\mathbf{5}} \, \mathbf{2}\sqrt{\pi} \int \left\{ \prod_{\mu} \mathbf{d}\xi^{\mu} \right\} \left\{ \prod_{\alpha} \mathbf{d}\chi_{\alpha} \right\} \Bigg/ \int \mathbf{d}\theta$$

$$\mathbf{N} = \mathbf{i}\,(2\pi)^{-5}\,2\sqrt{\pi}\int \left\{\prod_\mu \mathbf{d}\xi^\mu\right\}\left\{\prod_\alpha \mathbf{d}\chi_\alpha\right\}\Bigg/\int \mathbf{d}\theta$$

ξ^μ's are related to the location x^μ of the instanton in space-time.

Precise relation may be found by comparing

–world-sheet result for the coupling of ξ^μ to a string amplitude

– the expected coupling of x^μ via $e^{ip.x}$ factor

The gauge transformation parameter θ is related to the rigid U(1) gauge transformation parameter α on the D-instanton.

Relation between θ and α can be found by comparing string field theory gauge transformation and rigid U(1) transformation.

Result:

$$\xi^\mu = \mathbf{x}^\mu/(\mathbf{g_o}\,\pi\,\sqrt{\mathbf{2}}) \qquad \Rightarrow \qquad \prod_\mu \mathbf{d}\xi^\mu = (\pi\sqrt{\mathbf{2}}\mathbf{g_o})^{-\mathbf{10}} \prod_\mu \mathbf{dx}^\mu$$

g_o: open string coupling = $2^{-1}\pi^{-3/2}g_s^{1/2}$

$$\theta = \mathbf{2}\,\alpha/\mathbf{g_o} \quad \Rightarrow \quad \int \mathbf{d}\theta = (\mathbf{2}/\mathbf{g_o}) \int \mathbf{d}\alpha = \mathbf{4}\pi/\mathbf{g_o}$$

since α has period 2π.

$$N = i\,(2\pi)^{-5}\,2\sqrt{\pi}\,(\pi\sqrt{2}g_o)^{-10}\,g_o/(4\pi)\,\int \left\{\prod_\mu dx^\mu\right\}\left\{\prod_\alpha d\chi_\alpha\right\}$$

Integrations over the collective modes x^μ and χ_α have to be done at the end after computing the full amplitude, since the other world-sheet components also have x^μ and χ_α dependence.

x^μ integral eventually generates the $(2\pi)^{10}\delta^{(10)}(\sum_j p_j)$ factor.

Integration over the grassmann odd variables χ_α will vanish unless there are 16 insertions of χ_α in the rest of the amplitude.

Only non-vanishing contribution comes from inserting 4 fermion zero modes on each of the four disks

The χ_α integrals generate a factor of $\epsilon^{\alpha_1\cdots\alpha_{16}}$ multiplying the product of the four disk amplitudes with $\chi_{\alpha_1},\cdots,\chi_{\alpha_{16}}$ insertions.

– precisely reproduces Green-Gutperle prediction for k=1

k-instantons

The center of mass degrees freedom give the same integral as a single instanton except for some powers of k from Chan-Paton factors

The relative degrees of freedom have to be analyzed similarly by gauge 'unfixing' the Siegel gauge and explicitly integrating over the out of Siegel gauge modes.

The remaining part is a supersymmetric matrix integral which had already been computed while studying the index of multiple D0-branes Krauth, Nicolai, Staudacher; Moore, Nekrasov, Shatashvili

Putting these results together we reproduce precisely the leading term in the k-instanton amplitude as predicted by Green and Gutperle.

Conclusion

String (field) theory gives a systematic procedure for computing perturbative and D-instanton contribution to the amplitudes.

We should be able to apply this procedure to calculate D-instanton contribution in situations where S-duality may not be of help

e.g. semi-realistic string compactifications $\cdots$

$\cdots$ and other problems where D-instanton corrections might play a dominant role

$\cdots$ and perhaps also to find a non-perturbative definition of string theory.