Conservative & radiative binary dynamics from amplitudes

Mao Zeng, Higgs Centre for Theoretical Physics, University of Edinburgh

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arXiv:2005.04236 (JHEP), Parra-Martinez, Ruf, MZ,
arXiv:2101.07255 (PRL), 2104.03957 (accepted by JHEP), Herrmann, Parra-Martinez, Ruf, MZ

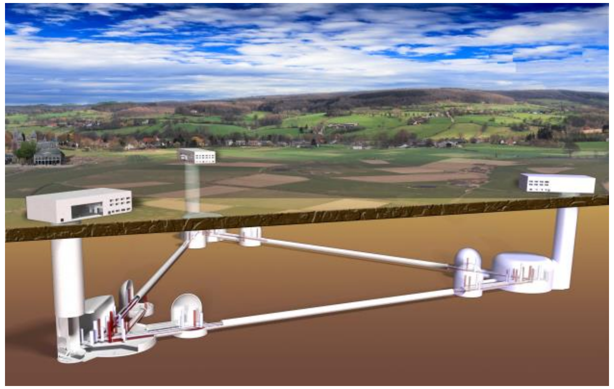
Outline

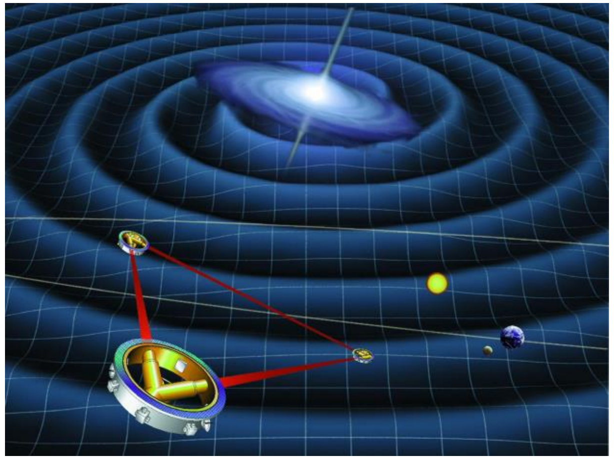
- Introduction
- Bridge from post-Newtonian to post-Minkowskian expansion
  - differential equations, method of regions
- Radiative dynamics from KMOC formalism & reverse unitarity
- Result: complete 3rd-post-Minkowskian binary dynamics
  - conservative, energy loss, radiation reaction

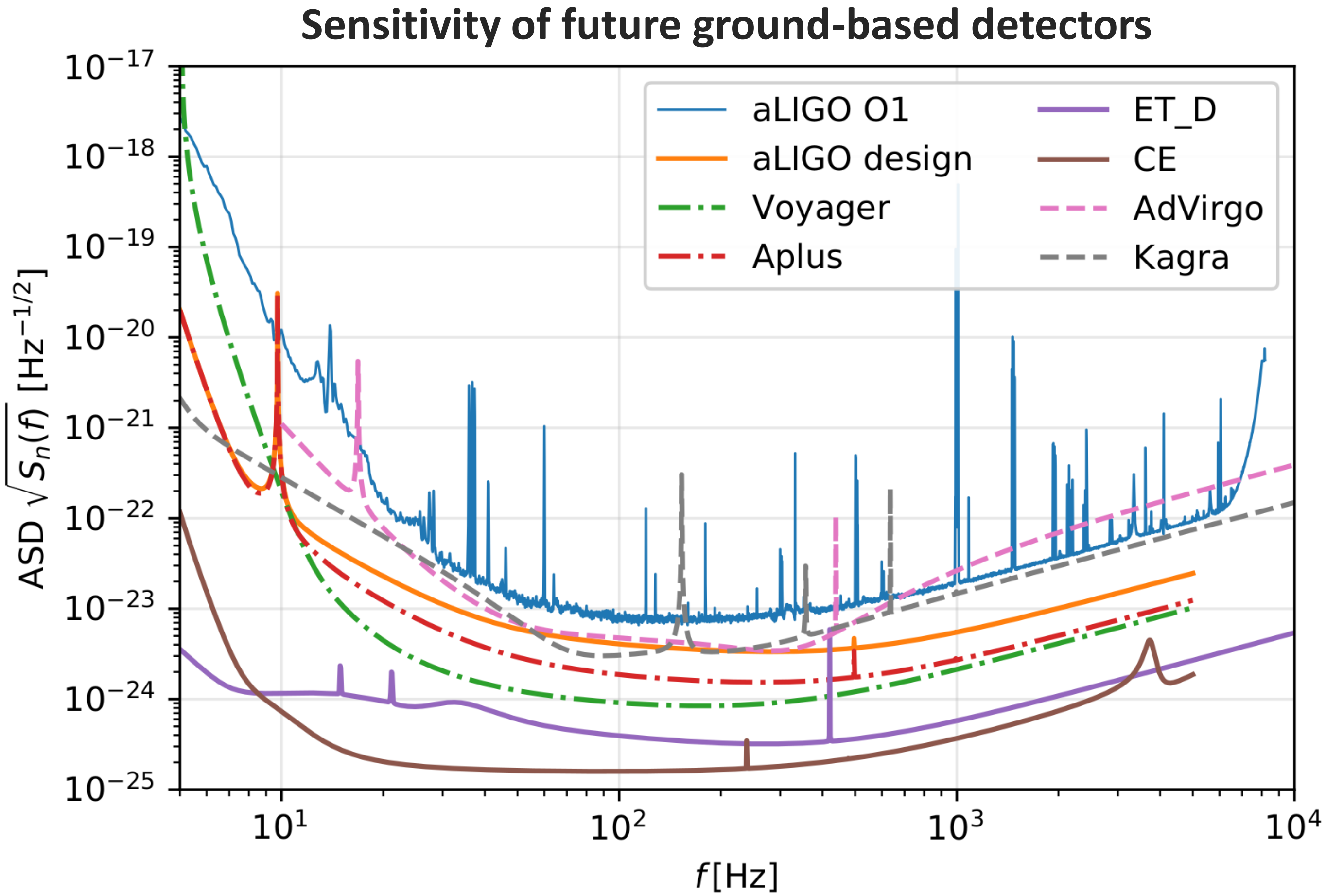
Introduction

Motivation – future detectors

- Theoretical predictions need vast improvements for future detectors!







Post-Newtonian (PN) expansion

- Joint expansion in GM/R and v^2, locked together by Virial's theorem.

$$m = m_A + m_B, \qquad \nu = \mu/m$$
$$\mu = m_A m_B/m$$

$$\frac{H}{\mu} = \boxed{\overset{\mathcal{O}(v^2)}{\frac{v^2}{2}} - \overset{\mathcal{O}(G)}{\frac{Gm}{R}}} + H_{1\mathrm{PN}} + H_{2\mathrm{PN}} + H_{3\mathrm{PN}} + H_{4\mathrm{PN}} \ldots$$

0PN, Newton

$H_{2\mathrm{PN}}$ (1980) $H_{3\mathrm{PN}}$ (2000) $H_{4\mathrm{PN}}$ (2014)

$$H_{1\mathrm{PN}} \longrightarrow \mathcal{O}(v^4) + \mathcal{O}(Gv^2) + \mathcal{O}(G^2)$$

$$\boxed{+\frac{1}{c^2}\left\{-\frac{v^4}{8} + \frac{3\nu v^4}{8} + \frac{Gm}{R}\left(-\frac{v_R^2 \nu}{2} - \frac{3v^2}{2} - \frac{\nu v^2}{2}\right) + \frac{G^2 m^2}{2R^2}\right\}}$$

1PN, Lorentz, Droste, 1917; Einstein, Infeld, Hoffman, 1938

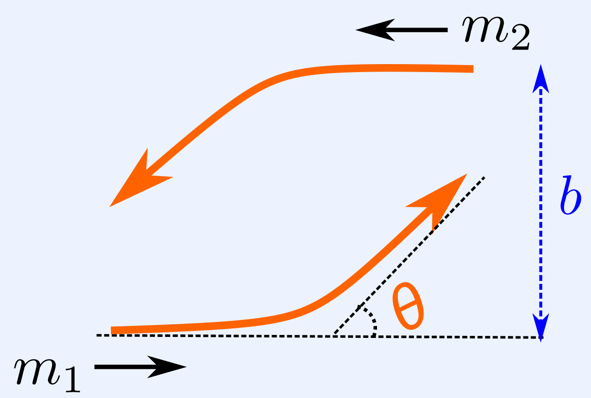
Post-Minkowskian (PM) expansion

- Expansion in coupling constant GM/R, **exact velocity dependence.** [Bertotti, Kerr, Plebanski, Portilla, Westpfahl, Gollder, Bel, Damour, Derulle, Ibanez, Martin, Ledvinka, Schaefer, Bicak…]

- Most accurate PM scattering angle until ~2019 [Westpfahl, '85]

Scattering angle of two black holes, as function of $m_1, m_2, b, \sigma \equiv \hat{p}_1 \cdot \hat{p}_2$.

$$\theta = \frac{4G(m_1+m_2)}{b}\frac{2\sigma^2-1}{2(\sigma^2-1)} + \frac{3\pi}{4}\frac{G^2(m_1+m_2)^2}{b^2}\frac{5\sigma^2-1}{\sigma^2-1} + \mathcal{O}(G^3)$$



New method: scattering amplitudes

Also: talks by Carlo Heissenberg, Mikhail Solon

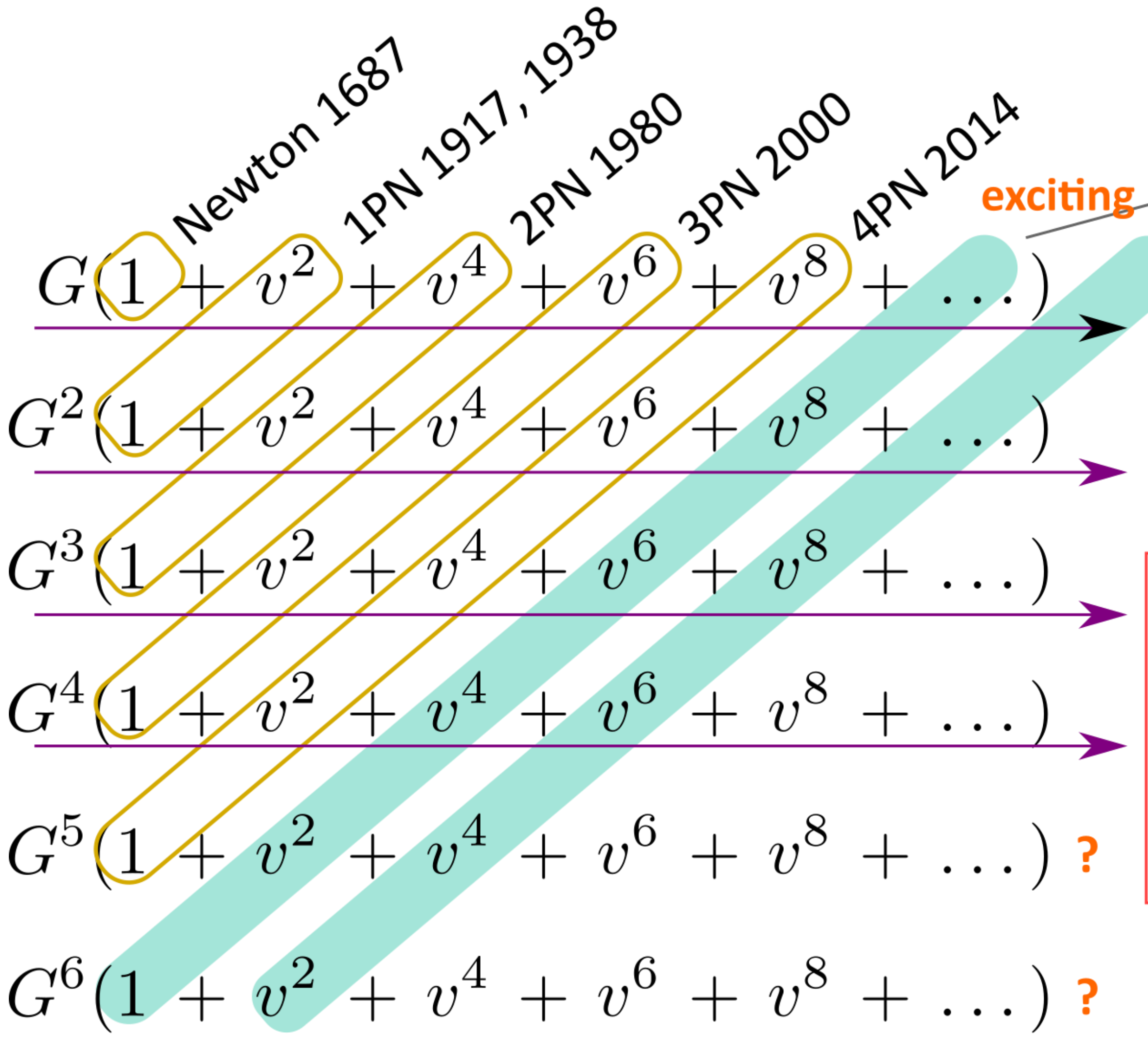
- Classical physics from quantum amplitudes through many methods: *eikonal* [Amati, Ciafaloni, Veneziano, '90], *NR EFT* [Cheung, Rothstein, Solon, '18], *observable formalism* [Kosower, Maybee, O'Connell, '18] ...

- Gauge-invariant scattering observables (deflection angle etc.) also encode **bound-state dynamics**, through matching to *effective-one-body Hamiltonian* [Buonanno, Damour, '99; Damour '16], *EFT potential* [Cheung, Rothstein, Solon, '18], or *direct analytic continuation* [Kalin, Porto, '20].

- Loop integrand tamed by modern methods: *generalized unitarity, double copy, nonlinear gauge fixing…*

- Advanced loop integration techniques developed for particle physics – adapted to cutting edge GR calculations.

New PM results – conservative spinless case



$$G(1 + v^2 + v^4 + v^6 + v^8 + \ldots)$$
$$G^2(1 + v^2 + v^4 + v^6 + v^8 + \ldots)$$
$$G^3(1 + v^2 + v^4 + v^6 + v^8 + \ldots)$$
$$G^4(1 + v^2 + v^4 + v^6 + v^8 + \ldots)$$
$$G^5(1 + v^2 + v^4 + v^6 + v^8 + \ldots)$$
$$G^6(1 + v^2 + v^4 + v^6 + v^8 + \ldots)$$

5PN: distinguish BH / NS

6PN: LISA, ET...

1PM

2PM [Westpfahl 1985]

3PM [Bern, Cheung, Roiban, Shen, Solon, **MZ**, 2019]

4PM (potential) [Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, **MZ**, 2021]

Both results reproduced by classical worldline EFT [Kalin, Liu, Porto, ’20; Dlapa, Kalin, Liu, Porto, ‘21]

New results after 30+ year hiatus.

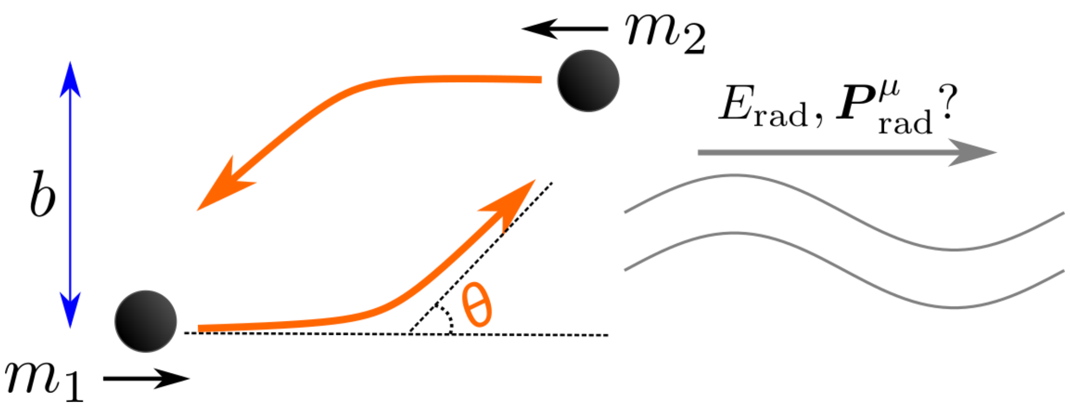
[adapted from Mikhail Solon's 2019 slide]

Next frontier – PM radiative dynamics

Also: talk by Thibault Damour

- What explains the high-energy divergence of the conservative scattering angle at $\mathcal{O}(G^3)$ [Bern, Cheung, Roiban, Shen, Solon, **MZ**, '2019] ?
  - ✓ Canceled by radiation reaction. [Damour, '20. Di Vecchia, Heissenberg, Russo, Veneziano, '20, '21. Herrmann, Parra-Martinez, Ruf, **MZ**, '21 (longer one)].

- What's the energy loss in hyperbolic black hole scattering, at the leading $\mathcal{O}(G^3)$ order but arbitrary velocity?
  - ✓ Computed from reverse unitarity. [Herrmann, Parra-Martinez, Ruf, **MZ**, '21 (PRL)]



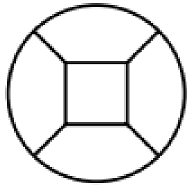
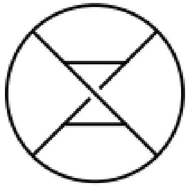
- Angular momentum loss, waveforms, higher orders… [Jakobsen, Mogull, Plefka, Steinhoff, '21. Mougiakakos, Riva, Vernizzi, '21. Cristofoli, Gonzo, Kosower, O'Connell, '21. Bini, Damour, Geralico, '21]

Bridge from PN to PM expansion

Powerful combo: BCJ + advanced loop integration

- Example 1: UV behavior of $N = 8$ supergravity at 5 loops.

  [Bern, Carrasco, Chen, Edison, Johansson, Parra-Martinez, Roiban, **MZ**, '18]

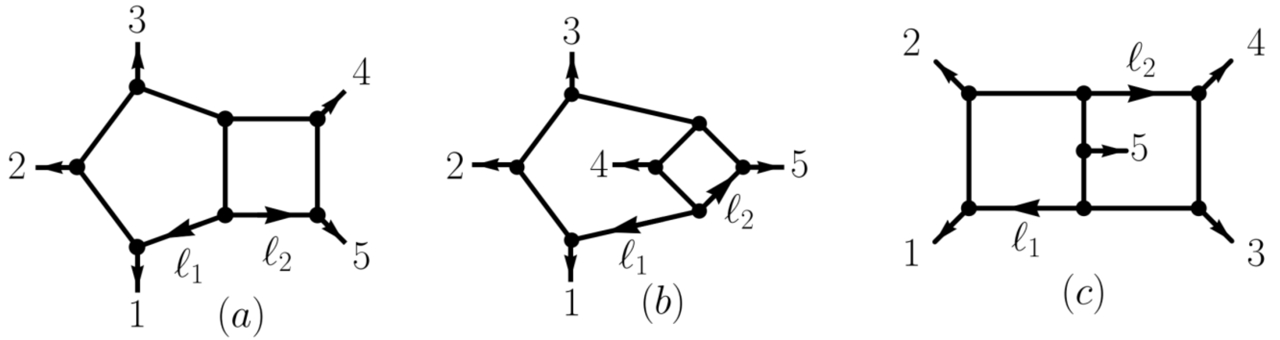
$M_4^{(5)}\Big|^{D=24/5}_{\text{leading}} \propto M_4^{\text{tree}} \Big(\frac{1}{48}$  $+ \frac{1}{16}$  $\Big)$

Result hints at integrated version of BCJ?

- Example 2: 2-loop 5-point amplitude of $N = 8$ supergravity.

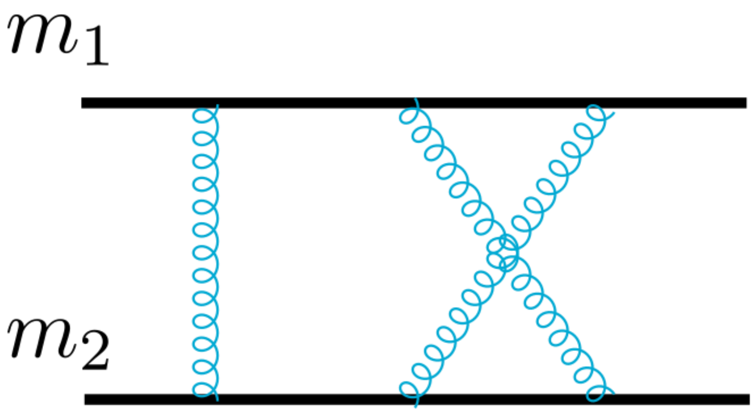
  [Carrasco, Johanson, '11; Mafra, Schlotterer, '15; Abreu, Dixon, Herrmann, Page, **MZ**, '19; Chicherin, Gehrmann, Henn, Wasser, Zhang, '19]

Same master integrals as e.g. $pp \to 3j$ for QCD.



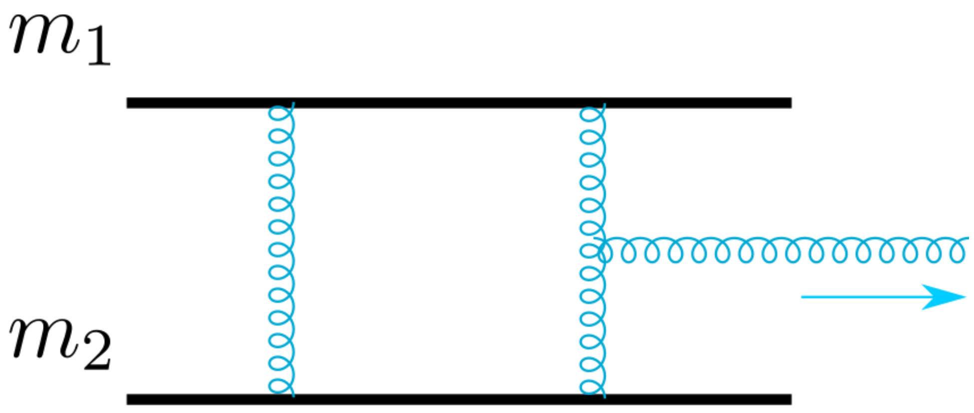
Rational structures in integrated amplitudes are double copies!

What amplitudes needed for GW physics?



$\Phi + \phi \to \Phi + \phi$

e.g. conservative potential.



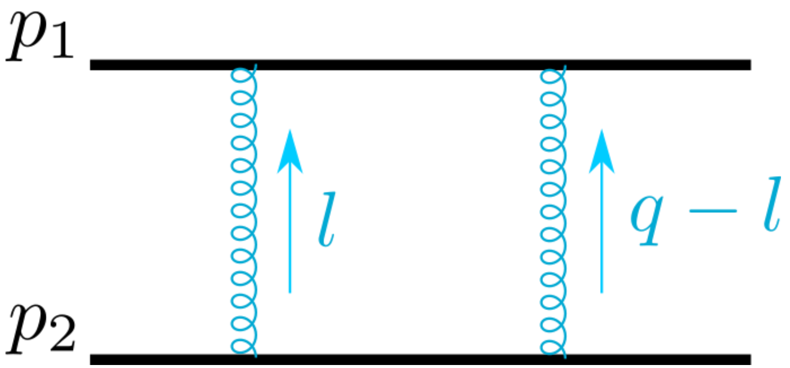
$\Phi + \phi \to \Phi + \phi + h$

e.g. radiative energy loss.

- Exact amplitudes *very hard* – need simplifications from classical limit.
  - but see exciting recent progess in e.g. $\mu e/ee$ scattering, [Kreer, Weinzierl, '21; Heller '21; Duhr, Smirnov, Tancredi, '21]

Method of regions [Beneke, Smirnov ’98]

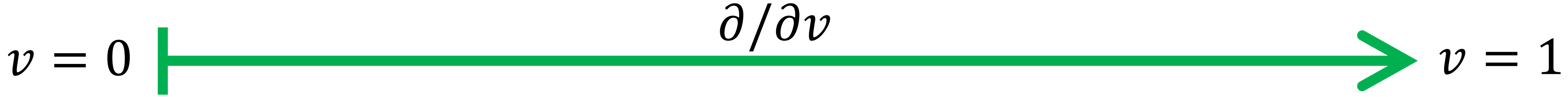
- Asymptotic series in $|q|/|p|$, to any order, is a sum of two expansions, both integrated over full $\mathbb{R}^d$:



1. **Soft region,** $|l| \sim |q| \ll |p|$, expand in small $|q|/|p|, |l|/|p|$.
2. Hard region, $|q| \ll |l| \sim |p|$, expand in small $|q|/|p|, |q|/|l|$, **not needed for classical physics.**

Soft integrals have *linearized matter propagators.* Values depend on normalized velocities u_1^μ, u_2^μ, through $u_1 \cdot u_2 \equiv y \equiv \sqrt{1+v^2}$.

Differential equations (DEs) for soft integrals



$v = 0$

$\partial / \partial v$

$v = 1$

PN boundary condition.
with ***canonical form*** *of DE [Henn, ’13]*
Smooth in potential region, at most logarithmic singularity in soft region.

PM: exact function in velocity ✓

Ultra-relativistic limit ✓

Integration by parts:
[Tkachov, ’81. Chetyrkin, ’81]

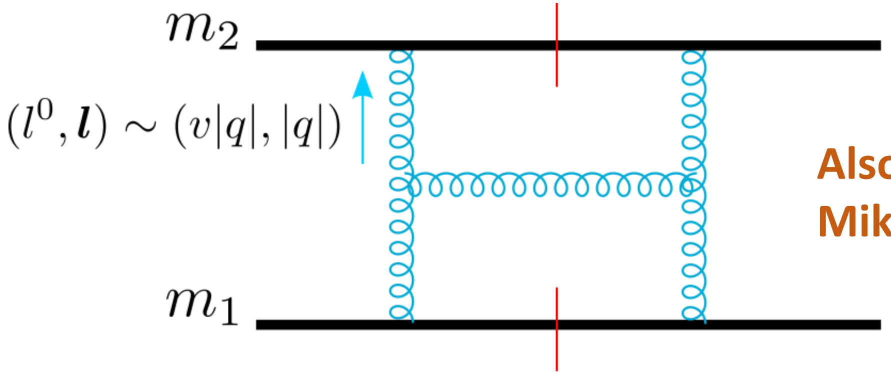
$$\mathcal{M} = \vec{c} \cdot \vec{I}, \quad \frac{\partial}{\partial v} \vec{I} = A \cdot \vec{I}$$

[Kotikov ’91. Bern, Dixon, Kosower, ’92. Gehrmann, Remiddi, ’99. Henn, ’13]

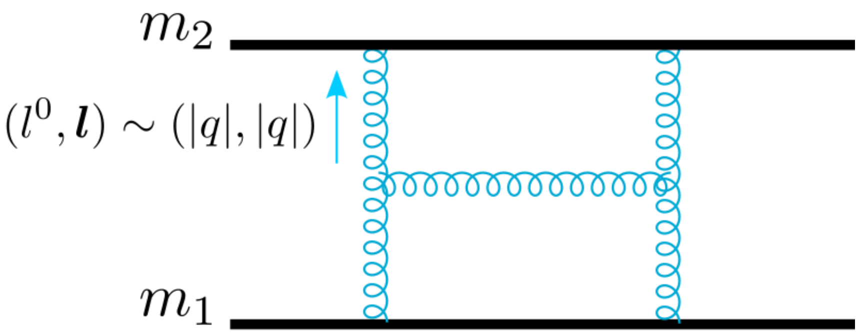
Automated algebraic transformation to canonical form [Lee, ’14; Prausa, ’17; Gituliar, Magerya, 17…]

Introduced to post-Minkowskian gravity in [Parra-Martinez, Ruf, **MZ**, ’20]. Already **widely adopted at 3PM** [Porto, Kalin, ’20; Di Vecchia, Heissenberg, Russo, Veneziano, ’21; Herrmann, Parra-Martinez, Ruf, **MZ**, ’21; Bjerrum-Bohr, Damgaard, Planté, Vanhove, ’21; Brandhuber, Chen, Travaglini, Wen, ’21] **and 4PM** [Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, **MZ**, ’21; Dlapa, Kalin, Liu, Porto, ’21].

3 kinds of boundary conditions, *same diff. eqs.*

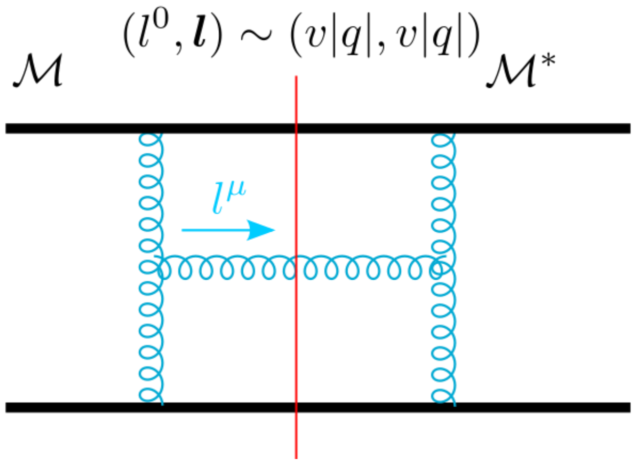


Also: talk by Mikhail Solon



1. **Potential region.** Localized on matter poles w/ appropriate symmetry factors. **Gives (resummed) conservative dynamics.** [Parra-Martinez, Ruf, **MZ**, '20]

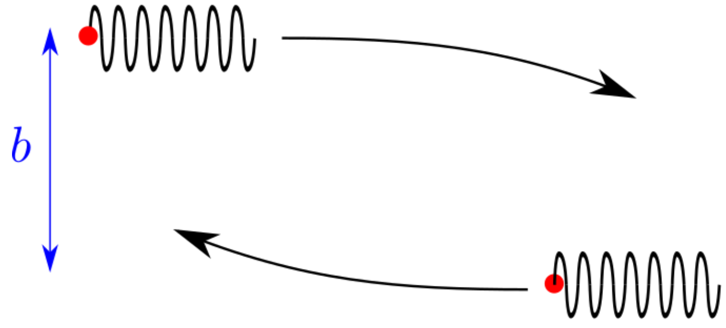
2. **Soft region** integrals integrated over conventional contour. **Incorporates radiation reaction.** [Herrmann, Parra-Martinez, Ruf, **MZ**, '21 (longer one)], in full agreement with [Di Vecchia, Heissenberg, Russo, Veneziano, '21].



3. **Phase space integration** in formalism of [Kosower, Maybee, O'Connell, '18]. **Gives 3PM-accurate energy loss.** [Herrman, Parra-Martinez, Ruf, **MZ**, '20 (PRL)]. Also deduced from relation to 4PM [Bern, Parra-Martinez, Roiban, Ruf, Solon, **MZ**, '21 (PRL)].

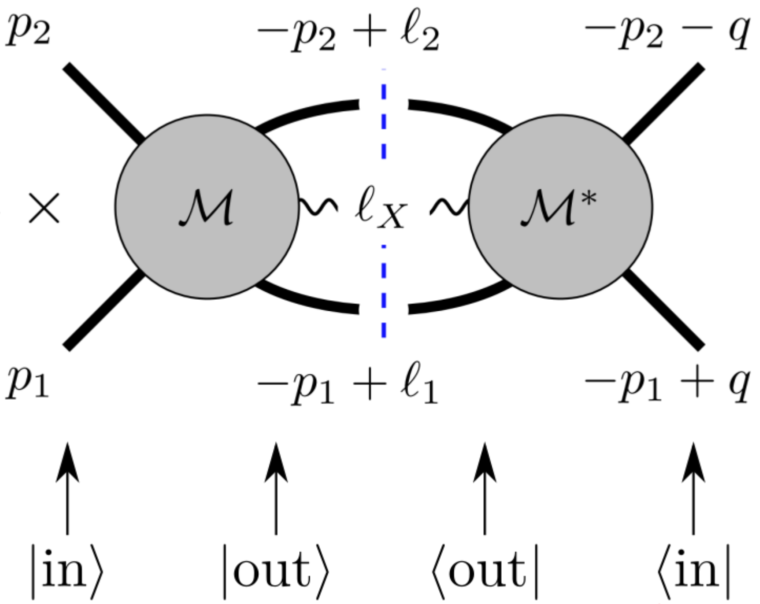
Radiated momentum from KMOC formalism

[Herrmann, Parra-Martinez, Ruf, MZ, ’21 (PRL)]. Follows formalism of [Kosower, Maybee, O’Connel, ’18]



$$|\mathrm{in}\rangle \sim \int_{\boldsymbol{q}_T} e^{i\boldsymbol{b}\cdot\boldsymbol{q}_T} |\boldsymbol{p} + \boldsymbol{q}_T\rangle$$

$$\sum_X \int \mathrm{d}\Phi_{2+X}\ \ell_X^\mu \times$$



Relative phase $\sim e^{i\boldsymbol{b}.\boldsymbol{q}}$ $\Longrightarrow$ Fourier transform over $\boldsymbol{q}$.

Phase space integrand from **cutting** virtual integrand. Integration by **reverse unitarity.**

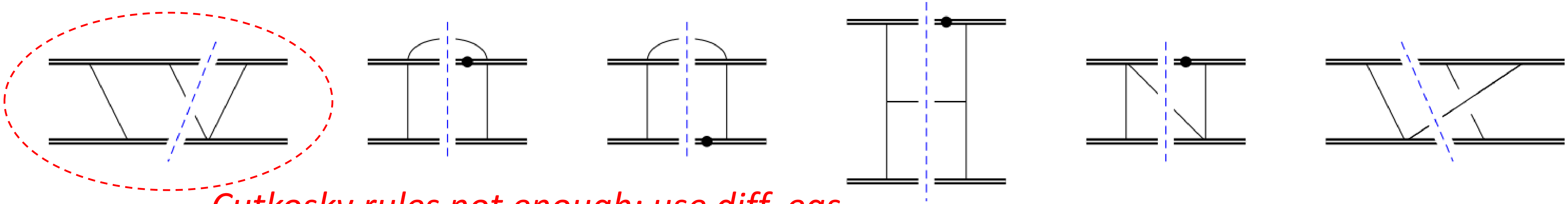
Reverse unitarity for energy / momentum loss

Higgs production cross section [Anastasiou, Melnikov, '02; Anastasiou, Dixon, Melnikov, '03; Anastasiou, Duhr, Dulat, Furlan, Herzog, Mistlberger, '15…], energy-energy correlator in e^+e^- annihilation [Dixon, Luo, Shtabovenko, Yang, Zhu, '18…].

Phase space integrals differ from loop integrals only by contour choice (+ve energy pole of Feynman propagator). **IBP and Diff. Eqs. unchanged.**

$$\frac{2\pi i}{(-1)^n n!}\delta^{(n)}(k^2) = \frac{1}{(k^2 - i0)^{1+n}} - \frac{1}{(k^2 + i0)^{1+n}}$$

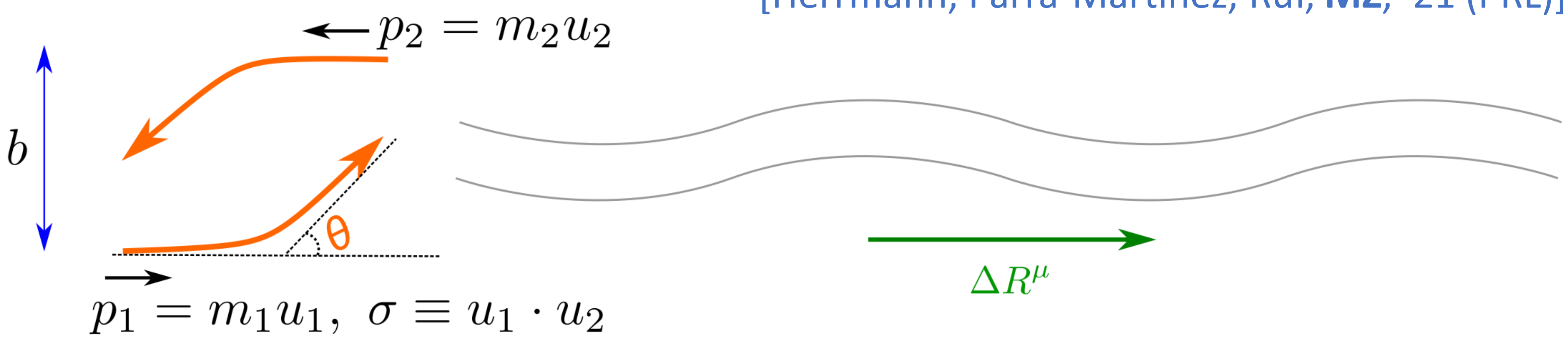
Master integrals for energy loss (blue dashed line = on-shell final states):



Cutkosky rules not enough; use diff. eqs.

Radiated momentum at leading $\mathcal{O}(G^3)$

[Herrmann, Parra-Martinez, Ruf, **MZ**, '21 (PRL)]



$$\Delta R^\mu = \frac{G^3 m_1^2 m_2^2}{|b|^3} \frac{u_1^\mu + u_2^\mu}{\sigma + 1} \mathcal{E}(\sigma) + \mathcal{O}(G^4) \, . \qquad \mathcal{E}(\sigma) = f_1 + f_2 \log\left(\frac{\sigma+1}{2}\right) + f_3 \frac{\sigma \operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} \, ,$$

"super-classical" terms canceled after summing diagrams

$$f_1 = \frac{210\sigma^6 - 552\sigma^5 + 339\sigma^4 - 912\sigma^3 + 3148\sigma^2 - 3336\sigma + 1151}{48\left(\sigma^2-1\right)^{3/2}} \, ,$$

$$f_2 = -\frac{35\sigma^4 + 60\sigma^3 - 150\sigma^2 + 76\sigma - 5}{8\sqrt{\sigma^2-1}} \, , \qquad f_3 = \frac{\left(2\sigma^2-3\right)\left(35\sigma^4 - 30\sigma^2 + 11\right)}{8(\sigma^2-1)^{3/2}}$$

Energy loss: comparisons

- **Consistent with "tail effect" divergence** of 4PM conservative result [Bern, Parra-Martinez, Roiban, Ruf, Solon, **MZ**, '21], using "cross-order" relation of [Bini, Damour, Geralico, '20; Bini, Damour, '17; Blanchet, Foffa, Larrouturou, Sturani, '19].

- *Ultra-relativistic limit, $\sigma \to \infty$:*

$$\mathcal{E}(\sigma) \sim \frac{35}{8}\pi(1+2\log 2)\sigma^3 \approx 32.79\,\sigma^3$$

coefficients agrees with numerical value 32.78(2) from [Bini, Damour, Geralico, '21]

- *Near-static limit, $\sigma \to 1$:*

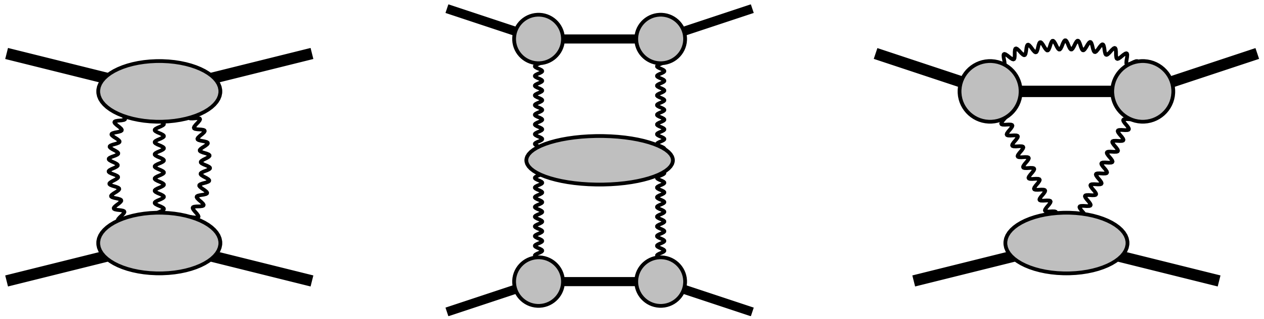
$$\mathcal{E}(\sigma) \sim \frac{37v}{15} + \frac{2392v^3}{840} + \frac{61603v^5}{10080} + \cdots + \frac{12755740946147v^{15}}{762814660608} + \mathcal{O}(v^{17})$$

post-Newtonian result reproduced by [Bini, Damour, Geralico, '21] for all terms shown.

- After **analytic continuation** [Kalin, Porto, '19 (×2); Bini, Damour, Geralico, '20] to **energy loss in elliptic orbit**, agree with previous results for terms that overlap [Bini, Damour, Geralico, '20; Kovacs, Thorne, '78; Blanchet, Schaefer, 89'; Peters, Matthews, '63; Peters, '64; Wagoner, Will, '77; Junker, Schaefer, '92; Gopakumar, Iyer, '97, '02; Arun, Blanchet, Iyer, Qusailah, '08; Blanchet, '13 (review)]

Complete 3PM dynamics [Herrmann, Parra-Martinez, Ruf, **MZ**, ’21 (×2)]

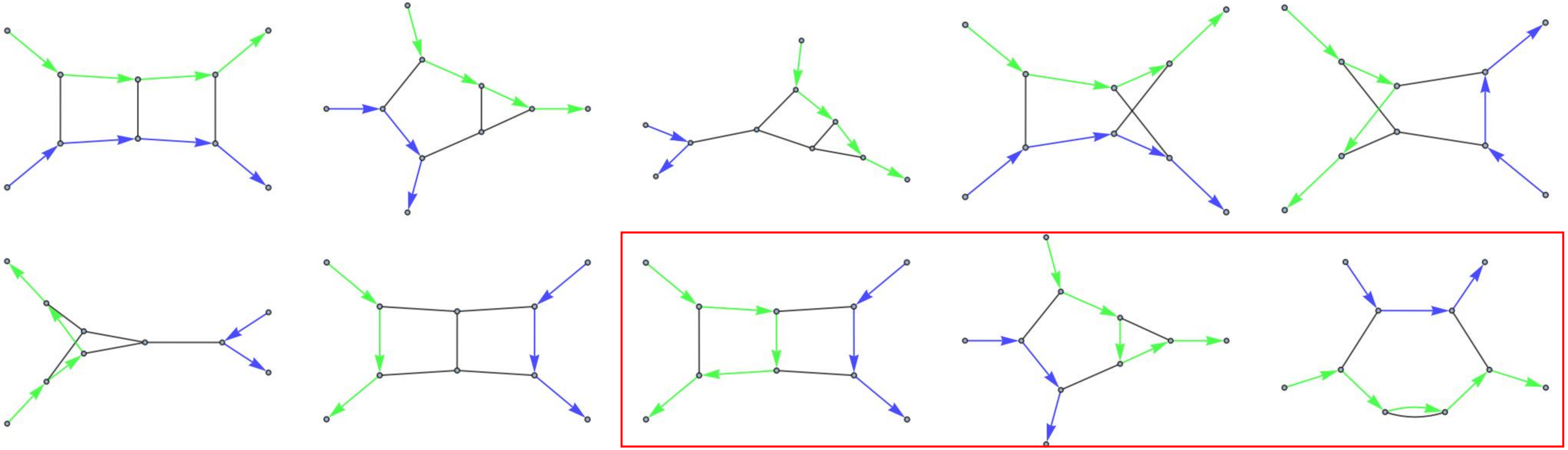
- Spanning cuts



A new cut beyond conservative case

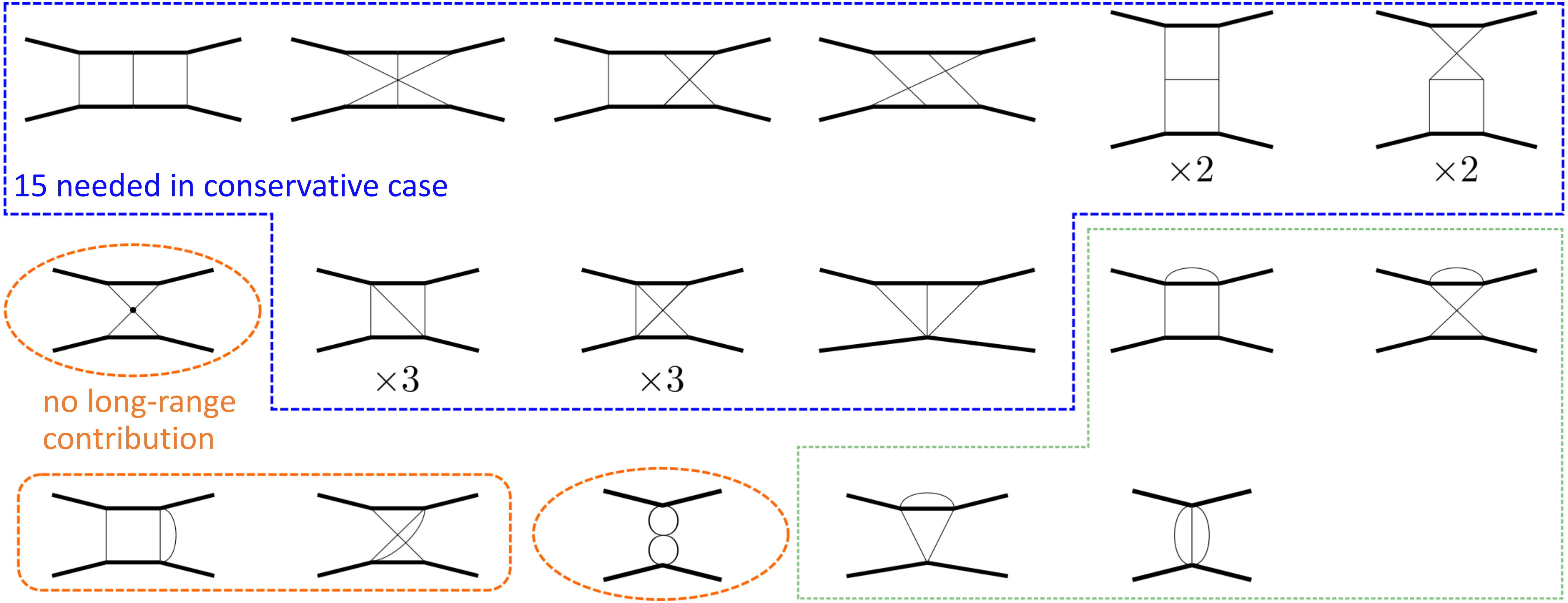
- Top level Feynman diagrams (non-conservative diagrams in red rectangle)

  simplified vertices: [Cheung, Remmen, ’16, ’17; Rafie-Zinedine, ’18]



Complete 3PM dynamics [Herrmann, Parra-Martinez, Ruf, **MZ**, ’21 (×2)]

Full set of master integrals for transverse impulse during scattering.



15 needed in conservative case

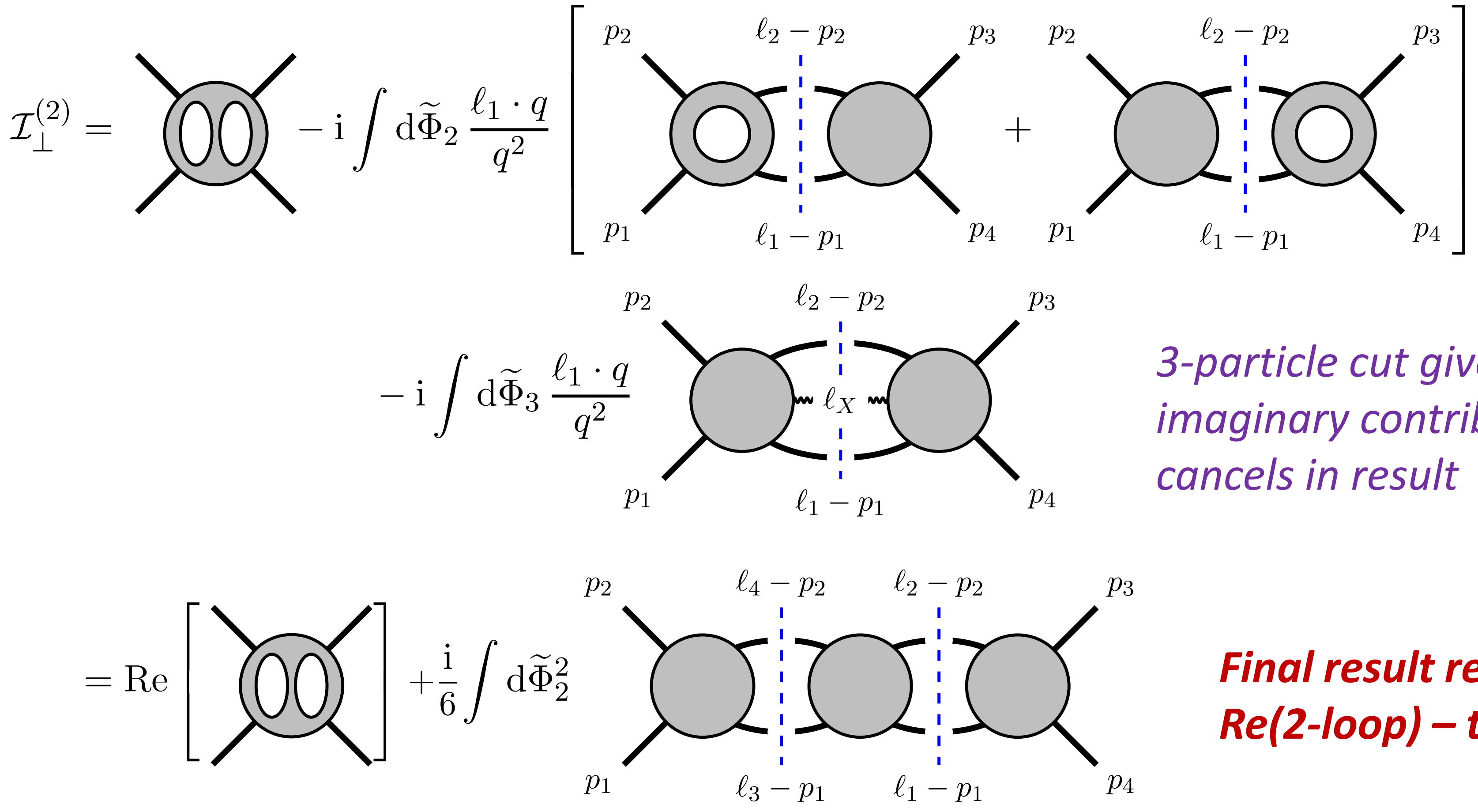
no long-range contribution

canceled by 2-particle cut in KMOC

4 more needed (3 plus crossing) for radiation reaction

Simplifications in KMOC formalism

Cutting rules and diagram symmetries ⇒



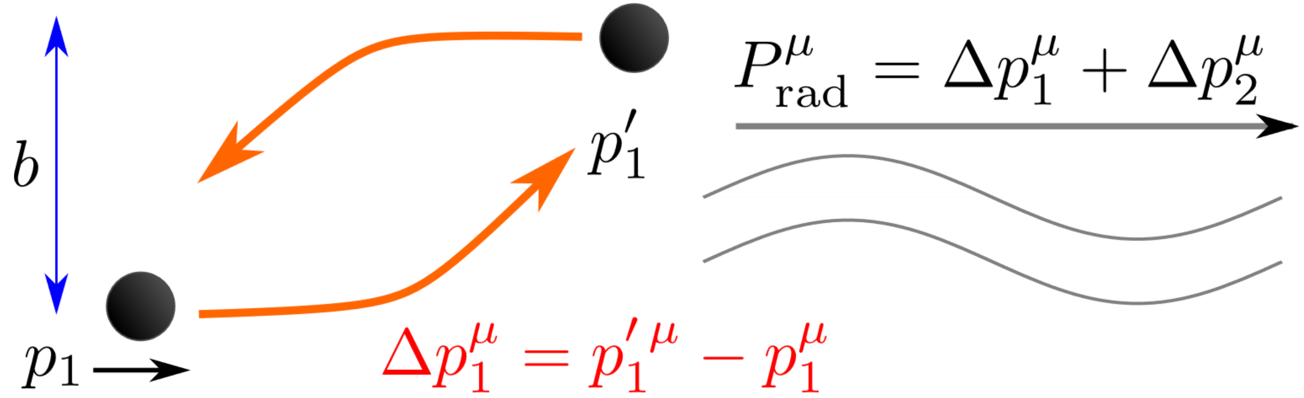
2-particle cuts almost cancel, leaving cut box contribution

3-particle cut gives purely imaginary contribution, cancels in result

Final result resembles eikonal calculation: Re(2-loop) – tree^3/3!

Complete 3PM result for impulse

[Herrmann, Parra-Martinez, Ruf, **MZ**, ’21 (×2)]



- **Potential region:**
  - **Transverse impulse** reproduces [Bern, Cheung, Roiban, Shen, Solon, MZ, ’19], using KMOC (or eikonal).
  - **Compensating longitudinal impulse** guarantees energy conservation, from 2-particle cuts.
- **Soft region:**
  - **Extra longitudinal part:** first exact result for total **radiated energy / momentum.**
  - **Extra transverse part: radiation reaction** cancels leading high-energy divergence, as observed in [Damour, ’20; Di Vecchia, Heissenberg, Russo, Veneziano, ’20, ’21]. Full agreement.

Discussions & open questions

- **Scattering amplitudes** address full spectrum of questions in binary dynamics, including **radiation effects** (previously: conservative dynamics, spin, finite sizes…). **Full 3PM spinless dynamics obtained**.
- **Collider methods** introduced: expansion by regions, diff. eqs., reverse unitarity…
- **Complementary approaches: NR EFT, eikonal, KMOC** all agreed for 3PM conservative dynamics. Latter two applied to and agreed on radiation reaction.
- Connection to **linear response theory** of radiation reaction [Damour, '20; Bini, Damour, Geralico, '21] ? Would need cross-order relation in amplitudes.
- Classical physics localizes on **matter poles** – how to best exploit wordline-like picture?
  - **Prescription defining individual diagrams on matter poles:** [Cheung, Rothstein, Solon, '18; Bern, Cheung, Roiban, Shen, Solon, MZ, '19; Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, MZ, '21]
  - **Combining integrand to localize on delta functions:** [Akhoury, Saotome, Sterman, '13; Bjerrum-Bohr, Damgaard, Festuccia, Plante, Vanhove, '18; Bjerrum-Bohr, Damgaard, Plante, Vanhove, '21; Brandhuber, Chen, Travaglini, Wen, '21.]

Thank you!

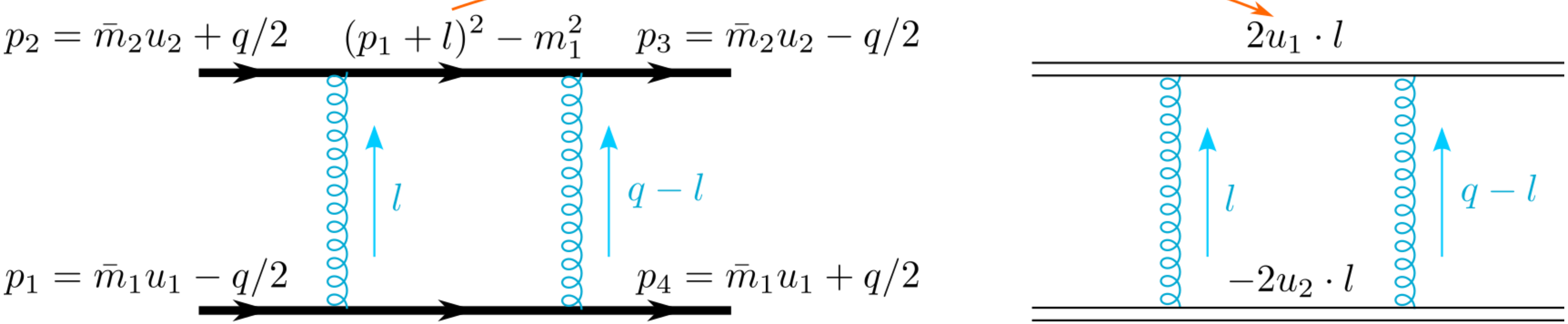
Expansion scheme for classical soft region

Parra-Martinez, Ruf, **MZ**, ’20,
Cristofoli, Damgaard, Di Vecchia, Heissenberg, ’20

- Symmetric parametrization

$$(p_1+l)^2 - m_1^2 = 2p_1 \cdot l + \cancel{l^2}$$
$$= \bar{m}_1(2u_1 \cdot l) + \ldots$$

c.f. Akhoury, Saotome, Sterman, 1308.5204v3



$$u_1^2 = u_2^2 = 1, \qquad u_1 \cdot q = u_2 \cdot q = 0, \qquad q^2 = t, \qquad u_1 \cdot u_2 = y.$$

normalized velocitiies

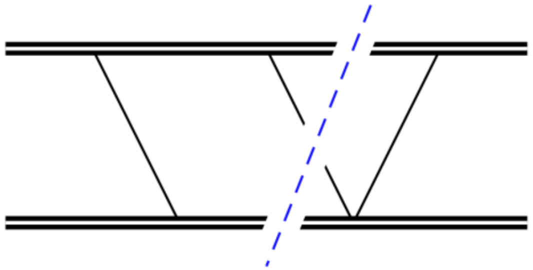
only scale, fixed by dimensional analysis

Velocity-like parameter. Non-trivial dependence (polylogs, elliptics…)

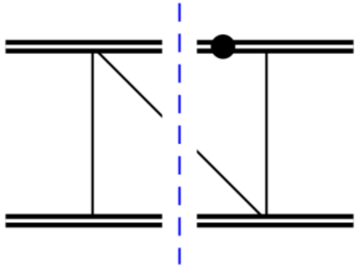
Example diff. eq. for phase space integral

Reverse unitarity: reuse differential equations *canonicalized* by `epsilon` [Prausa, '07]

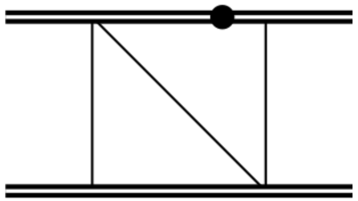
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1-x^2}{2x}\right.$$



$$\left.\vphantom{\frac{1}{x}}\right] = \frac{1}{x}$$

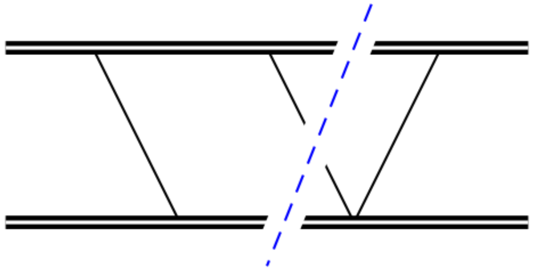


$$= \frac{1}{x}\, 2\,\mathrm{Im}$$



$$x = y - \sqrt{y^2-1}, \quad y \equiv \frac{p_1 \cdot p_2}{m_1 m_2}, \quad \underset{\substack{\uparrow \\ \text{ultra-relativistic}}}{0} < x < \underset{\substack{\uparrow \\ \text{static}}}{1}$$

$$\Longrightarrow$$



$$= \frac{2x}{1-x^2}\pi^3 \log(x) = -\frac{2x}{1-x^2}\pi^3 \operatorname{arccosh}(y)$$