Collinear Expansion of Cross Sections and the EEC at N3LL’ in QCD

Gherardo Vita

SLAC NATIONAL ACCELERATOR LABORATORY

Amplitudes 2021
Zoom, 18 August 2021

Based on:

“Collinear expansion for color singlet cross sections” [2006.03055]

“N-Jettiness Beam Functions at N3LO” [2006.03056]

“TMD PDFs at N3LO” [2006.05329]

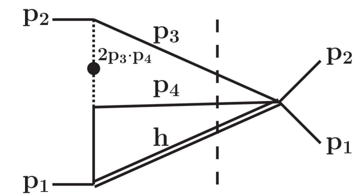
“TMD Fragmentation Functions at N3LO” [2012.07853]

“The EEC in the back-to-back limit at N3LO and N3LL’ ” [2012.07859]

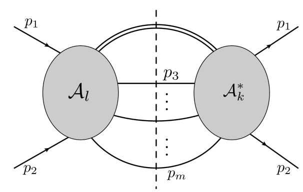
M.A.Ebert, B.Mistlberger, **GV**

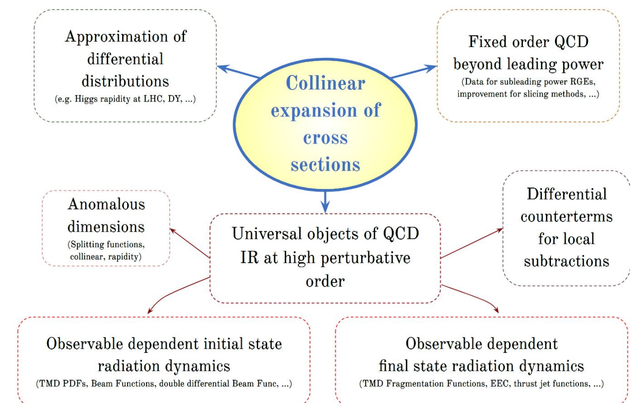
Outline

- **Introduction**
  - Kinematic limits of cross sections



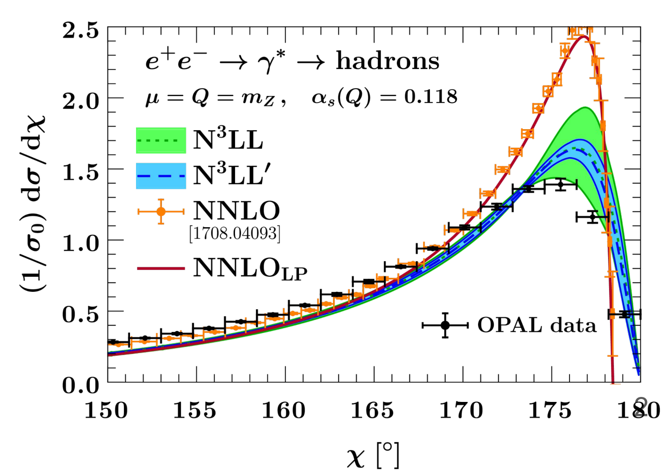
- **Collinear expansion of cross sections**
  - Matrix element expansion and IBPs



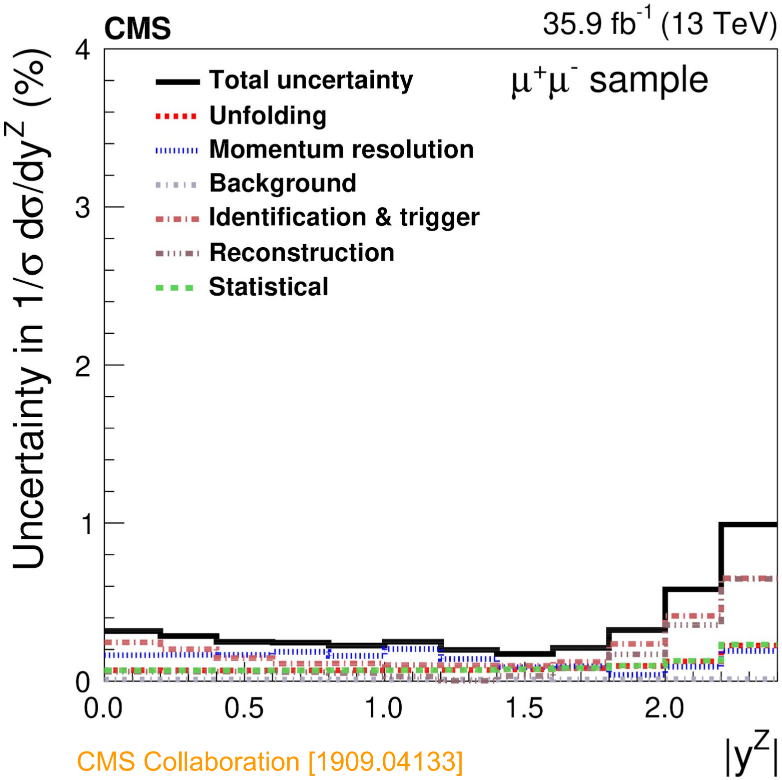


- **Applications**
  - Energy-Energy Correlation in the back-to-back limit at N3LL` in QCD

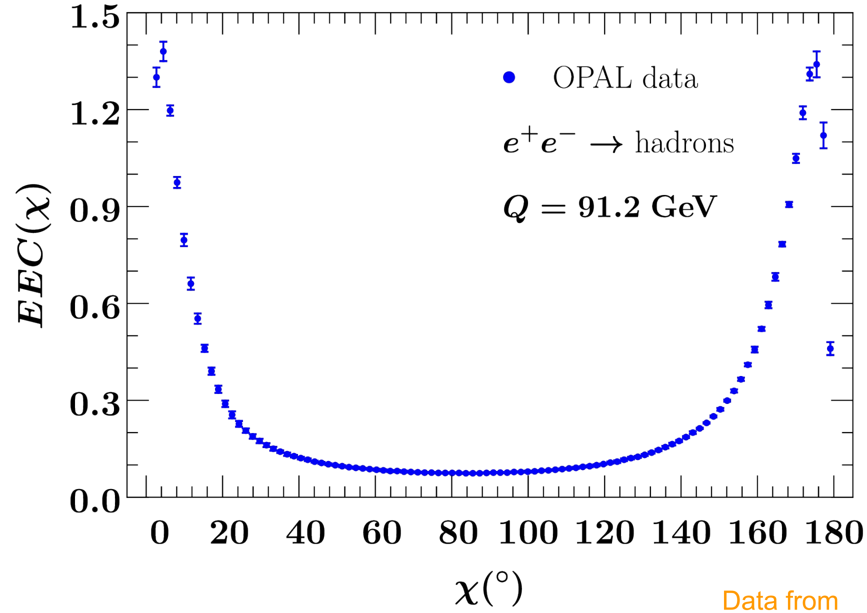
$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{\hat{\sigma}_0}{2} \overbrace{H_{q\bar{q}}(Q,\mu)}^{\text{Hard Function}} \int \frac{\mathrm{d}^2\vec{b}_T\,\mathrm{d}^2\vec{q}_T}{(2\pi)^2} e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T}\,\delta\left(1-z-\frac{q_T^2}{Q^2}\right) \underbrace{J_q\left(b_T,\mu,\frac{\nu}{Q}\right) J_{\bar{q}}\left(b_T,\mu,\frac{\nu}{Q}\right)}_{\text{EEC Jet Functions}} \overbrace{\tilde{S}_q(b_T,\mu,\nu)}^{\text{TMD Soft Function}}$$



Testing the Standard Model at Colliders



CMS Collaboration [1909.04133]



Data from [Z. Phys. C 59, 1-19 (1993)]

- Experimental measurements of key benchmark processes have reached astonishing level of precision.
- Situation will improve even further from future runs/experiments

Analytic predictions for collider observables

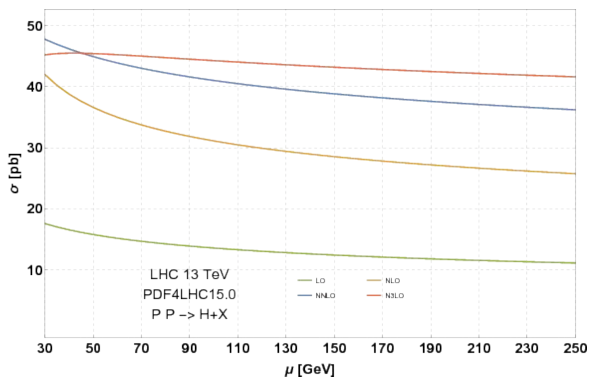
$$\hat{\sigma}_{ab\to X} = \underbrace{\sigma_0}_{\text{LO}} + \underbrace{\alpha_s\sigma_1}_{\text{NLO}} + \underbrace{\alpha_s^2\sigma_2}_{\text{NNLO}} + \underbrace{\alpha_s^3\sigma_3}_{\text{N}^3\text{LO}} + \dots$$

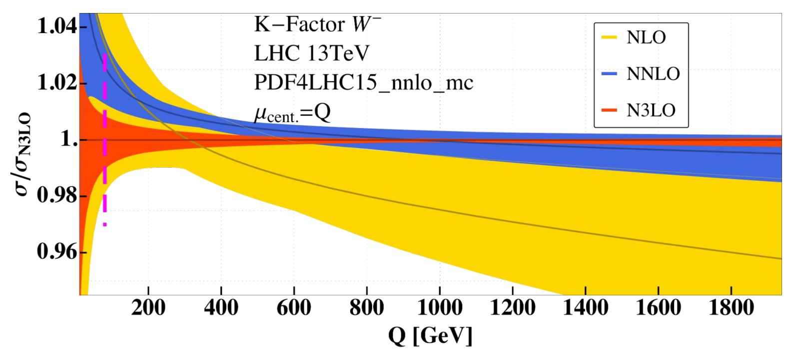
Analytic N3LO cross sections for LHC currently available for $2 \longrightarrow 1$ processes

- Inclusive Higgs production and Drell-Yan (γ, Z, $W^{\pm}$ production)

[Mistlberger 1802.00833] [Duhr, Dulat, Mistlberger 1904.09990] [Duhr, Dulat, Mistlberger 2007.13313]
[Duhr, Dulat, Hirschi, Mistlberger 2004.04752] [Duhr, Dulat, Mistlberger 2001.07717]

See yesterday's talk by Bernhard





At the **amplitude** level: Recent progress on N3LO $2 \longrightarrow 2$ and NNLO $2 \longrightarrow 3$

See talk by Simon

[Ahmed, Henn, Mistlberger] [Jin, Luo]
[Caola, von Manteuffel, Tancredi]

[Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia] [Chawdhry, Czakon, Mitov, Poncelet] [Kallweit, Sotnikov, Wiesemann] [Badger, Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia]

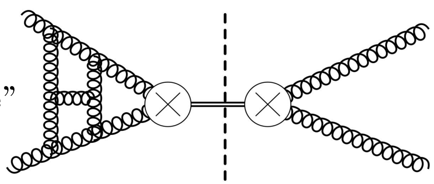
Analytic predictions for collider observables

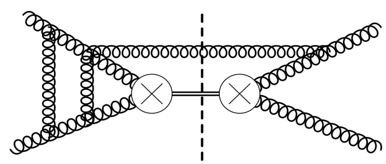
- **Important!** These are **analytic** computations of full **cross sections**, not amplitudes.
- Integral over **phase space** of final state particles
- Sum over all Real and Virtual corrections
- **Analytic** cancellation of IR divergences!

Example: Higgs production at N3LO in gg

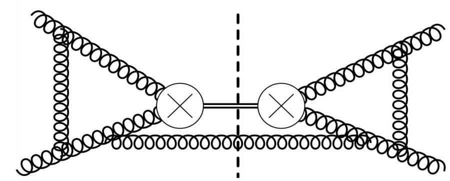
VVV

"Pure amplitude" contribution



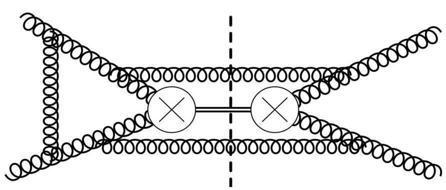


RVV

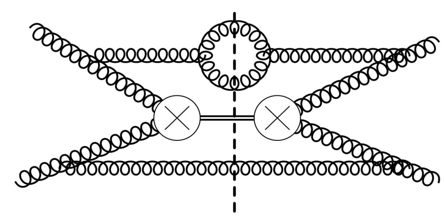


RVxRV

RRV



RRR



Analytic predictions for collider observables

- **Important!** These are **analytic** computations of full **cross sections**, not amplitudes.
- Integral over **phase space** of final state particles
- Make connection to modern multiloop techniques via **reverse unitarity**

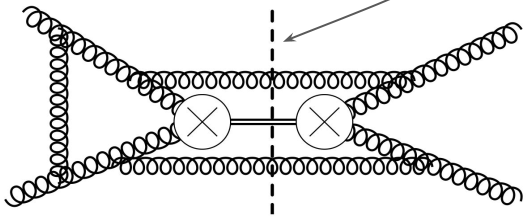
[Anastasiou, Melnikov]
[Anastasiou, Dixon, Melnikov]

$$\int \mathrm{d}^d p \, \delta_+(p^2) \qquad \delta_+(p^2) \sim \lim_{\varepsilon \to 0} \left[\frac{1}{p^2 + i\varepsilon} - \frac{1}{p^2 - i\varepsilon} \right]$$

See this as a loop integral

See phase space constraints as "cut" propagators

Note:
Final state measurements
(Higgs rapidity, transverse momentum, etc.)
= phase space constraints
= additional propagators



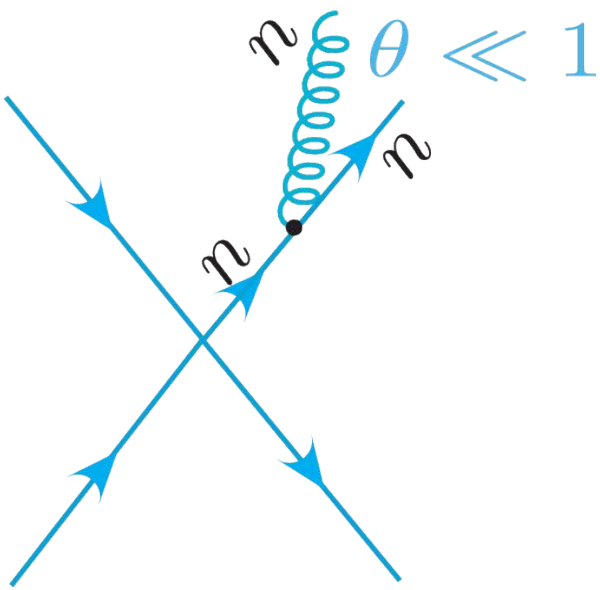
But...to go from inclusive $2 \to 1$ to **higher multiplicity** or **more differential** cross sections will require **huge** progress at each step of the calculation:

- IBPs
- Differential equations
- Space of functions

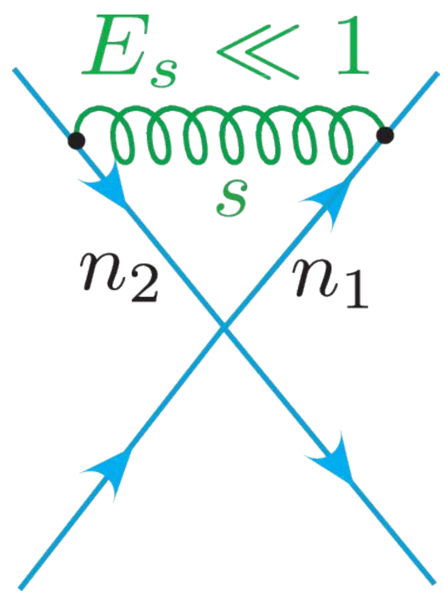
See talks by Klemm, Badger, Duhr, Weinzierl, ...

Expansion in kinematic limits

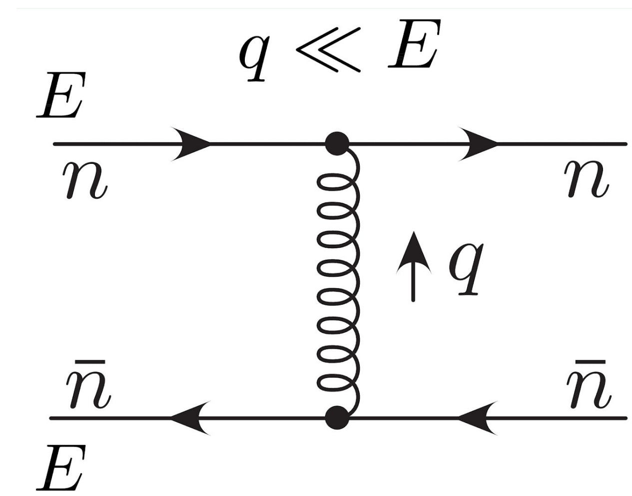
Collinear



Soft



Regge/ Forward



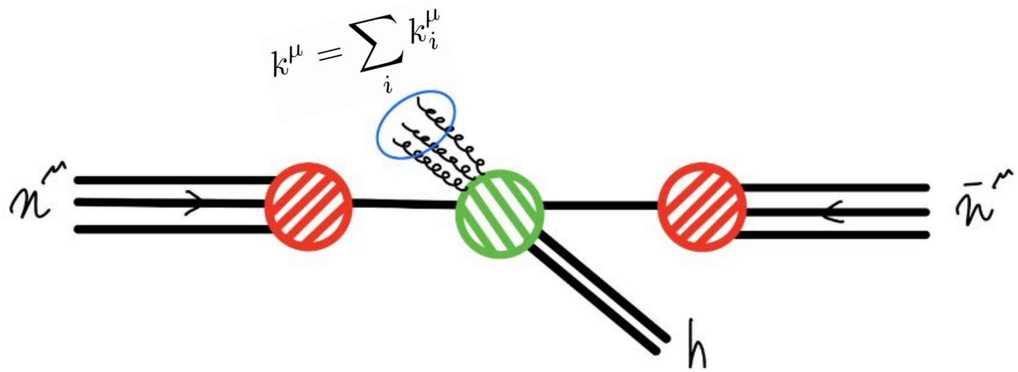
Much simpler structures arise in these limits

Expansion for Color Singlet Cross Sections

- Consider production of a <u>color singlet</u> state **h** in proton-proton collision
- **Measurements**: total momentum of radiation, color singlet Q and Y

Reverse Unitarity: think of **measurements** as cut propagators!

$$n^\mu = (1,0,0,1)$$

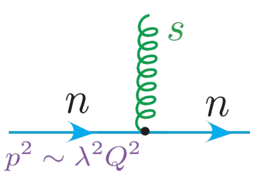


$$Y = \frac{1}{2}\log\left(\frac{\bar{n}\cdot p_h}{n\cdot p_h}\right)$$

$$\bar{n}^\mu = (1,0,0,-1)$$

$$Q^2 = p_h^2$$

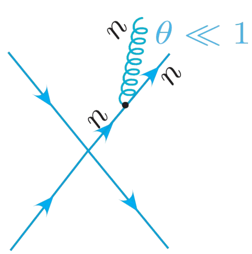
- **Limit** where total momentum of radiation is **soft** compared to Q



$$k^\mu \sim \lambda k^- \frac{n^\mu}{2} + \lambda k^+ \frac{\bar{n}^\mu}{2} + \lambda k_\perp^\mu\,, \quad \lambda \ll 1$$

Threshold expansion
(very well known in literature)

- **Limit** where total momentum of radiation is **collinear** to proton axis



$$k^\mu \sim k^- \frac{n^\mu}{2} + \lambda^2 k^+ \frac{\bar{n}^\mu}{2} + \lambda k_\perp^\mu\,, \quad \lambda \ll 1$$

Collinear Expansion
Our work!

Collinear Expansion for Matrix Elements

- Kinematic limit $\longrightarrow$ expansion of Feynman integrands appearing in the calculation of **partonic cross sections**

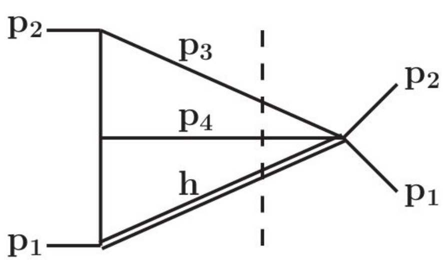
General idea has long history, see e.g.
Expansion by region [Beneke, Smirnov '97]

- Take for example double real emission (RR) scalar integral

$$w_1 = -\frac{\bar{n}\cdot k}{\bar{n}\cdot p_1}, \quad w_2 = -\frac{n\cdot k}{n\cdot p_2},$$

$$x = \frac{k^2}{(\bar{n}\cdot k)(n\cdot k)} = 1 - \frac{\vec{k}_\perp^2}{(\bar{n}\cdot k)(n\cdot k)}$$

$I_{\mathrm{RR}} =$



$$= \int \frac{d\Phi_{h+2}}{dw_1 dw_2 dx} \frac{1}{(p_2+p_3)^2(p_2+p_3+p_4)^2}$$

- Differential double real particle phase space scales homogeneously

In the **collinear** limit:

$$\int \frac{d\Phi_{h+2}}{dw_1 dw_2 dx} \rightarrow \lambda^{2-4\epsilon} \int \frac{d\Phi_{h+2}}{dw_1 dw_2 dx}$$

- Propagators can be **expanded** easily

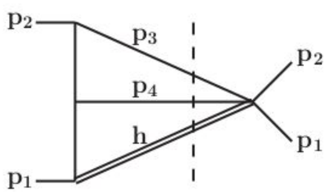
$$\frac{1}{(p_2+p_3+p_4)^2} \xrightarrow{\text{coll}} \frac{1}{2p_2\cdot(p_3+p_4) + \lambda^2\, 2p_3\cdot p_4} = \sum_{n=0}^{\infty} (\lambda^2)^n \frac{(-2p_3\cdot p_4)^n}{\left[p_2^+(p_3^- + p_4^-)\right]^{n+1}}$$

Collinear Expansion for double real graphs

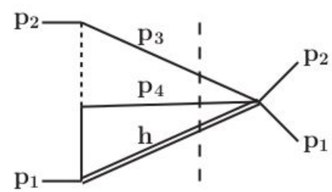
- We can perform a **collinear expansion** of the **integrand**

$$I_{\rm RR} \xrightarrow{\rm coll} \lambda^{2-4\epsilon} \int \frac{\mathrm{d}\Phi_{h+2}}{\mathrm{d}w_1 \mathrm{d}w_2 \mathrm{d}x} \left[\frac{1}{(p_2+p_3)^2 \left[p_2^+(p_3^- + p_4^-)\right]} + \lambda^2 \frac{(-2p_3\cdot p_4)}{(p_2+p_3)^2 \left[p_2^+(p_3^- + p_4^-)\right]^2} + \mathcal{O}(\lambda^3) \right]$$

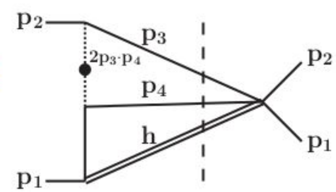
- Collinear expansion admits **diagrammatic** representation!



$\rightarrow \lambda^{2-4\epsilon} \Big[$



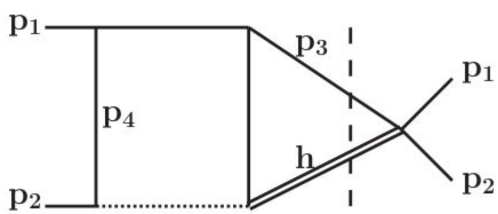
$- \lambda^2$



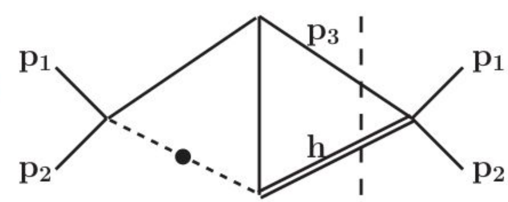
$+ \mathcal{O}(\lambda^3) \Big]$

- Same procedure can be applied for mixed loop/radiation integrals (like RV integrals at NNLO)

$I_{\rm RV} \xrightarrow{\rm coll} \lambda^{-2-4\epsilon} \Big[$



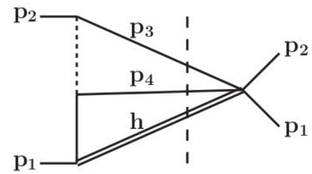
$- \lambda^2$



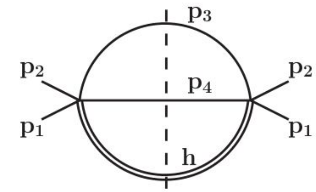
$+ \mathcal{O}(\lambda^3) \Big]$

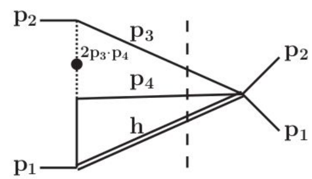
Collinear Expansion and IBPs

Key Point!
Expanded diagrams admit (simplified) integration by parts (**IBPs**) identities

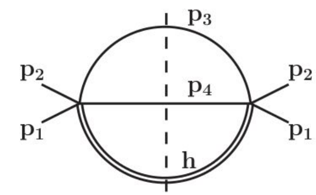


$$= -\frac{1-2\epsilon}{\epsilon(p_2^+ k^-)^2} \times$$





$$= -\frac{k^+ x}{p_2^+}\frac{1-2\epsilon}{\epsilon(p_2^+ k^-)^2} \times$$



- We can make use of modern technology for multiloop calculations with simplified kinematic dependence!

IBPs

Canonical Differential Equations

Reverse Unitarity

- **Simplifications w.r.t. full kinematics are huge** and enter at each step:
  - IBPs (smaller set of MI, smaller coefficients)
  - System of DE (e.g. ~ 10 MB for differential N3LO in collinear limit vs ~ 10 GB in full kinematics)
  - Space of functions (e.g. @N3LO: Elliptic functions for inclusive color singlet production in full kinematics vs only HPL for q_T distributions in collinear limit)

Collinear expansion of cross sections: **Applications**

Collinear expansion of cross sections: Applications

Collinear expansion of cross sections

Approximation of differential distributions
(e.g. Higgs rapidity at LHC, DY, ...)

Fixed order QCD beyond leading power
(Data for subleading power RGEs, improvement for slicing methods, ...)

Universal objects of QCD IR at high perturbative order

Anomalous dimensions
(Splitting functions, collinear, rapidity)

Differential counterterms for local subtractions

Observable dependent initial state radiation dynamics
(TMD PDFs, Beam Functions, double differential Beam Func, ...)

Observable dependent final state radiation dynamics
(TMD Fragmentation Functions, EEC, thrust jet functions, ...)

Applications:
EEC in the back-to-back limit

"TMD Fragmentation Functions at N3LO" [2012.07853]
"The EEC in the back-to-back limit at N3LO and N3LL' " [2012.07859]

Collinear expansion of cross sections

Approximation of differential distributions

Fixed order QCD beyond leading power

Universal objects of QCD IR at high perturbative order

Anomalous dimensions

Differential counterterms for local subtractions

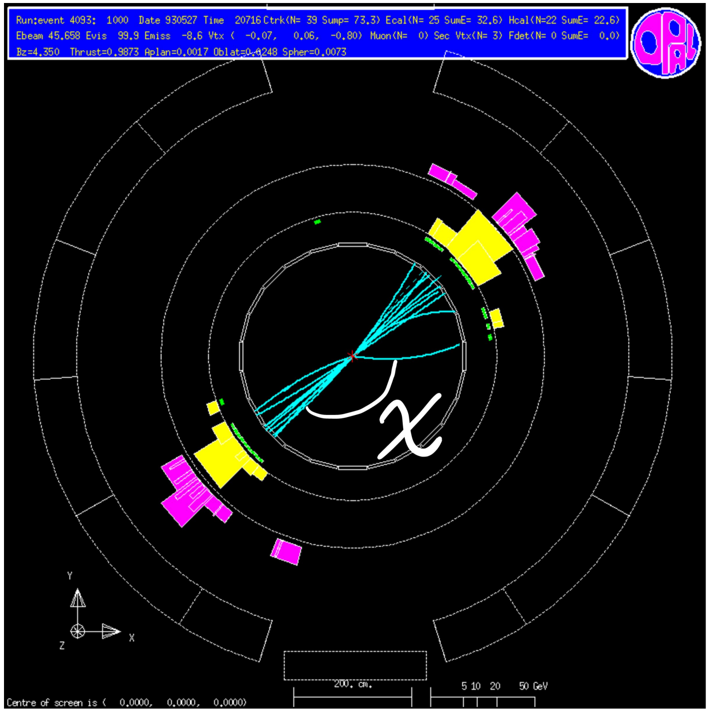
Observable dependent initial state radiation dynamics
(TMD PDFs, Beam Functions, double differential Beam Func, ...)

Observable dependent final state radiation dynamics
(TMD Fragmentation Functions, EEC, thrust jet functions, ...)

Energy-Energy Correlation

- Interesting observable involving measurements on QCD final state radiation is the Energy-Energy Correlation (EEC)

$$\mathrm{EEC}(\chi) = \frac{\mathrm{d}\sigma}{\mathrm{d}\chi} = \sum_{i,j} \int \mathrm{d}\sigma_{e^+e^- \to ij+X} \frac{E_i E_j}{Q^2} \delta(\cos\theta_{ij} - \cos\chi)$$



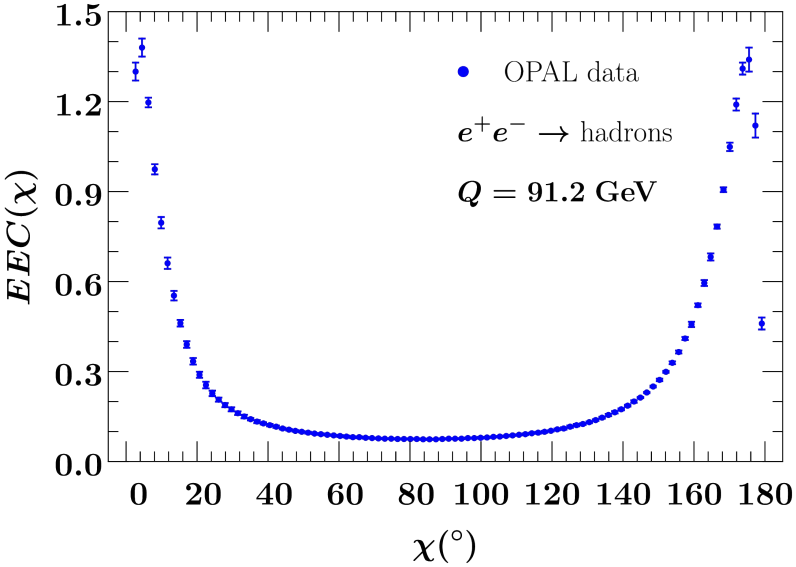
- Measures **angle** χ between pairs of color charged particles, weighted by energy

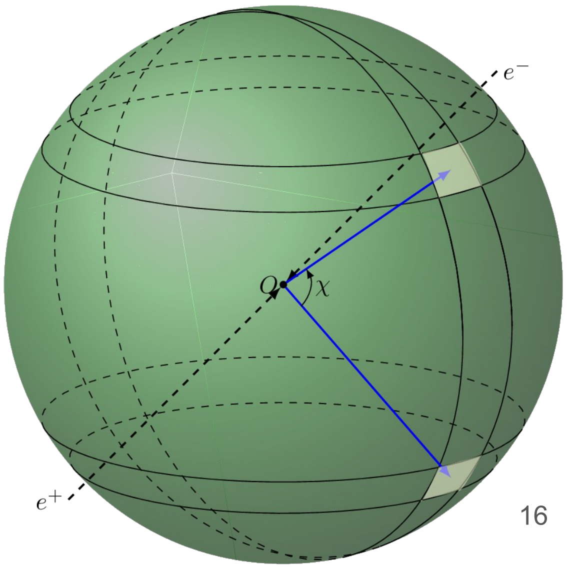
- One of the oldest IRC safe observables proposed [Basham, Brown, Ellis, Love, PRL **41**, 1585 **(1978)**]

Energy-Energy Correlation: Motivations

Interest in EEC for a variety of reasons:

- Very well measured event shape at electron-positron colliders **GOOD DATA!**
- Arguably the event shape with the simplest analytic structure **GOOD THEORY!**
- Can be expressed as a four point correlator in terms of energy flow operators [Maldacena, Hofman; Korchemsky;]
- Substantial recent progress in understanding it both in QCD and CFTs. [Moult, Dixon, Zhu; Korchemsky; Chicherin, Henn, Sokatchev, Yan; Simmons Duffin, Kologlu, Kravchuk, Zhiboedov; Moult, Vita, Yan; Luo, Shtabovenko, Yang;...]
- Natural playground for connections between $\mathcal{N}=4$ and QCD
- Allows precise extraction of α_s [OPAL Collaboration; Kardos, Kluth, Somogyi, Tulipant, Verbytskyi; ...]
- ...





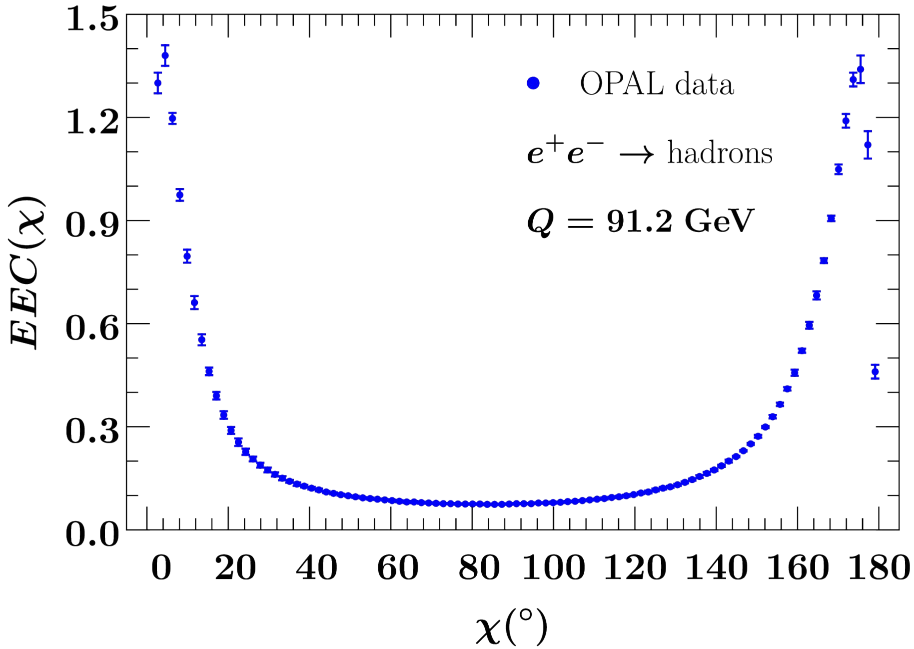
Energy-Energy Correlation: End Points

- It has singular structure and logarithmic enhancement at both end points

$\chi \to 0$
$z \to 0$
Collinear/Forward/ small angle Limit

$\boxed{z \equiv \frac{1}{2}(1 - \cos\chi)}$

$\chi \to \pi$
$z \to 1$
Back-to-Back Limit



- The two limits have very different structure (no symmetry between them)
- **Single logarithmic** series in small angle limit

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} \overset{z\to 0}{\sim} \sum_{L=1}^{\infty} \sum_{m=0}^{L-1} \left(\frac{\alpha_s}{4\pi}\right)^L c_{L,m} \frac{\log^m z}{z}$$

- **Double logarithmic** series at $z \to 1$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} \overset{z\to 1}{\sim} \sum_{L=1}^{\infty} \sum_{m=0}^{2L-1} \left(\frac{\alpha_s}{4\pi}\right)^L d_{L,m} \frac{\log^m(1-z)}{(1-z)}$$

- Plus **contact terms** $\sim \delta(1-z)\,,\ \delta(z)$

- We can derive factorization theorems at both ends in SCET for resummation

[Moult, Zhu] [Moult, Dixon, Zhu] [Ebert, Mistlberger, **GV**]

EEC in the back to back limit (in QCD)

- In the Back-to-Back Limit $\chi \longrightarrow \pi$ the EEC in QCD factorizes as

[Moult, Zhu] (2018)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{\hat{\sigma}_0}{2} \overbrace{H_{q\bar{q}}(Q,\mu)}^{\text{Hard Function}} \int \frac{\mathrm{d}^2\vec{b}_T\, \mathrm{d}^2\vec{q}_T}{(2\pi)^2} e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T}\, \delta\overbrace{\left(1-z-\frac{q_T^2}{Q^2}\right)}^{1-z\equiv(\cos\frac{\chi}{2})^2\approx\frac{q_T^2}{Q^2}} \underbrace{J_q\left(b_T,\mu,\frac{\nu}{Q}\right) J_{\bar{q}}\left(b_T,\mu,\frac{\nu}{Q}\right)}_{\text{EEC Jet Functions}} \overbrace{\tilde{S}_q(b_T,\mu,\nu)}^{\text{TMD Soft Function}}$$

Hard Function: IR finite form factor. (N3LO in 2010) [Gehrmann, Glover, Huber, Ikizlerli, Studerus]

Soft Function: Same as in TMD fact. (N3LO in 2016) [Li, Zhu]

Jet Function: Describe **collinear** dynamics of QCD radiation in the presence of EEC measurement.

Last missing ingredient at N3LO!

- EEC Jet Function (and TMDFFs) can be calculated using **collinear expansion of differential partonic cross sections!**

We have calculated TMDFF in **"TMD Fragmentation Functions at N3LO"** [2012.07853]

$$J_q\left(b_T,\mu,\frac{\nu}{Q}\right) = \int_0^1 \mathrm{d}\zeta\,\zeta \sum_i \boxed{\tilde{\mathcal{K}}_{qi}\left(\zeta,\vec{b}_T,\mu,\frac{\nu}{\omega_b}\right)} \longrightarrow \text{Matching Kernels for TMD Fragmentation Function}$$

$$\sim \sum_i \int_0^1 \mathrm{d}\zeta\,\zeta\,\mathcal{F}\left\{\int_0^1 \mathrm{d}x\,\mathrm{d}w_2 \lim_{\text{strict coll.}} \left[\delta\left[q_T^2 - Q^2(1-\zeta)w_2(1-x)\right] \frac{\mathrm{d}\hat{\eta}_{\bar{q}+h\to i+X}}{\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x}\bigg|_{w_1=-\frac{1-\zeta}{\zeta}}\right]\right\}$$

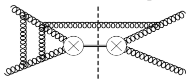
(After analytic continuation, UV, SCET-II, rapidity regularization and IR ...)

Fully differential calculation in collinear limit @N3LO

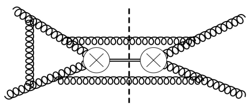
- We calculated the **collinear expansion of the partonic cross section** for DY and Higgs @N3LO **differential in (Q_T, τ, z)**

$$\frac{\mathrm{d}\eta_{j\bar{i}}}{\mathrm{d}Q^2\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x} \qquad w_1 = -\frac{\bar{n}\cdot k}{\bar{n}\cdot p_1}, \quad x = 1 - \frac{\vec{k}_\perp^2}{(\bar{n}\cdot k)(n\cdot k)}, \quad w_2 = -\frac{n\cdot k}{n\cdot p_2}$$

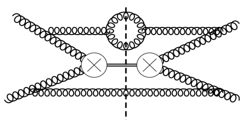
- RVV: known in full kinematics



- RRV: **170** Master Integrals



- RRR: **320** Master Integrals



- **2 scales** (dependence on small scale in collinear limit trivializes), **algebraic dependence** on variables
- Still well beyond the current level of technology for standard algorithmic techniques

How did we solve the system of DE?

- We constructed **dLog integrand basis** for coupled systems on the maximal cut
- We generalized to multiple variables with algebraic dependence Roman Lee's algorithm to obtain ε-form
- Boundaries from differential soft integrals
- Result has large non-rational alphabet

$$\mathcal{A} = \{1-x, x, z-x(z-1), x(z-1)+1, x(z-1)+2, x(z-1)-z+2, (z-1)x-x+1, x(z-1)^2+2(z-1)+1,$$
$$\left(x(z-1)^2+4x(z-1)+4\right)z+(z+1)\sqrt{x^2(z-1)^3+5x^2(z-1)^2+8x(z-1)+4}\sqrt{z},$$
$$z\sqrt{x(z-1)+1}+\sqrt{z}\sqrt{x(z-1)^2-3x(z-1)+z}, \sqrt{x(z-1)+1}\sqrt{z}\sqrt{x(z-1)^2-3x(z-1)+z}-(x(z-1)-1)z,$$

- "Mother" calculation. Project/integrate to obtain Beam Functions, TMDFF, EEC Jet Function

$$J_q\left(b_T,\mu,\frac{\nu}{Q}\right) = \int_0^1 \mathrm{d}\zeta\zeta\sum_i \tilde{\mathcal{K}}_{qi}\left(\zeta,\vec{b}_T,\mu,\frac{\nu}{\omega_b}\right)$$
$$\sim \sum_i \int_0^1 \mathrm{d}\zeta\zeta\mathcal{F}\left\{\int_0^1 \mathrm{d}x\,\mathrm{d}w_2 \lim_{\text{strict coll.}}\left[\delta\left[q_T^2-Q^2(1-\zeta)w_2(1-x)\right]\frac{\mathrm{d}\hat{\eta}_{\bar{q}+h\to i+X}}{\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x}\bigg|_{w_1=-\frac{1-\zeta}{\zeta}}\right]\right\}$$

EEC in the back to back limit in QCD @N3LO

- Thanks to our calculation we have the full behaviour of the EEC for $z \to 1$ in QCD up to $\mathcal{O}(\alpha_s^3)$ **analytically**

- This result is for electron-positron annihilation but we have derived also the case of gluon induced Higgs decay

- Having full analytic control of result we can compare to $\mathcal{N}=4$ where EEC is already known at this order. [Korchemsky; Chicherin, Henn, Sokatchev, Yan; Simmons Duffin, Kologlu, Kravchuk, Zhiboedov]

- We observe that **the principle of maximal transcendentality holds** also at this order!

- This also gives very non-trivial cross check involving Gluon EEC, Quark EEC and EEC in $\mathcal{N}=4$

$$
\begin{aligned}
\frac{1}{C_F}\frac{\mathrm{d}\bar{\sigma}^{(3)}}{\mathrm{d}z} &= -4C_F^2\mathcal{L}_5(\bar{z}) \\
&+\mathcal{L}_4(\bar{z})\Big[-30C_F^2-\frac{220}{9}C_FC_A+\frac{40}{9}C_Fn_f\Big] \\
&+\mathcal{L}_3(\bar{z})\Big[C_F^2(-16\zeta_2-104)+\frac{88}{9}C_Fn_f+C_FC_A\Big(-16\zeta_2-\frac{388}{9}\Big) \\
&\qquad -\frac{242}{9}C_A^2+\frac{88}{9}C_An_f-\frac{8}{9}n_f^2\Big] \\
&+\mathcal{L}_2(\bar{z})\Big[C_F^2(-144\zeta_2-16\zeta_3-189)+C_FC_A\Big(-\frac{592}{3}\zeta_2-72\zeta_3+\frac{244}{3}\Big) \\
&\qquad +C_Fn_f\Big(\frac{88}{3}\zeta_2-\frac{40}{3}\Big)+C_A^2\Big(\frac{2471}{27}-\frac{88}{3}\zeta_2\Big) \\
&\qquad +C_An_f\Big(\frac{16}{3}\zeta_2-\frac{760}{27}\Big)+\frac{44n_f^2}{27}\Big] \\
&+\mathcal{L}_1(\bar{z})\Big[C_F^2\Big(-\frac{542}{3}-412\zeta_2+224\zeta_3-192\zeta_4\Big) \\
&\qquad +C_FC_A\Big(-\frac{2900}{9}\zeta_2-\frac{1688}{3}\zeta_3-8\zeta_4+\frac{3797}{9}\Big) \\
&\qquad +C_Fn_f\Big(\frac{536}{9}\zeta_2+\frac{32}{3}\zeta_3-\frac{479}{9}\Big)+C_A^2\Big(-\frac{916}{9}\zeta_2-44\zeta_4-\frac{2354}{81}\Big) \\
&\qquad +C_An_f\Big(\frac{448}{9}\zeta_2+16\zeta_3-\frac{380}{81}\Big)+n_f^2\Big(\frac{124}{81}-\frac{16}{3}\zeta_2\Big)\Big] \\
&+\mathcal{L}_0(\bar{z})\Big[C_F^2\Big(64\zeta_3\zeta_2-402\zeta_2+332\zeta_3-552\zeta_4+48\zeta_5-\frac{169}{2}\Big) \\
&\qquad +C_FC_A\Big(-128\zeta_3\zeta_2+\frac{212}{3}\zeta_2-\frac{2812}{9}\zeta_3-\frac{1342}{3}\zeta_4-120\zeta_5+\frac{3358}{9}\Big) \\
&\qquad +C_Fn_f\Big(-\frac{20}{3}\zeta_2-\frac{296}{9}\zeta_3+\frac{244}{3}\zeta_4-\frac{623}{18}\Big) \\
&\qquad +C_A^2\Big(\frac{4420}{9}\zeta_2-\frac{560}{9}\zeta_3-\frac{326}{3}\zeta_4-40\zeta_5-\frac{4241}{27}\Big) \\
&\qquad +C_An_f\Big(-\frac{1508}{9}\zeta_2+\frac{184}{9}\zeta_3+\frac{56}{3}\zeta_4+\frac{1414}{27}\Big) \\
&\qquad +n_f^2\Big(\frac{112}{9}\zeta_2+\frac{16}{9}\zeta_3-\frac{98}{27}\Big)\Big] \\
&+\delta(\bar{z})\Big[C_F^2\Big(-\frac{337}{3}-\frac{1049}{3}\zeta_2+\frac{530}{3}\zeta_3+512\zeta_2\zeta_3-64\zeta_3^2-1396\zeta_4+\frac{3136}{3}\zeta_5-672\zeta_6\Big) \\
&\quad +C_FC_A\Big(\frac{10169}{27}+\frac{2729}{3}\zeta_2-\frac{22070}{9}\zeta_3+\frac{2176\zeta_4}{9}+528\zeta_5+22\zeta_6-288\zeta_2\zeta_3+64\zeta_3^2\Big) \\
&\quad +C_Fn_f\Big(-\frac{148}{27}-\frac{985\zeta_2}{9}+\frac{3340\zeta_3}{9}+\frac{58\zeta_4}{9}-\frac{368\zeta_5}{3}-\frac{224}{3}\zeta_2\zeta_3\Big) \\
&\quad +C_A^2\Big(-\frac{55504}{81}-\frac{3968}{81}\zeta_2+\frac{39337}{27}\zeta_3+\frac{3815}{18}\zeta_4-\frac{2720}{3}\zeta_5-\frac{700}{3}\zeta_2\zeta_3+59\zeta_6-56\zeta_3^2\Big) \\
&\quad +C_An_f\Big(\frac{15626}{81}-\frac{3326}{81}\zeta_2-\frac{3788}{27}\zeta_3-\frac{290}{9}\zeta_4+80\zeta_5+72\zeta_2\zeta_3\Big) \\
&\quad +n_f^2\Big(-\frac{1048}{81}+\frac{616}{81}\zeta_2-\frac{464}{27}\zeta_3-\frac{16}{9}\zeta_4\Big) \\
&\quad +N_{F,V}\frac{d_{abc}d^{abc}}{N_r}\big(2+5\zeta_2+\frac{7}{3}\zeta_3-\frac{\zeta_4}{2}-\frac{40}{3}\zeta_5\big)\Big].
\end{aligned}
$$

Ebert, Mistlberger, GV [2012.07859]

Peeking into the bulk of the EEC: N=2 moment

- Thanks to our calculation we have obtained the full $\delta(1-z)$ coefficient at $\mathcal{O}(\alpha_s^3)$

Total annihilation cross section to hadrons. Known to 4 loops.
[Baikov, Chetyrkin, Kuhn, Rittinger]

$$\boxed{\frac{R^{(n)}}{2}} = \boxed{V_1^{(n)}} + \boxed{\sum_{m=0}^{2n-1}(-1)^m m!\left[c_0^{(n,m)} - c_1^{(n,m)}\right]} + \boxed{\frac{1}{\hat{\sigma}_0}\int \mathrm{d}z\, z\, \frac{\mathrm{d}\sigma^{\mathrm{reg}}}{\mathrm{d}z}\bigg|_{\mathcal{O}(\alpha_s^n)}}$$

Relation from EEC **sum rules**

Log coefficients at end points.
Known to 3 loops.
[Moult, Zhu; Moult, Dixon, Zhu; Ebert, Mistlberger, GV]

New piece of information!

- Use **sum rule** and new calculation to extract moment of the entire EEC distribution.
- At $\mathcal{O}(\alpha_s^3)$ the EEC for electron-positron annihilation in QCD is only known numerically (with significant uncertainties)
- **First analytic information** about the bulk of the EEC in QCD at this order

$$\begin{aligned}
&\frac{1}{\hat{\sigma}_0}\int_0^1 \mathrm{d}z\, z\, \frac{\mathrm{d}\sigma^{\mathrm{reg}}_{e^+e^-}}{\mathrm{d}z}\bigg|_{\mathcal{O}(\alpha_s^3)} \\
&= C_F^3\left(64\zeta_3^2 + 672\zeta_6 - 496\zeta_2\zeta_3 - \frac{3616}{3}\zeta_5 + \frac{4778\zeta_4}{3} - 210\zeta_3 + \frac{99397}{216}\zeta_2 - \frac{3809015}{7776}\right) \\
&+ C_F n_f^2\left(\frac{16}{9}\zeta_4 + \frac{56}{27}\zeta_3 + \frac{3094}{405}\zeta_2 + \frac{156437}{13500}\right) \\
&+ C_F C_A^2\left(56\zeta_3^2 - 59\zeta_6 + \frac{628}{3}\zeta_2\zeta_3 + \frac{2212}{3}\zeta_5 - \frac{1768}{9}\zeta_4 - \frac{438601}{270}\zeta_3 + \frac{7930931}{8100}\zeta_2 - \frac{96056179}{180000}\right) \\
&- C_F^2 C_A\left(64\zeta_3^2 + 22\zeta_6 - 232\zeta_2\zeta_3 - 8\zeta_5 + \frac{19883}{18}\zeta_4 - \frac{74378}{45}\zeta_3 + \frac{392641}{216}\zeta_2 - \frac{113349701}{51840}\right) \\
&+ C_F C_A n_f\left(-72\zeta_3\zeta_2 - \frac{200}{3}\zeta_5 + \frac{812}{9}\zeta_4 + \frac{61169}{270}\zeta_3 - \frac{3933857}{16200}\zeta_2 + \frac{167350393}{2160000}\right) \\
&+ C_F^2 n_f\left(\frac{224}{3}\zeta_3\zeta_2 + \frac{128}{3}\zeta_5 - \frac{34}{9}\zeta_4 - \frac{7726}{45}\zeta_3 + \frac{98803}{360}\zeta_2 - \frac{406426043}{972000}\right) \\
&+ \frac{d_{abc}d^{abc}}{N_R} N_{F,V}\left(\frac{40}{3}\zeta_5 + \frac{\zeta_4}{2} - \frac{19}{3}\zeta_3 - 5\zeta_2 - \frac{1}{6}\right),
\end{aligned}$$

- We have presented results also for the Higgs EEC where there is not even a numerical result at this order

The EEC at N3LL` in the back-to-back limit

- The EEC Jet Function at **N3LO** we have calculated, was the last missing ingredient for resummation at **N3LL`** accuracy.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{\hat{\sigma}_0}{2} \overbrace{H_{q\bar{q}}(Q,\mu)}^{\text{Hard Function}} \int \frac{\mathrm{d}^2\vec{b}_T\,\mathrm{d}^2\vec{q}_T}{(2\pi)^2} e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T}\,\delta\left(1 - z - \frac{q_T^2}{Q^2}\right) \underbrace{J_q\left(b_T,\mu,\frac{\nu}{Q}\right) J_{\bar{q}}\left(b_T,\mu,\frac{\nu}{Q}\right)}_{\text{EEC Jet Functions}} \overbrace{\tilde{S}_q(b_T,\mu,\nu)}^{\text{TMD Soft Function}}$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} \ln H_i(Q,\mu) = \gamma_H^i(Q,\mu)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} \ln J_i(b_T,\mu,\nu/Q) = \tilde{\gamma}_J^i(\mu,\nu/Q)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} \ln \tilde{S}_i(b_T,\mu,\nu) = \tilde{\gamma}_S^i(\mu,\nu)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\nu} \ln J_i(b_T,\mu,\nu/Q) = -\frac{1}{2}\tilde{\gamma}_\nu^i(b_T,\mu)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\nu} \ln \tilde{S}_i(b_T,\mu,\nu) = \tilde{\gamma}_\nu^i(b_T,\mu)\,.$$

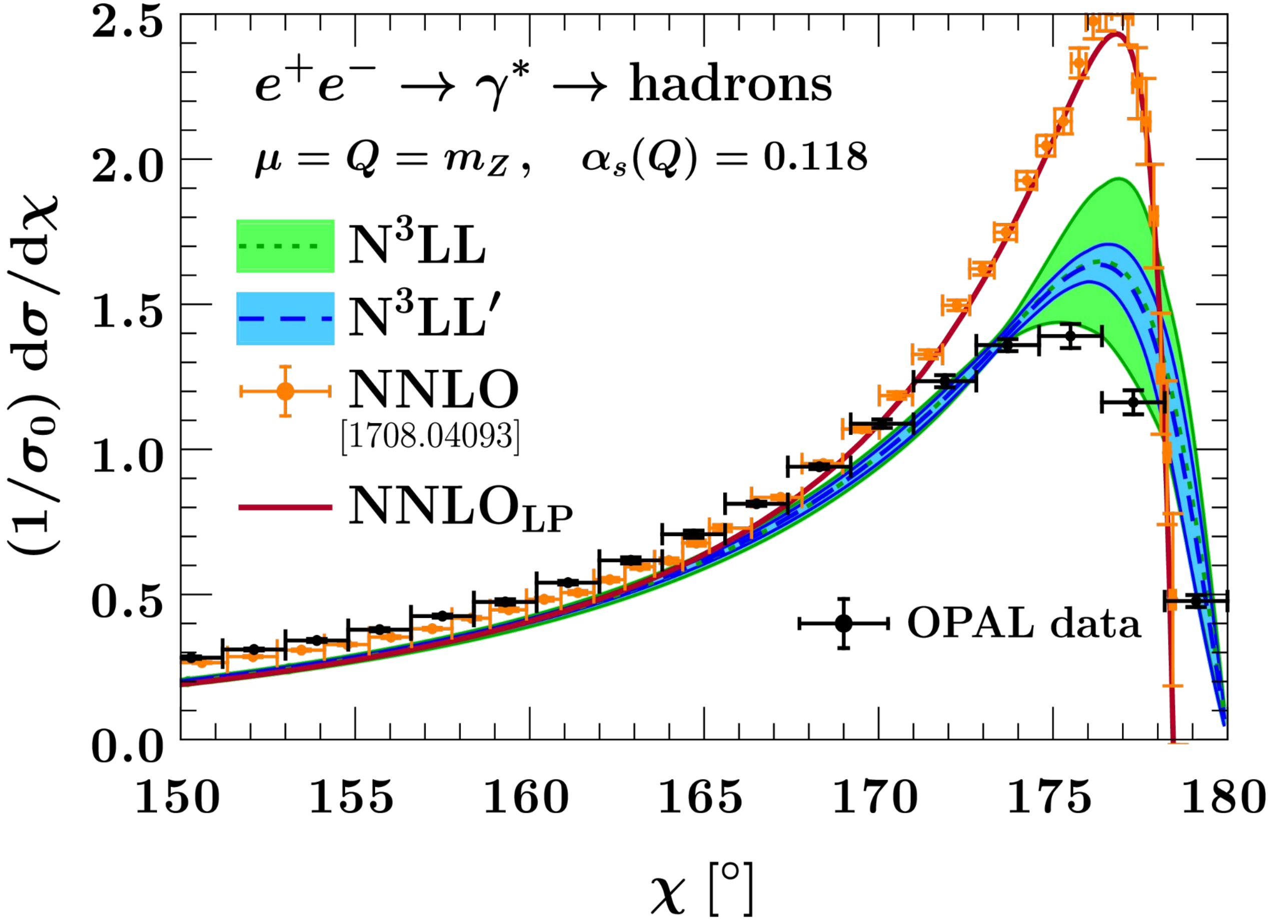
Accuracy	$H,\ J,\ S$	$\Gamma_{\text{cusp}}(\alpha_s)$	$\gamma(\alpha_s)$	$\beta(\alpha_s)$
LL	Tree level	1-loop	–	1-loop
NLL	Tree level	2-loop	1-loop	2-loop
NLL′	1-loop	2-loop	1-loop	2-loop
NNLL	1-loop	3-loop	2-loop	3-loop
NNLL′	2-loop	3-loop	2-loop	3-loop
N^3LL	2-loop	4-loop	3-loop	4-loop
N^3LL′	3-loop	4-loop	3-loop	4-loop

- H, J, S referred as "boundary" because they give the boundary conditions of the RGE differential equations. In conformal theories, no distinction of scales $\longrightarrow$ grouped into one object H

The EEC at N3LL` in the back-to-back limit

- Combining everything we get the EEC spectrum in the back-to-back limit at **N3LL`**

- N3LL` constitutes the **highest level of accuracy** for the resummation of an event shape in QCD to date.
- $\mathcal{O}(\alpha_s^3)$ analytic singular distribution matches numerical fixed order result as $\chi \to \pi$
- Resummation improves substantially the theory prediction at the peak
- **At N3LL` accuracy**, perturbative uncertainty finally comparable to experimental uncertainties on LEP data

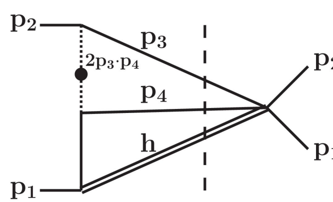


- Hadronization/non perturbative effects known to be sizable. Important to have control on perturbative result to disentangle effects for extraction of α_s

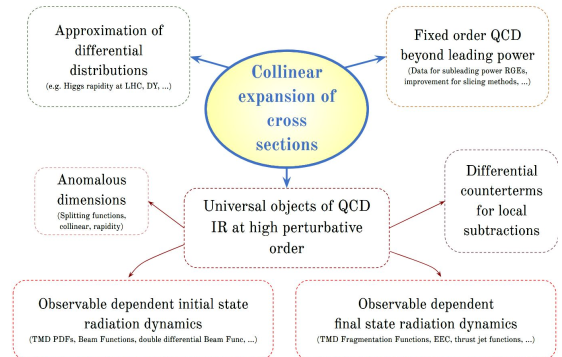
Conclusion

❖ Collinear expansion of cross sections

➢ Modern multiloop techniques can be used to efficiently compute **cross sections in collinear limit**



❖ Applications

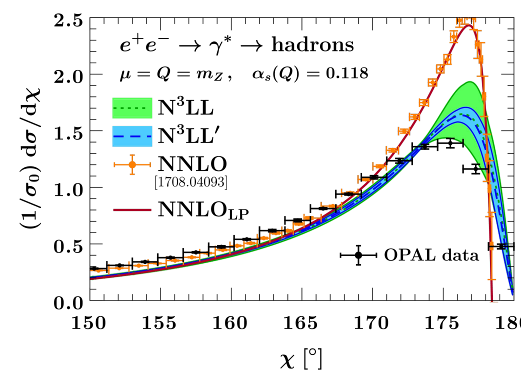


➢ Large variety of applications for this method:

- ➢ Approximation of differential distributions
  [Higgs Rapidity at the LHC, fully differential cross sections, ...]
- ➢ Anomalous dimensions, initial and final state collinear QCD building blocks
  [Beam Functions, TMDPDFs, TMD Fragmentation Functions, ...]

➢ Calculated **EEC Jet Function** at N3LO and obtained resummation at N3LL` in the back-to-back limit

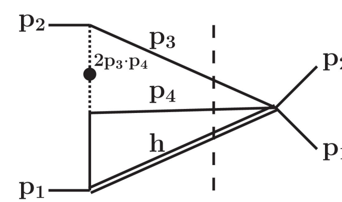
➢ Obtained moment of the full angular distribution at $\mathcal{O}(\alpha_s^3)$ analytically via EEC sum rules!



$$
\begin{aligned}
\frac{1}{C_F}\frac{\mathrm{d}\bar{\sigma}^{(3)}}{\mathrm{d}z} = & -4C_F^2\mathcal{L}_5(\bar{z}) \\
& + \mathcal{L}_4(\bar{z})\Big[-30C_F^2 - \frac{220}{9}C_FC_A + \frac{40}{9}C_Fn_f\Big] \\
& + \mathcal{L}_3(\bar{z})\Big[C_F^2(-16\zeta_2 - 104) + \frac{88}{9}C_Fn_f + C_FC_A\Big(-16\zeta_2 - \frac{388}{9}\Big) \\
& \qquad - \frac{242}{9}C_A^2 + \frac{88}{9}C_An_f - \frac{8}{9}n_f^2\Big] \\
& + \mathcal{L}_2(\bar{z})\Big[C_F^2(-144\zeta_2 - 16\zeta_3 - 189) + C_FC_A\Big(-\frac{592}{3}\zeta_2 - 72\zeta_3 + \frac{244}{3}\Big) \\
& \qquad + C_Fn_f\Big(\frac{88}{3}\zeta_2 - \frac{40}{3}\Big) + C_A^2\Big(\frac{2471}{27} - \frac{88}{3}\zeta_2\Big) \\
& \qquad + C_An_f\Big(\frac{16}{3}\zeta_2 - \frac{760}{27}\Big) + \frac{44n_f^2}{27}\Big] \\
& + \mathcal{L}_1(\bar{z})\Big[C_F^2\Big(-\frac{542}{3} - 412\zeta_2 + 224\zeta_3 - 192\zeta_4\Big) \\
& \qquad + C_FC_A\Big(-\frac{2900}{9}\zeta_2 - \frac{1688}{3}\zeta_3 - 8\zeta_4 + \frac{3797}{9}\Big) \\
& \qquad + C_Fn_f\Big(\frac{536}{9}\zeta_2 + \frac{32}{3}\zeta_3 - \frac{479}{9}\Big) + C_A^2\Big(-\frac{916}{9}\zeta_2 - 44\zeta_4 - \frac{2354}{81}\Big) \\
& \qquad + C_An_f\Big(\frac{448}{9}\zeta_2 + 16\zeta_3 - \frac{380}{81}\Big) + n_f^2\Big(\frac{124}{81} - \frac{16}{3}\zeta_2\Big)\Big] \\
& + \mathcal{L}_0(\bar{z})\Big[C_F^2\Big(64\zeta_3\zeta_2 - 402\zeta_2 + 332\zeta_3 - 552\zeta_4 + 48\zeta_5 - \frac{169}{2}\Big) \\
& \qquad + C_FC_A\Big(-128\zeta_3\zeta_2 + \frac{212}{3}\zeta_2 - \frac{2812}{9}\zeta_3 - \frac{1342}{3}\zeta_4 - 120\zeta_5 + \frac{3358}{9}\Big) \\
& \qquad + C_Fn_f\Big(-\frac{20}{3}\zeta_2 - \frac{296}{9}\zeta_3 + \frac{244}{3}\zeta_4 - \frac{623}{18}\Big) \\
& \qquad + C_A^2\Big(\frac{4420}{9}\zeta_2 - \frac{560}{9}\zeta_3 - \frac{326}{3}\zeta_4 - 40\zeta_5 - \frac{4241}{27}\Big) \\
& \qquad + C_An_f\Big(-\frac{1508}{9}\zeta_2 + \frac{184}{9}\zeta_3 + \frac{56}{3}\zeta_4 + \frac{1414}{27}\Big) \\
& \qquad + n_f^2\Big(\frac{112}{9}\zeta_2 + \frac{16}{9}\zeta_3 - \frac{98}{27}\Big)\Big] \\
& + \delta(\bar{z})\Big[C_F^2\Big(-\frac{337}{3} - \frac{1049}{3}\zeta_2 + \frac{530}{3}\zeta_3 + 512\zeta_2\zeta_3 - 64\zeta_3^2 - 1396\zeta_4 + \frac{3136}{3}\zeta_5 - 672\zeta_6\Big) \\
& \qquad + C_FC_A\Big(\frac{10169}{27} + \frac{2729}{3}\zeta_2 - \frac{22070}{9}\zeta_3 + \frac{2176\zeta_4}{9} + 528\zeta_5 + 22\zeta_6 - 288\zeta_2\zeta_3 + 64\zeta_3^2\Big) \\
& \qquad + C_Fn_f\Big(-\frac{148}{27} - \frac{985\zeta_2}{9} + \frac{3340\zeta_3}{9} + \frac{58\zeta_4}{9} - \frac{368\zeta_5}{3} - \frac{224}{3}\zeta_2\zeta_3\Big) \\
& \qquad + C_A^2\Big(-\frac{55504}{81} - \frac{3968}{81}\zeta_2 + \frac{39337}{27}\zeta_3 + \frac{3815}{18}\zeta_4 - \frac{2720}{3}\zeta_5 - \frac{700}{3}\zeta_2\zeta_3 + 59\zeta_6 - 56\zeta_3^2\Big) \\
& \qquad + C_An_f\Big(\frac{15626}{81} - \frac{3326}{81}\zeta_2 - \frac{3788}{27}\zeta_3 - \frac{290}{9}\zeta_4 + 80\zeta_5 + 72\zeta_2\zeta_3\Big) \\
& \qquad + n_f^2\Big(-\frac{1048}{81} + \frac{616}{81}\zeta_2 - \frac{464}{27}\zeta_3 - \frac{16}{9}\zeta_4\Big) \\
& \qquad + N_{F,V}\frac{d_{abc}d^{abc}}{N_r}\Big(2 + 5\zeta_2 + \frac{7}{3}\zeta_3 - \frac{\zeta_4}{2} - \frac{40}{3}\zeta_5\Big)\Big].
\end{aligned}
$$

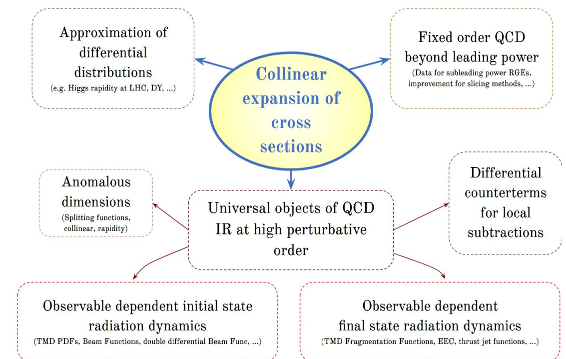
Conclusion

❖ Collinear expansion of cross sections



➢ Modern multiloop techniques can be used to efficiently compute **cross sections in collinear limit**

❖ Applications



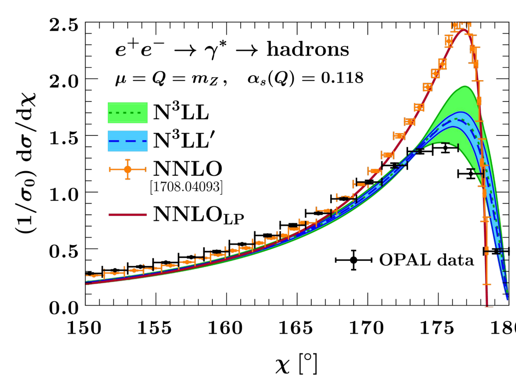
➢ Large variety of applications for this method:

- ➢ Approximation of differential distributions
  [Higgs Rapidity at the LHC, fully differential cross sections, ...]
- ➢ Anomalous dimensions, initial and final state collinear QCD building blocks
  [Beam Functions, TMDPDFs, TMD Fragmentation Functions, ...]

➢ Calculated **EEC Jet Function** at N3LO and obtained resummation at N3LL` in the back-to-back limit

$$
\begin{aligned}
\frac{1}{C_F}\frac{\mathrm{d}\bar\sigma^{(3)}}{\mathrm{d}z} = &-4C_F^2\mathcal{L}_5(\bar z)\\
&+\mathcal{L}_4(\bar z)\Big[-30C_F^2-\frac{220}{9}C_FC_A+\frac{40}{9}C_Fn_f\Big]\\
&+\mathcal{L}_3(\bar z)\Big[C_F^2(-16\zeta_2-104)+\frac{88}{9}C_Fn_f+C_FC_A\Big(-16\zeta_2-\frac{388}{9}\Big)\\
&\qquad-\frac{242}{9}C_A^2+\frac{88}{9}C_An_f-\frac{8}{9}n_f^2\Big]\\
&+\mathcal{L}_2(\bar z)\Big[C_F^2(-144\zeta_2-16\zeta_3-189)+C_FC_A\Big(-\frac{592}{3}\zeta_2-72\zeta_3+\frac{244}{3}\Big)\\
&\qquad+C_Fn_f\Big(\frac{88}{3}\zeta_2-\frac{40}{3}\Big)+C_A^2\Big(\frac{2471}{27}-\frac{88}{3}\zeta_2\Big)\\
&\qquad+C_An_f\Big(\frac{16}{3}\zeta_2-\frac{760}{27}\Big)+\frac{44n_f^2}{27}\Big]\\
&+\mathcal{L}_1(\bar z)\Big[C_F^2\Big(-\frac{542}{3}-412\zeta_2+224\zeta_3-192\zeta_4\Big)\\
&\qquad+C_FC_A\Big(-\frac{2900}{9}\zeta_2-\frac{1688}{3}\zeta_3-8\zeta_4+\frac{3797}{9}\Big)\\
&\qquad+C_Fn_f\Big(\frac{536}{9}\zeta_2+\frac{32}{3}\zeta_3-\frac{479}{9}\Big)+C_A^2\Big(-\frac{916}{9}\zeta_2-44\zeta_4-\frac{2354}{81}\Big)\\
&\qquad+C_An_f\Big(\frac{448}{9}\zeta_2+16\zeta_3-\frac{380}{81}\Big)+n_f^2\Big(\frac{124}{81}-\frac{16}{3}\zeta_2\Big)\Big]\\
&+\mathcal{L}_0(\bar z)\Big[C_F^2\Big(64\zeta_3\zeta_2-402\zeta_2+332\zeta_3-552\zeta_4+48\zeta_5-\frac{169}{2}\Big)\\
&\qquad+C_FC_A\Big(-128\zeta_3\zeta_2+\frac{212}{3}\zeta_2-\frac{2812}{9}\zeta_3-\frac{1342}{3}\zeta_4-120\zeta_5+\frac{3358}{9}\Big)\\
&\qquad+C_Fn_f\Big(-\frac{20}{3}\zeta_2-\frac{296}{9}\zeta_3+\frac{244}{3}\zeta_4-\frac{623}{18}\Big)\\
&\qquad+C_A^2\Big(\frac{4420}{9}\zeta_2-\frac{560}{9}\zeta_3-\frac{326}{3}\zeta_4-40\zeta_5-\frac{4241}{27}\Big)\\
&\qquad+C_An_f\Big(-\frac{1508}{9}\zeta_2+\frac{184}{9}\zeta_3+\frac{56}{3}\zeta_4+\frac{1414}{27}\Big)\\
&\qquad+n_f^2\Big(\frac{112}{9}\zeta_2+\frac{16}{9}\zeta_3-\frac{98}{27}\Big)\Big]\\
&+\delta(\bar z)\Big[C_F^2\Big(-\frac{337}{3}-\frac{1049}{3}\zeta_2+\frac{530}{3}\zeta_3+512\zeta_2\zeta_3-64\zeta_3^2-1396\zeta_4+\frac{3136}{3}\zeta_5-672\zeta_6\Big)\\
&\qquad+C_FC_A\Big(\frac{10169}{27}+\frac{2729}{3}\zeta_2-\frac{22070}{9}\zeta_3+\frac{2176\zeta_4}{9}+528\zeta_5+22\zeta_6-288\zeta_2\zeta_3+64\zeta_3^2\Big)\\
&\qquad+C_Fn_f\Big(-\frac{148}{27}-\frac{985\zeta_2}{9}+\frac{3340\zeta_3}{9}+\frac{58\zeta_4}{9}-\frac{368\zeta_5}{3}-\frac{224}{3}\zeta_2\zeta_3\Big)\\
&\qquad+C_A^2\Big(-\frac{55504}{81}-\frac{3968}{81}\zeta_2+\frac{39337}{27}\zeta_3+\frac{3815}{18}\zeta_4-\frac{2720}{3}\zeta_5-\frac{700}{3}\zeta_2\zeta_3+59\zeta_6-56\zeta_3^2\Big)\\
&\qquad+C_An_f\Big(\frac{15626}{81}-\frac{3326}{81}\zeta_2-\frac{3788}{27}\zeta_3-\frac{290}{9}\zeta_4+80\zeta_5+72\zeta_2\zeta_3\Big)\\
&\qquad+n_f^2\Big(-\frac{1048}{81}+\frac{616}{81}\zeta_2-\frac{464}{27}\zeta_3-\frac{16}{9}\zeta_4\Big)\\
&\qquad+N_{F,V}\frac{d_{abc}d^{abc}}{N_r}\Big(2+5\zeta_2+\frac{7}{3}\zeta_3-\frac{\zeta_4}{2}-\frac{40}{3}\zeta_5\Big)\Big].
\end{aligned}
$$

➢ Obtained moment of the full angular distribution at $\mathcal{O}(\alpha_s^3)$ analytically via EEC sum rules!



Thank You!

Backup

Expansion in kinematic limits

Kinematic limits can be exploited in different ways

Factorization theorems

- Understand **all order** structure of QCD in that limit
- Identify **universal objects**
- Improves predictions via **resummation** of infinite towers of terms

[See all SCET literature and beyond]

- Expansion of resummed results ⟶ data /cross checks for higher order calculations

Fixed order calculations

- Ingredients entering the calculation **dramatically simplify** (see for example first N3LO results for inclusive XS from Threshold expansion [Anastasiou, Duhr, Dulat, Herzog, Mistlberger 1503.06056])
- Inclusion of subleading terms is not particularly difficult ([Anastasiou, Duhr, Dulat, Herzog, Mistlberger] had >30 terms in threshold expansion, all order factorization at NLP in threshold still in progress)
- Obtain predictions at a perturbative order not reachable in full kinematics

- Kinematic expansions of XS ⟶ data for **subleading power** factorization theorems

They feed into each other: use fixed order in the kinematic limit to obtain universal objects identified by EFT. Examples: Beam Functions, TMDFFs, EEC Jet Functions at N3LO [Ebert, Mistlberger, **GV**]

Collinear Expansion of cross sections

1. Repeat procedure for all diagrams to obtain expansion of **partonic** cross section
2. With the expanded partonic cross section, construct an **expansion** of the **hadronic** cross section around collinear limit of radiation

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\mathrm{d}Y\mathrm{d}\mathcal{T}} = \lambda^{-2}\frac{\mathrm{d}\sigma^{(0)}}{\mathrm{d}Q^2\mathrm{d}Y\mathrm{d}\mathcal{T}} + \lambda^{-1}\frac{\mathrm{d}\sigma^{(1)}}{\mathrm{d}Q^2\mathrm{d}Y\mathrm{d}\mathcal{T}} + \ldots .$$

- **Note:** Translation to expansion of **hadronic** cross section is straightforward knowing relations to hadronic variables

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\mathrm{d}Y\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x} = \frac{\sigma_0}{\tau}\sum_{i,j} x_1 f_i\left(x_1\right) x_2 f_j\left(x_2\right)\frac{\mathrm{d}\eta_{ij}}{\mathrm{d}Q^2\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x}$$

$$w_1 = -\frac{\bar{n}\cdot k}{\bar{n}\cdot p_1}, \quad w_2 = -\frac{n\cdot k}{n\cdot p_2}$$

$$x = \frac{k^2}{(\bar{n}\cdot k)(n\cdot k)} = 1 - \frac{\vec{k}_\perp^2}{(\bar{n}\cdot k)(n\cdot k)}$$

$$x_1 = \frac{x_1^B}{z_1} = x_1^B\left[\sqrt{1+(k_T/Q)^2} - \frac{\bar{n}\cdot k}{Q}e^{-Y}\right]$$
$$x_2 = \frac{x_2^B}{z_2} = x_2^B\left[\sqrt{1+(k_T/Q)^2} - \frac{n\cdot k}{Q}e^{+Y}\right]$$

$$\frac{\mathrm{d}\eta_{ij}(y_1,y_2)}{\mathrm{d}Q^2\mathrm{d}Y\mathrm{d}\mathcal{T}} = \int_0^1 \mathrm{d}x \int_0^\infty \mathrm{d}w_1\mathrm{d}w_2\,\delta\left(y_1-z_1\right)\delta\left(y_2-z_2\right) \times\delta\left[\mathcal{T}-\mathcal{T}(Q,Y,w_1,w_2,x)\right]\frac{\mathrm{d}\eta_{ij}}{\mathrm{d}Q^2\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x}$$

Applications:
Higgs rapidity distribution at the LHC

Approximation of differential distributions

Collinear expansion of cross sections

Fixed order QCD beyond leading power

Anomalous dimensions

Universal objects of QCD IR at high perturbative order

Differential counterterms for local subtractions

Observable dependent initial state radiation dynamics
(TMD PDFs, Beam Functions, double differential Beam Func, ...)

Observable dependent final state radiation dynamics
(TMD Fragmentation Functions, EEC, thrust jet functions, ...)

Collinear expansion of cross sections: Applications

- We can build **approximation of hadronic differential cross section** by combining the expansions in different limits!

$$\frac{\sigma^{\text{appr.}}}{\mathrm{d}Q^2\mathrm{d}Y\,\mathrm{d}\mathcal{T}} \sim \frac{\sigma^{(n)}}{\mathrm{d}Q^2\mathrm{d}Y\,\mathrm{d}\mathcal{T}} + \frac{\sigma^{(\bar{n})}}{\mathrm{d}Q^2\mathrm{d}Y\,\mathrm{d}\mathcal{T}} - \frac{\sigma^{(s)}}{\mathrm{d}Q^2\mathrm{d}Y\,\mathrm{d}\mathcal{T}} + \cdots$$

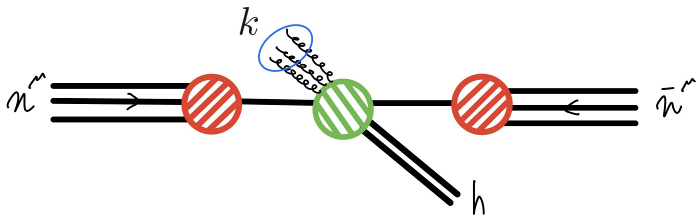
- Expansion in opposite collinear direction is obtained via simple swap

- Soft expansion in general simpler than (or subset of) collinear one

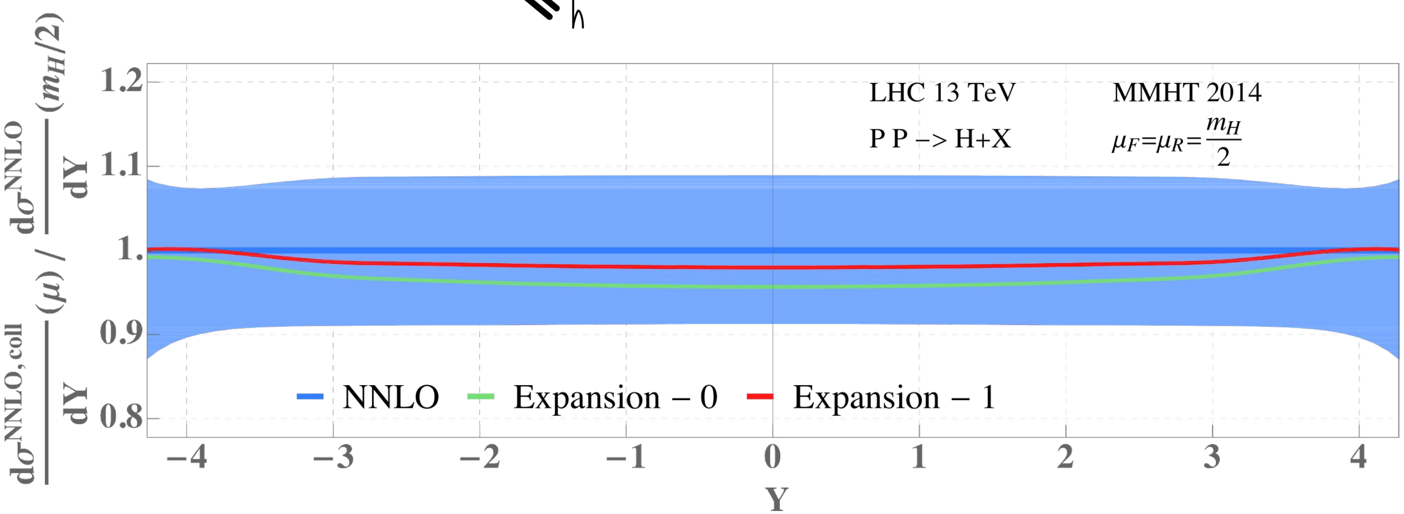
- Soft expansion necessary to remove overlap **[zero bin subtraction]**
(more complications at cross section level if observable $\mathcal{T}$ admits different expansion in soft vs collinear limit)

Approximation of Higgs Rapidity Spectrum at LHC

- Use this framework to compute an approximation to the Higgs Rapidity distribution



$$Y = \frac{1}{2}\log\left(\frac{\bar{n}\cdot p_h}{n\cdot p_h}\right)$$



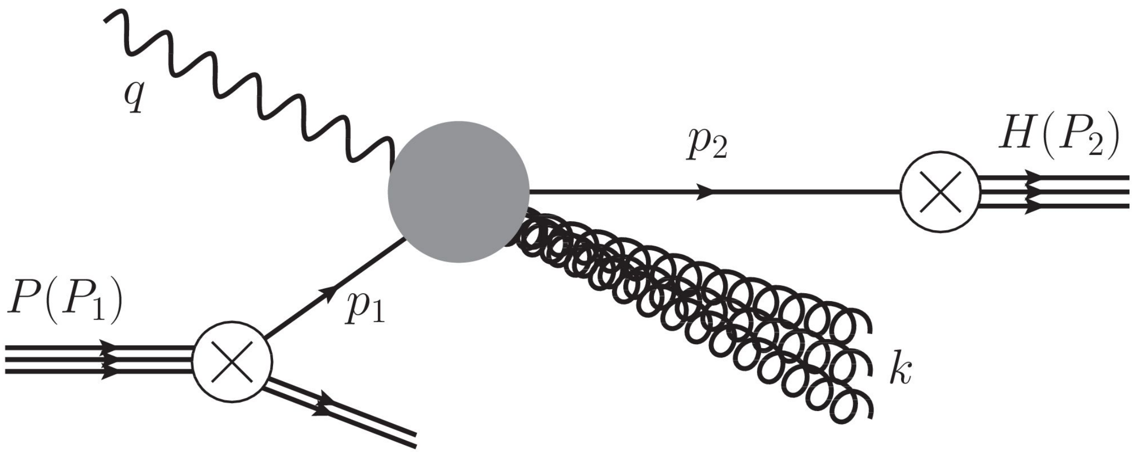
Just 2 orders in the **collinear expansion** give better than 2% accuracy over the entire spectrum!

- NNLO rapidity spectrum is known exactly, we can compare how quickly our approximation converges to exact result.
- Collinear expansion of rapidity distribution works really well!
- **Outlook**: compute collinear approximation of rapidity distribution at N3LO where no exact result is available. Supplement N3LO threshold approximation

[Dulat, Mistlberger, Pelloni]

Extension to Final State Radiation

- Can we generalize our method beyond color singlet production in proton-proton collisions to study **final state** radiation?
- Look at fully differential Semi-Inclusive Deep Inelastic Scattering **(SIDIS)**, production of an identified hadron H in DIS

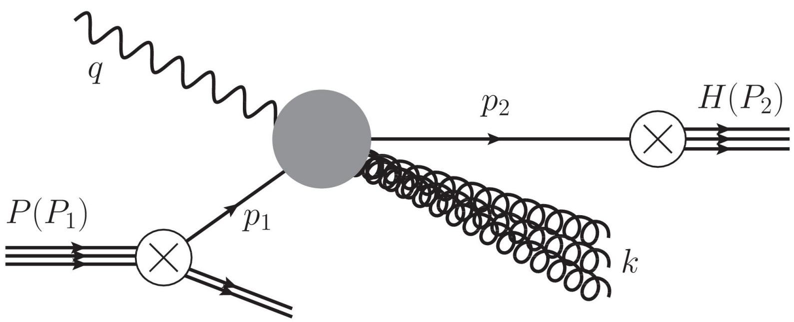


- Study the case of radiation collinear to **final state** hadron
- Can we relate it via crossing symmetry to the expansion of color-singlet production cross sections in pp?

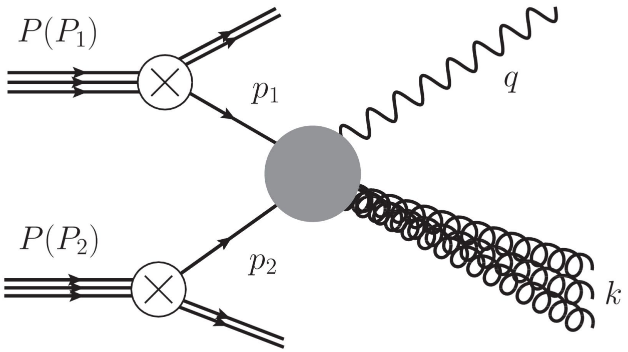
$$\underbrace{p(p_1) + h(q) \to p(-p_2) + X(-k)}_{\text{Semi-Inclusive DIS}} \quad \longleftrightarrow \quad \underbrace{p(p_1) + p(p_2) \to h(-q) + X(-k)}_{\text{DY/Higgs production in pp}}$$

SIDIS from production in pp

- **Yes!** In our formalism we can obtain SIDIS **from** color singlet production at the LHC



Semi-Inclusive Deep Inelastic Scattering (SIDIS)



Drell-Yan/Higgs production in pp

- Phase spaces are simply related (this is because we are differential in these variables)

$$\frac{\mathrm{d}\Phi_{1+m}}{\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x} \sim (1-w_1)^{-3+2\epsilon}\frac{\mathrm{d}\Phi_{h+m}}{\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x}$$

- Partonic Cross Sections related by crossing symmetry (needs analytic continuation)

$$\frac{\mathrm{d}\hat{\eta}_{i+h\to j+X}}{\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x} \sim (1-w_1)^{-3+2\epsilon}\left.\frac{\mathrm{d}\hat{\eta}_{i+h\to j+X}}{\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x}\right|_{\text{analyt.cont.}}$$

Analytic Continuation

- Crossing symmetry for SIDIS (and similarly for decay processes)

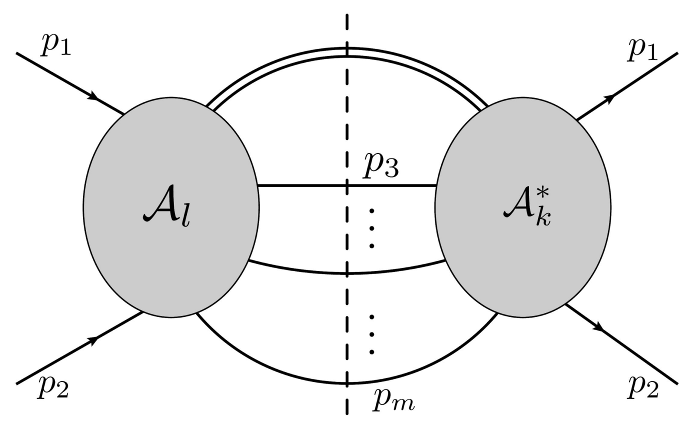
$$\underbrace{p(p_1) + h(q) \to p(-p_2) + X(-k)}_{\text{Semi-Inclusive DIS}} \quad \longleftrightarrow \quad \underbrace{p(p_1) + p(p_2) \to h(-q) + X(-k)}_{\text{DY/Higgs production in pp}}$$

- Variables we are differential in change sign under crossing!
- Therefore, crossing means analytically continuing cross section in those variables
- Mandelstam equipped with prescription $\quad (p_i + p_j)^2 \quad \to \quad (p_i + p_j)^2 + i0$
- Partonic cross section branch cut structure can be tracked by organizing it as

Difficult, but universal part
(no branch cuts at 0)

$$\frac{\mathrm{d}\eta_{ij}^{(m+l+k)}}{\mathrm{d}Q^2\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x} = (sw_1w_2)^{-m\epsilon} \times \left\{ \sum_{i_1,i_2=0}^{l} \sum_{j_1,j_2=0}^{k} \frac{\mathrm{d}\eta_{ij}^{(m+l+k,i_1,i_2,j_1,j_2)}}{\mathrm{d}Q^2\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x} \times \Re\left\{ \left[(-s)^{(i_1+i_2-l)\epsilon}(sw_1)^{-i_1\epsilon}(sw_2)^{-i_2\epsilon}\right]\left[(-s)^{(j_1+j_2-k)\epsilon}(sw_1)^{-j_1\epsilon}(sw_2)^{-j_2\epsilon}\right]^* \right\} \right\}$$

Mandelstam variables, easy to analytically continue

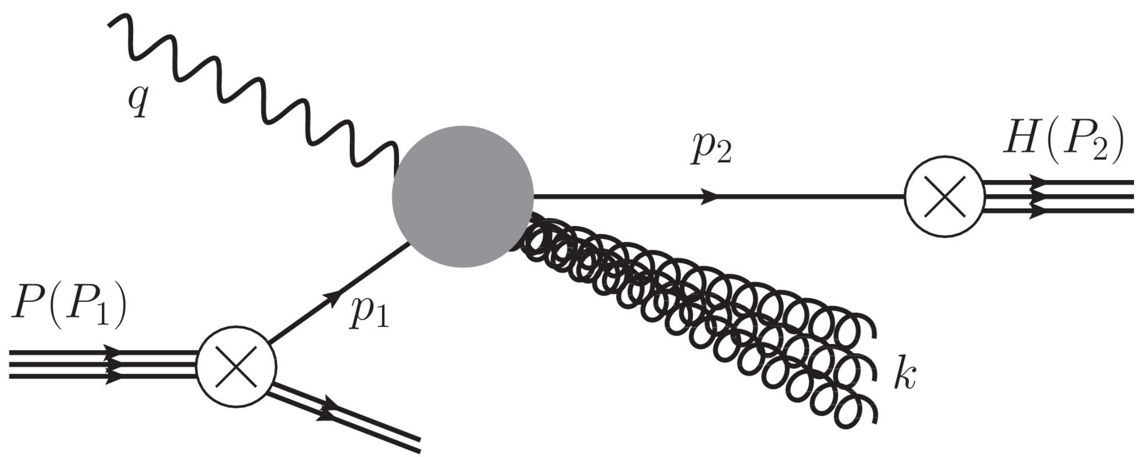


Simplest l-loop example:

$$\underbrace{\Re\left[(-s - i0)^{-l\epsilon}\right]}_{\text{DIS}} \qquad \longleftrightarrow \qquad \underbrace{\cos(l\pi\epsilon)\,\Re\left[(s + i0)^{-l\epsilon}\right]}_{\text{production}}$$

Extension to Final State Radiation

- Can we generalize our method beyond color singlet production in proton-proton collisions to study **final state** radiation? **YES!**
- Look at fully differential **Semi-Inclusive Deep Inelastic Scattering (SIDIS)**, production of an identified hadron H in DIS



- Study the case of radiation collinear to **final state** hadron
- Can we relate it via crossing symmetry to the expansion of color-singlet production cross sections in pp? **YES!**

$$\underbrace{p(p_1) + h(q) \to p(-p_2) + X(-k)}_{\text{Semi-Inclusive DIS}} \quad \longleftrightarrow \quad \underbrace{p(p_1) + p(p_2) \to h(-q) + X(-k)}_{\text{DY/Higgs production in pp}}$$

Resummation of the EEC in the back-to-back limit

- Extending method of collinear expansion of cross sections to processes with final state color charged particles we were able to calculate EEC Jet Function at **N3LO**

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{\hat{\sigma}_0}{2} \overbrace{H_{q\bar{q}}(Q,\mu)}^{\text{Hard Function}} \int \frac{\mathrm{d}^2\vec{b}_T\,\mathrm{d}^2\vec{q}_T}{(2\pi)^2} e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T}\,\delta\left(1-z-\frac{q_T^2}{Q^2}\right) \underbrace{J_q\left(b_T,\mu,\frac{\nu}{Q}\right) J_{\bar{q}}\left(b_T,\mu,\frac{\nu}{Q}\right)}_{\text{EEC Jet Functions}} \overbrace{\tilde{S}_q(b_T,\mu,\nu)}^{\text{TMD Soft Function}}$$

- SCET allows to resum large logs appearing in this limit.
- Each function obeys renormalization group equations (RGEs)

Anomalous dimensions obtained by poles of calculation in the EFT (known in the literature, rechecked in our calculation)

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\ln H_i(Q,\mu) = \gamma_H^i(Q,\mu)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\ln J_i(b_T,\mu,\nu/Q) = \tilde{\gamma}_J^i(\mu,\nu/Q)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\ln \tilde{S}_i(b_T,\mu,\nu) = \tilde{\gamma}_S^i(\mu,\nu)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\nu}\ln J_i(b_T,\mu,\nu/Q) = -\frac{1}{2}\tilde{\gamma}_\nu^i(b_T,\mu)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\nu}\ln \tilde{S}_i(b_T,\mu,\nu) = \tilde{\gamma}_\nu^i(b_T,\mu)\,.$$

Rapidity Renormalization Group Equations

- Running of operators resum logs as for running coupling in standard QFT

The EEC at N3LL`: Resummation

- Resummed cross section takes the following form

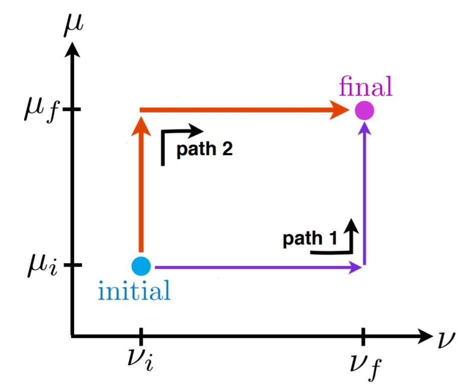
$$\frac{d\sigma}{dz} = \frac{\hat{\sigma}_0}{8} \overbrace{\int_0^\infty d(b_T Q)^2\, J_0\big(b_T Q\sqrt{1-z}\big)}^{\text{Fourier Transform. Nothing important}} \overbrace{H_{q\bar{q}}(Q,\mu_H) J_q\Big(b_T,\mu_J,\frac{\nu_J}{Q}\Big) J_{\bar{q}}\Big(b_T,\mu_J,\frac{\nu_J}{Q}\Big)\tilde{S}_q(b_T,\mu_S,\nu_S)}^{H,\ J,\ S\ \text{boundaries}}$$

$$\times \exp\Bigg[\underbrace{\int_{\mu_H}^{\mu}\frac{d\mu'}{\mu'}\gamma_H^q(Q,\mu')}_{\text{running of Hard function}} + 2\underbrace{\int_{\mu_J}^{\mu}\frac{d\mu'}{\mu'}\tilde{\gamma}_J^q(\mu',\nu_J/Q)}_{\mu\text{-running of Jet function}} + \underbrace{\int_{\mu_S}^{\mu}\frac{d\mu'}{\mu'}\tilde{\gamma}_S^q(\mu',\nu_S)}_{\mu\text{-running of Soft function}}\Bigg] \underbrace{\left(\frac{\nu_J}{\nu_S}\right)^{\tilde{\gamma}_\nu^q(b_T,\mu)}}_{\text{running in rapidity}}$$

"Canonical" choice of scales for the boundaries is the one that minimizes all the logs in them

$$\mu_H \sim Q\,, \quad \mu_J \sim \frac{b_0}{b_T}\,, \quad \mu_S \sim \frac{b_0}{b_T}\,,$$

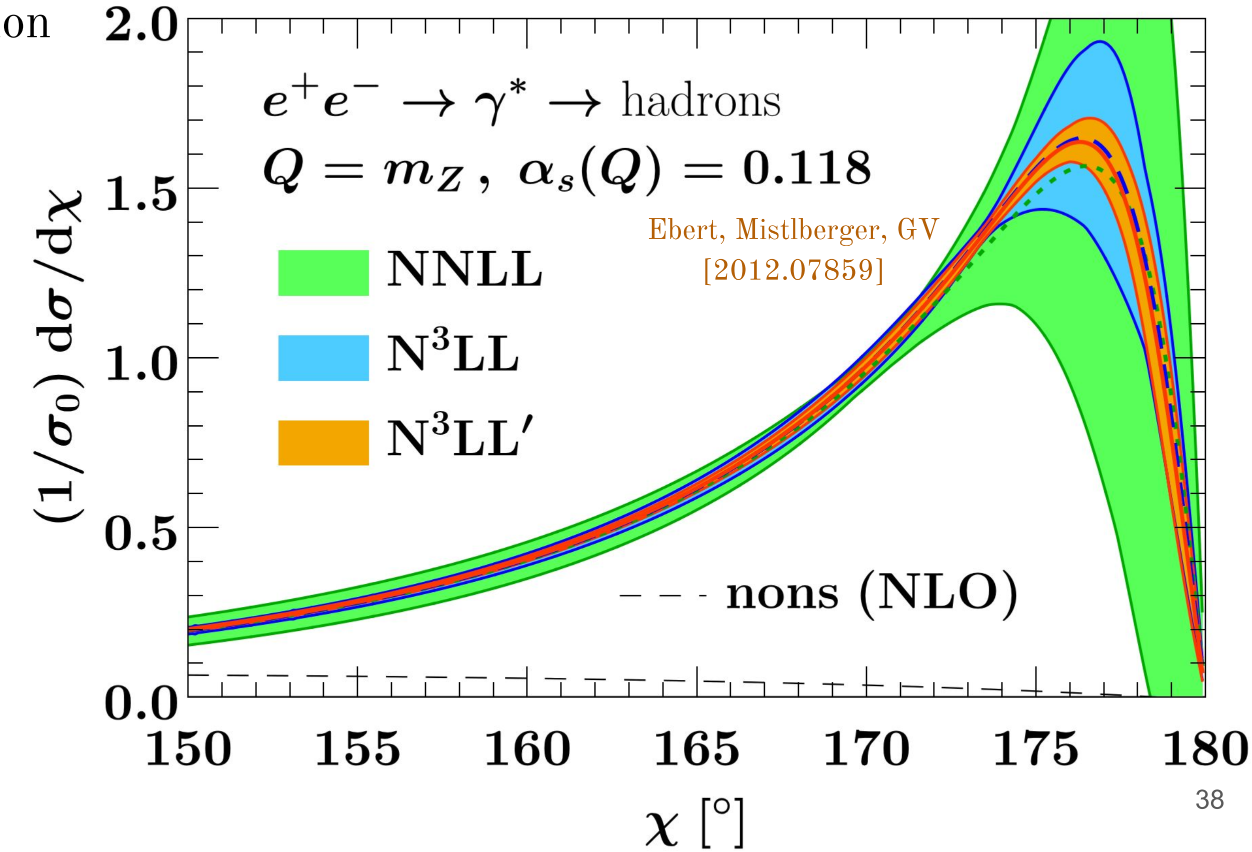
$$\nu_J \sim Q\,, \quad \nu_S \sim \frac{b_0}{b_T}\,,$$



Running with Rapidity RGE is in 2-d plane, but it is path independent [Chiu, Jain, Neill, Rothstein]

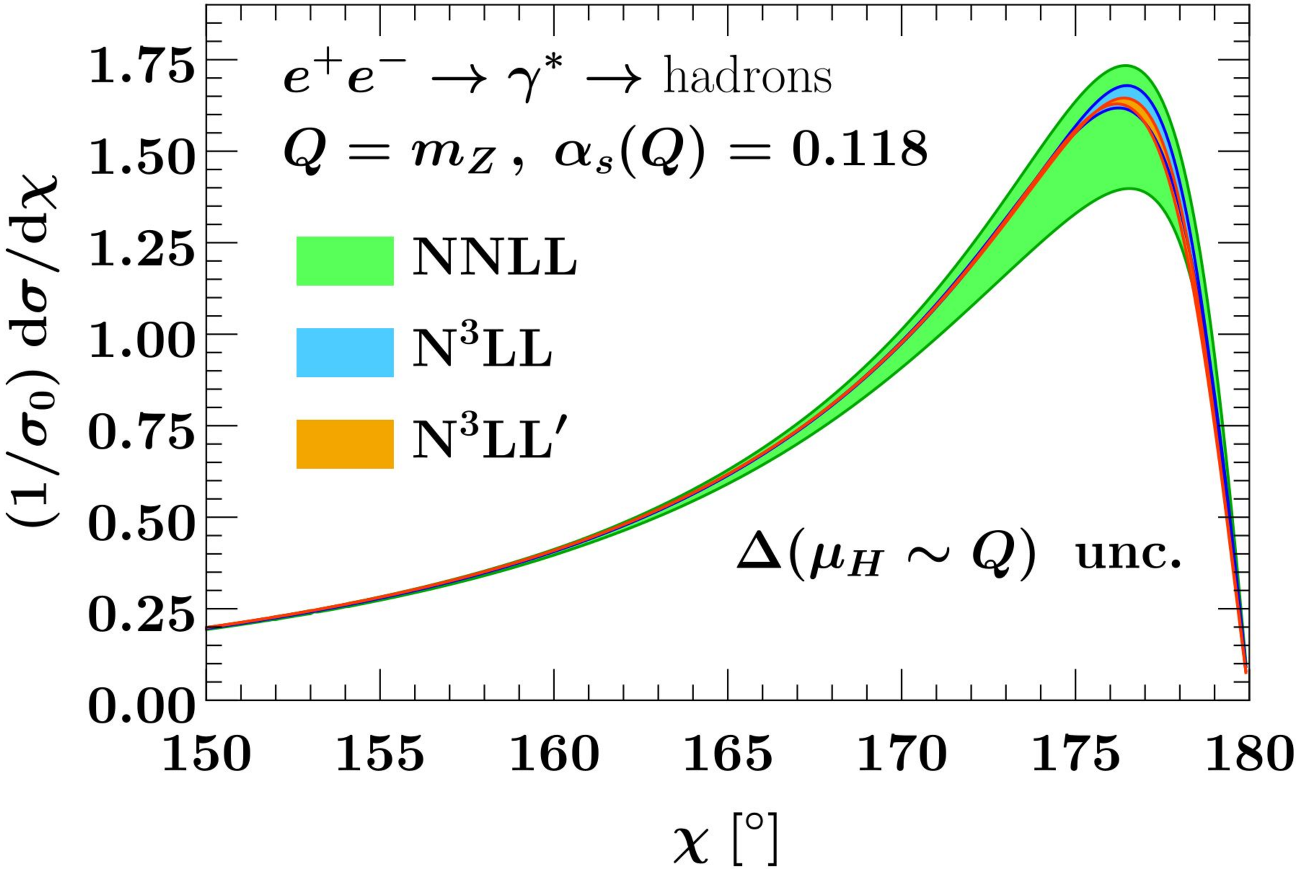
The EEC at N3LL`: Uncertainty estimation

- Estimate of uncertainty obtained by:
  - Hard, jet, soft (and rapidities) scale variation by a factor of 2.
    Envelope of 35 scale variations (Configurations with ratios of scales >2 or <1/2 discarded)
    (arguably standard procedure for multiscale resummaton in SCET) [Stewart, Tackmann, Walsh, Zuberi] (2013)
  - Variation of b* prescription
- Roughly 4% uncertainty around peak region
- Significant reduction of uncertainty compared to lower orders



Literature Comparison on Uncertainties

- Previous works in the literature make use of *quite non conservative* scheme to estimate resummation uncertainties and obtain ~ 8% uncertainty at NNLL.
- Applying a similar scheme to our N3LL` result we obtain a 0.5% uncertainty



Peeking into the bulk of the EEC: Sum rules

- Remarkably, the EEC obeys interesting sum rules

$$\int_0^1 \mathrm{d}z \frac{\mathrm{d}\sigma}{\mathrm{d}z} = \sigma$$

[Simmons Duffin, Kologlu, Kravchuk, Zhiboedov; Korchemsky; Moult, Dixon, Zhu]

$$\int_0^1 \mathrm{d}z\, z \frac{\mathrm{d}\sigma}{\mathrm{d}z} = \int_0^1 \mathrm{d}z (1-z) \frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{1}{2}\sigma$$

- General structure of the endpoints is know order by order

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = V_0 \delta(z) + \left[\frac{\phi_0(z)}{z}\right]_+ + V_1 \delta(1-z) + \left[\frac{\phi_1(z)}{1-z}\right]_+ + \frac{\mathrm{d}\sigma^{\mathrm{reg}}}{\mathrm{d}z}$$

$$\sigma = \hat{\sigma}_0 \sum_n \left(\frac{\alpha_s}{4\pi}\right)^n R^{(n)}$$

$$V_0 = \hat{\sigma}_0 \sum_n \left(\frac{\alpha_s}{4\pi}\right)^n V_0^{(n)}$$

$$V_1 = \hat{\sigma}_0 \sum_n \left(\frac{\alpha_s}{4\pi}\right)^n V_1^{(n)}$$

$$\phi_0(z) = \hat{\sigma}_0 \sum_n \left(\frac{\alpha_s}{4\pi}\right)^n \sum_{m=0}^{n-1} c_0^{(n,m)} \log^m z$$

$$\phi_1(z) = \hat{\sigma}_0 \sum_n \left(\frac{\alpha_s}{4\pi}\right)^n \sum_{m=0}^{2n-1} c_1^{(n,m)} \log^m (1-z)\,.$$

- Derive powerful relations between ingredients living at different angles of the EEC

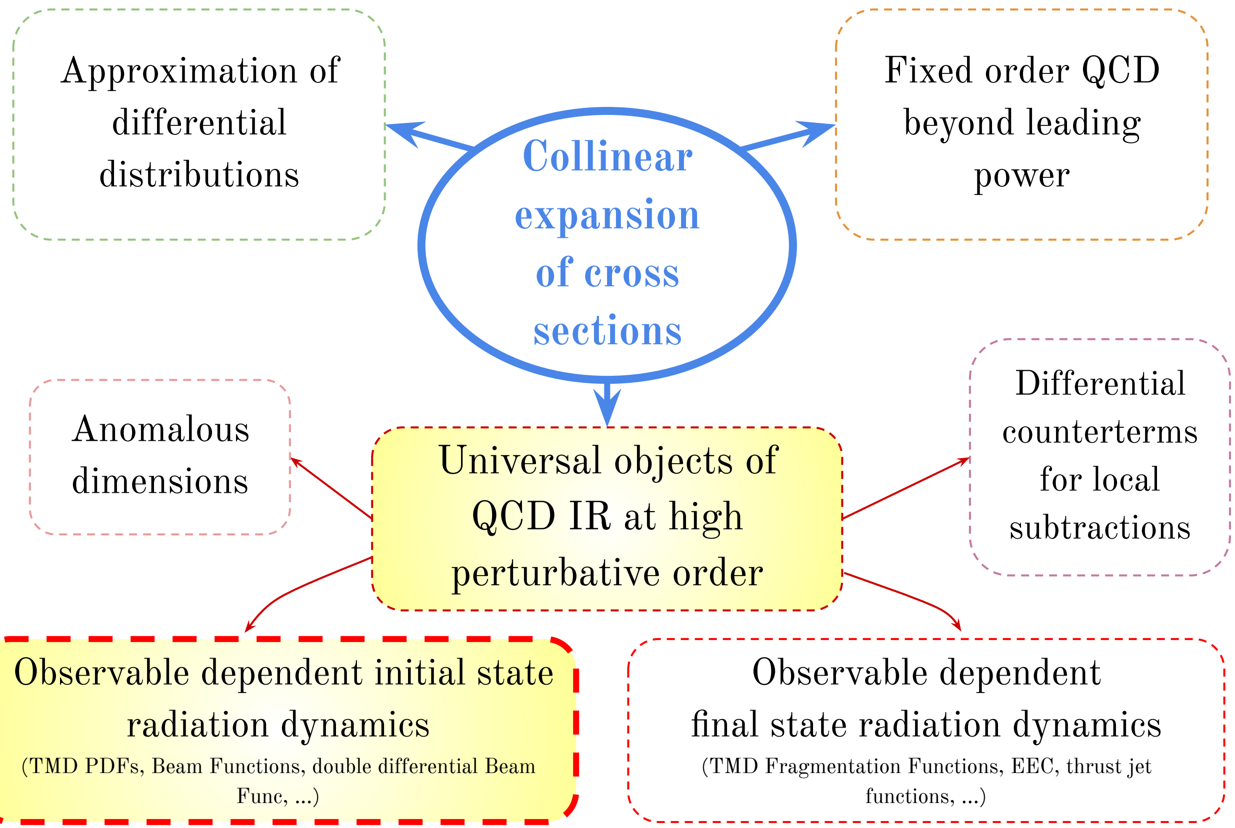
$$R^{(n)} = V_0^{(n)} + V_1^{(n)} + \frac{1}{\hat{\sigma}_0} \int \mathrm{d}z \left.\frac{\mathrm{d}\sigma^{\mathrm{reg}}}{\mathrm{d}z}\right|_{\mathcal{O}(\alpha_s^n)}$$

$$\frac{R^{(n)}}{2} = V_1^{(n)} + \sum_{m=0}^{2n-1} (-1)^m m! \left[c_0^{(n,m)} - c_1^{(n,m)}\right] + \frac{1}{\hat{\sigma}_0} \int \mathrm{d}z\, z \left.\frac{\mathrm{d}\sigma^{\mathrm{reg}}}{\mathrm{d}z}\right|_{\mathcal{O}(\alpha_s^n)}$$

$$\frac{R^{(n)}}{2} = V_0^{(n)} + \sum_{m=0}^{2n-1} (-1)^m m! \left[c_1^{(n,m)} - c_0^{(n,m)}\right] + \frac{1}{\hat{\sigma}_0} \int \mathrm{d}z\, (1-z) \left.\frac{\mathrm{d}\sigma^{\mathrm{reg}}}{\mathrm{d}z}\right|_{\mathcal{O}(\alpha_s^n)}$$

Applications:
Beam Functions at N3LO

"N-Jettiness Beam Functions at N3LO" [2006.03056]
"Transverse momentum dependent PDFs at N3LO" [2006.05329]
"Calculation of Differential Collinear Expansions at N3LO" [in preparation]



Beam Functions

- **Beam Functions**: **universal gauge invariant matrix elements** encoding collinear dynamics of protons and initial state radiation in the presence of multiple measurements

- Can be understood as a generalization of a **Parton Distribution Functions** (PDFs)

PDF: $$f_q(x) = \langle p_n | \bar{\chi}_n \frac{\bar{n}\!\!\!/}{2} \left[\delta(p^- - \bar{n}\cdot\mathcal{P})\, \chi_n \right] | p_n \rangle$$

Beam Function: $$B_q(x, \mathcal{T}) = \langle p_n | \bar{\chi}_n \frac{\bar{n}\!\!\!/}{2} \left[\delta(p^- - \bar{n}\cdot\mathcal{P})\, \delta(\mathcal{T} - \hat{\mathcal{T}})\, \chi_n \right] | p_n \rangle$$

$$\sigma_{pp\to X} \sim \int f_a(x_1) f_b(x_2) \otimes \hat{\sigma}_{ab\to X}$$

Inclusive

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\mathcal{T}} \sim \int H \mathcal{B}_n \otimes \mathcal{B}_{\bar{n}} \otimes S + \dots$$

More differential

Beam Functions

- **Beam Functions** enter the factorization for many LHC observables
  (N-Jettiness, Transverse momentum distributions [TMDPDFs], Transverse Energy-Energy Correlator, Double Differential Distributions...)

Leading power **factorization** for **Transverse-Momentum Distribution**

$$\frac{d\sigma}{dQ^2 dY d^2\vec{q}_T} = \sigma_0 \sum_{i,j} H_{ij}(Q^2,\mu) \int d^2\vec{b}_T\, e^{i\vec{q}_T\cdot\vec{b}_T}\, \tilde{B}_i\left(x_1^B, b_T, \mu, \frac{\nu}{\omega_a}\right) \tilde{B}_j\left(x_2^B, b_T, \mu, \frac{\nu}{\omega_b}\right) \tilde{S}(b_T,\mu,\nu)$$

q_T Beam Functions

Leading power **factorization** for **Beam Thrust**

$$\frac{d\sigma}{dQ^2 dY d\mathcal{T}_0} = \sigma_0 \sum_{i,j} H_{ab}(Q^2,\mu) \int dt_a\, dt_b\, B_a(t_a, x_1^B, \mu) B_b(t_b, x_2^B, \mu)\, S\left(\mathcal{T}_0 - \frac{t_a}{Q_a} - \frac{t_b}{Q_b}, \mu\right)$$

τ Beam Functions

- Factorization theorems encode singular structure of color singlet diff. distr. at LHC
- Knowing them at N3LO allows to obtain **fully differential distributions at N3LO** via slicing methods

Q_T Subtraction: [Catani, Grazzini]
N-Jettiness Subtraction:
[Gaunt, Stahlhofen, Tackmann, Walsh]
[Boughezal, Focke, Petriello, Liu]

Subtractions can be systematically improved by analytically computing power corrections

From Expanded XS to BFs: Projections and checks

N-Jettiness Beam Function

TMD Beam Function

$$B_a(t_a, x_1^B, \mu) \xleftarrow{\text{project to } \boldsymbol{\tau}} \frac{d\eta_{j\bar{i}}}{dQ^2 dw_1 dw_2 dx} \xrightarrow{\text{project to } \mathbf{q_T}} \tilde{B}_i\left(x_1^B, b_T, \mu, \frac{\nu}{\omega_a}\right)$$

N-Jettiness Beam Function:

- Coupling renormalization
- SCET_I renormalization
- IR poles subtracted via N3LO PDF counterterms

TMD Beam Function:

- Coupling renormalization
- Zero-bin subtraction via Soft Function
- SCET_II renormalization
- IR poles subtracted via N3LO PDF counterterms

Checks

N-Jettiness Beam Function:

- 6 orders of poles cancel in all channels
- Terms involving $\mathcal{L}_n\left(\frac{t}{\mu^2}\right)$ $n = 0, \ldots, 5$ vs RGE prediction [Billis, Ebert, Michel and Tackmann]
- Eikonal limit vs threshold consistency
- Generalized leading color approx [Behring, Melnikov, Rietkerk, Tancredi, Wever]

TMD Beam Function:

- All rapidity divergences regulated
- 3 orders of $\boldsymbol{\varepsilon}$ poles cancel for all channels
- Log terms vs RGE prediction [Billis, Ebert, Michel and Tackmann]
- Eikonal limit vs threshold consistency
- Quark channels vs [Luo, Yang, Zhu, Zhu 1912.05778]

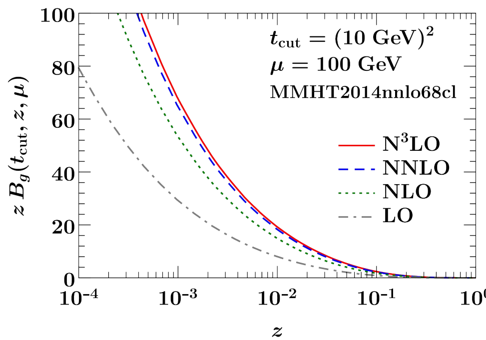
Beam Functions Results

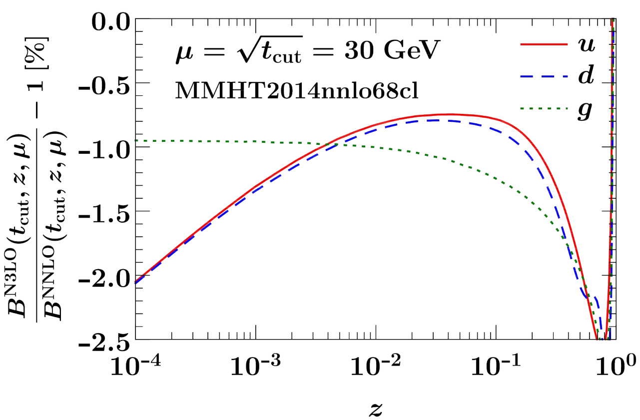
We have obtained full matching kernel for all **quark** and **gluon** channels

N-Jettiness Beam Function

- **New space of functions:** HPL are not sufficient. Iterated integrals with non-rational alphabet:

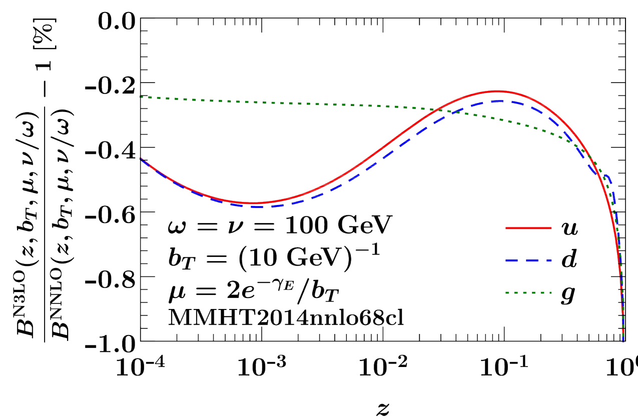
$$\mathcal{A} = \left\{ \frac{1}{z}, \frac{1}{1-z}, \frac{1}{2-z}, \frac{1}{1+z}, \frac{1}{z}, \frac{1}{\sqrt{4-z}\sqrt{z}} \right\}$$

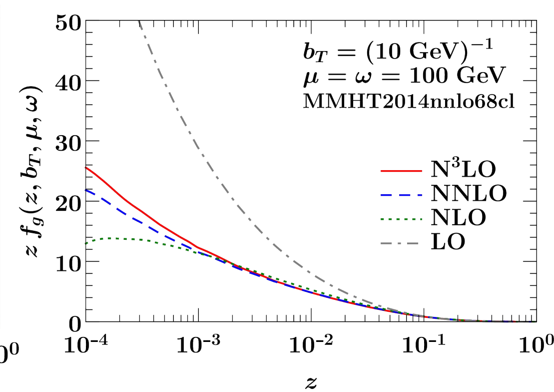




TMD Beam Function

- Quark result available in the literature [Luo, Yang, Zhu, Zhu 1912.05778]
  - discrepancy for one color structure in all quark-quark channels
  - Agreement for all the other terms
- Both quark and gluon matching kernels can be expressed in terms of only HPLs

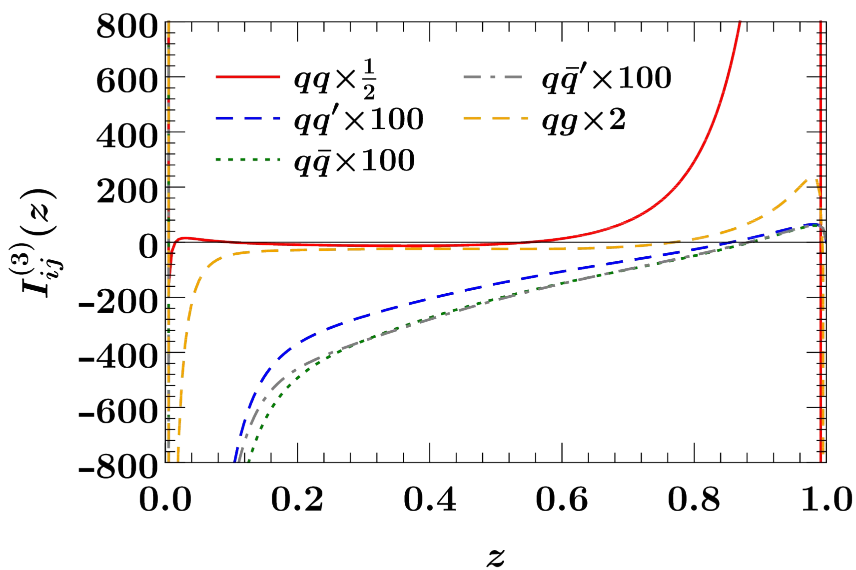


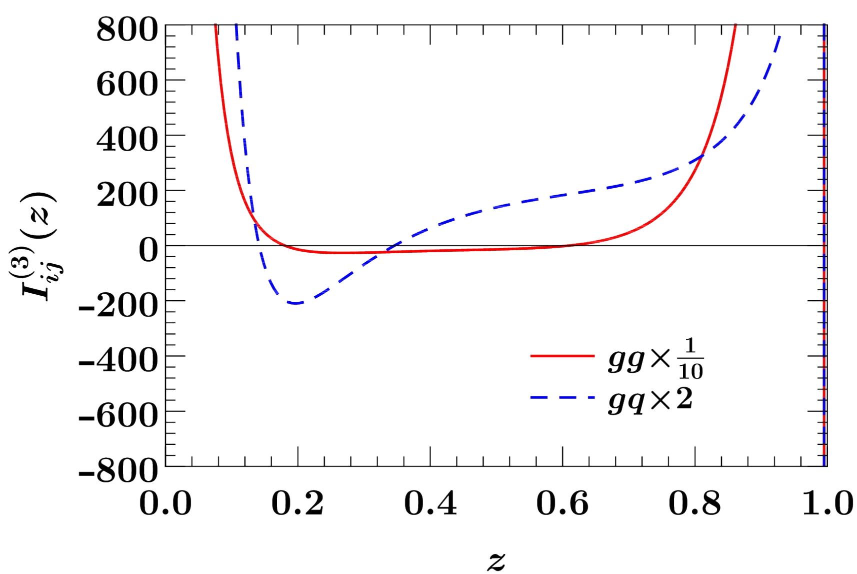


- N3LO corrections have non trivial z dependence
- Channels give roughly 0.5-2% correction on top of NNLO
- More complete analysis needed to assess impact of N3LO terms

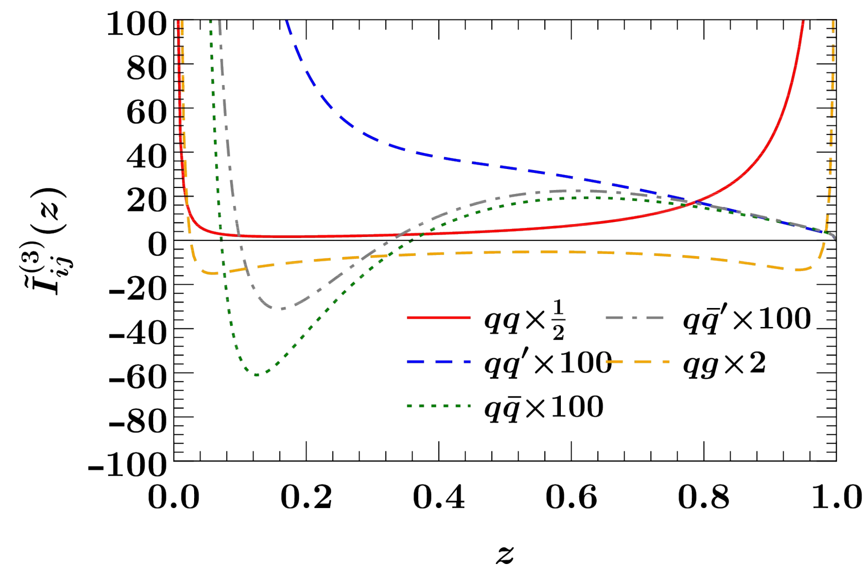
Beam Function Matching kernels

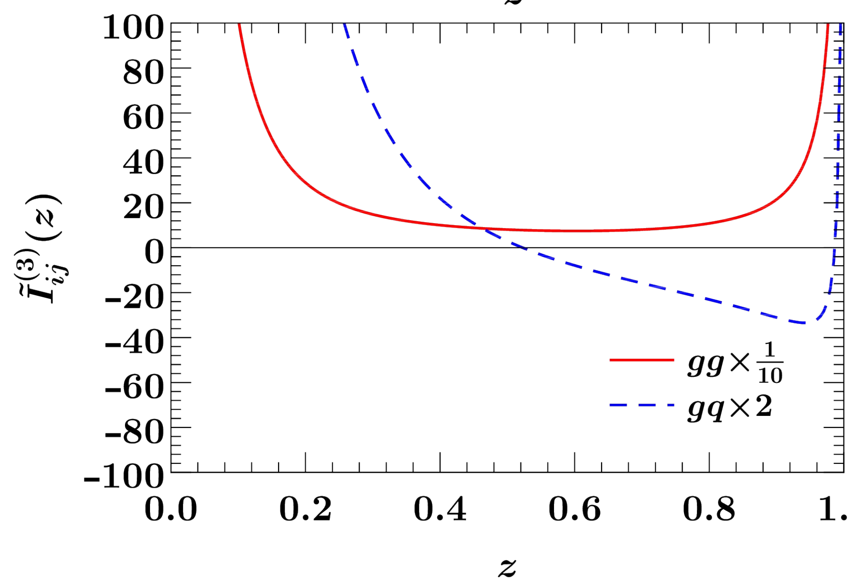
N-Jettiness Beam Function





TMD Beam Function



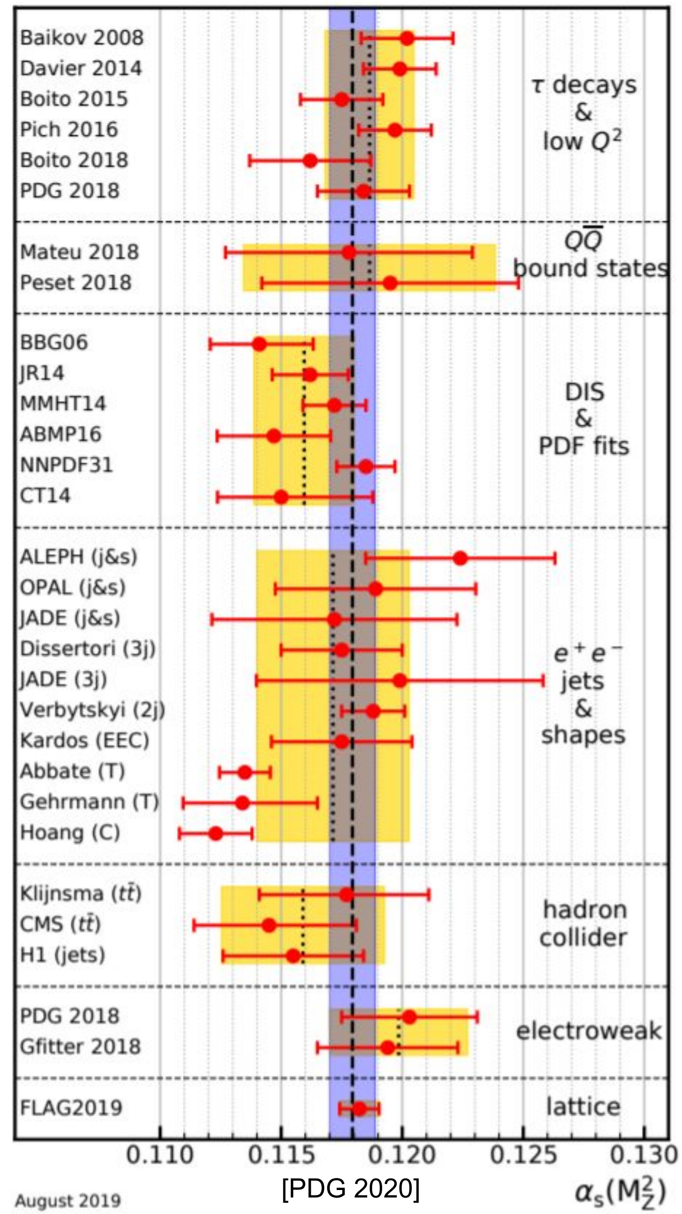


Strong coupling extraction

- Reference point for the value is conventionally chosen to be 91.2 GeV, the mass of the Z boson.
- Many different ways proposed, but most precise measurements come from:
  - PDF Fits
  - Event shapes
  - Lattice
- Precision is at the (half) **percent** level
  (to be compared to the 10^{-8} of the QED coupling)!
- World average combining everything is

$$\alpha_s(M_Z^2) = 0.1176 \pm 0.0011$$

[PDG 2020]



[PDG 2020]